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Abstract

A regularization procedure developed in [1] for integral curvature invariants on manifolds with conical singularities is generalized to the case of squashed cones. In general, the squashed conical singularities do not have rotational $O(2)$ symmetry in a subspace orthogonal to a singular surface Σ so that the surface is allowed to have extrinsic curvatures. A new feature of squashed conical singularities is that the surface terms in the integral invariants, in the limit of small angle deficit, now depend also on the extrinsic curvatures of Σ. A case of invariants which are quadratic polynomials of the Riemann curvature is elaborated in different dimensions and applied to several problems related to entanglement entropy. The results are in complete agreement with computations of the logarithmic terms in entanglement entropy of 4D conformal theories [2]. Among other applications of the suggested method are logarithmic terms in entanglement entropy of non-conformal theories and a holographic formula of entanglement entropy for theories with gravity duals.

Contents

1 Introduction and main results

The fact that the curvature of manifolds with conical singularities has a distributional nature is well known since the work by Sokolov and Starobinsky in 1977 [3] who studied a spacetime around a straight cosmic string. Later, this result was extended to certain invariant curvature polynomials such as the Euler number and the Lovelock gravity [4]. A completely general description of delta-function terms in the curvature has been developed in [1]. The idea here was to employ a regularization procedure which treats a conical space as limit of a sequence of regular manifolds.

Let $2\pi\alpha$ be a length of a unit radius circle around a tip of a conical space $\mathcal{C}_\alpha$. Near a conical singularity a manifold $\mathcal{M}$ has locally the structure of a direct product $\mathcal{C}_\alpha \times \Sigma$. We call codimension 2 hypersurface Σ a singular surface. As was shown in [1] for integrals of some power of curvature tensor, if α is close to 1, the leading terms proportional to $(1-\alpha)$ do not depend on the regularization procedure while terms proportional to $(1-\alpha)^k$, $k \geq 2$, depend on the regularization and are ill defined. The analysis of [1] was restricted by an assumption about a Killing vector field (and a corresponding $O(2)$ isometry) for which the conical singularities are fixed points. This condition implies that Σ is embedded in $\mathcal{M}$ with vanishing extrinsic curvatures. The assumption about the Killing field was related to applications of the conical singularity method to (classical and quantum) entropy of black holes, see [5]. In this case Σ is an analog of the bifurcation surface of Killing horizons of a stationary black hole which a set of fixed points for an Abelian group of isometry.

In more general applications, there is no $O(2)$ isometry group for which the singular surface Σ is a fixed point set (although, as we will see, Σ is a fixed set for a discrete group and in some cases Σ may be also interpreted as a bifurcation surface for event past and future horizons). In computations of entanglement entropy in field theories by a replica method Σ has a meaning of an arbitrary entangling surface embedded in a simplest case in a flat spacetime. Indeed, choosing appropriate polar-like coordinates near the surface the Minkowski spacetime metric can be written in the form

$$ds^2 = dr^2 + r^2 d\tau^2 + (a + r\cos\tau)^2 d\varphi^2 + dz^2 \,, \tag{1.1}$$

if Σ is a cylinder of radius a, and

$$ds^2 = dr^2 + r^2 d\tau^2 + (a + r\cos\tau)^2 (d\theta^2 + \sin^2\theta d\varphi^2) \,, \tag{1.2}$$

if Σ is a sphere of radius a. In the both cases the position of Σ is at $r = 0$. Suppose that τ ranges from 0 to $2\pi n$, where $n = 2, 3, ...$ A physical motivation for this step is described in Sec. 2. As a mathematical consequence one can easily see that the geometry acquires conical singularities located at $r = 0$ with a usual structure $\mathcal{C}_n \times \Sigma$. However, here this structure is local and is not extended to a global product of two spaces. An important feature which should be emphasized is that instead of a continuous $O(2)$ symmetry of a conical space one has a discrete group of transformations $\tau \to \tau + 2\pi k$. We call conical singularities of this type *squashed cones* by following a terminology first used by J. Dowker in [6].

The aim of this paper is to develop a method of calculating integrals of plynomial curvature invariants in case of squashed conical singularities. We do it by purely geometric methods similar (but not equivalent) to those used in [1]. First of all, we test the new method for quadratic polynomials in different dimensions. This allows us to set the stage for future extensions.

To summarize our main results we consider a set of orbifold constructions $\mathcal{M}_n$ made by gluing n identical replicas obtained by cutting a Riemannian manifold $\mathcal{M}$. The cuts are made along a codimension 1 hypersurface in $\mathcal{M}$ which ends on a codimension 2 hypersurface Σ. When n replicas are glued together in $\mathcal{M}_n$ one gets conical singularities on Σ with the local structure $\mathcal{C}_n \times \Sigma$. We suppose that in integrals on regularized manifolds $\tilde{\mathcal{M}}_n$ there is a way to go to continuous values of n and consider the limit $n \to 1$.

In this limit we first extend application of the known integral formula

$$\int_{\tilde{\mathcal{M}}_n} \sqrt{g} d^d x \; R \to n \int_{\mathcal{M}} \sqrt{g} d^d x \; R + 4\pi(1-n) A(\Sigma) + \dots . \tag{1.3}$$

where $A(\Sigma)$ is the area of Σ. The difference with $O(2)$ symmetric conical singularities is in appearance of regularization dependent $O((1-n)^2)$ terms denoted by dotes in the r.h.s. of (1.3). In two dimensions, $d = 2$, the $O((1-n)^2)$ terms are absent and the formula is exact.

To present more non-trivial results we define two extrinsic curvature tensors of Σ

$$k^{(i)}_{\mu\nu} = h^{\lambda}_{\mu} h^{\rho}_{\nu} (n_i)_{\lambda;\rho} \quad , \quad h^{\lambda}_{\mu} = \delta^{\lambda}_{\mu} - \sum_i (n_i)_\mu (n_i)^\lambda \quad ,$$

where n_i are two unit mutually orthogonal normal vectors to Σ. It is convenient to introduce two invariants

$$\text{Tr } k^2 = \sum_i (k^{(i)})_{\mu\nu} (k^{(i)})^{\mu\nu} \quad , \quad k^2 = \sum_i (\text{Tr } k^{(i)})^2 \quad ,$$

which do not depend on the particular choice of the pair of normals n_i. Then in the limit $n \to 1$ we have the following behaviour of the regularized integrals:

$$\int_{\tilde{\mathcal{M}}_n} \sqrt{g} d^d x \; R^2 \to n \int_{\mathcal{M}} \sqrt{g} d^d x \; R^2 + 8\pi(1-n) \int_{\Sigma} \sqrt{\gamma} d^{d-2} y \; R + \dots \quad , \tag{1.4}$$

$$\int_{\tilde{\mathcal{M}}_n} \sqrt{g} d^d x \; R^2_{\mu\nu} \to n \int_{\mathcal{M}} \sqrt{g} d^d x \; R^2_{\mu\nu} + 4\pi(1-n) \int_{\Sigma} \sqrt{\gamma} d^{d-2} y \; \left(R_{ii} - \frac{1}{2} k^2 \right) + \dots \quad , \tag{1.5}$$

$$\int_{\tilde{\mathcal{M}}_n} \sqrt{g} d^d x \; R^2_{\mu\nu\alpha\beta} \to n \int_{\mathcal{M}} \sqrt{g} d^d x \; R^2_{\mu\nu\alpha\beta} + 8\pi(1-n) \int_{\Sigma} \sqrt{\gamma} d^{d-2} y \; (R_{ijij} - \text{Tr } k^2) + \dots \tag{1.6}$$

The dotes in the r.h.s. of (1.4)-(1.6) represent regularization dependent $O((1-n)^2)$ terms. Quantities R_{ijij} and R_{ii} have been introduced in [1] and are invariant projections on a subspace orthogonal to Σ of the corresponding components of the Riemann tensor. If Σ has zero extrinsic curvatures Eqs. (1.4)-(1.6) reduce to formulas obtained in [1].

One of main applications of our results is the calculation of entanglement entropy, see [5] for reviews. In particular, the integrals of the Euler density and of the Weyl squared appear in the logarithmic terms in the entropy. The corresponding contributions on Σ and their dependence on the extrinsic curvature have been found in [2] using the conformal symmetry and the holography. It was one of motivations for the present work to derive these surface terms in a purely geometric way thus stressing their uniqueness and universality.

The rest of the paper is organized as follows. In Section 2 we describe physical motivations for orbifold geometries (1.1), (1.2), since these two examples play a crucial role in the subsequent analysis. The main idea is that cylindrical and spherical singular surfaces may be viewed as entangling surfaces and as bifurcation surfaces of past and future horizons for sets of specially chosen Rindler observers (although this is not the case of Killing horizons).

In Section 3 we describe our regularization method and present the calculation of integrals (1.4)-(1.6) over squashed cones, first for spaces (1.1), (1.2) and then for a general geometry. For dimensional reasons, there are only two terms which are quadratic in the extrinsic curvature, $\mathrm{Tr}k^2$ and $(\mathrm{Tr}k)^2$, that may contribute to the integral over Σ. Therefore, in order to fix unknown coefficients at these terms it is enough to do calculations for two typical cases (1.1) and (1.2). An important feature here is that a naive application of the regularization used in [1] produces a metric which is singular at $r=0$. This can be cured by introduction of an extra regularization parameter depending on n. We choose a simplest option and replacing factors $a+r\cos\tau$ in (1.1), (1.2) with $a+r^n\cos\tau$. In this Section we also check consistency of (1.4)-(1.6) by calculating in $d=4$ integrals of the Euler density and the square of the Weyl tensor.

Section 4 demonstrates a number of immediate applications of these results. It starts with discussion of properties of coefficients in heat trace asymptotics of Laplace operators on base manifolds with squashed conical singularities. We then discuss calculation of entanglement entropy by a replica method. We do this in terms of an effective action in the presence of squashed cones, and offer some general predictions for non-conformal field theories. We observe that there is no extrinsic curvature contribution to the integral of the square of the Ricci scalar, see (1.4). This is the only possibility in the context of the entanglement entropy calculation. It indicates that the logarithmic term in the entropy is the same for a minimal and for a conformally coupled scalar field. Indeed, the non-minimal coupling in the scalar field operator, $-\Box+\xi R$, is irrelevant as soon as spacetime is flat or Ricci flat and thus it should not appear in the logarithmic term in the entropy for these spaces. This also agrees with the direct numerical evaluations [7] of the logarithmic term available in the literature, which shows that the logarithmic term is indeed the same for the conformally coupled scalar field and for a minimal scalar field.

In subsection 4.5 we discuss a holographic formula for entanglement entropies in conformal field theories which allow a dual description in terms of AdS gravity one dimension higher. We make a proposal for the holographic formula when the gravity theory includes arbitrary terms quadratic in curvatures. This proposal extends the holographic formula beyond the Gauss-Bonnet AdS gravities discussed earlier.

Conclusions and perspectives are presented in Section 5. In Appendix we present calculations of the regularized integrals in 5 and 6 dimensions.

2 Motivations and a geometrical setup

2.1 Cylindrical and spherical Rindler horizons

One of physical applications of distributional properties of the Riemannian geometry in the presence of squashed conical singularities is related to the study of the entanglement of quantum correlations across surfaces of different shapes. Such surfaces are called entangling surfaces. In a framework of our discussion entangling surfaces correspond to singular surfaces, so we denote these surfaces by the same letter Σ. To avoid unnecessary definitions at this stage and to make a direct link with geometries (1.1), (1.2), we begin with particular case of entangling surfaces in Minkowski spacetime,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad , \tag{2.1}$$

which can be related to properties of event horizons for certain classes of Rindler observers. A Rindler observer moves in (2.1) along one of the axis with a constant acceleration vector square $w^2 = w_\mu w^\mu$. As is known, the Rindler observer perceives the Minkowski vacuum as a thermal bath with the Unruh temperature $T_U = w/(2\pi)$. These thermal properties result from an information loss behind the Rindler horizons. The trajectory of a Rindler observer moving along the x-coordinate is

$$x(\lambda) = w^{-1}\cosh\lambda w \quad , \quad t(\lambda) = w^{-1}\sinh\lambda w \quad , \tag{2.2}$$

where λ is a proper time and the acceleration is w.

A well known set of Rindler observers are those which *all* move in the same direction and make the so called Rindler frame of reference

$$ds^2 = -r^2 d\tau^2 + dr^2 + dy^2 + dz^2 \quad . \tag{2.3}$$

The transition from (2.1) to coordinates (2.3) is motivated by (2.2) and has the form

$$x = r\cosh\tau \quad , \quad t = r\sinh\tau \quad . \tag{2.4}$$

All observers which are at rest with respect to (2.3) are the Rindler observers. The future and past event horizons are null hyperplanes which intersect (or bifurcate) at a codimension 2 plane Σ with coordinates $x = 0$, $t = 0$. Σ divides the states of the theory defined on a constant time hypersurface $\mathcal{H}$ $(t = 0)$ on those which are located on the same side of the horizon ($\mathcal{H}_R$, $x > 0$) and can be measured and unobservable states on the opposite side ($\mathcal{H}_L$, $x < 0$). One denotes these states with letters 'R' and 'L', respectively. The trace of the Minkowski vacuum over the left (unobservable states) yields the reduced density matrix of the Rindler observers $\hat{\rho}_R = \text{Tr}_L\ |0\rangle\langle 0|$.

A replica method to study the quantum entanglement across Σ is based on calculation of quantities $\text{Tr}_R\hat{\rho}_R^n$ where n is a natural number n. By ignoring the technicalities we give a geometrical construction associated with $\text{Tr}\hat{\rho}_R^n$. In a path integral representation of the Minkowski vacuum $|0\rangle$ field configurations are set on a half of the Euclidean space

$$ds^2 = dt^2 + dx^2 + dy^2 + dz^2 \ \ , \tag{2.5}$$

below the hypersurface $t = 0$ (i.e. for the values $t < 0$). The density matrix $\hat{\rho}_R$ is obtained by gluing two identical half planes along their $\mathcal{H}_L$ parts. This yields a plane with a cut along $\mathcal{H}_R$. The arguments ϕ_+, ϕ_- in the matrix elements $\langle\phi_+|\hat{\rho}_R|\phi_-\rangle$ are defined on the 'upper' and 'lower' parts of the cut, which we denote as $\mathcal{H}_R^+$ and $\mathcal{H}_R^-$, respectively. The space for the product of n matrices $\hat{\rho}_R^n$ is obtained from n replicas by gluing $\mathcal{H}_R^-$ cut of k-th replica with $\mathcal{H}_R^+$ cut of $k+1$-th replica. To get the trace $\text{Tr}\hat{\rho}_R^n$ one glues together the remaining open ends. The corresponding space is nothing but a higher dimensional generalization of a Riemann surface which can easily be defined in the Euclidean Rindler coordinates

$$ds^2 = r^2 d\tau^2 + dr^2 + dy^2 + dz^2 \ \ , \tag{2.6}$$

where τ varies from 0 to $2\pi n$. We denote this space $\mathcal{M}_n$. There is a conical singularity at the Euclidean horizon $x = 0$ $(r = 0)$. More rigorous arguments can be given which show that $\text{Tr}\rho_R^n$ has a path integral representation where field configurations live on $\mathcal{M}_n$.

It is interesting to note that the planar Rindler horizon associated to the Rindler coordinates (2.3) allows us to make a straightforward generalization to curved horizon (entangling) surfaces in Minkowski spacetime. Consider, for example, a set of Rindler observers which move radially with respect to the axis $x = y = 0$ and have trajectories

$$\begin{aligned} &x(\lambda) = a + w^{-1}\cosh(\lambda w)\cos\varphi \ \ , \ \ y(\lambda) = a + w^{-1}\cosh(\lambda w)\sin\varphi\,, \\ &t(\lambda) = w^{-1}\sinh\lambda w \ \ , \ \ z = \text{const}\,, \end{aligned} \tag{2.7}$$

where a is constant and w is an acceleration. The corresponding coordinate transformations

$$x = a + r\cosh\tau\cos\varphi \ \ , \ \ y = a + r\cosh\tau\sin\varphi \ \ , \ \ t = r\sinh\tau \tag{2.8}$$

change (2.1) to

$$ds^2 = -r^2 d\tau^2 + dr^2 + (a + r\cosh\tau)^2 d\varphi^2 + dz^2 \quad . \tag{2.9}$$

It is easy to see that any observer who is at rest with respect to the coordinate ρ in (2.9) is a Rindler observer with the acceleration $w = 1/r$. We call (2.9) the *cylindrical Rindler coordinates*, the corresponding observers are called the *cylindrical Rindler observers* to distinguish them from standard (planar) Rindler observers. The coordinates (2.9) have a future event horizon which is topologically $\mathcal{C} \times R^1$, where $\mathcal{C}$ is a light cone with the apex at $t = -a$ that crosses $t = 0$ surface at a circle $x^2 + y^2 = a^2$. Signals sent from inside of the future event horizon reach none of the cylindrical Rindler observers. This follows from the fact that the Rindler horizon of each particular observer is tangent to the event horizon in the cylindrical coordinates. One can also define a past even horizon for this set of observes by reflection with respect to the surface $t = 0$. The bifurcation surface of these horizons, Σ is a cylinder.

One can introduce the reduced density matrix $\hat{\rho}$ of the cylindrical Rindler observers by taking the trace over degrees of freedom inside the cylinder Σ. If Σ is considered as an entangling surface the computations of quantities like $\mathrm{Tr}\hat{\rho}_R^n$ require the field theory to live on a Euclidean manifold $\mathcal{M}_n$ which is obtained (in a complete analogy with the case of planar Rindler coordinates (2.3)) by the Wick rotation in (2.9). This yields

$$ds^2 = r^2 d\tau^2 + dr^2 + (a + r\cos\tau)^2 d\varphi^2 + dz^2 \,, \tag{2.10}$$

where $0 \leq \tau < 2\pi n$ and the condition $0 < r \leq b < a$ is implied. Metric (2.10) coincides with (1.1) and it has a conical singularity on the cylinder $r = 0$ (i.e. on Σ).

On the other hand, one can also make the coordinate transformation

$$\begin{aligned} &x = a + r\cosh\tau\sin\theta\cos\varphi \,, \;\; y = a + r\cosh\tau\sin\theta\sin\varphi \,, \\ &z = a + r\cosh\tau\cos\theta \,, \;\; t = r\sinh\tau \,, \end{aligned} \tag{2.11}$$

which changes the metric (2.1) to the form

$$ds^2 = -r^2 d\tau^2 + dr^2 + (a + r\cosh\tau)^2 (d\theta^2 + \sin^2\theta d\varphi^2) \,. \tag{2.12}$$

These coordinates can be called *spherical Rindler coordinates* since the observers which are at rest with respect to (2.11) are the Rindler observes moving radially toward or out of the center $x = y = z = 0$. We call these observers the *spherical Rindler observers.* The future event horizon of spherical Rindler observers is a three-dimensional cone crossing the sphere $x^2 + y^2 + z^2 = a^2$ at $t = 0$ and having the apex at $t = -a$. The past event horizon is obtained by the reflection with respect to the plane $t = 0$. The bifurcation surface of these horizons, Σ, is a 2-sphere with the position at $t = 0$. Studying of the quantum entanglement across Σ with the help of the replica method leads to the Euclidean space (1.2).

In the standard Rindler metric (2.3) the coordinates belong to the class of Killing frames of reference. The reduced density matrix then allows a representation $\hat{\rho}_R \sim \exp(-\hat{H}_R/T_U)$ in terms of a local Rindler Hamiltonian $\hat{H}_R$ which generates translations along the Killing time τ. Because of this property one can consider non-integers powers $\hat{\rho}_R^\alpha \sim \exp(-\alpha H_R/T_U)$ which are described by the metric (2.6) where the period of τ is $2\pi\alpha$.

On the other hand, the metrics (2.9) and (2.12) (and their Euclidean counterparts (1.1) and (1.2)) are not static. Therefore, the symmetry under the time translation $t \to t + b$ is missing. However, in the Euclidean version of the metric (1.1) and (1.2) there exists a symmetry under the global translations, $\tau \to \tau + 2\pi k$, for any integer k. This is all we need in order to construct $\text{Tr}\hat{\rho}_R^n$. As a result of this feature only integer powers of the reduced density matrices of the cylindrical and spherical Rindler observers allow a geometrical representation. One can define these matrices in terms of a modular Hamiltonian $\hat{H}$ as $\rho = \exp(-2\pi\hat{H})$. However, the modular Hamiltonians for cylindrical and spherical Rindler observers are essentially non-local operators. That the metric is not static is closely related to the fact that the bifurcation surface Σ has the non-vanishing extrinsic curvatures. The latter property is in the main focus of our study here.

2.2 General definition of squashed cones

Metrics (2.9) and (2.12), where a complete rotation of τ is $2\pi n$, correspond to orbifolds $\mathcal{M}_n$ obtained by gluing n copies of the flat space with cuts. The cuts are made in $t = 0$ plane and end on Σ. Our aim now is to specify a general class of orbifolds $\mathcal{M}_n$ which we are dealing with. We start with spaces which appear when the replica procedure is applied to a static spacetime $\mathcal{M}$ with the metric

$$ds^2 = B(x)dt^2 + h_{ab}(x)dx^a dx^b \quad , \tag{2.13}$$

where $a, b = 1, .., d-1$. This case is most interesting from the point of view its applications. From now on we consider the Euclidean signature manifolds, so $B(x) > 0$. We choose an entangling surface Σ in a constant time section $\mathcal{H}$. The surface Σ is a co-dimension two surface which, in the case of a static spacetime, has only one non-vanishing extrinsic curvature. The extrinsic curvature for a normal vector directed along the Killing vector ∂_t is zero.

We suppose that Σ divides a constant t hypersurface $\mathcal{H}$ on two parts, say, $\mathcal{H}_L$ and $\mathcal{H}_R$, as in the case of the Rindler spacetimes. The construction of $\mathcal{M}_n$ requires, first, the preparation of a single replica by cutting $\mathcal{M}$ along $\mathcal{H}_L$ (or $\mathcal{H}_R$). Then n replicas are glued together along the cuts as was described above.

We would like to understand the properties of the metric of $\mathcal{M}_n$ near Σ in a suitably chosen coordinates and then generalize these properties beyond the class of static spacetimes. Consider in $\mathcal{H}$ the normal Riemann coordinates r, y^i $(i = 1,..,d-2)$ with the origin on Σ. One has

$$h_{ab}(x)dx^a dx^b = d\varrho^2 + (\gamma_{ij}(y) + 2\varrho k_{ij}(y) + O(\varrho^2))dy^i dy^j \quad , \tag{2.14}$$

Coordinate ϱ is the geodesic distance from a point on the hypersurface $\mathcal{H}$ to Σ. The element $\gamma_{ij}(y)dy^i dy^j$ is a metric on Σ. It is easy to show that $k_{ij}(y)$ is the extrinsic curvature tensor of Σ for the unit normal vector $n_a = \delta^r_a$. ($k_{ij}(y)$ is also an extrinsic curvature of Σ in $\mathcal{M}$.) It is convenient to introduce a coordinate $\zeta = \sqrt{B}t$,

$$d\zeta = \sqrt{B}dt + \zeta w_a dx^a \quad , \tag{2.15}$$

where $w_a = \frac{1}{2}\partial_a B/B$ are the components of the acceleration vector of the coordinate frame. Up to terms of the second order in ϱ and ζ metric (2.13) near Σ takes the form

$$ds^2 \simeq d\zeta^2 + d\varrho^2 + (\gamma_{ij}(y) + 2\varrho k_{ij}(y))dy^i dy^j - 2\zeta w_\varrho(y)d\zeta d\varrho - 2\zeta d\zeta w_i(y)dy^i \quad . \tag{2.16}$$

This asymptotic behaviour depends on the choice of coordinates. One can make an additional coordinate transformation

$$v^i = y^i - \frac{1}{2}\zeta^2 w^i(y) \quad , \quad \bar{\varrho} = \varrho - \frac{1}{2}\zeta^2 w_\varrho(y) \tag{2.17}$$

to bring (2.16) to a simpler form

$$ds^2 \simeq dx_1^2 + dx_2^2 + (\gamma_{ij}(v) + 2x_2\, k_{ij}(v))dv^i dv^j \,, \tag{2.18}$$

where we introduced $x_1 = \zeta$ and $x_2 = \bar{\varrho}$ and omitted the terms proportional to $\zeta^2, \bar{\varrho}^2, \bar{\varrho}\zeta$. The next coordinate transformation

$$x_1 = r\sin\tau \,,\ x_2 = r\cos\tau \tag{2.19}$$

brings (2.18) to the form

$$ds^2 \simeq r^2 d\tau^2 + dr^2 + (\gamma_{ij}(v) + 2r\cos\tau k_{ij}(v))dv^i dv^j \quad . \tag{2.20}$$

One can compare it with (1.1), (1.2) to see that these metrics are particular cases of (2.20).

In the case of a static spacetime the entangling surface Σ has a single non-vanishing extrinsic curvature. The generalization of (2.18) to surfaces with two non-trivial curvatures is straightforward

$$ds^2 \simeq dx_1^2 + dx_2^2 + (\gamma_{ij}(v) + 2x_p\, k^{(p)}_{ij}(v))dv^i dv^j \quad , \tag{2.21}$$

where $p = 1, 2,$ $k^{(p)}_{ij}$ are extrinsic curvatures of Σ for normals n_p $((n_p)_\mu = \delta^p_\mu)$. After the coordinate transformation (2.19) to the polar coordinate system this metric becomes

$$ds^2 \simeq r^2 d\tau^2 + dr^2 + \left(\gamma_{ij}(v) + 2r\cos\tau k^{(1)}_{ij}(v) + 2r\sin\tau k^{(2)}_{ij}(v)\right) dv^i dv^j \,, \tag{2.22}$$

which is a generalization of (2.20). The metrics of the type (2.22) were recently considered in [15] with a similar motivation to generalize the conical singularity method to the metrics which are not static.

3 Squashed cones, regularization and curvature invariants

3.1 Regularization of symmetric cones

Let us start with a brief review of the regularization method used in [1] in order to calculate the curvature invariants on a manifold with conical singularities with the rotational symmetry. Consider a static Euclidean metric of the following type

$$ds^2 = g(r)d\tau^2 + dr^2 + \gamma_{ij}(r,v)dv^i dv^j\ ,$$
$$g(r) = r^2 + O(r^4)\ , \quad \gamma_{ij}(r,v) = \gamma_{ij}(v) + O(r^2)\ . \tag{3.1}$$

By setting the period $2\pi n$ for the angular coordinate τ we get an orbifold $\mathcal{M}_n$ with conical singularities at $r=0$, the singular surface Σ is equipped with intrinsic coordinates $\{v^i\}$ and intrinsic metric $\gamma_{ij}(v)$. $\mathcal{M}_n$ can be considered as a limit in the sequence of regular manifolds $\tilde{\mathcal{M}}_n$ parametrized by a regularization parameter b. The family of regular metrics can be obtained by replacing in (3.1) the g_{rr} component to $\tilde{g}_{rr} = f_n(r,b)$, where the regularization function takes the form

$$f_n(r,b) = \frac{r^2 + b^2 n^2}{r^2 + b^2}\ . \tag{3.2}$$

Metric (3.1) is then recovered in the limit when $b \to 0$. The limit however should be taken in certain order. Consider in $\tilde{\mathcal{M}}_n$ a domain $\Omega(r_0)$ around Σ, defined as $0 < r < r_0$, and change the radial variable, $r = bx$. In terms of the new variable x the regularization function becomes independent of the parameter b, $f_n(x) = \frac{x^2+n^2}{x^2+1}$. During the limiting procedure $f_n(x)$ is fixed but integral curvature invariants depend on b. This allows one to consider asymptotic series with respect to the parameter b. The integral over the given domain of an invariant $\mathcal{R}$ which is an m order a polynomial of components of the Riemann tensor behaves, when $b \to 0$, as

$$\int_{\Omega(r_0)} \sqrt{g} d^d x\ \mathcal{R} = \int_0^{2\pi n} d\tau \int_0^{r_0/b} dx \int_\Sigma \sqrt{\gamma} d^{d-2}x\, \mathcal{R}$$
$$= \frac{A_k}{b^k} + \frac{A_{k-1}}{b^{k-1}} + .. + A_0 + O(b)\ . \tag{3.3}$$

In the first line in (3.3) the integration over x in this limit can be extended to infinity. The highest power k of the singular terms in these asymptotic series depends on the order m of the polynomial $\mathcal{R}$. Since spacetime is static integrals (3.3) can be considered at arbitrary values of n. If one takes the limit $n \to 1$, the terms A_k with $k > 0$ can be shown to be $O((1-n)^2)$, while A_0 is proportional $(1-n)$. In applications we are only interested in the A_0 terms. Since the size of the domain r_0 can be made arbitrary small, A_0 represents the contribution of the conical singularities and is determined by local (intrinsic and extrinsic) geometry on Σ.

3.2 Regularization of squashed cones

The same regularization procedure can be applied to metrics (1.1) and (1.2). However, if one only replaces the component g_{rr} with the regularization function $f_n(r,b)$, there appears a curvature singularity at $r=0$ in the regularized metric. It can be seen, for instance, from the behaviour of the Ricci scalar, which diverges as $1/r$. This feature is a manifestation of the fact that the squashed conical singularities are not a direct product of a two-dimensional cone and a surface Σ. These geometries near Σ have a structure of a warped product of the two spaces. In order to overcome the difficulty we have to introduce an additional regularization parameter p in the metric. This can be done by changing the power of r in the warped factor $(a+r\cos\tau)$ to $(a+r^p b^{1-p}\cos\tau)$. The singularity at $r=0$ in the regularized metric goes away if $p>2$. In the present paper we apply the regularization method to integrals quadratic in curvatures. Such integrals are regular if $p>1$.

The option we choose is not to introduce p as an independent parameter. It is enough to assume that it is a function $p(n)$ of n, such that $p(n)=n+O(n-1)^2$. The simplest choice which we use for the further computations is $p(n)=n$. Thus, regularized metrics (1.1), (1.2) look as

$$ds^2 = r^2 d\tau^2 + f_n(r,b)dr^2 + (a + r^n c^{1-n}\cos\tau)^2 d\varphi^2 + dz^2 \ \ , \tag{3.4}$$

$$ds^2 = r^2 d\tau^2 + f_n(r,b)dr^2 + (a + r^n c^{1-n}\cos\tau)^2 (d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{3.5}$$

where c is an irrelevant constant which will appear in the final result in the limit $n\to 1$.

A note of caution regarding the limit n to 1 should be added. Since regularized metrics (3.4), (3.5) depend explicitly on $\cos\tau$, they cannot be considered at arbitrary non-integer values of n. At non-integer n there is a jump in extrinsic curvatures on the hypersurfaces $\tau=0$ and $\tau=2\pi n$. Therefore, our prescription is to do first the computations for an integer n and then analytically continue n to 1. This is the essence of the replica method, as it is used in statistical physics. Since $n\geq 2$ for the orbifolds the regularization (3.4), (3.5) makes finite all integrals quadratic in curvatures.

3.3 Integrals for cylindrical and spherical singular surfaces

With these remarks the calculations are pretty straightforward although a bit tedious. For the integral of the Ricci scalar we do reproduce the known formula (1.3) earlier established in the case symmetric conical singularities. Below we summarize the results of the calculation for the quadratic combinations of curvature for regularized metrics (3.4), (3.5). When the singular surface is cylinder (metric (3.4)) we obtain in the limits described above

$$\begin{aligned}
&\int_{\tilde{\mathcal{M}}_n} \sqrt{g}d^4x\ R^2 \to O(n-1)^2 \\
&\int_{\tilde{\mathcal{M}}_n} \sqrt{g}d^4x\ R_{\mu\nu}R^{\mu\nu} \to 4\pi^2\frac{L}{a}(n-1) + O(n-1)^2\,, \\
&\int_{\tilde{\mathcal{M}}_n} \sqrt{g}d^4x\ R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \to 16\pi^2\frac{L}{a}(n-1) + O(n-1)^2\,,
\end{aligned} \tag{3.6}$$

where L is the length of cylinder in direction z. When Σ is a sphere (metric (3.5)) we find

$$\begin{aligned}
&\int_{\tilde{\mathcal{M}}_n} \sqrt{g}d^4x\ R^2 \to O(n-1)^2 \\
&\int_{\tilde{\mathcal{M}}_n} \sqrt{g}d^4x\ R_{\mu\nu}R^{\mu\nu} \to 32\pi^2(n-1) + O(n-1)^2\,, \\
&\int_{\tilde{\mathcal{M}}_n} \sqrt{g}d^4x\ R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \to 64\pi^2(n-1) + O(n-1)^2\,.
\end{aligned} \tag{3.7}$$

We note that terms $O(1-n)^2$ contain divergences when b is taken to zero. These terms are thus non-universal and depend on the regularization.

3.4 A general case of integrals quadratic in curvature

The above results for a sphere and a cylinder can be used to derive general formulas for the curvatures for conical singularities on orbifolds. One starts with the case of orbifolds which are locally flat. The key observation here is that there exist only two combinations of the extrinsic curvature which can contribute to the surface integrals, k^2 and $\mathrm{Tr}k^2$. Therefore, the integral of any combination $\mathcal{R}$ quadratic in Riemann curvature can be expressed as linear combination of integrals of these two invariants over Σ

$$\int_{\tilde{\mathcal{M}}_n} \sqrt{g}dx^4\ \mathcal{R} \to (n-1)\left(\alpha_1\int_\Sigma \sqrt{\gamma}d^2y\ k^2 + \alpha_2\int_\Sigma \sqrt{\gamma}d^2y\ \mathrm{Tr}\ k^2\right) + O(n-1)^2\,. \tag{3.8}$$

The unknown coefficients α_1 and α_2 can be determined by applying (3.8) to cases considered in sec. 3.3. When Σ is a cylinder we have

$$\int_\Sigma \sqrt{\gamma}d^2y\ k^2 = \int_\Sigma \sqrt{\gamma}d^2y\ \mathrm{Tr}\ k^2 = 2\pi\frac{L}{a}\,, \tag{3.9}$$

while for a sphere

$$\int_\Sigma \sqrt{\gamma}d^2y\ k^2 = 16\pi\,, \quad \int_\Sigma \sqrt{\gamma}d^2y\ \mathrm{Tr}\ k^2 = 8\pi\,. \tag{3.10}$$

If these results are used together with (3.6) and (3.7) we find for the squashed conical singularities in locally flat geometry the following result

$$\int_{\tilde{\mathcal{M}}_n} \sqrt{g} d^d x \; R^2 \to O(n-1)^2 \, ,$$
$$\int_{\tilde{\mathcal{M}}_n} \sqrt{g} d^d x \; R_{\mu\nu} R^{\mu\nu} \to 2\pi(n-1) \int_\Sigma \sqrt{\gamma} d^{d-2} y \; k^2 + O(n-1)^2 \, ,$$
$$\int_{\tilde{\mathcal{M}}_n} \sqrt{g} d^d x \; R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \to 8\pi(n-1) \int_\Sigma \sqrt{\gamma} d^{d-2} y \; \text{Tr } k^2 + O(n-1)^2 \, . \tag{3.11}$$

This result can be combined with the one obtained earlier in [1] in the case of the symmetric conical singularities, when the extrinsic curvatures are vanishing. We immediately arrive at formulas (1.4)-(1.6) announced in the Introduction. It should be noted that in (1.4)-(1.6) the cross-terms of the form $\int_\Sigma \mathcal{R}^m k^l$, where k is extrinsic curvature and $\mathcal{R}$ is the Riemann curvature, are not allowed by dimensional reasons. These terms may appear for integrals of cubic combinations of curvature and higher.

3.5 Topological and conformal invariants

In order to demonstrate that our results are robust, and as a consistency check, we apply obtained relations to the Euler characteristics and conformal invariants on manifolds with squashed conical singularities. Let us define

$$E_4 = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \quad , \tag{3.12}$$

$$W^2 = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \quad . \tag{3.13}$$

On a closed Riemannian 4-dimensional manifold the integral of E_4 yields the Euler characteristics. In four dimensions W^2 is the square of the Weyl tensor. Therefore, the integral of W^2 is invariant with respect to the local conformal transformations of the metric in four dimensions. Since quantities (3.12),(3.13) are quadratic polynomials we can define their integrals on $\mathcal{M}_n$ by applying (1.4)-(1.6).

The integral of the Euler density over manifold with a squashed conical singularity is

$$\int_{\tilde{\mathcal{M}}_n \to \mathcal{M}_n} \sqrt{g} d^4 x \; E_4 = n \int_{\mathcal{M}} \sqrt{g} d^4 x \; E_4 + 8\pi(1-n) \int_\Sigma \sqrt{\gamma} d^2 y \; R_\Sigma \, , \tag{3.14}$$

where R_Σ is intrinsic curvature at the 2-surface Σ. To get (3.14) we used the Gauss-Codazzi equations which imply that

$$R_\Sigma = R - 2R_{ii} + R_{ijij} + k^2 - \text{Tr } k^2 \quad . \tag{3.15}$$

For closed manifolds Eq. (3.14) can be written as

$$\chi_4[\mathcal{M}_n] = n\chi_4[\mathcal{M}] + (1-n)\chi_2[\Sigma] \ \ , \tag{3.16}$$

where $\chi_4[\mathcal{M}_n]$, $\chi_4[\mathcal{M}]$, $\chi_2[\Sigma]$ are the Euler characteristics of $\mathcal{M}_n$, $\mathcal{M}$ and Σ, respectively. Equation (3.16) coincides with corresponding relation for $O(2)$ symmetric conical singularities found in [1]. One can also use (3.16) for orbifolds which are locally flat. This requires adding corresponding boundary terms to integral (3.14) which make topological characteristics non-trivial even when bulk curvatures vanish.

Consider now the integral of the square of the Weyl tensor. In the limit $n \to 1$ we find

$$\int_{\tilde{\mathcal{M}}_n} \sqrt{g} d^4x \ W^2 \to n \int_{\mathcal{M}} \sqrt{g} d^4x \ W^2 + 8\pi(1-n) \int_{\Sigma} \sqrt{\gamma} d^2y \ K_\Sigma + ... \, , \tag{3.17}$$

where we introduced a conformal invariant

$$K_\Sigma = R_{ijij} - R_{ii} + \frac{1}{3}R - \left(\text{Tr } k^2 - \frac{1}{2}k^2\right) \, . \tag{3.18}$$

The conformal invariance of the l.h.s. of (3.17) implies that the term proportional to $(n-1)$ in the r.h.s is invariant as well (the bulk term in (3.17) is a conformal invariant). To see the conformal invariance of the terms on Σ we note that the combination $R_{ijij} - R_{ii} + R/3$ is expressed as a normal projection of the Weyl tensor. One can also show that the integral of $(\text{Tr } k^2 - k^2/2)$ yields another conformal invariant.

Both the Euler number and the Weyl tensor squared appear in the logarithmic terms in entanglement entropy of a four-dimensional conformal field theory. The respective surface contributions (3.14) and (3.17) have been obtained in [2] by using the arguments that involve the holography. Later, for simple geometries in Minkowski spacetime, it was confirmed both by numerical calculations (for a sphere and a cylinder) [7] and by an analytic analysis (for a sphere) [8]. We thus confirm this analysis by a direct purely geometric computation which does not involve any extra assumptions.

3.6 Integrals in higher dimensions

In higher dimensions, again by the dimensional reasons, there are still only two possible terms, $\text{Tr}k^2$ and k^2, constructed from the extrinsic curvature that may contribute to the integrals of quadratic combinations of the Riemann curvature. Thus, the general structure (3.6) is valid in a higher dimension d although the exact coefficients may depend on dimension d. These coefficients again can be determined by doing the computation for two test surfaces, S^{d-2} and $S^1 \times R^{d-3}$. We have done these calculations (see details in the Appendix) and we have checked that in dimensions $d = 5$ and $d = 6$ the formulas (3.11) are not changed. We conclude that, likely, the coefficients in (3.11) do not dependent on d. This is similar to what we had in the case of Σ with an $O(2)$ symmetry when the terms are universal. The exceptional case is $d=3$ when Σ is a curve for which the extrinsic curvature is not defined.

4 Applications

4.1 Heat kernel coefficients for squashed cones

With respect to the reduced density matrix $\hat{\rho}_R$ one can define an effective action $W(n) = -\ln \mathrm{Tr}\hat{\rho}_R^n$. For a free field with a wave operator $\hat{\mathcal{D}}$

$$W(n) = -\frac{1}{2}\int_{\epsilon^2}^{\infty}\frac{ds}{s}\mathrm{Tr}\; e^{-s\hat{\mathcal{D}}}\,, \tag{4.1}$$

where ϵ is a UV cut-off. The effective action here is defined on an orbifold $\mathcal{M}_n$.

The operator $\hat{\mathcal{D}}$ is of a Laplace type and one expects that it is well defined on a Hilbert space of fields on $\mathcal{M}_n$ so that the heat trace of this operator has a standard asymptotic behaviour

$$\mathrm{Tr}\; e^{-s\hat{\mathcal{D}}} \sim \frac{1}{(4\pi s)^{d/2}}\sum_{p=0} A_p(\hat{\mathcal{D}})\; s^p \tag{4.2}$$

at small s. Here $A_p(\hat{\mathcal{D}}) = A_p(n)$ are the heat coefficients which are given by integrals of local invariant structures on $\mathcal{M}_n$. In the presence of conical singularities the heat coefficient are represented as

$$A_p(n) = A_p^{(\mathrm{reg})}(n) + A_p^{(\mathrm{surf})}(n) \quad , \tag{4.3}$$

where $A_p^{(\mathrm{reg})}(n)$ is given by an integral over a regular domain of $\mathcal{M}_n$, and $A_p^{(\mathrm{surf})}(n)$ is a contribution from the conical singularities. This contribution is given by an integral over the singular surface Σ. The regular part has the same form as in the absence of the conical singularities. Therefore, $A_p^{(\mathrm{reg})}(n) = nA_p(1)$.

By taking into account the structure of $A_p^{(\mathrm{surf})}(n)$ in the case of the conical singularities with $O(2)$ isometry we assume that $A_p^{(\mathrm{surf})}(n)$ has a simple analytical dependence on n such that one can go to continuous values of n and consider the limit $n \to 1$. Our aim is to determine $A_p^{(\mathrm{surf})}(n)$ in the limit $n \to 1$ when the entangling surface is an arbitrary co-dimension 2 surface.

In order to illustrate our procedure we do it first for the case a scalar field with the operator $\hat{\mathcal{D}} = -\Delta + V$, where V is a potential. In this case the regular bulk part of the heat kernel coefficients is well known,

$$\begin{aligned} &A_0^{(\mathrm{reg})}(n) = n\,\mathrm{vol}(\mathcal{M})\,,\;\; A_1^{(\mathrm{reg})}(n) = n\int_{\mathcal{M}}\sqrt{g}d^dx\left(\frac{1}{6}R - V\right)\,, \\ &A_2^{(\mathrm{reg})}(n) = n\int_{\mathcal{M}}\sqrt{g}d^dx\left(\frac{1}{180}R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - \frac{1}{180}R_{\mu\nu}R^{\mu\nu} + \frac{1}{2}\left(\frac{1}{6}R - V\right)^2\right)\,. \end{aligned} \tag{4.4}$$

The surface part of the coefficients was calculated in [11] in the case when Σ is a fixed point of an Abelian isometry, so that the extrinsic curvatures of Σ vanish. In particular, in [11] the exact dependence on n of the surface heat kernel coefficients was determined. If we are however interested in the leading in $(1-n)$ behaviour of the coefficients then an important observation made in [10] is the following:

$$A_p^{(\mathrm{reg})}(n) + A_p^{(\mathrm{surf})}(n) = A_p^{(\mathrm{reg})}(\mathcal{M}_n) + O(1-n)^2 \,. \tag{4.5}$$

The left hand side contains an exact hear kernel coefficient computed on the conical space while on the right hand side one has the regular, bulk, part of the coefficient extended to a conical space using the formulas (1.4)-(1.6). The right hand side should be understood in the sense of the limiting procedure explained above. Ref. [10] dealt with the case of vanishing extrinsic curvatures. Here we suggest that this result can be extended to a general case and determine the surface heat kernel coefficients to leading order in $(1-n)$ using our generalized formulas (1.3) and (1.4)-(1.6),

$$\begin{aligned} &A_0^{(\mathrm{surf})}(n) = 0\,,\ A_1^{(\mathrm{surf})}(n) = \frac{2\pi}{3}(1-n)\int_\Sigma 1 + O(1-n)^2\,, \\ &A_2^{(\mathrm{surf})}(n) = \frac{2\pi}{3}(1-n)\int_\Sigma \left(\frac{1}{6}R - V + \frac{1}{30}(2R_{ijij} - R_{ii} - 2\mathrm{Tr}k^2 + \frac{1}{2}(\mathrm{Tr}k)^2)\right) + O(1-n)^2\,. \end{aligned} \tag{4.6}$$

The validity of (4.6) in the case of $O(2)$ isometry (extrinsic curvature is zero) can be checked by direct comparison of (4.6) with the exact expressions obtained in [11]. The suggestion about the dependence of the heat kernel coefficients on the extrinsic curvature of the singular surface is our new result. We hope that direct computations can be done to confirm it.

The validity of (4.6) in case of A_2 coefficient has been also established in case of massless spinor fields and for the gauge invariant combination of the heat coefficients in case of gauge fields. Thus, our suggestion is as well applicable to heat coefficient of Laplace operators on $\mathcal{M}_n$.

Going back to scalar fields one can note that in the conformally invariant case, $V = \frac{1}{6}R$, the coefficient $A_2^{(\mathrm{surf})}(n)$ is conformally invariant. Below we shall consider the conformal theories in more detail.

4.2 Surface terms in conformal anomalies

In a conformal field theory in $d = 4$ the heat kernel coefficient A_2, both the bulk and the surface parts, is supposed to be conformal invariant. By conformal invariance we here mean the invariance under the transformations, $g_{\mu\nu} \to e^{2\sigma} g_{\mu\nu}$ and $\gamma_{ij} \to e^{2\sigma}\gamma_{ij}$, of both the bulk metric $g_{\mu\nu}$ and of the induced surface metric γ_{ij}. In general, for a conformal field theory, the bulk part of the coefficient is a linear combination of the Euler density and the Weyl tensor squared

$$A_2 = \int_{\mathcal{M}} \sqrt{g} d^4x(-aE_4 + bW^2)\,, \tag{4.7}$$

where E_4 and W^2 are defined in (3.12), (3.13). The coefficients a and b depend on the spin of the field in question. For a real conformal scalar field $a = 1/360$ and $b = 3/360$. The coefficient (4.7) determines what is called the conformal anomaly. Under a global rescaling $g \to \lambda^2 g$, $\gamma \to \lambda^2 \gamma$ the UV finite part of the effective action changes as

$$W_{\rm fin}(\lambda^2 g, \lambda^2 \gamma) = W_{\rm fin}(g,\gamma) + \frac{A_2}{16\pi^2} \ln\lambda\,. \tag{4.8}$$

On a conical space the coefficient A_2 has a surface part so that the anomaly in (4.8) contains both the bulk and the surface contributions. Property (4.6) implies that to determine the surface part of A_2 we can use formulas (3.14) and (3.17). This yields

$$\begin{aligned} &A_2^{\rm (surf)}(n) = 8\pi(1-n)\int_\Sigma \sqrt{\gamma} d^2y\,(-aR_\Sigma + bK_\Sigma) + O(1-n)^2\,, \\ &K_\Sigma = R_{ijij} - R_{ii} + \frac{1}{3}R - ({\rm Tr}k^2 - \frac{1}{2}({\rm Tr}k)^2\,, \end{aligned} \tag{4.9}$$

where K_Σ is conformal invariant. This expression is our result for the surface conformal anomaly. It confirms the form proposed earlier in [2] on the basis of conformal invariance and the holography. Let us emphasize that this result should hold for conformal scalar fields, massless spinor and gauge fields.

4.3 Non-conformal field theories

In non-conformal massless field theories the heat kernel coefficient A_2 is modified by addition of the Ricci scalar squared,

$$A_2 = \int_{\mathcal{M}} \sqrt{g} d^4x(-aE_4 + bW^2 + cR^2) \quad . \tag{4.10}$$

We expect that the surface part of the coefficient should be

$$A_2^{\rm (surf)}(n) = 8\pi(1-n)\int_\Sigma \sqrt{\gamma} d^2y\,(-aR_\Sigma + bK_\Sigma + cR) + O(1-n)^2\,, \tag{4.11}$$

where we used our main result (1.4)-(1.6) . We see that the non-conformally invariant term R^2 in the bulk coefficient does not produce any extra extrinsic curvature term in the surface part of the coefficient. In particular, this implies that in the Ricci flat spacetime the surface heat kernel coefficient is the same as in a conformal field theory. If the quantum field in question has a mass m then the heat kernel of operator $(\hat{\mathcal{D}} + m^2)$ is simply the product $e^{-m^2 s}e^{-s\hat{\mathcal{D}}}$ so that the mass dependence of the surface coefficients and the surface effective action can be easily restored. We shall not do this here as it is a trivial exercise.

4.4 Logarithmic term in entanglement entropy of conformal and non-conformal theories

It is apparent that for these surfaces the extrinsic curvature is non-vanishing and thus should be expected to contribute to the entanglement entropy. This problem was first analyzed in [2] for the logarithmic terms in the entropy in a conformal field theory in four dimensions. This analysis is based on the conformal symmetry and the holography and can be summarized as follows.

Entanglement entropy can be computed using the replica method by differentiating the effective action $W(n)$ with respect to n,

$$S = -\mathrm{Tr}\hat{\rho}\ln\hat{\rho} = (n\partial_n - 1)W(n)|_{n=1}\,. \tag{4.12}$$

The UV divergences of the entropy can be deduced from the surface heat kernel coefficients we have just derived. For a quantum field of mass m we find that

$$S = \frac{N_s\, A(\Sigma)}{48\pi\epsilon^2} + \frac{1}{2\pi}\int_\Sigma \sqrt{\gamma}d^2y(aR_\Sigma - bK_\Sigma - cR + \frac{1}{12}m^2 D_s)\ln\,\epsilon\,, \tag{4.13}$$

where N_s is the total number of physical degrees of freedom in the theory of spin s while D_s is the dimension of the representation of spin s. We have included here the mass term since its contribution appears to be universal for fields of any spin. In the conformal case ($c = 0$ and $m = 0$) equation (4.13) agrees with the result derived in [2]. The logarithmic term in (4.13) for a non-conformal field theory is our new result. We see from (4.13) that in a Ricci flat spacetime and for a massless field the non-conformal term in (4.13) vanishes and the logarithmic term is exactly the same as in a conformal field theory. This agrees with the numerical computation of the logarithmic term for a scalar field in Minkowski spacetime made in [7].

4.5 Holographic formula of entanglement entropy for generic AdS gravities

Entanglement entropy $S(\Sigma)$ in conformal field theories (CFT) which admit a dual description in terms of anti-de Sitter gravity allow a remarkable representation known as a holographic formula [12]

$$S(\Sigma) = \frac{A(\tilde{\Sigma})}{4G_{(d+1)}}\quad . \tag{4.14}$$

Here G is the Newton constant in the dual gravity theory. The dual theory has one dimension higher than the conformal field theory. The holographic formula is defined in terms of the volume $A(\tilde{\Sigma})$ of a minimal codimension 2 hypersurface $\tilde{\Sigma}$ in the bulk AdS spacetime with the condition that the asymptotic boundary of $\tilde{\Sigma}$ belongs to a conformal class of Σ. There are infrared divergences in (4.14) which according to the AdS/CFT dictionary [13] are related to the ultraviolet divergences of the entanglement entropy on the CFT side.

Formula (4.14) is valid for theories where the dual gravity action $I[\mathcal{M}^{(d+1)}]$ has the Einstein form with a negative cosmological constant. Our method allows one to make the prediction about holographic formula when the bulk gravity includes terms quadratic in curvature

$$I[\mathcal{M}^{(d+1)}] = -\int_{\mathcal{M}^{(d+1)}} \sqrt{g} d^{d+1}x \ \left[\frac{R}{16\pi G_{(d+1)}} + 2\Lambda + a\ R^2 + b\ R_{\mu\nu}R^{\mu\nu} + c\ R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right] \quad (4.15)$$

In what follows we look for a modification of (4.14) for 5D gravity (4D CFT). We use arguments of [14] for the origin of the holographic formula The basic idea here is that in the replica method the n-th power of the reduced CFT density matrix Tr $\hat{\rho}^n$ is determined by an AdS 'partition function' which in the semiclassical approximation is just an action on an orbifold $\mathcal{M}_n^{(d+1)}$ constructed from n replicas of $\mathcal{M}^{(d+1)}$. Conical singularities of $\mathcal{M}_n^{(d+1)}$ are located on a codimension 2 hypersurface $\tilde{\Sigma}$ whose boundary is conformal to the entangling surface Σ. According to this prescription

$$S(\Sigma) = -(n\partial_n - 1)\ln Z(n)_{n\to 1} \quad , \tag{4.16}$$

where in the semiclassic approximation

$$-\ln Z(n) \sim I[\mathcal{M}_n^{(d+1)}] \quad . \tag{4.17}$$

One can use now results (1.4)-(1.6) for the action on squashed conical singularities to get from (4.16),(4.17) the following formula

$$S(\Sigma) = \frac{A(\tilde{\Sigma})}{4G_{(5)}} + 4\pi\int_{\tilde{\Sigma}} \sqrt{\gamma} d^3y \ \left[2aR + b\left(R_{ii} - \frac{1}{2}k^2\right) + 2c(R_{ijij} - \text{Tr}\ k^2)\right] \quad . \tag{4.18}$$

which reduces to (4.14) when $a = b = c = 0$. Subadditivity property of entanglement entropy and arguments of [14] require that $\tilde{\Sigma}$ is a hypersurface which where the functional in the r.h.s of (4.18) has a minimum.

We will leave studying consequences of Eq. (4.18) for a future work and end this section with a discussion of the Gauss-Bonnet gravity which is a particular case of (4.15). For this type of the theory the constants a, b, c are related to each other and expressed in terms of a single parameter λ as $a = 2\lambda,\ b = -4\lambda,\ c = \lambda$. Combination of the quadratic terms is just extension to 5 dimensions of the Euler density (3.12). The holographic formula for the bulk Gauss-Bonnet gravity

$$S(\Sigma) = \frac{A(\tilde{\Sigma})}{4G_{(5)}} + 8\pi\lambda\int_{\tilde{\Sigma}} \sqrt{\gamma} d^3y\ \hat{R} \quad , \tag{4.19}$$

where $\hat{R}$ is the scalar curvature of $\tilde{\Sigma}$. To get (4.19) from (4.18) we used the Gauss-Codazzi equations (3.15). Formula (4.19) has been suggested in [14] for surfaces with vanishing extrinsic curvatures and for a generic surfaces in the Gauss-Bonnet gravity in [16],[17]. It is also known as the Jacobson-Myers functional.

The fact that our method reproduces modified holographic formula for the Gauss-Bonnet gravity gives a further support to this formula.

5 Conclusions

The aim of our work was to check whether one can define geometry on manifolds with conical singularities in a distributional sense. We presented a method how it can be done, at least in case of orbifold constructions which are locally flat but may have singular surfaces of different kinds. We checked consistency of the method for topological invariants and conformal invariants. In case of studies of entanglement entropy it was possible to find an agreement of some holographic predictions with the distributional nature of squashed cones. This give a further support for the holographic formula of entanglement entropy. One of immediate by-products of the our analysis is a suggestion in Sec. 4.5 of a holographic formula for entanglement entropy in theories with gravity duals which has an arbitrary combination of terms quadratic in curvature in the action.

There is an essential difference in squashed and symmetric cones. For symmetrical conical singularities there is a useful formula suggested in [1] for the Riemann tensor

$$ {}^{(n)}R^{\mu\nu}{}_{\lambda\rho} = R^{\mu\nu}{}_{\lambda\rho} + 2\pi(n-1)((n^\mu \cdot n_\lambda)((n^\nu \cdot n_\rho) - (n^\nu \cdot n_\lambda)((n^\mu \cdot n_\rho))\delta_\Sigma + O((n-2)^2) \quad , \quad (5.20) $$

Here $R^{\mu\nu}{}_{\lambda\rho}$ is a regular part of the curvature, δ_Σ is a covariant delta-function with a support on the singular surface Σ.

As we said, (5.20) is quite useful in applications. Unfortunately, as one can conclude from our results this simple factorization is not applicable to squashed conical singularities. It also cannot be modified to describe correctly terms in the integrals depending on extrinsic curvatures of Σ. This is the reason why integrals require a separate consideration in each particular case.

We carried out the computations for integrals quadratic in curvatures in different dimensions. Since the suggested method yields self-consistent results we believe it can be applied to more complicated integrals which contain combinations of higher power terms. Some work in this direction is in progress.

Let us finish with comments on some recent papers which appeared while the present work was in the final stages of preparation. Two other new papers appeared which overlap with some results reported here. Spherical Rindler horizons discussed in Sec. 2.1 have been also introduced in [18]. Asymptotic form of the metric (2.2) near a codimension 2 hypersurface with non-vanishing extrinsic curvatures, see Sec. 2.2, was also used in [15]. The authors of

[15] also suggest an interesting alternative derivation of the holographic formula (4.14). They use a conformal transformation from singular boundary manifolds which appear in the replica method on the CFT side to non-singular ones. This yields the gravity partition function with non-singular boundary conditions and non-singular background geometries in the bulk. It would be interesting to find predictions of [15] for bulk gravities in the form (4.15) and compare them with (4.18).

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A Integrals in 5 and 6 dimensions

Here we present calculations of the regularized integrals which correspond to the case when $\mathcal{M}_n$ is obtained from n copies of the Minkowski spacetime. The copies are glued along cuts which meet on a codimension 2 hypersurface Σ. Thus, Σ is a singular surfrace where conical singularities are located. We present results for different choices of Σ in five dimensions (in the present section) and in sex dimensions (next section). The regularized geometry is denoted as $\tilde{\mathcal{M}}_n$. It is convenient to introduce the following notations:

$$I_1 = \int_{\tilde{\mathcal{M}}_n} \sqrt{g} d^d x \ R^2 \quad , \quad I_2 = \int_{\tilde{\mathcal{M}}_n} \sqrt{g} d^d x \ R_{\mu\nu} R^{\mu\nu} \quad , \quad I_3 = \int_{\tilde{\mathcal{M}}_n} \sqrt{g} d^d x \ R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \quad . \tag{A.1}$$

$$K_1 = \int_\Sigma \sqrt{\gamma} d^{d-2} y \ k^2 \quad , \quad K_2 = \int_\Sigma \sqrt{\gamma} d^{d-2} y \ \mathrm{Tr} \ k^2 \quad . \tag{A.2}$$

The dimensionality d is 5 or 6. In all the cases given below relations (3.11) are satisfied.

A1. $d = 5$. Let Σ be a hypersphere $\Sigma = S^3$ of the radius a, then

$$\begin{aligned} &I_1 \to O(n-1)^2 \\ &I_2 \to 36\pi^3 a(n-1) + O(n-1)^2 \,, \\ &I_3 \to 48\pi^3 a(n-1) + O(n-1)^2 \,, \end{aligned} \tag{A.3}$$

$$K_1 = 18\pi^2 a \quad , \quad K_2 = 6\pi^2 a \quad . \tag{A.4}$$

A2. $d = 5$. Let Σ be a hypercylinder $\Sigma = S^2 \times R^1$ of the radius a and the length L, then

$$\begin{aligned} &I_1 \to O(n-1)^2 \\ &I_2 \to 32\pi^2 L(n-1) + O(n-1)^2 \,, \\ &I_3 \to 64\pi^2 L(n-1) + O(n-1)^2 \,, \end{aligned} \tag{A.5}$$

$$K_1 = 16\pi L \quad , \quad K_2 = 8\pi L \quad , \tag{A.6}$$

Note that the radius a of S^2 does not appear directly in these formulas.

A3. $d = 5$. For $\Sigma = S^1 \times R^2$, where S^1 has the radius a, and R^2 is a square of the area L^2 one finds

$$\begin{aligned} &I_1 \to O(n-1)^2 \\ &I_2 \to 4\pi^2 \frac{L^2}{a}(n-1) + O(n-1)^2 \,, \\ &I_3 \to 16\pi^2 \frac{L^2}{a}(n-1) + O(n-1)^2 \,, \end{aligned} \tag{A.7}$$

$$K_1 = 2\pi \frac{L^2}{a} \quad , \quad K_2 = 2\pi \frac{L^2}{a} \quad . \tag{A.8}$$

A4. $d = 6$. If Σ is a hypersphere $\Sigma = S^4$ of radius a

$$\begin{aligned} I_1 &\to O(n-1)^2 \\ I_2 &\to \frac{256}{3}\pi^3 a^2 (n-1) + O(n-1)^2 \, , \\ I_3 &\to \frac{256}{3} a^2 \pi^3 (n-1) + O(n-1)^2 \, , \end{aligned} \tag{A.9}$$

$$K_1 = \frac{128}{3}\pi^2 a^2 \quad , \quad K_2 = \frac{32}{3}\pi^2 a^2 \quad . \tag{A.10}$$

A5. $d = 6$. If Σ is a hypercylinder $\Sigma = S^3 \times R^1$ of the radius a and the length L

$$\begin{aligned} I_1 &\to O(n-1)^2 \, , \\ I_2 &\to 36\pi^3 a L (n-1) + O(n-1)^2 \, , \\ I_3 &\to 48\pi^3 a L (n-1) + O(n-1)^2 \, , \end{aligned} \tag{A.11}$$

$$K_1 = 18\pi^2 a L \quad , \quad K_2 = 6\pi^2 a L \quad . \tag{A.12}$$

A6. $d = 6$. If Σ is a product $\Sigma = S^2 \times R^2$ of a 2-sphere of radius a and a square of length L

$$\begin{aligned} I_1 &\to O(n-1)^2 \, , \\ I_2 &\to 32\pi^2 L^2 (n-1) + O(n-1)^2 \, , \\ I_3 &\to 64\pi^3 L^2 (n-1) + O(n-1)^2 \, , \end{aligned} \tag{A.13}$$

$$K_1 = 16\pi L^2 \quad , \quad K_2 = 8\pi L^2 \quad . \tag{A.14}$$

Note that the radius a of S^2 does not appear directly in the formulas.

A7. $d = 6$. For Σ being a product $\Sigma = S^1 \times R^3$ of a circle of radius a and a cube of length L one finds

$$\begin{aligned} I_1 &\to O(n-1)^2 \\ I_2 &\to 4\pi^2 \frac{L^3}{a} (n-1) + O(n-1)^2 \, , \\ I_3 &\to 16\pi^2 \frac{L^3}{a} (n-1) + O(n-1)^2 \, , \end{aligned} \tag{A.15}$$

$$K_1 = 2\pi \frac{L^3}{a} \quad , \quad K_2 = 2\pi \frac{L^3}{a} \quad . \tag{A.16}$$

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