

# 2-CY-tilted algebras that are not Jacobian

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## 2-CY-TILTED ALGEBRAS THAT ARE NOT JACOBIAN

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ABSTRACT. Over any field of positive characteristic we construct 2-CY-tilted algebras that are not Jacobian algebras of quivers with potentials. As a remedy, we propose an extension of the notion of a potential, called hyperpotential, that allows to prove that certain algebras defined over fields of positive characteristic are 2-CY-tilted even if they do not arise from potentials.

In another direction, we compute the fractionally Calabi-Yau dimensions of certain orbit categories of fractionally CY triangulated categories. As an application, we construct a cluster category of type  $G_2$ .

### 1. INTRODUCTION

A 2-CY-tilted algebra is an endomorphism algebra of a cluster-tilting object in a 2-Calabi-Yau triangulated category. There are close connections between 2-CY-tilted algebras and Jacobian algebras of quivers with potentials as introduced by Derksen, Weyman and Zelevinsky [13]. On the one hand, already in [13] it is shown that cluster-tilted algebras of Dynkin type, which are particular kind of 2-CY-tilted algebras, are Jacobian algebras. Later, Buan, Iyama, Reiten and Smith have shown in [9] that all cluster-tilted algebras, and more generally the 2-CY-tilted algebras arising from cluster categories associated in [8] to words in Coxeter groups are Jacobian. Moreover, they have shown that under some conditions the notions of mutation of cluster-tilting objects in a 2-CY category and mutation of quivers with potentials are compatible.

On the other hand, by the work of Amiot [2], any finite-dimensional Jacobian algebra is 2-CY-tilted. It is therefore natural to ask whether any 2-CY-tilted algebra is a Jacobian algebra of a quiver with potential [3, Question 2.20]. The purpose of this note is twofold. First, we provide a negative answer to this question over any field of positive characteristic. Our examples are given by certain self-injective Nakayama algebras which are also known as truncated cycle algebras. Second, we show that it is actually possible to slightly extend the notion of a potential in order to exclude this kind of examples. Let us explain the motivation behind such extension.

Since 2-CY-tilted algebras have some remarkable homological and structural properties [27], it is of interest to know that certain finite-dimensional algebras defined in a uniform way over all fields (e.g. as quivers with relations “over  $\mathbb{Z}$ ”) are 2-CY-tilted. Often this is done by “integrating” the defining relations to give a potential so that the algebra could be seen as a Jacobian algebra. However, there are cases where such “integration” is only possible provided we restrict the characteristic of the field.

Consider for example the algebra  $\Lambda_K = K[x]/(x^{n-1})$  over a field  $K$  for some  $n > 2$ , which could be described as a quiver with one vertex, one loop  $x$  and a relation  $x^{n-1}$ .

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As long as the characteristic of  $K$  does not divide  $n$ , this algebra is Jacobian (take the potential  $x^n$ ) and hence 2-CY-tilted. However, the AR-quiver of  $\Lambda_K$  and the fact that it is symmetric do not depend on the field  $K$ , so one would like to say that  $\Lambda_K$  is 2-CY-tilted regardless of the characteristic of  $K$ . Another example of the same kind is given by the remark of Ringel [34, §14] that barbell algebras with two loops are 2-CY-tilted, provided that one assumes that the characteristic of the ground field is not 3.

As some of the constructions involving a quiver with potential rely only on its cyclic derivatives (see e.g. the definitions of the Ginzburg dg-algebra or the Jacobian algebra), our idea is to replace these derivatives with arbitrary elements and to concentrate on the required conditions that these elements have to satisfy in order for such constructions to make sense. This will avoid the need to “integrate” relations into one potential and will allow to prove in a characteristic-free manner that certain algebras are 2-CY-tilted.

**1.1. Hyperpotentials.** Recall that the construction of Amiot [2] starts with a dg-algebra  $\Gamma$  which is concentrated in non-positive degrees, homologically smooth, bimodule 3-CY and whose 0-th cohomology  $H^0(\Gamma)$  is finite-dimensional, and produces a 2-CY triangulated category with a cluster-tilting object whose endomorphism algebra is  $H^0(\Gamma)$ . This construction is applied to a quiver with potential  $(Q, W)$  over a field  $K$  by considering its Ginzburg dg-algebra defined in [20]. Keller proves in [26] that the Ginzburg dg-algebra has the required properties by showing that it is quasi-isomorphic to the deformed 3-Calabi-Yau completion of the path algebra  $KQ$  by an element in  $\mathrm{HH}_1(KQ)$  which is the image of the potential  $W$  under Connes’ map  $\mathrm{HC}_0(KQ) \rightarrow \mathrm{HH}_1(KQ)$ .

These considerations raise the possibility of working from the outset with elements in  $\mathrm{HH}_1$  (and not in  $\mathrm{HC}_0$ ) and motivate the following definition. Indeed, Ginzburg’s original definition in [20, §5] starts with a cyclic 1-form satisfying certain conditions, which is not necessarily a differential of a potential.

**Definition.** Let  $K$  be a commutative ring and let  $Q$  be a quiver. Denote by  $Q_0, Q_1$  the sets of vertices and arrows of  $Q$  and by  $A = \widehat{KQ}$  the completed path algebra of  $Q$  over  $K$  (i.e. its elements are infinite  $K$ -linear combinations of paths in  $Q$ ). For any  $i \in Q_0$  let  $e_i \in A$  be the idempotent corresponding to the path of length 0 starting at  $i$ .

A *hyperpotential* on  $Q$  is a collection of elements  $(\rho_\alpha)_{\alpha \in Q_1}$  in  $A$  indexed by the arrows of  $Q$  satisfying the following conditions:

- (i) If  $\alpha: i \rightarrow j$  then  $\rho_\alpha \in e_j A e_i$ . In other words,  $\rho_\alpha$  is a (possibly infinite) linear combination of paths starting at  $j$  and ending at  $i$ .
- (ii)  $\sum_{\alpha \in Q_1} [\alpha, \rho_\alpha] = 0$  in  $A$ .

Hyperpotentials represent elements in  $\mathrm{HH}_1(A)$ , and any potential  $W \in \mathrm{HC}_0(A)$  gives rise to a hyperpotential by considering its cyclic derivatives  $(\partial_\alpha W)_{\alpha \in Q_1}$ . Conversely, when the ring  $K$  contains  $\mathbb{Q}$ , any hyperpotential arises in this way, so there is nothing new. However, when  $K$  does not contain  $\mathbb{Q}$  (e.g. when it is a field of positive characteristic) there are hyperpotentials that do not arise from potentials but nevertheless one would like to attach to them suitable 3-CY and 2-CY categories.

In order to do that, one defines the *Ginzburg dg-algebra*  $\Gamma(Q, (\rho_\alpha))$  of a hyperpotential  $(Q, (\rho_\alpha))$  in the usual way, see [20, §5.2] and [28, §2.6]; Let  $\widetilde{Q}$  be the graded quiver whose set of vertices is  $Q_0$  and whose arrows are the arrows of  $Q$  (in degree 0) together with an arrow  $\alpha^*: j \rightarrow i$  of degree  $-1$  for each arrow  $\alpha: i \rightarrow j$  in  $Q_1$  and a loop  $t_i$  of degree

$-2$  at each vertex  $i \in Q_0$ . As a graded algebra,  $\Gamma(Q, (\rho_\alpha))$  is the completion of the graded path algebra  $\widehat{K\tilde{Q}}$  with respect to path length (so that each graded piece consists of the infinite linear combinations of paths of a given degree). Its differential is defined as the continuous linear map homogeneous of degree 1 which satisfies the Leibniz rule and whose values on the generators are given by

$$d(\alpha) = 0, \quad d(\alpha^*) = \rho_\alpha, \quad d(t_i) = e_i \left( \sum_{\beta \in Q_1} [\beta, \beta^*] \right) e_i$$

for each  $i \in Q_0$  and  $\alpha \in Q_1$ . Note that the condition  $d^2 = 0$  is equivalent to the condition (ii) in the definition of hyperpotential. The *Jacobian algebra* of a hyperpotential  $(Q, (\rho_\alpha))$  is defined as the 0-th cohomology of its Ginzburg dg-algebra. Equivalently, it is the quotient of  $\widehat{K\tilde{Q}}$  by the closure of the ideal generated by the elements  $\rho_\alpha$  for  $\alpha \in Q_1$ .

By following the proof of [26, Theorem 6.3] by Keller we deduce:

**Proposition 1.** *The Ginzburg dg-algebra of a hyperpotential is (topologically) homologically smooth and 3-CY.*

Note that [26] treats the non-completed version of the Ginzburg algebra. The corresponding statement for the completed version appears in [28, Theorem A.17].

When  $K$  is a field, the results of Amiot [2] (see also [28, §A.20] for the completed case) imply the following.

**Corollary.** *If the Jacobian algebra of a hyperpotential is finite-dimensional, then it is 2-CY-tilted.*

It may be possible to develop a theory of mutations for hyperpotentials as done for potentials by Derksen, Weyman and Zelevinsky in [13]. However, since our original motivation was to show that certain algebras are 2-CY-tilted, we will not pursue this direction here.

In Section 2.2 we define the notions of *right equivalence* and *weak right equivalence* for hyperpotentials and characterize them as arising from isomorphisms of the corresponding Ginzburg dg-algebras having certain prescribed properties. In particular, the categorical constructions do not distinguish between weakly equivalent hyperpotentials. We show that (weakly) right equivalent potentials as defined in [13] and [19] are so also when considered as hyperpotentials.

**1.2. The examples.** We demonstrate the usefulness of the notion of a hyperpotential by presenting the following class of examples providing a negative answer to Question 2.20 in [3]. Let  $K$  be a field and let  $m, e \geq 1$  such that  $me \geq 3$ . Consider the  $K$ -algebra  $\Lambda_{m,e}$  given as the path algebra of the quiver  $Q_m$  which is a cycle with  $m$  vertices and arrows  $\alpha_1, \dots, \alpha_m$  (as shown in Figure 1) modulo the ideal generated by all paths of length  $me - 1$ . It is well known that  $\Lambda_{m,e}$  is a finite-dimensional self-injective Nakayama algebra over  $K$ . Denote by  $\text{char } K$  the characteristic of  $K$ .

**Proposition 2.** *Let  $K$  be a field and let  $m, e \geq 1$  such that  $me \geq 3$ .*

(a)  $\Lambda_{m,e}$  is the Jacobian algebra of the hyperpotential  $(\rho_{\alpha_i})_{i=1}^m$  on  $Q_m$  given by

$$\rho_{\alpha_i} = \alpha_{i+1} \dots \alpha_{i-1} (\alpha_i \alpha_{i+1} \dots \alpha_{i-1})^{e-1} \quad (1 \leq i \leq m)$$

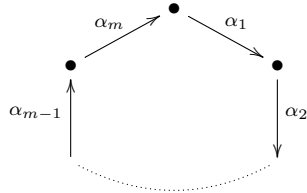


FIGURE 1. The quiver  $Q_m$  which is a cycle on  $m$  vertices.

- and hence it is always 2-CY-tilted.
- (b) Let  $W$  be any potential on the quiver  $Q_m$ . Then the Jacobian algebra of  $(Q_m, W)$  is either the completed path algebra  $\widehat{KQ_m}$  or the algebra  $\Lambda_{m,d}$  for some  $d \geq 1$  not divisible by  $\text{char } K$ .
  - (c) Conversely, if  $\text{char } K$  does not divide  $e$ , then  $\Lambda_{m,e}$  is the Jacobian algebra of the quiver with potential  $(Q_m, W_{m,e})$ , where  $W_{m,e} = (\alpha_1 \alpha_2 \dots \alpha_m)^e$ .
  - (d) If  $\text{char } K$  divides  $e$ , then  $\Lambda_{m,e}$  is not a Jacobian algebra of a quiver with potential.

**1.3. Orbit categories.** Another approach to the construction of 2-CY triangulated categories is via the machinery of triangulated orbit categories developed by Keller [24]. Based on this approach, the next statement provides a construction, which is independent on the characteristic of  $K$ , of an ambient 2-CY category for the algebras  $\Lambda_{m,e}$ . Denote by  $D_n$  an orientation of the Dynkin diagram of type  $D$  with  $n$  vertices (where for  $n = 3$  we use the convention that  $D_3 = A_3$ ).

**Proposition 3.** *Let  $m, e$  be as in Proposition 2 and assume in addition that  $m$  is even or that  $e$  is odd.*

- (a) *There is an auto-equivalence  $F$  of the bounded derived category  $\mathcal{D}^b(\text{mod } KD_{me})$  such that the orbit category*

$$\mathcal{C}_{m,e} = \mathcal{D}^b(\text{mod } KD_{me})/F$$

*is a 2-CY triangulated category with a cluster-tilting object whose endomorphism algebra is  $\Lambda_{m,e}$ .*

- (b) *The shape of the AR-quiver of  $\mathcal{C}_{m,e}$  is  $\mathbb{Z}D_{me}/\langle(\phi\tau)^m\rangle$  where  $\phi$  is the automorphism of order 2 of the Dynkin diagram underlying  $D_{me}$ .*

The proof of Proposition 3 relies on the next observation dealing more generally with orbit categories of fractionally Calabi-Yau categories, whose proof uses a result of Keller [24] on triangulated orbit categories together with calculations of Calabi-Yau dimensions by Dugas [14, §9]. Recall that a triangulated  $K$ -linear category  $\mathcal{C}$  with suspension functor  $\Sigma$  and a Serre functor  $S$  is *fractionally Calabi-Yau of dimension*  $(d, e)$  for some  $(d, e) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$  (or  $(d, e)$ -CY for short) if  $S^e \simeq \Sigma^d$ . Observe that the set of pairs  $(d, e) \in \mathbb{Z}^2$  satisfying  $S^e \simeq \Sigma^d$  forms a lattice.

**Proposition 4.** *Let  $\mathcal{H}$  be a hereditary  $K$ -linear category such that its derived category  $\mathcal{D}^b(\mathcal{H})$  is equivalent to  $\mathcal{D}^b(\text{mod } A)$  for some finite-dimensional  $K$ -algebra  $A$ . Assume that  $\mathcal{D}^b(\mathcal{H})$  is  $(d_1, e_1)$ -CY for some  $(d_1, e_1) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ .*

Consider the orbit category  $\mathcal{C} = \mathcal{D}^b(\mathcal{H})/F$  where  $F = S^{e_2}\Sigma^{-d_2}$  for some  $(d_2, e_2) \in \mathbb{Z}^2$ . Then the following conditions are equivalent:

- (i)  $e_1d_2 - e_2d_1 \neq 0$ .
- (ii)  $\text{rank } L = 2$ , where  $L$  is the lattice  $L = \mathbb{Z}(d_1, e_1) + \mathbb{Z}(d_2, e_2) \subseteq \mathbb{Z}^2$ .
- (iii) The category  $\mathcal{C}$  is Hom-finite.

Moreover, if any of these conditions holds then the orbit category  $\mathcal{C}$  is triangulated. If  $d_2$  is even or  $e_2$  is odd and in addition  $d_1 - d_2$  is even or  $e_1 - e_2$  is odd, then  $\mathcal{C}$  is  $(d, e)$ -CY for any  $(d, e) \in L$ . In particular, if  $(d, 1) \in L$  then  $\mathcal{C}$  is  $d$ -Calabi-Yau.

**Corollary.** Under the assumptions of the proposition, for any rational number  $r \in \mathbb{Q}$  there is a fraction  $(d, e)$  such that the orbit category  $\mathcal{C}$  is  $(d, e)$ -CY with  $d/e = r$ .

**1.4. Remarks.** We make a few remarks on these observations.

**Remark 1.** The assumptions on the parity of  $d_1$ ,  $d_2$ ,  $e_1$  and  $e_2$  in Proposition 4 are needed in order to apply the results of Dugas [14], who observed that otherwise there may be delicate sign issues. They could be dropped at the expense of replacing the lattice  $L$  by its sublattice  $2L \subseteq L$  of index 4.

**Remark 2.** By taking  $\mathcal{H}$  to be the category of representations of a Dynkin quiver, Proposition 4 and its corollary apply in particular to the cluster categories [10], higher cluster categories [35] and repetitive higher cluster categories [29] associated with Dynkin quivers. Moreover, by Amiot's classification [1] of triangulated categories with finitely many indecomposables, it applies also to many stable categories of self-injective algebras of finite representation type, see for example [14]. Many authors [14, 15, 23] have considered only the intersection of the lattice  $L$  with the ray  $\{(n, 1) : n \geq 0\}$ .

One could take  $\mathcal{H}$  to be any other hereditary category whose derived category is fractionally Calabi-Yau. Over an algebraically closed field, such categories were classified by van Roosmalen [36]. In particular, the proposition and its corollary apply also to the tubular cluster categories studied by Barot and Geiss [5] which are special cases of the cluster categories associated with canonical algebras [4]. Here,  $\mathcal{H}$  is the category of sheaves over a weighted projective line in the sense of Geigle and Lenzing [17].

**Example.** Let  $\mathcal{C}$  be the cluster category of tubular type  $(2, 2, 2, 2; \lambda)$  for  $\lambda \neq 0, 1$ . In this case, the hereditary category  $\mathcal{H}$  is  $(2, 2)$ -CY and  $(d_2, e_2) = (2, 1)$ , hence from  $(0, 1) = (2, 2) - (2, 1)$  we see that  $\mathcal{C}$  is not only 2-CY, but 0-CY as well. It follows that the endomorphism algebra of any object in  $\mathcal{C}$  is symmetric, and in particular all the cluster-tilted algebras of tubular type  $(2, 2, 2, 2; \lambda)$ , whose description as quivers with potentials can be found in [18, Figure 1], are symmetric.

**Remark 3.** By using Amiot's description [1] of triangulated categories with finitely many indecomposables and the classification in [11, Appendix A] of the possible shapes of the AR-quivers of such 2-CY categories with cluster-tilting objects, Bertani-Økland and Oppermann have classified all representation-finite 2-CY-tilted algebras arising from standard algebraic 2-CY triangulated categories over an algebraically closed field [7, Theorem 5.7]. Proposition 3 can be seen as a special case of their classification, and in the notations of [11], the category  $\mathcal{C}_{m,e}$  corresponds to type  $D_{me}$  with generator  $(m, \bar{m}) \in \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

However, our proofs will not rely on [11, Appendix A]. As an illustration of our methods, let us give the following example of a 2-CY-tilted algebra of finite representation type which seems not to appear in [7, 11].

**Example** (Cluster category of type  $G_2$ ). The algebra  $\Lambda$  given as the quiver

$$\bullet \longrightarrow \bullet \begin{array}{c} \circlearrowright \\ \beta \end{array}$$

with the relation  $\beta^3 = 0$  is of finite representation type and its AR-quiver is shown in the lecture notes of Gabriel [16, Fig. 19]. It has 30 vertices arranged in a cylinder, and by inserting 2 additional vertices one gets the translation quiver  $\mathbb{Z}E_8/\langle\tau^4\rangle$ .

Indeed, there is an auto-equivalence of  $\mathcal{D}^b(\text{mod } KE_8)$  such that the corresponding orbit category  $\mathcal{C}_{G_2}$  is 2-CY triangulated with a cluster-tilting object whose endomorphism algebra is isomorphic to  $\Lambda$ . The AR-quiver of  $\mathcal{C}_{G_2}$  is  $\mathbb{Z}E_8/\langle\tau^4\rangle$  and the exchange graph of cluster-tilting objects is an octagon. The algebra  $\Lambda$  is Jacobian precisely when the characteristic of  $K$  is not 2, and in this case a potential is  $\beta^4$  (note that  $\beta^3$  is always a hyperpotential). The category  $\mathcal{C}_{G_2}$  models a cluster algebra of type  $G_2$ ; we give more details in Section 2.6.

**Remark 4.** When  $e = 1$ , the category  $\mathcal{C}_{m,1}$  in Proposition 3 is the cluster category, as introduced in [10], of the Dynkin quiver  $D_m$  and it is well known that  $\Lambda_{m,1}$  is a cluster-tilted algebra of type  $D_m$ . In particular, it appears in Ringel's classification of the self-injective cluster-tilted algebras [33]. Observe that  $\Lambda_{m,1}$  is a Jacobian algebra in any characteristic.

**Remark 5.** The quiver  $Q_m$  belongs to the mutation class of the Dynkin quiver  $D_m$  which is acyclic, thus it has a unique non-degenerate potential up to right equivalence. It follows that the potential  $W_{m,e}$  in Proposition 2 is non-degenerate if and only if  $e = 1$ . Therefore the categories  $\mathcal{C}_{m,e}$  for  $e > 1$  will not properly model the corresponding cluster algebra.

**Remark 6.** Self-injective Jacobian algebras were studied by Herschend and Iyama [22]. However, observe that unless  $e = 1$ , the quiver with potential  $(Q_m, W_{m,e})$  has no cut in their sense, so that the algebras  $\Lambda_{m,e}$  for  $e > 1$  do not arise as 3-preprojective algebras of 2-representation-finite algebras of global dimension 2.

**Remark 7.** If  $\Lambda$  is a self-injective 2-CY-tilted algebra with  $m$  simple modules arising from an ambient 2-CY category  $\mathcal{C}$ , we can also consider its stable module category  $\underline{\text{mod}} \Lambda$  which is triangulated. There are stabilization functors

$$\mathcal{C} \rightarrow \text{mod } \Lambda \rightarrow \underline{\text{mod}} \Lambda$$

where the left functor was considered by Keller and Reiten [27], who also showed that  $\underline{\text{mod}} \Lambda$  is 3-CY. At each stage, the AR-quiver of the next category is obtained from that of the previous one by deleting  $m$  vertices (corresponding to indecomposable summands of a suitable cluster-tilting object in  $\mathcal{C}$ , or to the indecomposable projectives in  $\text{mod } \Lambda$ , respectively). In our case we get a sequence

$$\mathcal{C}_{m,e} \rightarrow \text{mod } \Lambda_{m,e} \rightarrow \underline{\text{mod}} \Lambda_{m,e}$$

where  $\mathcal{C}_{m,e}$  has  $m^2e$  indecomposables and the AR-quiver of  $\underline{\text{mod}} \Lambda_{m,e}$  has the shape of the translation quiver  $\mathbb{Z}A_{me-2}/\langle\tau^m\rangle$ .

The self-injective algebras of finite representation type whose stable module categories are higher cluster categories have been classified by Holm and Jorgensen [23]. In particular,  $\underline{\text{mod}} \Lambda_{m,1}$  is the 3-cluster category of type  $A_{m-2}$ .

**Remark 8.** The algebras  $\Lambda_{m,e}$  are symmetric precisely when  $m \leq 2$  (i.e. when the quiver is a loop or a 2-cycle). In this case, they have been shown to be 2-CY-tilted (at least in characteristic zero) by Burban, Iyama, Keller and Reiten [11]. The ambient 2-CY triangulated categories considered there are the stable categories of maximal Cohen-Macaulay modules over simple hypersurface singularities of odd dimension.

## 2. THE PROOFS

**2.1. Hochschild and cyclic homology for completed path algebras.** In this section we explain why hyperpotentials are the elements of the first (continuous) Hochschild homology of the completed path algebra. Let  $K$  be a commutative ring. Let  $Q$  be a finite quiver and denote by  $Q_0$  its set of vertices and by  $Q_1$  its set of arrows. Let  $A = \widehat{KQ}$  be the completed path algebra of  $Q$  over  $K$ . For any  $i \in Q_0$ , let  $e_i \in A$  be the idempotent corresponding to the trivial path at  $i$ , and let  $A_+$  be the subspace of (infinite) linear combinations of non-trivial paths (it is a two-sided ideal in  $A$ ).

The path algebra  $KQ$  is a tensor algebra over the commutative ring  $R = \bigoplus_{i \in Q_0} Ke_i$  of the projective  $R$ -bimodule  $\bigoplus_{\alpha \in Q_1} K\alpha$ . The Hochschild and cyclic homology of tensor algebras were computed by Loday and Quillen [30, §5]. Let us recall the result in our case.

Let  $\sigma: A \rightarrow A$  be the continuous linear map defined by  $\sigma(e_i) = e_i$  for  $i \in Q_0$  and by  $\sigma(\alpha_1 \dots \alpha_n) = \alpha_n \alpha_1 \dots \alpha_{n-1}$  for a path  $\alpha_1 \dots \alpha_n$ . Obviously,  $\sigma$  vanishes on paths that are not cycles. Then

$$\begin{aligned} \text{HH}_0(A) &= \text{Coker}(\text{id} - \sigma) = A_\sigma \\ \text{HH}_1(A) &= \ker((\text{id} - \sigma)|_{A_+}) = A_+^\sigma \end{aligned}$$

and  $\text{HH}_n(A) = 0$  for  $n \geq 2$ . The elements of the space  $A_\sigma$  of  $\sigma$ -coinvariants are infinite linear combinations of cycles modulo rotation (each cycle can be rotated independently). Indeed, if  $x = \sum c_n x_n$  is a sum of cycles  $x_n$  and  $i_n \geq 0$  are arbitrary, then  $x$  and  $\sum c_n \sigma^{i_n} x_n$  differ by  $(\text{id} - \sigma)(y)$  for  $y = \sum c_n y_n$  with  $y_n = (\text{id} + \sigma + \dots + \sigma^{i_n-1})(x_n)$  (here,  $y_n = 0$  if  $i_n = 0$ ). Thus, the space of potentials is precisely  $(A_+)^\sigma$ .

The space  $A_+^\sigma$  of  $\sigma$ -invariants consists of infinite linear combinations of non-trivial cycles that are invariant under rotation. If  $\alpha \in Q_1$  and  $\rho$  is such that  $\alpha\rho$  is a cycle, then  $\sigma(\alpha\rho) = \rho\alpha$ . Now, any linear combination of cycles in  $A_+$  can be written in a unique way as  $\sum_{\alpha \in Q_1} \alpha\rho_\alpha$ . Since  $\sigma(\sum_{\alpha \in Q_1} \alpha\rho_\alpha) = \sum_{\alpha \in Q_1} \rho_\alpha\alpha$ , we see that  $\sum_{\alpha \in Q_1} \alpha\rho_\alpha \in A_+^\sigma$  if and only if  $(\rho_\alpha)_{\alpha \in Q_1}$  is a hyperpotential. In this way we get an identification between hyperpotentials and elements in  $\text{HH}_1(A)$ .

From the Connes' exact sequence

$$\dots \rightarrow \text{HH}_n(A) \rightarrow \text{HC}_n(A) \rightarrow \text{HC}_{n-2}(A) \rightarrow \text{HH}_{n-1}(A) \rightarrow \dots$$

we see that  $\text{HC}_0(A) \simeq \text{HH}_0(A)$  and

$$0 \rightarrow \text{HC}_2(A) \rightarrow \text{HC}_0(A) \xrightarrow{B} \text{HH}_1(A) \rightarrow \text{HC}_1(A) \rightarrow 0.$$



The Connes' map  $B$  is induced by the norm map  $N: A \rightarrow A$  which is the continuous linear map defined by  $N(e_i) = 0$  for  $i \in Q_0$  and by  $N(\alpha_1 \dots \alpha_n) = \sum_{j=1}^n \alpha_j \alpha_{j+1} \dots \alpha_{j-1}$  for a path  $\alpha_1 \dots \alpha_n$ . Comparing this with the definition of the cyclic derivative  $\partial_\alpha$  with respect to an arrow  $\alpha \in Q_1$ ,

$$\partial_\alpha(\alpha_1 \dots \alpha_n) = \sum_{j: \alpha_j = \alpha} \alpha_{j+1} \dots \alpha_{j-1}$$

we see that an element  $W \in \text{HC}_0(A)$  is sent by  $B$  to  $\sum_{\alpha \in Q_1} \alpha \partial_\alpha W$ , and hence any potential  $W$  can be regarded as the hyperpotential  $(\partial_\alpha W)_{\alpha \in Q_1}$ .

Let us write an explicit resolution which is a special case of the treatment in [26, §6.1]. For an arrow  $\alpha: i \rightarrow j$ , set  $s(\alpha) = i$  and  $t(\alpha) = j$ . Denote by  $\widehat{\otimes}$  the completed tensor product over  $K$  and let  $A^e = A^{op} \widehat{\otimes} A$ . As an  $A^e$ -module,  $A$  has a projective resolution

$$0 \rightarrow \bigoplus_{\alpha \in Q_1} A e_{s(\alpha)} \widehat{\otimes} e_{t(\alpha)} A \rightarrow \bigoplus_{i \in Q_0} A e_i \widehat{\otimes} e_i A \rightarrow A \rightarrow 0$$

where the right map sends an element  $p \otimes q$  to  $pq$  and the left map sends an element  $p \otimes q$  in the  $\alpha$  component to  $p\alpha \otimes q - p \otimes \alpha q$ . Indeed, this complex is contractible as a complex of right  $A$ -modules via the homotopy defined by the continuous maps sending a path  $p$  starting at  $i$  to  $e_i \otimes p$  and an element  $\alpha_1 \dots \alpha_n \otimes q$  to  $(\sum_{j: \alpha_j = \alpha} \alpha_1 \dots \alpha_{j-1} \otimes \alpha_{j+1} \dots \alpha_n q)_{\alpha \in Q_1}$ .

Applying the isomorphism  $(A \widehat{\otimes} A) \otimes_{A^e} A \simeq A$  given by  $(a \otimes b) \otimes x \mapsto bxa$ , we get the following complex which computes Hochschild homology

$$0 \rightarrow \bigoplus_{\alpha \in Q_1} e_{t(\alpha)} A e_{s(\alpha)} \rightarrow \bigoplus_{i \in Q_0} e_i A e_i \rightarrow 0$$

where the middle map sends  $(x_\alpha)_{\alpha \in Q_1}$  to  $\sum_{\alpha \in Q_1} [x_\alpha, \alpha]$ , from which we identify  $\text{HH}_1(A)$  with the space of hyperpotentials.

**2.2. Equivalences of hyperpotentials.** Let  $Q, Q'$  be two quivers on the same set of vertices, and let  $A = \widehat{KQ}$ ,  $A' = \widehat{KQ'}$  be their completed path algebras. We will denote by  $e'_i$  the idempotent in  $A'$  corresponding to the vertex  $i \in Q_0$ . Consider a continuous homomorphism of algebras  $\varphi: A \rightarrow A'$  satisfying  $\varphi(e_i) = e'_i$  for all  $i \in Q_0$ . It induces a map  $\varphi_*: \text{HH}_1(A) \rightarrow \text{HH}_1(A')$  which we now explicitly compute in terms of hyperpotentials.

For an arrow  $\alpha \in Q_1$ , let  $\Delta_\alpha: A \rightarrow A \widehat{\otimes} A$  be the continuous (double) derivation taking the values  $\Delta_\alpha(e_i) = 0$  for  $i \in Q_0$  and

$$\Delta_\alpha(\beta) = \begin{cases} e_{s(\alpha)} \otimes e_{t(\alpha)} & \text{if } \beta = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

for  $\beta \in Q_1$ . By induction we get

$$\Delta_\alpha(\alpha_1 \dots \alpha_n) = \sum_{j: \alpha_j = \alpha} \alpha_1 \dots \alpha_{j-1} \otimes \alpha_{j+1} \dots \alpha_n$$

for any path  $\alpha_1 \dots \alpha_n$ . The isomorphism  $(A \widehat{\otimes} A) \otimes_{A^e} A \simeq A$  is induced by the operation  $\diamond$  whose values on (topological) generators are  $(a \otimes b) \diamond x = bxa$  for  $a, b, x \in A$ , and with

these notations we have  $\partial_\alpha x = \Delta_\alpha(x) \diamond 1$  for every  $x \in A$  and  $\alpha \in Q_1$ . Observe that always  $\Delta_\alpha(x) \diamond y \in e_{t(\alpha)} A e_{s(\alpha)}$ .

**Lemma 2.1.** *Let  $x \in A_+$ ,  $y \in A$ . Then*

$$\sum_{\alpha \in Q_1} [\alpha, \Delta_\alpha(x) \diamond y] = [x, y]$$

*Proof.* By continuity and linearity we may assume that  $x = \alpha_1 \dots \alpha_n$  is a non-trivial path, so that

$$\begin{aligned} \sum_{\alpha \in Q_1} [\alpha, \Delta_\alpha(x) \diamond y] &= \sum_{j=1}^n \alpha_j \alpha_{j+1} \dots \alpha_n y \alpha_1 \dots \alpha_{j-1} - \sum_{j=1}^n \alpha_{j+1} \dots \alpha_n y \alpha_1 \dots \alpha_{j-1} \alpha_j \\ &= \alpha_1 \dots \alpha_n y - y \alpha_1 \dots \alpha_n = [x, y]. \end{aligned}$$

□

In particular, by taking  $y = 1$  we get the well-known identity  $\sum_{\alpha \in Q_1} [\alpha, \partial_\alpha x] = 0$  which justifies why we can consider a potential  $x$  as a hyperpotential  $(\partial_\alpha x)_{\alpha \in Q_1}$ .

Now let  $(\rho_\alpha)_{\alpha \in Q_1}$  be a hyperpotential on  $Q$  and let  $\varphi: A \rightarrow A'$  as in the beginning of this section. We define a hyperpotential  $(\rho'_\beta)_{\beta \in Q'_1}$  by

$$(2.1) \quad \rho'_\beta = \sum_{\alpha \in Q_1} \Delta_\beta(\varphi(\alpha)) \diamond \varphi(\rho_\alpha) \quad (\beta \in Q'_1)$$

This is indeed well-defined, since by Lemma 2.1 we have

$$\begin{aligned} \sum_{\beta \in Q'_1} [\beta, \rho'_\beta] &= \sum_{\beta \in Q'_1} \left[ \beta, \sum_{\alpha \in Q_1} \Delta_\beta(\varphi(\alpha)) \diamond \varphi(\rho_\alpha) \right] = \sum_{\alpha \in Q_1} \sum_{\beta \in Q'_1} [\beta, \Delta_\beta(\varphi(\alpha)) \diamond \varphi(\rho_\alpha)] \\ &= \sum_{\alpha \in Q_1} [\varphi(\alpha), \varphi(\rho_\alpha)] = \varphi \left( \sum_{\alpha \in Q_1} [\alpha, \rho_\alpha] \right) = 0. \end{aligned}$$

**Lemma 2.2.** *We have  $\varphi_*((\rho_\alpha)_{\alpha \in Q_1}) = (\rho'_\beta)_{\beta \in Q'_1}$ .*

For the proof, we start by considering the Hochschild complex

$$\dots \rightarrow A \widehat{\otimes} A \widehat{\otimes} A \xrightarrow{d_2} A \widehat{\otimes} A \xrightarrow{d_1} A$$

with  $d_1(a \otimes b) = ab - ba$  and  $d_2(a \otimes b \otimes c) = ab \otimes c - a \otimes bc + ca \otimes b$ . Denote by  $\sim$  the equivalence relation on  $A \widehat{\otimes} A$  defined by  $\text{Im } d_2$  (i.e. two elements are equivalent if their difference lies in  $\text{Im } d_2$ ).

**Lemma 2.3.** *Any element in  $A \widehat{\otimes} A$  is equivalent to an element of the form  $\sum_{\alpha \in Q_1} \rho_\alpha \otimes \alpha$  with  $\rho_\alpha \in e_{t(\alpha)} A e_{s(\alpha)}$  for each  $\alpha \in Q_1$ .*

*Proof.* Consider an element  $y \otimes x \in A \widehat{\otimes} A$  for some  $x, y \in A$ . From  $e_i \otimes e_j = e_i e_i \otimes e_j \sim e_i \otimes e_j e_j - e_j e_i \otimes e_i$  we see that  $e_i \otimes e_j \sim 0$  for all  $i, j \in Q_0$ . Moreover,  $y \otimes e_i = 1 \cdot y \otimes e_i \sim 1 \otimes y e_i - e_i \otimes y$  and hence we may assume that  $x \in A_+$ .

If  $\alpha_1 \dots \alpha_n$  is a path, then  $y \otimes \alpha_1 \dots \alpha_n \sim y \alpha_1 \dots \alpha_{n-1} \otimes \alpha_n + \alpha_n y \otimes \alpha_1 \dots \alpha_{n-1}$ , so by induction

$$y \otimes \alpha_1 \dots \alpha_n \sim \sum_{j=1}^n \alpha_{j+1} \dots \alpha_n y \alpha_1 \dots \alpha_{j-1} \otimes \alpha_j = \sum_{\alpha \in Q_1} (\Delta_\alpha(\alpha_1 \dots \alpha_n) \diamond y) \otimes \alpha$$

and by linearity and continuity

$$(2.2) \quad y \otimes x \sim \sum_{\alpha \in Q_1} (\Delta_\alpha(x) \diamond y) \otimes \alpha$$

for all  $x \in A_+$ ,  $y \in A$ . □

To complete the proof of Lemma 2.2, note that the hyperpotential  $(\rho_\alpha)$  could be seen as the element  $\sum_{\alpha \in Q_1} \rho_\alpha \otimes \alpha \in \ker d_1 \subseteq A \widehat{\otimes} A$ . Applying  $\varphi_*$  and using (2.2), we get

$$\varphi_* \left( \sum_{\alpha \in Q_1} \rho_\alpha \otimes \alpha \right) = \sum_{\alpha \in Q_1} \varphi(\rho_\alpha) \otimes \varphi(\alpha) \sim \sum_{\alpha \in Q_1} \sum_{\beta \in Q'_1} (\Delta_\beta(\varphi(\alpha)) \diamond \varphi(\rho_\alpha)) \otimes \beta = \sum_{\beta \in Q'_1} \rho'_\beta \otimes \beta.$$

The notion of right equivalence for potentials was defined in [13], and weak right equivalence was introduced in [19]. Let us consider analogous notions for hyperpotentials.

**Definition 2.4.** Let  $Q$  and  $Q'$  be two quivers on the same set of vertices.

Two hyperpotentials  $(Q, (\rho_\alpha)_{\alpha \in Q_1})$  and  $(Q', (\rho'_\beta)_{\beta \in Q'_1})$  are *right equivalent* if there exists a continuous isomorphism of algebras  $\varphi: \widehat{KQ} \rightarrow \widehat{KQ'}$  with  $\varphi(e_i) = e'_i$  for all  $i \in Q_0$  such that  $\varphi_*((\rho_\alpha)_{\alpha \in Q_1}) = (\rho'_\beta)_{\beta \in Q'_1}$ .

They are *weakly right equivalent* if there exists  $c \in K^\times$  such that  $(Q, (c\rho_\alpha)_{\alpha \in Q_1})$  and  $(Q', (\rho'_\beta)_{\beta \in Q'_1})$  are right equivalent.

The original definitions of these notions for two quivers with potentials  $(Q, W)$  and  $(Q', W')$  could be rephrased in terms of the map  $\varphi_*: \mathrm{HH}_0(A) \rightarrow \mathrm{HH}_0(A')$ , namely as  $\varphi_*(W) = W'$  and  $\varphi_*(cW) = W'$ , respectively.

**Lemma 2.5.** *Two potentials that are (weakly) right equivalent are also (weakly) right equivalent as hyperpotentials. Conversely, if  $K$  contains  $\mathbb{Q}$ , then two potentials that are (weakly) right equivalent as hyperpotentials are so also as potentials.*

*Proof.* Combining the chain rule [13, Lemma 3.9], Lemma 2.2 and Eq. (2.1), we see that for any  $W \in \mathrm{HH}_0(A)$  and continuous algebra homomorphism  $\varphi: A \rightarrow A'$  such that  $\varphi(e_i) = e'_i$  for all  $i \in Q_0$  we have

$$\varphi_*((\partial_\alpha W)_{\alpha \in Q_1}) = (\partial_\beta \varphi(W))_{\beta \in Q'_1},$$

hence the first part. For the second part, note that the kernel of the Connes' map  $\mathrm{HC}_0(A') \rightarrow \mathrm{HH}_1(A')$  is  $\mathrm{HC}_2(A')$ , which is spanned by the trivial paths if  $K$  contains  $\mathbb{Q}$ . □

The second part of Lemma 2.5 is not true in positive characteristic.

**Example 2.6.** Let  $K$  be a field of characteristic  $p$  and let  $n > p$ . Consider a quiver with one vertex and one loop  $\alpha$  (this example could be extended to cycles of longer length). Then the potentials  $W = \alpha^n$  and  $W' = \alpha^n + \alpha^p$  are not (weakly) right equivalent since any automorphism would map  $W$  to a power series in  $\alpha$  whose terms have all degree at least  $n$ . However,  $W$  and  $W'$  yield the same hyperpotential, namely,  $n\alpha^{n-1}$ , hence as hyperpotentials they are right equivalent.

The next proposition characterizes (weakly) right equivalent hyperpotentials in terms of the existence of isomorphisms between their Ginzburg dg-algebras having certain prescribed values. It follows that the 3-CY triangulated categories as well as the generalized cluster categories associated to weakly equivalent hyperpotentials are equivalent. For potentials, the “only if” direction in part (a) has been shown in [28, Lemma 2.9].

**Proposition 2.7.** *Let  $Q$  and  $Q'$  be two quivers on the same set of vertices, let  $(Q, (\rho_\alpha))$  and  $(Q', (\rho'_\beta))$  be hyperpotentials and let  $\Gamma, \Gamma'$  be the corresponding Ginzburg dg-algebras.*

- (a)  *$(Q, (\rho_\alpha))$  and  $(Q', (\rho'_\beta))$  are right equivalent if and only if there exists a continuous isomorphism of dg-algebras  $\Phi: \Gamma \rightarrow \Gamma'$  such that  $\Phi(e_i) = e'_i$  and  $\Phi(t_i) = t'_i$  for all  $i \in Q_0$ .*
- (b)  *$(Q, (\rho_\alpha))$  and  $(Q', (\rho'_\beta))$  are weakly right equivalent if and only if there exist a continuous isomorphism of dg-algebras  $\Phi: \Gamma \rightarrow \Gamma'$  and an element  $c \in K^\times$  such that  $\Phi(e_i) = e'_i$  and  $\Phi(t_i) = ct'_i$  for all  $i \in Q_0$ .*

*Proof.* Let  $\Gamma = \Gamma(Q, (\rho_\alpha))$  and  $\Gamma' = \Gamma(Q', (\rho'_\beta))$  be the Ginzburg dg-algebras of two hyperpotentials. We determine the possible form of a continuous isomorphism  $\Phi: \Gamma \rightarrow \Gamma'$  such that  $\Phi(e_i) = e'_i$  and  $\Phi(t_i) = t'_i$  for all  $i \in Q_0$ . First, by looking at the degree 0 part,  $\Phi$  induces a continuous isomorphism  $\varphi: A \rightarrow A'$  with values  $\varphi(e_i) = e'_i$  and  $\varphi(\alpha) = \Phi(\alpha)$  for  $i \in Q_0$  and  $\alpha \in Q_1$ .

Now, the set of elements  $\varphi(u)\beta^*\varphi(v)$  where  $\beta \in Q'_1$  and  $u, v$  are paths in  $Q$  such that  $v$  starts at  $s(\beta)$  and  $u$  ends at  $t(\beta)$  forms a (topological) basis of the degree  $-1$  part of  $\Gamma'$ . Hence we can write for  $\alpha \in Q_1$

$$\Phi(\alpha^*) = \sum_{u,v,\beta} b_{u,v}^{\alpha,\beta} \varphi(u)\beta^*\varphi(v)$$

for some scalars  $b_{u,v}^{\alpha,\beta} \in K$ . Since  $\Phi(e_i) = e'_i$  for all  $i \in Q_0$ , the coefficient  $b_{u,v}^{\alpha,\beta}$  can be non-zero only when  $v$  ends at  $s(\alpha)$  and  $u$  starts at  $t(\alpha)$ . Thus, we can rewrite this sum as ranging over all the non-trivial paths  $p$  in  $Q$  and all their factorizations  $p = v\alpha u$  as

$$(2.3) \quad \Phi(\alpha^*) = \sum_{\beta \in Q'_1} \sum_{p=\alpha_1 \dots \alpha_n} \sum_{j: \alpha_j=\alpha} b_{p,j}^\beta \varphi(\alpha_{j+1} \dots \alpha_n) \beta^* \varphi(\alpha_1 \dots \alpha_{j-1})$$

where  $b_{p,j}^\beta \in K$  are some coefficients defined for the paths  $p = \alpha_1 \dots \alpha_n$  and  $1 \leq j \leq n$ .

We use now our assumption that  $\Phi(t_i) = t'_i$  for all  $i \in Q_0$  and the fact that  $\Phi$  should commute with the differentials, hence

$$\sum_{\beta \in Q'_1} [\beta, \beta^*] = \Phi \left( \sum_{\alpha \in Q_1} [\alpha, \alpha^*] \right) = \sum_{\alpha \in Q_1} [\varphi(\alpha), \Phi(\alpha^*)].$$

Plugging in the expression (2.3) and comparing coefficients, we deduce that  $\sum_p b_{p,1}^\beta = \beta$  and  $b_{p,j+1}^\beta = b_{p,j}^\beta$  for all  $1 \leq j < n$ , so we can rewrite (2.3) as

$$(2.4) \quad \Phi(\alpha^*) = \sum_{\beta \in Q'_1} \sum_{p=\alpha_1 \dots \alpha_n} b_p^\beta \sum_{j: \alpha_j=\alpha} \varphi(\alpha_{j+1} \dots \alpha_n) \beta^* \varphi(\alpha_1 \dots \alpha_{j-1})$$

where the coefficients  $b_p^\beta$  are defined by the equations  $\varphi^{-1}(\beta) = \sum_p b_p^\beta p$  for all  $\beta \in Q'_1$ .

Taking differentials of (2.4), noting that  $\Phi(d\alpha^*) = \varphi(\rho_\alpha)$ , we get

$$\begin{aligned} \varphi(\rho_\alpha) &= \sum_{\beta \in Q'_1} \sum_{p=\alpha_1 \dots \alpha_n} b_p^\beta \sum_{j: \alpha_j=\alpha} \varphi(\alpha_{j+1} \dots \alpha_n) \rho_\beta \varphi(\alpha_1 \dots \alpha_{j-1}) \\ &= \sum_{\beta \in Q'_1} \sum_p b_p^\beta \varphi(\Delta_\alpha(p) \diamond \varphi^{-1}(\rho_\beta)) = \sum_{\beta \in Q'_1} \varphi(\Delta_\alpha(\varphi^{-1}(\beta)) \diamond \varphi^{-1}(\rho_\beta)) \end{aligned}$$

or in other words,  $(\rho_\alpha)_{\alpha \in Q_1} = \varphi_*^{-1}((\rho_\beta)_{\beta \in Q'_1})$ .

This proves one direction in part (a). For the other direction, we define  $\Phi: \Gamma \rightarrow \Gamma'$  by specifying its values on  $e_i$ ,  $\alpha$ ,  $\alpha^*$  and  $t_i$ , namely by setting  $\Phi(e_i) = e'_i$ ,  $\Phi(t_i) = t'_i$ ,  $\Phi(\alpha) = \varphi(\alpha)$  and finally  $\Phi(\alpha^*)$  according to (2.4). One then shows that such  $\Phi$  is actually an isomorphism in the same way as in [28, Lemma 2.9].

Part (b) of the proposition now follows from the next statement which can be verified by direct calculation.  $\square$

**Lemma 2.8.** *Let  $(Q, (\rho_\alpha)_{\alpha \in Q_1})$  be a hyperpotential and let  $c \in K^\times$ . Then the continuous map of  $K$ -algebras  $\Phi: \Gamma(Q, (c\rho_\alpha)) \rightarrow \Gamma(Q, (\rho_\alpha))$  whose values on the generators are*

$$\Phi(e_i) = e_i, \quad \Phi(\alpha) = \alpha, \quad \Phi(\alpha^*) = c\alpha^*, \quad \Phi(t_i) = ct_i$$

for  $i \in Q_0$  and  $\alpha \in Q_1$ , is an isomorphism of dg-algebras.

**2.3. Proof of Proposition 2.** We will freely use some notions from the theory of quivers with potentials. For details and explanations, we refer the reader to the paper [13].

For part (a), just note that

$$\sum_{i=1}^n \alpha_i \rho_{\alpha_i} = \sum_{i=1}^n (\alpha_i \alpha_{i+1} \dots \alpha_{i-1})^e = \sum_{i=1}^n (\alpha_{i+1} \dots \alpha_{i-1} \alpha_i)^e = \sum_{i=1}^n \rho_{\alpha_i} \alpha_i$$

(indices are taken modulo  $n$ , i.e.  $\alpha_{n+1} = \alpha_1$  and  $\alpha_0 = \alpha_n$ ).

Let  $W$  be a potential on  $Q_m$ . For an arrow  $\alpha = \alpha_i$ , set

$$\omega_\alpha = \alpha_i \alpha_{i+1} \dots \alpha_{i-1} \quad \omega'_\alpha = \alpha_{i+1} \dots \alpha_{i-1}$$

so that  $\omega_\alpha = \alpha \omega'_\alpha$ . Since the cycles  $\omega_\alpha^r$  and  $\omega'_\beta^r$  are cyclically equivalent for any two arrows  $\alpha, \beta$  and  $r \geq 1$ , the potential  $W$  is cyclically equivalent to  $P(\omega_\alpha)$  for some power series  $P(x) \in K[[x]]$  which is independent on the arrow  $\alpha$ , hence we may assume that  $W = P(\omega_\alpha)$ . Taking cyclic derivatives, we see that  $\partial_\alpha W = \partial_\alpha P(\omega_\alpha) = P'(\omega_\alpha) \omega'_\alpha$  for any arrow  $\alpha$ .

If  $P'(x) = 0$ , then the Jacobian ideal vanishes and the Jacobian algebra equals the completed path algebra  $\widehat{KQ}_m$ . Otherwise,

$$P'(x) = a_0 x^{d-1} + a_1 x^d + \dots$$

for some  $a_0 \neq 0$  and  $d \geq 2$ . Note that  $d$  cannot be divisible by  $\text{char } K$ , as otherwise the derivative of  $x^d$  would vanish and so  $a_0 = 0$ , a contradiction.

We deduce that in the Jacobian algebra, for any arrow  $\alpha$

$$\rho_\alpha := \omega_\alpha^{d-1} \omega'_\alpha = -a_0^{-1} \left( \sum_{i=1}^{\infty} \omega_\alpha^i (\omega_\alpha^{d-1} \omega'_\alpha) \right) = -a_0^{-1} \left( \sum_{i=1}^{\infty} \omega_\alpha^i \right) \rho_\alpha$$

hence the path  $\rho_\alpha$  of length  $md - 1$  equals a linear combination of paths of length at least  $m(d+1) - 1$ . Since we can repeatedly substitute for  $\rho_\alpha$  the expression in the RHS, we get that for any  $N \geq 1$ , the element  $\rho_\alpha$  equals a linear combination of paths of length at least  $N$ , and hence it vanishes. Since no shorter paths are involved in any relation, we deduce that the Jacobian algebra equals  $\Lambda_{m,d}$ . This proves part (b). Part (c) follows by taking  $P(x) = x^e$ , observing that  $P'(x) = ex^{e-1}$  does not vanish by the assumption on the characteristic of  $K$ .

To show part (d), observe that by the Splitting Theorem of [13], we may assume that the potential is reduced, and hence that the quiver equals  $Q_m$ . In view of part (b), the algebra  $\Lambda_{m,e}$  is not a Jacobian algebra of any potential on  $Q_m$ .

**2.4. Proof of Proposition 3.** We recall some facts on  $\mathcal{D}^b(\text{mod } KD_n)$ , the bounded derived category of the path algebra of the Dynkin quiver  $D_n$  on  $n \geq 3$  vertices. It has a translation functor  $\Sigma$  and a Serre functor  $S$ , and they are related by  $S^{n-1} \simeq \Sigma^{n-2}$  if  $n$  is even and  $S^{2n-2} \simeq \Sigma^{2n-4}$  if  $n$  is odd (for this fractionally Calabi-Yau property, see [31]). The AR-translation on  $\mathcal{D}^b(\text{mod } KD_n)$  is given by  $\tau = S\Sigma^{-1}$ , and its AR-quiver is  $\mathbb{Z}D_n$ , see Happel [21]. The effect of  $\Sigma$  on the AR-quiver is given by  $\tau^{-n+1}$  if  $n$  is even and by  $\tau^{-n+1}\phi$  if  $n$  is odd.

Consider the auto-equivalence  $F = S^{1-m(e-1)}\Sigma^{m(e-1)-2}$  on  $\mathcal{D}^b(\text{mod } KD_{me})$  and let  $\mathcal{C}_{m,e} = \mathcal{D}^b(\text{mod } KD_{me})/F$  be the corresponding orbit category. In order to show that  $\mathcal{C}_{m,e}$  is a triangulated 2-Calabi-Yau category, we use Proposition 4 and distinguish two cases. If  $m$  is even, then, in the notation of Proposition 4,

$$(d_1, e_1) = (me - 2, me - 1), \quad (d_2, e_2) = (2 - m(e - 1), 1 - m(e - 1))$$

hence  $d_1, d_2$  are both even and  $e_1d_2 - e_2d_1 = m \neq 0$ . Therefore Proposition 4 applies and the claim follows since  $(2, 1) = (e - 1)(d_1, e_1) + e(d_2, e_2) \in L$ .

If  $m$  is odd, then by our assumption  $e$  is odd and

$$(d_1, e_1) = (2me - 4, 2me - 2), \quad (d_2, e_2) = (2 - m(e - 1), 1 - m(e - 1))$$

so again  $d_1, d_2$  are both even and  $e_1d_2 \neq e_2d_1$ . Therefore Proposition 4 applies and the claim follows from  $(2, 1) = \frac{e-1}{2}(d_1, e_1) + e(d_2, e_2) \in L$ .

The effect of  $F$  on the AR-quiver of  $\mathcal{D}^b(\text{mod } KD_{me})$  is given by that of

$$\tau^{1-m(e-1)}\Sigma^{-1} = \tau^{1-m(e-1)}\tau^{me-1}\phi^{me \bmod 2} = \tau^m\phi^{m \bmod 2} = (\tau\phi)^m$$

(using our assumption that  $m(e-1)$  is even), so by [10, Prop. 1.2] and [10, Prop. 1.3] the category  $\mathcal{C}_{m,e}$  is Krull-Schmidt and the shape of its AR-quiver is  $\mathbb{Z}D_{me}/\langle(\tau\phi)^m\rangle$ .

Choose a vertex  $v$  with  $\phi(v) \neq v$ . Since  $\mathcal{C}$  is 2-CY, the suspension of any object is isomorphic to its AR-translate. Using this, one can verify from the mesh relations in  $\mathbb{Z}D_{me}$  that the sum of the  $m$  indecomposables corresponding to the vertices

$$v, (\phi\tau)v, (\phi\tau)^2v \dots, (\phi\tau)^{m-1}v$$

is a cluster-tilting object in  $\mathcal{C}_{m,e}$  whose endomorphism algebra is isomorphic to  $\Lambda_{m,e}$ . An example for  $m = 4$  and  $e = 2$  is shown in Figure 2. Similar pictures appear in §2.1 and §2.2 of [33].

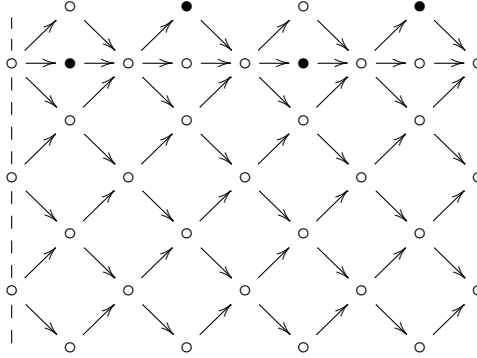


FIGURE 2. The AR-quiver of  $\mathcal{C}_{4,2}$ , where the left and right columns have to be identified along the dashed lines. The sum of the 4 indecomposables marked in  $\bullet$  is a cluster-tilting object with endomorphism algebra  $\Lambda_{4,2}$ .

2.5. **Proof of Proposition 4 and its corollary.** Let  $D = d_1e_2 - d_2e_1$ . We have

$$F^{e_1} = S^{e_1e_2}\Sigma^{-e_1d_2} \simeq \Sigma^{d_1e_2-e_1d_2} = \Sigma^D,$$

hence if  $D = 0$  then  $F^{e_1}$  is isomorphic to the identity functor and the endomorphism algebra  $\text{End}_{\mathcal{C}}(X) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\mathcal{H})}(X, F^n X)$  is infinite-dimensional for any  $X \neq 0$ .

Conversely, assume that  $D \neq 0$  and let  $X, Y \in \mathcal{D}^b(\mathcal{H})$ . Since  $\mathcal{H}$  has finite global dimension, we have

$$\text{Hom}_{\mathcal{D}^b(\mathcal{H})}(X, \Sigma^m F^s Y) = 0$$

for all  $0 \leq s < |e_1|$  and  $|m| \gg 0$ . Let  $n \in \mathbb{Z}$  and write it as  $n = e_1m + s$  with  $0 \leq s < |e_1|$ . Then

$$\text{Hom}_{\mathcal{D}^b(\mathcal{H})}(X, F^n Y) = \text{Hom}_{\mathcal{D}^b(\mathcal{H})}(X, F^{e_1m+s} Y) = \text{Hom}_{\mathcal{D}^b(\mathcal{H})}(X, \Sigma^{Dm} F^s Y)$$

vanishes when  $|n| \gg 0$  and therefore  $\text{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\mathcal{H})}(X, F^n Y)$  is finite-dimensional. This shows the equivalence of the conditions in the proposition.

From now on assume that  $D \neq 0$ . In order to show that the orbit category  $\mathcal{C}$  is triangulated we use a result of Keller [24, Theorem 1]. We have to verify conditions (2) and (3) in that theorem. Indeed, condition (2) holds since if  $U \in \mathcal{H}$ , then  $F^{e_1}U \simeq \Sigma^D U$  hence only finitely many objects  $F^i U$  ( $i \in \mathbb{Z}$ ) could lie in  $\mathcal{H}$ . Moreover, since  $\mathcal{H}$  is hereditary, any indecomposable of  $\mathcal{D}^b(\mathcal{H})$  is of the form  $\Sigma^n U$  for an indecomposable  $U$  of  $\mathcal{H}$  and some  $n \in \mathbb{Z}$ . Writing  $n = Dq + t$  with  $0 \leq t < |D|$  we see that  $F^{-e_1q}(\Sigma^n U) \simeq \Sigma^t U$  hence condition (3) is satisfied with the integer  $|D|$ .

In order to show that  $\mathcal{C}$  is  $(d, e)$ -CY for any  $(d, e) \in L$  we use a result of Dugas [14, §9]. The suspension  $\bar{\Sigma}$  and the Serre functor  $\bar{S}$  on  $\mathcal{C}$  are induced by  $\Sigma$  and  $S$ , respectively, and by our assumption  $d_2(e_2 - d_2)$  is even. We can therefore use Theorem 9.5 of [14] to deduce that  $\bar{S}^{e_2} \simeq \bar{\Sigma}^{d_2}$ . Now rewrite  $F$  as

$$F = S^{e_2-e_1} S^{e_1} \Sigma^{-d_2} \simeq S^{e_2-e_1} \Sigma^{d_1-d_2}$$

and since  $(d_1 - d_2)(d_1 - d_2 + e_2 - e_1)$  is even, by the same theorem we deduce that  $\bar{S}^{e_2 - e_1} \simeq \bar{\Sigma}^{d_2 - d_1}$ . Since  $L$  is generated by  $(d_2, e_2)$  and  $(d_2 - d_1, e_2 - e_1)$ , we conclude that  $\mathcal{C}$  is  $(d, e)$ -CY for any  $(d, e) \in L$ .

Note that in any case, even if we do not assume any parity restrictions on  $d_1, d_2, e_1$  and  $e_2$ , Theorem 9.5 of [14] implies that  $\bar{S}^{2e_2} \simeq \bar{\Sigma}^{2d_2}$  and  $\bar{S}^{2(e_2 - e_1)} \simeq \bar{\Sigma}^{2(d_2 - d_1)}$ , hence  $\mathcal{C}$  is  $(d, e)$ -CY for any  $(d, e) \in 2L$ .

The corollary follows from the following observation. We have  $L \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}(d_1, e_1) + \mathbb{Q}(d_2, e_2) = \mathbb{Q}^2$  since  $\text{rank } L = 2$ . Therefore for any  $r \in \mathbb{Q}$  there are rationals  $s_1, s_2 \in \mathbb{Q}$  such that  $(r, 1) = s_1(d_1, e_1) + s_2(d_2, e_2)$ . Multiplying by a common denominator of  $s_1$  and  $s_2$  we deduce that  $(nr, n) \in L$  for some integer  $n \neq 0$ .

**2.6. A cluster category of type  $G_2$ .** There are several approaches to categorify cluster algebras corresponding to non simply-laced Dynkin diagrams, see for example the work by Demonet [12].

We proceed as in the proof of Proposition 3, starting with a Dynkin quiver of type  $E_8$ . Here,  $\mathcal{D}^b(\text{mod } KE_8)$  has translation functor  $\Sigma$  and Serre functor  $S$  which are related by  $S^{15} \simeq \Sigma^{14}$ . Consider the auto-equivalence  $F = S^4 \Sigma^{-4}$  and let  $\mathcal{C} = \mathcal{D}^b(\text{mod } KE_8)/F$  be the corresponding orbit category. From Proposition 4 we deduce that  $\mathcal{C}$  is a triangulated 2-CY category, as we have  $(2, 1) = 4 \cdot (4, 4) - (14, 15)$ . The effect of  $F$  on the AR-quiver of  $\mathcal{D}^b(\text{mod } KE_8)$ , which is  $\mathbb{Z}E_8$ , is given by  $\tau^4$ , hence by [10, Prop. 1.3] the AR-quiver of  $\mathcal{C}$  has the shape  $\mathbb{Z}E_8/\langle \tau^4 \rangle$ .

In order to determine the cluster-tilting objects in  $\mathcal{C}$ , we make the following observations which could be verified on a computer, and refer to Figure 3 for details. We denote by  $\bar{\Sigma}$  the translation on  $\mathcal{C}$  and observe that  $\bar{\Sigma}^4$  is the identity on  $\mathcal{C}$  since  $(4, 0) = 15 \cdot (4, 4) - 4 \cdot (14, 15)$ .

1. The only indecomposable objects  $Z$  which are rigid (i.e.  $\text{Hom}_{\mathcal{C}}(Z, \bar{\Sigma}Z) = 0$ ) are of the form  $\bar{\Sigma}^i X$  or  $\bar{\Sigma}^i Y$  ( $0 \leq i < 4$ ), where  $X$  and  $Y$  are as in Figure 3.
2. The indecomposables  $Z$  such that  $\text{Hom}_{\mathcal{C}}(X, Z) = 0$  are shown in Fig. 3(a).
3. The indecomposables  $Z$  such that  $\text{Hom}_{\mathcal{C}}(Y, Z) = 0$  are shown in Fig. 3(b).

It follows that the cluster-tilting objects in  $\mathcal{C}$  are  $\bar{\Sigma}^i X \oplus \bar{\Sigma}^j Y$ , where  $0 \leq i < 4$  and  $j \in \{i, i + 1\}$ , their endomorphism algebras are given by the quivers

$$(\star) \quad \bullet \xrightarrow{\alpha} \bullet \begin{array}{c} \circlearrowleft \\ \beta \end{array}$$

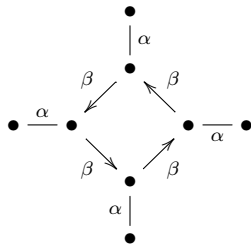
(where the edge  $\alpha$  can be oriented arbitrarily) with the relation  $\beta^3 = 0$ , and their exchange graph is an octagon as shown in Figure 4.

The exchange graph of the cluster algebra of type  $G_2$  is also an octagon (see e.g. the last example in [25, §2]) and the connection is explained by the following observation. It is possible to compute the cluster character in the sense of Palu [32] corresponding to the cluster-tilting object  $X \oplus \bar{\Sigma}Y$ . With this choice, the AR-quiver of the resulting 2-CY-tilted algebra is the one shown in [16, Fig. 19], so we can use the dimension vectors listed there to aid in the calculations. More calculations are simplified by the properties of a cluster character. The resulting values on the rigid indecomposable objects of  $\mathcal{C}$  are shown in Figure 5, so there is a bijection compatible with mutations between the rigid indecomposable objects of  $\mathcal{C}$  and the cluster variables in the cluster algebra of type  $G_2$  such that (basic) cluster-tilting objects in  $\mathcal{C}$  correspond to the clusters.





Note that  $F^4 = S^{16}\Sigma^{-16} \simeq S\Sigma^{-2}$ , hence as in the cases treated by Bertani-Økland and Oppermann in [7], there is a covering functor from the cluster category of type  $E_8$  to  $\mathcal{C}$ . Each of the algebras depicted in  $(\star)$  is obtained from a corresponding cluster-tilted algebra of type  $E_8$  whose quiver is shown below (where the edges labeled  $\alpha$  should have the same orientation either pointing inwards or outwards with respect to the inner square)



by identifying all the arrows with the same label (equivalently, by dividing modulo the action of the cyclic group  $\mathbb{Z}/4\mathbb{Z}$  on the quiver by rotations). These algebras appear as items 1562 and 1574 in the lists supplementing the paper [6].

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