Quantum aspects of the Liénard II equation and Jacobi's Last Multiplier-II

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Abstract

This is a continuation of the paper [J. Phys. A: Math. Theor. 46 (2013) 165202], in which we mapped the Liénard II equation $\ddot{x} + f(x)\dot{x}^2 + g(x) = 0$ to a position dependent mass system and carried out its quantization using the von Roos ordering technique. In this paper we present further results on the construction of three sets of exactly solvable potentials giving rise to bound-state solutions of the Schrödinger equation using the (exceptional) Jacobi polynomials.

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1 Introduction

There has been a lot of interest lately in PT-symmetric Hamiltonians as they provide examples of non-hermitian Hamiltonians with a real spectrum. Interest in quantization of dynamical systems especially dissipative ones has also grown over the last few years. In this context we consider the equation

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \tag{1.1}$$

which is called the Liénard II equation [1] since it involves a quadratic damping term in contrast to the usual Liénard equation, $\ddot{x} + f(x)\dot{x} + g(x) = 0$. Here the over dot denotes differentiation with respect to time t.

In the context of single particle dynamics an equation of the Liénard II type arises naturally whenever the mass is position dependent because assuming the linear momentum, $p = m(x)\dot{x}$, it follows that

$$\frac{dp}{dt} = m(x)\ddot{x} + m'(x)\dot{x}^2. \tag{1.2}$$

Consequently if we set the force to be proportional to $m(x)$, i.e., take $\mathcal{F}(x) = -m(x)g(x)$ one obtains from Newton's second law the equation of motion,

$$\ddot{x} + \frac{m'(x)}{m(x)}\dot{x}^2 + g(x) = 0. \tag{1.3}$$

This is obviously a special case of (1.1) with $f(x) = m'(x)/m(x)$. Besides the above, from a more mechanical perspective, such equations with a quadratic damping term often result from the movement of an object through a fluid medium as with an automobile pushing through air or a boat through water [2].

In a recent article [3] we have shown that one can associate with (1.1) a certain mass function which is essentially identical to the Jacobi's last multiplier (JLM) of the underlying dynamical system [4, 5, 6, 7, 8]. Since a knowledge of the JLM allows us to derive a Lagrangian for the system one is therefore able to construct a corresponding Hamiltonian using a Legendre transformation. Such a Hamiltonian was shown to quite generally involve a position-dependent mass (PDM).

The occurrence of such PDM terms is not completely unexpected for they have appeared in several nonlinear oscillators [9, 10] and in the PT-symmetric cubic anharmonic oscillator [11]. In recent years the study of the exact solutions of the position-dependent mass Schrödinger equation (PDMSE) (see for example [12, 13, 14, 15, 16, 17]) using the method of point canonical transformations [18, 19, 20, 21] or supersymmetric quantum mechanics [22] has gained a certain degree of importance owing to their relevance in diverse areas of physics ranging from quantum dots [23], quantum liquids [24], metal clusters [25], compositionally graded crystals [26] etc. thereby providing sufficient motivation for the study of the Liénard II equation from a quantum mechanical perspective.

The quantization of such PDM Hamiltonians is, however, beset with a number of problems foremost among which is the issue of ordering. In the coordinate representation the situation was resolved by Von Roos [27] who developed a novel scheme which we have put to use here. As for the issue of requiring PT-symmetry it is easy to show that the Liénard II equation respects this symmetry provided the functions $f(x)$ and $g(x)$ are odd.

Bhattacharjie and Sudarshan in [18] introduced a method for determining classes of potentials appearing in the Schrödinger equation whose solutions corresponded to the classical orthogonal polynomials. These polynomials are characterized as the polynomial solutions of a Sturm-Liouville problem in connection with the celebrated theorem of Bochner [28]. One of the major recent developments in quantum mechanics has been the construction of a class of exactly solvable potentials associated with the new family of exceptional orthogonal polynomials [29, 30, 31]. Gomez-Ullate *et al* [30] extended Bochner's result by discarding the assumption that the first element of the orthogonal polynomial sequence be a constant. In other words, in stark contrast to the families of classical orthogonal polynomials which start with a constant, the families of the exceptional orthogonal polynomials begin with some polynomial of degree greater than or equal to one, and yet still form complete, orthogonal sets with respect to some positive-definite measure.

Motivation, result and plan: Our primary motivation is to study the quantization of the Liénard II equation. In this context we note that exactly solvable potentials and their corresponding solutions in terms of the exceptional polynomials provide us with a useful technique for determining suitable forms of the functions $f(x)$ and $g(x)$ appearing in the Liénard II equation for which one can solve the Schrödinger equation and determine the eigenspectrum using the point canonical transformation method [18, 19]. The recent discovery of the exceptional classes of X_1-polynomials [29] for the Laguerre and Jacobi polynomial class has further enlarged the class of potentials and in turn the classes of functions $f(x)$ and $g(x)$ which may be tackled. Exceptional X_1 Laguerre or Jacobi type polynomials were shown to be the eigenfunctions of the rationally extended radial oscillator or Scarf I potentials by using the point canonical transformation method in [32] and by the methods of supersymmetric quantum mechanics in [33]. Construction of two distinct families of Laguerre and Jacobi type X_m exceptional orthogonal polynomials as eigenfunctions of infinitely many shape invariant potentials by deforming the radial oscillator, the hyperbolic (or trigonometric) Poschl Teller potentials and hyperbolic (or trigonometric) Scarf potentials was done in [34, 35, 36]. Recently Quesne [32] has constructed certain exactly solvable potentials giving rise to bound-state solutions to the Schrödinger equation, which are new and can be written in terms of the Jacobi-type X_1 exceptional orthogonal polynomials. The present paper is a continuation of our previous work [3], where we deduced eigenfunctions and the associated spectrum in terms of associated Laguerre and exceptional Laguerre polynomials. Here we show how the eigenfunctions and eigenspectrum of the quantum Liénard II can be obtained in terms of the exceptional Jacobi polynomials. In particular we extend the results obtained in [32] and derive three sets of exactly solvable one-dimensional quantum mechanical potentials. The organization of the paper is as follows. In Section 2 we briefly introduce the notion of Jacobi's last multiplier and use it to deduce the Hamiltonian of the Liénard II equation. We also consider the Schrödinger equation in the coordinate representation and combine the approaches of Von

Roos and Bhattacharjie and Sudarshan in this section. In Section 3 we derive new potentials using the X_1- Jacobi equation and compute their associated eigenfunctions and eigenvalues.

2 JLM and the Hamiltonian of the Liénard II equation

Given a second-order ordinary differential equation (ODE)

$$\ddot{x} = F(x, \dot{x}) \tag{2.1}$$

we define the Jacobi last multiplier M as a solution of the following ODE [8]

$$\frac{d \log M}{dt} + \frac{\partial F(x, \dot{x})}{\partial \dot{x}} = 0. \tag{2.2}$$

Assuming (2.1) to be derivable from the Euler-Lagrange equation one can show that the JLM is related to the Lagrangian by the following equation

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}. \tag{2.3}$$

In case of the Liénard II equation

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \tag{2.4}$$

one can show that the solution of the JLM is given by [37]

$$M(x) = e^{2F(x)}, \quad F(x) := \int^x f(s)ds. \tag{2.5}$$

Furthermore it follows from (2.3) that its Lagrangian is

$$L(x, \dot{x}) = \frac{1}{2} e^{2F(x)} \dot{x}^2 - V(x), \tag{2.6}$$

where the potential term

$$V(x) = \int^x e^{2F(s)} g(s) ds. \tag{2.7}$$

Clearly the conjugate momentum

$$p := \frac{\partial L}{\partial \dot{x}} = \dot{x} e^{2F(x)} \text{ implies } \dot{x} = p e^{-2F(x)}, \tag{2.8}$$

so that the final expression for the Hamiltonian is

$$H = \frac{p^2}{2M(x)} + \int^x M(s) g(s) ds, \tag{2.9}$$

where we have purposely written it in terms of the last multiplier $M(x)$ to highlight its role as a position dependent mass term.

2.1 Formulation of the method

Using the von Roos decomposition for position dependent mass (PDM) we write the Hamiltonian (2.9) as follows:

$$H(\hat{x},\hat{p}) = \frac{1}{4}[M^{\alpha}(\hat{x})\hat{p}M^{\beta}(\hat{x})\hat{p}M^{\gamma}(\hat{x}) + M^{\gamma}(\hat{x})\hat{p}M^{\beta}(\hat{x})\hat{p}M^{\alpha}(\hat{x})] + V(\hat{x}). \tag{2.10}$$

Here the parameters α, β and γ are required to satisfy the condition

$$\alpha + \beta + \gamma = -1 \tag{2.11}$$

in order to ensure dimensional correctness of the PDM term while the potential term is given by (2.7). Then in the coordinate representation with $\hat{p} = -i\hbar d/dx$ the Schrödinger equation

$$H\psi = E\psi \tag{2.12}$$

implies

$$(E - V(x))\psi(x) = -\frac{\hbar^2}{2M(x)}\left[\psi'' - \frac{M'}{M}\psi' + \frac{\beta+1}{2}\left(2\frac{M'^2}{M^2} - \frac{M''}{M}\right)\psi + \alpha(\alpha+\beta+1)\frac{M'^2}{M^2}\psi\right], \tag{2.13}$$

where the $'$ denotes differentiation with respect to the argument x. Using (2.5) the last equation has the appearance (setting $\hbar = 1$)

$$-2(E-V(x))\psi(x)e^{2F(x)} = \psi''(x) - 2f(x)\psi'(x) + [(\beta+1)(2f^2(x) - f'(x)) + 4\alpha(\alpha+\beta+1)f^2(x)]\psi(x). \tag{2.14}$$

Next using point canonical transformation we assume that the wave function $\psi(x)$ is of the form

$$\psi(x) = w(x)G(u(x)), \tag{2.15}$$

such that G satisfies the second-order ODE

$$\frac{d^2G}{du^2} + Q(u)\frac{dG}{du} + R(u)G(u) = 0. \tag{2.16}$$

Substituting (2.15) in (2.14) and comparing with (2.16) leads to the identifications

$$Q(u) = \frac{u''}{u'^2} + \frac{2w' - 2fw}{wu'}, \tag{2.17}$$

and

$$u'^2R(u) = 2(E-V)e^{2F(x)} + \frac{w'' - 2fw'}{w} + (\beta+1)(2f^2 - f') + 4\alpha(\alpha+\beta+1)f^2. \tag{2.18}$$

From (2.17) it follows that

$$w(x) = (u')^{-1/2}e^{F(x)}e^{\frac{1}{2}\int Q(u)du}. \tag{2.19}$$

A simple way of finding the unknown function $u(x)$ has been first proposed by Bhattacharjie and Sudarshan [18]. A particular choice of the special function $G(u)$ provides the complete functional forms of the first two unknowns $Q(u)$ and $R(u)$. A specific choice of the special function $G(u)$ and a clever choice of $u(x)$ make the Schrödinger equation an exactly solvable potential $V(x)$. Using the expression for $w(x)$ from (2.19) to simplify (2.18) we finally arrive at

$$2(E-V)e^{2F(x)} = \frac{u'''}{2u'} - \frac{3}{4}\left(\frac{u''}{u'}\right)^2 + u'^2\left[R(u) - \frac{1}{2}Q'(u) - \frac{Q^2}{4}\right] + \beta f' - (2\beta+1+4\alpha(\alpha+\beta+1))f^2. \tag{2.20}$$

This equation is central to our present analysis. It is clear that the choice of the second-order ODE in (2.16) must be such that its coefficients $R(u)$ and $Q(u)$, which appear in (2.20), together with their argument $u = u(x)$ cause the right hand side to have a term proportional to $e^{2F(x)}$; whose coefficient can then to identified with the energy eigenvalue occurring on the left. The remaining terms, depending on the variable x, can then be viewed as representing the potential function $V(x)$. In general this expression involves both the functions $f(x)$ and $g(x)$ as the latter occurs explicitly in the definition of the potential function (2.7). For an appropriate choice of $u(x)$ additional terms may arise from its derivatives which are proportional to f^2 and f' so that the expression for the potential, resulting from (2.20), can be simplified by making suitable choices for the parameters α, β and γ in order to ensure that the coefficients of f^2 and f' vanish. Note that the choice of the parameters α, β and γ, which are often called the ambiguity parameters, is not unique. Several possibilities have been explored in the literature [38, 39, 40, 41]. In our case however the choice of these parameters is dictated more by our endeavour to map the Liénard II equation to an exactly solvable quantum mechanical problem, in particular to an exactly solvable potential. Moreover as our motivation is based on exploiting an appropriate linear ordinary differential equation for this purpose the values of the ambiguity parameters follow as a consequence of this requirement instead of being assigned *ab intio*.

2.2 The linear harmonic oscillator and the Jacobi last multiplier

In order to illustrate these possibilities we consider the case when (2.16) is taken to be the Hermite differential equation

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2ny = 0, \quad n = 0, 1, 2, ... \tag{2.21}$$

so that

$$Q(u) = -2u, \quad R(u) = 2n, \;\; n \in \mathbb{N}_0. \tag{2.22}$$

If we now set

$$u' = e^{F(x)} \;\; \text{which implies} \;\; u(x) = \int^x e^{F(s)}ds, \tag{2.23}$$

it follows that

$$2(E-V)e^{2F(x)} = e^{2F(x)}[2n + 1 - u^2] + (\beta + \frac{1}{2})f' - (2\beta + \frac{5}{4} + 4\alpha(\alpha+\beta+1))f^2. \tag{2.24}$$

Choosing the coefficients of f' and f^2 to be zero yields the following values of the ambiguity parameters:

$$\alpha = -\frac{1}{4}, \quad \beta = -\frac{1}{2}, \quad \gamma = -\frac{1}{4}. \tag{2.25}$$

Next equating the constant terms in (2.25) implies that the energy eigenvalues are given by

$$E_n = \left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, ... \tag{2.26}$$

It remains to obtain the expression for the potential which is given by

$$V(x) = \frac{1}{2}u^2 = \frac{1}{2}\left(\int^x \sqrt{M(s)}ds\right)^2. \tag{2.27}$$

Each integrable function $M(x) \geq 0$ defines a point canonical transformation from the variable x onto a new variable u by the formula

$$u = q(x) := \int \sqrt{M(x)}dx. \tag{2.28}$$

The inverse transformation is also well defined. Thus one essentially has a quadratic potential in terms of the new coordinate as is expected for a harmonic oscillator potential. The choice of $M(x) = \lambda u'^2$ has been used in various places (for example, [13, 15]) where λ is a constant parameter.

The new coordinate there naturally fixes the nature of the function $g(x)$ because from (2.7) we have

$$V(x) = \int M(x)g(x)dx = \frac{1}{2}(\int \sqrt{M(x)}dx)^2$$

which determine

$$g(x) = \frac{1}{\sqrt{M(x)}} \int \sqrt{M(x)}dx, \tag{2.29}$$

entirely in terms of the JLM or PDM term $M(x)$.

3 The exceptional Jacobi equation

The X_1-Jacobi polynomials $\hat{P}_n^{(a,b)}(x)$, with $n = 1, 2, ...$ and $a, b > -1$ $(a \neq b)$ are the solutions of the second-order ODE

$$\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

with

$$Q(x) = -\frac{(b + a + 2)x - (b - a)}{1 - x^2} - \frac{2(b - a)}{(b - a)x - (b + a)}, \tag{3.1}$$

$$R(x) = -\frac{(b-a)x-(n-1)(n+b+a)}{1-x^2} - \frac{(b-a)^2}{(b-a)x-(b+a)}. \tag{3.2}$$

In order to make use of (2.20) we note that

$$X := R(u) - \frac{1}{2}Q' - \frac{1}{4}Q^2(u) = \frac{Cu+D}{1-u^2} + \frac{Gu+J}{(1-u^2)^2} + \frac{K}{(b-a)u-(b+a)} + \frac{L}{[(b-a)u-(b+a)]^2} \tag{3.3}$$

with [32]

$$C = \frac{(b-a)(b+a)}{2ab}, \tag{3.4}$$

$$D = n^2 + (b+a-1)n + \frac{1}{4}[(b+a)^2 - 2(b+a) - 4] + \frac{b^2+a^2}{2ab}, \tag{3.5}$$

$$G = \frac{(b-a)(b+a)}{2}, \quad J = -\frac{1}{2}(b^2+a^2-2), \tag{3.6}$$

$$K = \frac{(b-a)^2(b+a)}{2ab}, \quad L = -2(b-a)^2. \tag{3.7}$$

We shall now explore certain possibilities.

Case I: The Scarf-I potential

In the first case we propose to derive or rather reproduce the previous result of the Scarf-I potential [32]. To this end we set

$$u'e^{-F(x)} = \lambda\sqrt{1-u^2}, \quad \lambda = \text{const}. \tag{3.8}$$

then it immediately follows that

$$u(x) = \sin\theta, \;\; \theta := (\lambda \int^x e^{F(s)} ds). \tag{3.9}$$

As a result (2.20) becomes

$$2(E - V(x)) = \frac{\lambda^2}{4} - \frac{3\lambda^2}{4}\sec^2\theta + e^{-2F(x)}\left[(\beta+\frac{1}{2})f' - (2\beta + \frac{5}{4} + 4\alpha(\alpha+\beta+1))f^2\right]$$

$$+\lambda^2\left[Cu + D + \frac{Gu+J}{1-u^2}\right] + \lambda^2(1-u^2)\left[\frac{K}{(b-a)u-(b+a)} + \frac{L}{[(b-a)u-(b+a)]^2}\right]. \tag{3.10}$$

In order to simplify this expression we may set the coefficients of f' and f^2 to be zero which once again yield the values given in (2.25) for the ambiguity parameters. Next equating the coefficients of the constant terms on both sides we find after defining the change of parameters:

$$a = A - B - \frac{1}{2}, \;\; b = A + B - \frac{1}{2} \;\; \Rightarrow \;\; A = \frac{1}{2}(b+a+1), \;\; B = \frac{1}{2}(b-a),$$

that the eigenvalue may be expressed as

$$2E = \lambda^2(n-1+A)^2, \quad n = 1, 2, 3, ... \tag{3.11}$$

which upon scaling, ($\lambda^2 = 2$), can be simply written as

$$E_\nu = (\nu + A)^2, \quad \nu = 0, 1, 2, ...$$

On the other hand the potential $V(x)$ can be expressed as

$$V(x) = V_1(x) + V_2(x)$$

where

$$V_1(x) = [A(A-1) + B^2] \sec^2\theta - B(2A-1) \sec\theta \tan\theta, \tag{3.12}$$

$$V_2(x) = \frac{2(2A-1)}{[2B\sin\theta - (2A-1)]} + \frac{2[(2A-1)^2 - 4B(2A-1)\sin\theta + 4B^2]}{[2B\sin\theta - (2A-1)]^2}. \tag{3.13}$$

The potential $V_1(x)$ represents the Scarf-I potential with the value of θ usually restricted to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Here however as, $\theta = 2\int^x e^{F(s)} ds$, we have

$$0 < \int^x e^{F(s)} ds < \frac{\pi}{4}.$$

Notice that as, $\theta \to \pm\frac{\pi}{2}$, the second potential i.e., $V_2(x)$ approaches a constant value so that overall the potential $V(x)$ behaves like a Scarf-I potential. The explicit form of the wave function follows from (2.15) and (2.19) and is given by

$$\psi_\nu(x) = N_\nu \frac{e^{F(x)/2}}{\sqrt{2}} \frac{(1-\sin\theta)^{\frac{A-B}{2}} (1+\sin\theta)^{\frac{A+B}{2}}}{[2B\sin\theta - (2A-1)]} P_{\nu+1}^{(A-B-\frac{1}{2}, A+B-\frac{1}{2})}(\sin\theta). \tag{3.14}$$

Up to this point we have basically reproduced the essential results of [32] regarding the exceptional Jacobi polyomials and it follows that the normalization factor is given, in this case, by

$$\frac{N_\nu}{2} = \frac{B}{2^{A-2}} \left(\frac{\nu!(2\nu + 2A)\Gamma(\nu + 2A)}{(\nu + A - B + \frac{1}{2})((\nu + A + B + \frac{1}{2})\Gamma((\nu + A - B + \frac{1}{2})\Gamma((\nu + A + B + \frac{1}{2})} \right)^{1/2}. \tag{3.15}$$

Case II: A new potential

A second possibility follows from the choice

$$u' e^{-F(x)} = \lambda(1 - u^2), \tag{3.16}$$

which gives

$$u(x) = \tanh\theta(x), \tag{3.17}$$

where, as before, $\theta(x) = \lambda \int^x e^{F(s) ds}$. It now follows from (2.20) that

$$2(E - V(x)) = e^{-2F(x)} \left[-\lambda^2 e^{2F(x)} + (\beta + \frac{1}{2}) f' - (2\beta + \frac{5}{4} + 4\alpha(\alpha + \beta + 1)) f^2 \right]$$

$$+\lambda^2\left[(Cu+D)(1-u^2)+(Gu+J)+\frac{K(1-u^2)^2}{(b-a)u-(b+a)}+\frac{L(1-u^2)^2}{[(b-a)u-(b+a)]^2}\right]. \quad (3.18)$$

The choice $\beta=-1/2, \alpha=\gamma=-1/4$ causes the coefficients of f' and f^2 to vanish and upon equating the coefficient of the constant term (after setting $\lambda^2=2$) we obtain the energy eigenvalue as

$$E_\nu(A,\delta)=\left[(\nu+A)^2-\frac{1}{4}\right]-\left(1+\frac{1}{\delta^2}\right)\left(A-\frac{1}{2}\right)^2+2(1-2\delta^2), \quad \nu=0,1,2... \quad (3.19)$$

In arriving at this expression we have made use of the definitions (3.4)-(3.7) and have redefined the parameters a and b by the following

$$A=\frac{1}{2}(b+a+1), \quad \delta=\frac{b+a}{b-a}, \quad \nu=n-1.$$

As $a\neq b$ therefore the presence of the factor $(b-a)$ in the denominators is not a cause of undue concern. Unlike the previous case we note here the presence of both the parameters a and b (or alternatively A and δ) in the expression for the energy eigenvalue. The explicit form of the corresponding potential is

$$V(x)=U_1(x)+U_2(x)$$

where

$$U_1(x)=\left[(\nu+A)^2-\frac{1}{4}\right]\tanh^2\theta+\left[\frac{1}{2\delta}(2A-1)^2-2\delta\right]\tanh\theta \quad (3.20)$$

$$U_2(x)=\frac{6\delta(1-\delta^2)}{(\tanh\theta-\delta)}-\frac{2(1-\delta^2)^2}{(\tanh\theta-\delta)^2}. \quad (3.21)$$

The corresponding wave function is

$$\psi_\nu(x)=N_\nu\frac{e^{F(x)/2}}{2^{1/4}}\frac{(1+\tanh\theta)^{\frac{1}{2}(A+B-\frac{1}{2})}(1-\tanh\theta)^{\frac{1}{2}(A-B-\frac{1}{2})}}{[2B\tanh\theta-(2A-1)]}P_{\nu+1}^{(A-B-\frac{1}{2},A+B-\frac{1}{2})}(\tanh\theta), \quad (3.22)$$

whence the normalization factor follows from the requirement

$$\frac{|N_\nu|^2}{2}\int dy\frac{(1-\sin y)^{A-B-1}(1+\sin y)^{A+B-1}}{[2B\sin y-(2A-1)]^2}\left[P_{\nu+1}^{(A-B-\frac{1}{2},A+B-\frac{1}{2})}(\sin y)\right]^2=1, \quad (3.23)$$

where we have made the change of variables, $\tanh\theta=\sin y$, keeping in mind that $\theta(x)=\sqrt{2}\int^x e^{F(s)}ds$.

Case III: Another new type of potential
A third possibility consists in setting

$$u'(x)=\lambda e^{F(x)}[(b-a)u-(b+a)], \quad (3.24)$$

which implies $u(x) = \delta + e^{\theta}$ where $\delta = (b+a)/(b-a)$ and $\theta = \lambda(b-a)\int^x e^{F(s)}ds$. Upon substitution in (2.20) and using (3.3) the energy eigenvalue after simplification has the following form

$$\frac{2E}{\lambda^2} = -(D+\frac{1}{4})(b-a)^2 + 2C(b^2-a^2) + L - K(b+a).$$

Using the values of the constants C, D, L and K as stated in (3.4)-(3.7) to simplify this expression we obtain finally

$$E_\nu = -\frac{\lambda^2}{2}(b-a)^2(\nu+A)^2, \quad \nu = 0, 1, 2, ... \tag{3.25}$$

where as before $A = (b+a+1)/2$. On the other hand from the non constant terms it follows that the potential is a rational function of $u(x)$ and is given by

$$V(x) = -\frac{\lambda^2(b-a)^2}{2(1-u^2)^2}\left[B_0 + B_1 u + B - 2u^2 + B_3 u^3\right], \tag{3.26}$$

with

$$B_0 = J\delta^2 + w_2, \quad B_1 = G\delta^2 - 2\delta J + w_1, \tag{3.27}$$

$$B_2 = J - w_2 - 2\delta G, \quad B_3 = G - w_1, \tag{3.28}$$

$$w_1 = C(\delta^2+1) - 2\delta D, \quad w_2 = D(\delta^2+1) - 2\delta C. \tag{3.29}$$

Once again the values of the constants appearing in the above equations are explicitly given in terms of the parameters a and b by eqns.(3.4)-(3.7).

4 Conclusion and outlook

An alert reader would have noticed the inherent flexibility existing in our approach for the construction of potentials employing Jacobi's last multiplier. At the same time the exceptional orthogonal polynomials have allowed us to compute the eigenspectrum given the functions $f(x)$ and $g(x)$ of the Liénard equation. Thus this paper serves two purpose: firstly, to unveil the contribution of the Jacobi last multiplier to the study of exactly solvable position-dependent mass models, and secondly, to describe a procedure for quantization of the Liénard type II equation. With regard to the first we have shown here how the JLM can be used to express the Hamiltonian of the Liénard II as an exactly solvable position-dependent mass system. This leads us naturally to our second goal in which we have proposed an adaption of the techniques for exactly solvable systems to quantize the Liénard type II equation. The eigenfunctions being obtained in terms of (exceptional) Jacobi polynomials.

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