Algebras of quasi-quaternion type

Sefi LADKANI

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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SEFI LADKANI

ABSTRACT. We define algebras of quasi-quaternion type, which are symmetric algebras of tame representation type whose stable module category has certain structure similar to that of the algebras of quaternion type introduced by Erdmann. We observe that symmetric tame algebras that are also 2-CY-tilted are of quasi-quaternion type.

We present a combinatorial construction of such algebras by introducing the notion of triangulation quivers. The class of algebras that we get contains Erdmann's algebras of quaternion type on the one hand and the Jacobian algebras of the quivers with potentials associated by Labardini to triangulations of closed surfaces with punctures on the other hand, hence it serves as a bridge between modular representation theory of finite groups and cluster algebras.

1. INTRODUCTION

The purpose of this note is to report on some connections between representation theory of groups and cluster algebras, more precisely, between algebras of quaternion type introduced and studied by Erdmann [7] and others and 2-CY-tilted algebras which are endomorphism algebras of cluster-tilting objects in 2-Calabi-Yau categories arising in the additive categorification of cluster algebras.

Recall that an algebra of quaternion type is a tame, symmetric, indecomposable algebra whose non-projective modules are Ω-periodic with period dividing 4 and its Cartan matrix is non-singular. The possible quivers with relations of such algebras were classified by Erdmann [7], and later works of Holm [13] and Erdmann-Skowronski [8] established that the algebras given in those lists are actually of quaternion type.

It seems natural to remove the condition that the Cartan matrix is non-singular and to consider tame, symmetric, indecomposable algebras whose non-projective modules are Ω-periodic of period dividing 4. In terms of the stable module category, the last condition means that the 4-th power of the suspension (shift) functor acts as the identity on objects. Such algebras will be called *algebras of quasi-quaternion type*.

We construct a large class of algebras of quasi-quaternion type that are also 2-CY-tilted. It turns out that this class contains in particular:

- All the algebras appearing in Erdmann's lists of algebras of quaternion type [7];
- All the Jacobian algebras of the quivers with potentials associated by Labardini to triangulations of closed surfaces with punctures [18].

Our construction has several consequences, both for the representation theory of finite-dimensional algebras as well as for theory of quivers with potentials. Namely, we obtain:

1. A new proof that the algebras in Erdmann's lists are of quaternion type;

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2. New tame symmetric algebras with periodic modules which seem not to appear in the classification announced by Erdmann and Skowronski [9];
3. New symmetric 2-CY-tilted algebras in addition to the ones arising from odd-dimensional isolated hypersurface singularities [4];
4. Infinitely many non-degenerate potentials with pairwise non-isomorphic Jacobian algebras on the adjacency quiver [10] of any triangulation of a closed surface with exactly one puncture (and arbitrary genus).

We observe that the property of being of quasi-quaternion type is preserved under derived equivalences (see below), hence our strategy is to construct some of these algebras from combinatorial data and then produce more algebras using derived equivalences. To this end we introduce *triangulation quivers*. These are quivers having the property that at any vertex there are exactly two incoming arrows and two outgoing arrows, together with the data of a permutation f on the set of arrows such that $f(\alpha)$ starts where an arrow α ends, subject to the condition that f^3 is the identity (this last condition justifies the term "triangulation"). These data give rise to another permutation g and an involution $\alpha \mapsto \bar{\alpha}$ on the set of arrows, see Section 2.1.

A triangulation quiver can be dually encoded as a ribbon graph whose nodes are the cycles of the permutation g, its edges are the vertices of the quiver and the cyclic ordering of the edges around each node is induced by g. Thus functions on the nodes can be viewed as functions on the arrows that are constant on g-cycles. Given multiplicities and scalars associated to the nodes, one can construct from such data a Brauer graph algebra. We construct another algebra which we call *triangulation algebra* and prefer to work in a complete setting; each arrow of the quiver gives rise to a certain commutativity relation and the algebra is defined as the quotient of the complete path algebra by the closure of the ideal generated by these commutativity relations. *A-priori* it is not clear that the triangulation algebra is finite-dimensional, but it turns out that for most triangulation quivers and multiplicities, the triangulation algebra satisfies certain additional zero-relations of length 3 which allow to prove that it is finite-dimensional.

Our main results concerning triangulation algebras are summarized in the next theorem. For the precise definitions of the terms occurring in the formulation, we refer the reader to Section 2.2.

Theorem 1.1. *Let (Q, f) be a connected triangulation quiver, let K be a field, let $m\colon Q_1 \to \mathbb{Z}_{>0}$ and $c\colon Q_1 \to K^{\times}$ be g-invariant functions of multiplicities and scalars, and assume that m is admissible. Assume further that the associated ribbon graph with multiplicities is not one of the two exceptional cases shown in Figure 2 and consider the corresponding triangulation algebra Λ defined by*

$$\Lambda = \widehat{KQ}/\overline{\langle \bar{\alpha} \cdot f(\bar{\alpha}) - c_\alpha \omega_\alpha^{m_\alpha - 1} \cdot \omega'_\alpha \rangle}_{\alpha \in Q_1}.$$

(a) *Λ is finite dimensional; it has a presentation as quiver with relations*

$$\Lambda \simeq KQ/\langle \alpha \cdot f(\alpha) \cdot gf(\alpha)\, ,\, \bar{\alpha} \cdot f(\bar{\alpha}) - c_\alpha \omega_\alpha^{m_\alpha - 1} \cdot \omega'_\alpha \rangle_{\alpha \in Q_1} \tag{1.1}$$

(b) *Λ is symmetric.*

(c) *Λ degenerates to the corresponding Brauer graph algebra Γ given by*

$$\Gamma = KQ/\langle \alpha \cdot f(\alpha)\, ,\, c_\alpha \omega_\alpha^{m_\alpha} - c_{\bar{\alpha}} \omega_{\bar{\alpha}}^{m_{\bar{\alpha}}} \rangle_{\alpha \in Q_1}$$

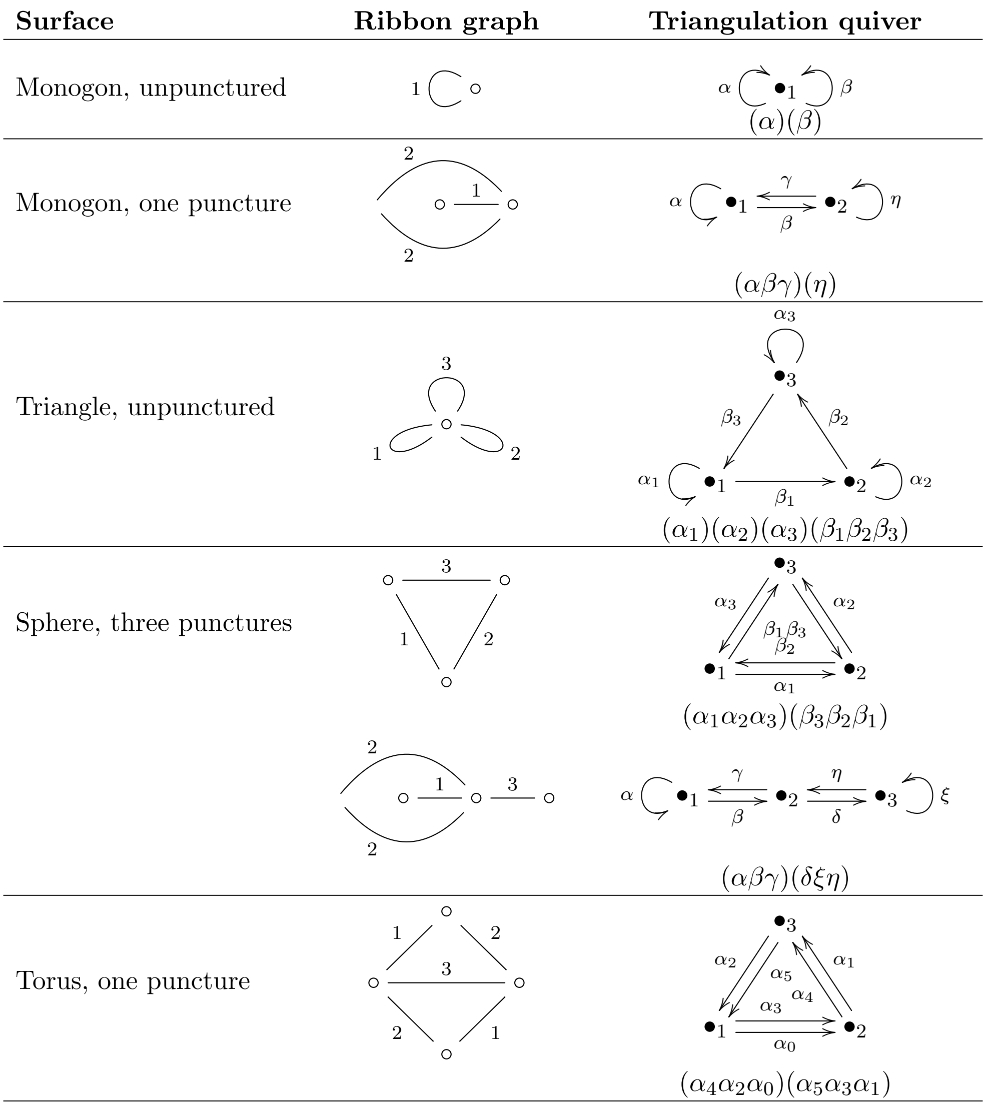

Figure 1. The triangulation quivers with at most 3 vertices. We list the marked surface, the ribbon graph(s) corresponding to its triangulation(s) and the associated triangulation quivers, where we write the permutation f in cycle form below each quiver. For the torus, all nodes in the ribbon graph should be identified and edges with the same label are also identified.

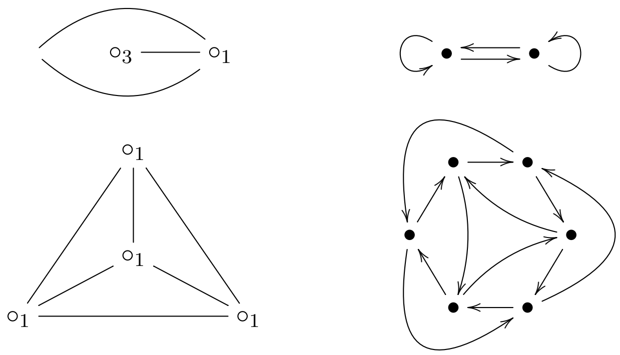

FIGURE 2. Exceptional ribbon graphs with multiplicities: A monogon with one puncture (top) and a tetrahedron, which is a triangulation of a sphere with 4 punctures (bottom). The ribbon graph is shown on the left, where at each node we indicate its multiplicity. The corresponding triangulation quiver is shown on the right.

and hence Λ *is of tame representation type.*

(d) *The elements* $\rho_\alpha = f(\alpha)f^2(\alpha) - c_\alpha \omega_{g(\alpha)}^{m_\alpha - 1}\omega'_{g(\alpha)}$ *satisfy* $\sum_{\alpha \in Q_1}[\alpha, \rho_\alpha] = 0$ *in* KQ*, hence* Λ *is a Jacobian algebra of a hyperpotential (see* [19] *for the definition) and therefore it is 2-CY-tilted, i.e. there is a 2-Calabi-Yau triangulated category* $\mathcal{C}$ *and a cluster-tilting object* T *in* $\mathcal{C}$ *such that* $\Lambda \simeq \operatorname{End}_{\mathcal{C}}(T)$.

(e) Λ *is of quasi-quaternion type.*

(f) *More generally, for any cluster-tilting object* T' *in* $\mathcal{C}$ *which is reachable from* T *by a sequence of mutations, the algebra* $\operatorname{End}_{\mathcal{C}}(T')$ *is derived equivalent to* Λ *and of quasi-quaternion type.*

The exceptional cases are dealt with in the next proposition.

Proposition 1.2. *Let* (Q, f) *be a connected triangulation quiver, let* K *be a field, let* $m\colon Q_1 \to \mathbb{Z}_{>0}$ *and* $c\colon Q_1 \to K^\times$ *be* g*-invariant functions of multiplicities and scalars. Assume that the associated ribbon graph with multiplicities is one of the two exceptional cases shown in Figure 2 and that moreover:*

- $\prod_{\alpha \in Q_1} c_\alpha \neq 1$ *in the punctured monogon case; or*
- $c_\alpha c_{\bar{\alpha}} c_{f(\alpha)} c_{f(\bar{\alpha})} \neq 1$ *for some* $\alpha \in Q_1$ *in the tetrahedron case.*

Then the statements of Theorem 1.1 hold for the triangulation algebra Λ *with the following modifications of claims* (a) *and* (c)*:*

(a′) Λ *is finite dimensional; it has a presentation as quiver with relations*

$$\Lambda \simeq KQ/\langle \bar{\alpha} \cdot f(\bar{\alpha}) - c_\alpha \omega_\alpha^{m_\alpha - 1} \cdot \omega'_\alpha \rangle_{\alpha \in Q_1}$$

and the zero relations $\alpha \cdot f(\alpha) \cdot gf(\alpha)$ *follow from the commutativity relations.*

(c′) Λ *is of tame representation type.*

One could also formulate a slightly more general version of Theorem 1.1 and Proposition 1.2 by replacing the multiplicities and the scalars by power series in $K[[x]]$, i.e. replace the functions m and c with a g-invariant function $q\colon Q_1 \to K[[x]]$ and consider the algebra

$$\Lambda = \widehat{KQ}/\overline{\langle \bar{\alpha} \cdot f(\bar{\alpha}) - q_\alpha(\omega_\alpha) \cdot \omega'_\alpha \rangle}_{\alpha \in Q_1}, \tag{1.2}$$

so that the case treated here corresponds to the choice of $q_\alpha(x) = c_\alpha x^{m_\alpha - 1}$. However, in most cases the algebra Λ in (1.2) depends only on the leading term of each power series $q_\alpha(x)$, so for simplicity we chose not to formulate the results in full generality. We hope to report on the general case in a later version.

Any triangulation of a marked surface in the sense of Fomin, Shapiro and Thurston [10] gives rise to a triangulation quiver (see Section 2.1) and hence, by choosing multiplicities and scalars, to algebras of quasi-quaternion type. Hence, as opposed to algebras of quaternion type, there are algebras of quasi-quaternion type with arbitrarily many non-isomorphic simple modules.

The triangulation quivers with small number of vertices can be enumerated, see Figure 1 for the quivers with at most three vertices. In particular, some algebras of quaternion type with 1, 2 or 3 vertices arise from a monogon, a punctured monogon or a sphere with three punctures, respectively, see Example 2.11. Some of the triangulation algebras arising from a punctured monogon or a sphere with 3 punctures arise also from minimally elliptic curve singularities, see Section 7 of [4].

In general, the triangulation quiver constructed from a triangulation differs from the adjacency quiver constructed in [10]. However, for a triangulation of a closed surface satisfying a technical condition called (T3) in our work [20] these two quivers coincide and the triangulation algebra (where all multiplicities are set to 1) coincides with the Jacobian algebra of the potential constructed by Labardini [18]. Since any closed surface considered in [18] admits such a triangulation and any other triangulation can be obtained from it by a sequence of flips, by using the facts that a flip of triangulations results in a mutation of the corresponding quivers with potentials [18] and that mutation of quivers with potentials is compatible with mutation of cluster-tilting objects [3] we get the following result.

Corollary 1.3. *Consider a closed surface which is not a sphere with less than* 4 *punctures. Then the Jacobian algebras arising from its triangulations are of quasi-quaternion type and they are all derived equivalent to each other. Moreover, they arise as algebras in part* (f) *of Theorem 1.1 for a suitable triangulation quiver.*

Note that for the proof of this result one does not need to know that the potentials are non-degenerate. Note also that for a sphere with 4 punctures one has to use Proposition 1.2 and impose the corresponding restriction on the scalars.

The proof of parts (a) and (b) of Theorem 1.1 is similar to our proofs in the case of the Jacobian algebras of the quivers with potentials arising from triangulations of closed surfaces [20]. We note that in the presentation (1.1) it is enough to require only one zero relation $\alpha \cdot f(\alpha) \cdot gf(\alpha)$, as the rest would follow from that relation and the commutativity relations.

In [13] Holm establishes the tameness of the algebras of quaternion type by showing that some of them degenerate to algebras of dihedral type and then applying a result of Geiss [11]. Part (c) can be seen as a generalization of this statement to arbitrary triangulation quivers. We note that connections between Brauer graph algebras and cluster mutations have also been discovered by Marsh and Schroll [22]. For the two exceptional cases considered in Proposition 1.2, the statement (c′) holds since the corresponding triangulation algebras are of tubular type [2].

In part (d) we use the notion of a hyperpotential introduced in [19] in order to formulate the results in a characteristic-free form. In particular, we get that 2-blocks with quaternion defect group are 2-CY-tilted. The triangulation algebra Λ is a Jacobian algebra of a quiver with potential as defined in [5] when the characteristic of the ground field K is zero or does not divide any of the multiplicities m_α. In that case a potential can be written as

$$\sum_{\alpha} \alpha \cdot f(\alpha) \cdot f^2(\alpha) - \sum_{\beta} m_\beta^{-1} c_\beta \omega_\beta^{m_\beta} \tag{1.3}$$

where the sums run over representatives of f-cycles and g-cycles, respectively.

Parts (e) and (f) are consequences of parts (b), (c) and (d). In fact, the statements therein hold more generally for any tame symmetric 2-CY-tilted algebra Λ. Part (e) follows from the next proposition which records some observations on symmetric algebras that are also 2-CY-tilted.

Proposition 1.4. *Let Λ be a finite-dimensional symmetric algebra that is also 2-CY-tilted, i.e. $\Lambda = \mathrm{End}_{\mathcal{C}}(T)$ for some cluster-tilting object T within a triangulated 2-Calabi-Yau category $\mathcal{C}$ with suspension functor Σ.*

(a) *The functor Ω^4 on the stable module category $\underline{\mathrm{mod}}\,\Lambda$ is isomorphic to the identity, hence all non-projective Λ-modules are Ω-periodic with period dividing 4.*
(b) *The functor Σ^2 acts as the identity on the objects of $\mathcal{C}$.*
(c) *Assume that Λ is a Jacobian algebra of a hyperpotential. Then it is rigid if and only if Λ is semi-simple.*

Here, by *rigid* we mean that $\mathrm{HH}_0(\Lambda) = \Lambda/[\Lambda,\Lambda]$ is spanned by the images of the primitive idempotents corresponding to the vertices. This definition is equivalent to the one in [5] for finite-dimensional Jacobian algebras of quivers with potentials. Parts (a) and (b) of the proposition have also been recently observed by Valdivieso-Diaz [24].

The derived equivalences in part (f) are instances of (refined version of) good mutations introduced in our previous work [21]. They follow from a more general statement concerning the derived equivalences of neighboring 2-CY-tilted algebras which is an improvement of [21, Theorem 5.3]. Before stating the theorem, we recall some relevant notions.

Let Λ be a basic algebra and P an indecomposable projective Λ-module and write $\Lambda = P \oplus Q$. Consider the silting mutations in the sense of Aihara and Iyama [1] of Λ at P within the triangulated category $\mathrm{per}\,\Lambda$ of perfect complexes, which are the following two-term complexes

$$U_P^-(\Lambda) = (P \to Q') \oplus Q, \qquad\qquad U_P^+(\Lambda) = (Q'' \to P) \oplus Q,$$

where $Q', Q'' \in \mathrm{add}\,Q$, the maps are left/right $(\mathrm{add}\,Q)$-approximations and Q, Q', Q'' are in degree 0. These two-term complexes of projective modules are known also as Okuyama-Rickard complexes. In [21] we considered these complexes in relation with our definition of mutations of algebras.

Theorem 1.5. *Let T be a cluster-tilting object in a 2-Calabi-Yau category $\mathcal{C}$, let X be an indecomposable summand of T and let T' be the cluster-tilting object which is the Iyama-Yoshino mutation* [15] *of T at X.*

Consider the 2-CY-tilted algebras $\Lambda = \operatorname{End}_{\mathcal{C}}(T)$ and $\Lambda' = \operatorname{End}_{\mathcal{C}}(T')$. Let P be the indecomposable projective Λ-module corresponding to X and let P' be the indecomposable projective Λ'-module corresponding to X.

(a) *If $U_P^-(\Lambda)$ and $U_{P'}^+(\Lambda')$ are tilting complexes, then*
$$\operatorname{End}_{\operatorname{per}\Lambda} U_P^-(\Lambda) \simeq \Lambda' \qquad \text{and} \qquad \operatorname{End}_{\operatorname{per}\Lambda'} U_{P'}^+(\Lambda') \simeq \Lambda.$$
(b) *If $U_P^+(\Lambda)$ and $U_{P'}^-(\Lambda')$ are tilting complexes, then*
$$\operatorname{End}_{\operatorname{per}\Lambda} U_P^+(\Lambda) \simeq \Lambda' \qquad \text{and} \qquad \operatorname{End}_{\operatorname{per}\Lambda'} U_{P'}^-(\Lambda') \simeq \Lambda.$$
(c) *If Λ is weakly symmetric, then by* [12] *Λ' is also weakly symmetric, hence all the complexes $U_P^-(\Lambda)$, $U_P^+(\Lambda)$, $U_{P'}^-(\Lambda')$ and $U_{P'}^+(\Lambda')$ are tilting complexes and*
$$\operatorname{End}_{\operatorname{per}\Lambda} U_P^-(\Lambda) \simeq \Lambda' \simeq \operatorname{End}_{\operatorname{per}\Lambda} U_P^+(\Lambda).$$
In particular, Λ and Λ' are derived equivalent.

(d) *If Λ is symmetric then Λ' is symmetric.*

We note that there are related works by Dugas [6] concerning derived equivalences of symmetric algebras and by Mizuno [23] concerning derived equivalences of self-injective quivers with potential.

The category of perfect complexes over a symmetric algebra is 0-Calabi-Yau, hence the derived equivalences in part (c) can be considered as 0-CY analogs of the derived equivalences of Iyama-Reiten [14] and Keller-Yang [17] for 3-CY-algebras.

Rephrasing part (c), we see that if Λ is a (weakly) symmetric 2-CY-tilted algebra and P an indecomposable projective Λ-module, then the algebras $\operatorname{End}_{\operatorname{per}\Lambda} U_P^-(\Lambda)$ and $\operatorname{End}_{\operatorname{per}\Lambda} U_P^+(\Lambda)$ are isomorphic to each other, 2-CY-tilted and derived equivalent to Λ. A careful look at the derived equivalences constructed by Holm [13] for algebras of quaternion type shows that all of them arise from tilting complexes of the form appearing in part (c) above. Since the representatives of the derived classes are triangulation algebras and hence 2-CY-tilted, we deduce that all the algebras of quaternion type are of the form given in Theorem 1.1(f) and in particular they are 2-CY-tilted.

Many of the algebras occurring in part (f) of Theorem 1.1 are themselves triangulation algebras. In fact, one can define a notion of mutation of triangulation quivers that will lead to mutation of the potentials (1.3), see Section 2.3.

Finally, we note that an argument as in Prop. 2.1 and Prop. 2.2 of [13] yields the following observation.

Proposition 1.6. *Any algebra which is derived equivalent to an algebra of quasi-quaternion type is also of quasi-quaternion type.*

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The report was written during my visit to IHES at Bures-sur-Yvette. I would like to thank the IHES for the hospitality and the inspiring atmosphere. This report has been completed within the last days of my long pleasant postdoctoral stay in Europe for over 6 years. I hope to find some time in the future to produce a more detailed version.

I discussed various aspects of this work with Thorsten Holm, Maxim Kontsevich, Robert Marsh and Andrzej Skowronski. I thank them for their interest.

2. Combinatorial construction of algebras of quasi-quaternion type

2.1. **Ribbon quivers and triangulation quivers.** A *quiver* is a finite directed graph. More precisely, it is a quadruple $Q = (Q_0, Q_1, s, t)$ where Q_0 and Q_1 are finite sets (of *vertices* and *arrows*, respectively) and $s, t \colon Q_1 \to Q_0$ are functions specifying for each arrow its starting and terminating vertex, respectively.

Definition 2.1. A *ribbon quiver* is a pair (Q, f) consisting of a quiver Q and a permutation $f \colon Q_1 \to Q_1$ on its set of arrows satisfying the following conditions:

(i) At each vertex $i \in Q_0$ there are exactly two arrows starting at i and two arrows ending at i;

(ii) For each arrow $\alpha \in Q_1$, the arrow $f(\alpha)$ starts where α ends.

Note that loops are allowed in Q. A loop at a vertex is counted both as an incoming and outgoing arrow at that vertex.

Let (Q, f) be a ribbon quiver. Since at each vertex of Q there are exactly two outgoing arrows, there is an involution $\alpha \mapsto \bar{\alpha}$ on Q_1 mapping each arrow α to the other arrow starting at the vertex $s(\alpha)$. Composing it with f gives rise to the permutation $g \colon Q_1 \to Q_1$ given by $g(\alpha) = \overline{f(\alpha)}$ so that for each arrow α, the set $\{f(\alpha), g(\alpha)\}$ consists of the two arrows starting at the vertex which α ends at.

Given a quiver Q satisfying condition (i) in the definition, the data of the permutation f is equivalent to the data of the permutation g. Thus from now on when considering a ribbon quiver (Q, f) we will freely refer to the involution $\alpha \mapsto \bar{\alpha}$ and the permutation g as defined above.

Ribbon quivers are closely related to ribbon graphs. Informally speaking, a ribbon graph is a graph consisting of nodes and edges together with a cyclic ordering of the edges around each node. This can be made more formal in the next definition.

Definition 2.2. A *ribbon graph* is a triple (H, ι, σ) where H is a finite set, ι is an involution on H without fixed points and σ is a permutation on H.

The elements of H are called *half edges*. A ribbon graph gives rise to a graph (V, E) (possibly with loops and multiple edges between nodes) as follows. The set E of edges consists of the cycles of ι and the set V of nodes consists of the cycles of σ. An edge $e \in E$ can be written as $(h\, \iota(h))$ for some $h \in H$. The σ-cycles that h and $\iota(h)$ belong to are the nodes that e is incident to. Finally, the cyclic ordering around each node is induced by σ.

Proposition 2.3. *The notions of ribbon quiver and ribbon graph are equivalent.*

Proof. A ribbon quiver (Q, f) gives rise to a ribbon graph (H, ι, σ) by taking $H = Q_1$ and defining $\iota(\alpha) = \bar{\alpha}$ and $\sigma(\alpha) = \overline{f(\alpha)}$ for each $\alpha \in Q_1$.

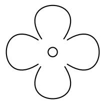

FIGURE 3. A node with 4 loops corresponds to a boundary component with 4 marked points.

Conversely, a ribbon graph (H, ι, σ) gives rise to a ribbon quiver (Q, f) as follows. Set $Q_1 = H$ and take Q_0 to be the set of cycles of ι. Define the maps $s, t \colon Q_1 \to Q_0$ and the permutation $f \colon Q_1 \to Q_1$ by letting, for $h \in H$, $s(h)$ to be the ι-cycle that h belongs to and setting $t = s\sigma$ and $f = \iota\sigma$.

We finally note that these two constructions are inverses of each other. □

We will focus on a subclass of ribbon quivers formed by what we call triangulation quivers.

Definition 2.4. A *triangulation quiver* is a ribbon quiver (Q, f) such that f^3 is the identity on the set of arrows.

As their name suggests, triangulation quivers naturally arise from triangulations of marked surfaces. Following Fomin, Shapiro and Thurston [10], a *marked surface* is a pair (S, M) consisting of a compact, connected, oriented, Riemann surface S (possibly with boundary) and a finite set M of points in S, called *marked points*, such that each connected component of the boundary of S contains at least one point from M. The points in M which are not on the boundary of S are called *punctures*.

We refer to [10] for the notion of (ideal) *triangulation* of a marked surface.

Proposition 2.5. *A triangulation of a marked surface gives rise to a ribbon graph whose associated ribbon quiver is a triangulation quiver.*

Proof. Consider a triangulation τ of a marked surface (S, M). We associate to τ a ribbon graph as follows: the nodes are the punctures in M and the connected components of the boundary of S, and the edges are the arcs of τ as well as the boundary segments (sides of triangles which are part of the boundary).

For each boundary segment on a boundary component C we draw the corresponding edge as a loop incident to the node corresponding to C. In this way each marked point p on C could be identified with the "space" between the consecutive loops corresponding to the two boundary segments which have p as endpoint, see an example in Figure 3.

Using this identification, we can now draw the edges corresponding to arcs, placing them correctly between the loops (if an endpoint of the arc is on a boundary). The cyclic ordering at each node is the counterclockwise ordering induced by the orientation of S.

The vertices of the corresponding ribbon quiver are the arcs of τ as well as the boundary segments. At each vertex corresponding to a boundary segment where is a loop δ with $f(\delta) = \delta$, and each triangle in τ with sides v_1, v_2, v_3 (which may be arcs or boundary segments) arranged in a clockwise order gives rise to three arrows $v_1 \xrightarrow{\alpha} v_2$, $v_2 \xrightarrow{\beta} v_3$ and $v_3 \xrightarrow{\gamma} v_1$ with $f(\alpha) = \beta$, $f(\beta) = \gamma$ and $f(\gamma) = \alpha$. □

The construction of the triangulation quiver of an ideal triangulation resembles that of the adjacency quiver defined in [10], however there are several differences:

1. In the triangulation quiver there are vertices corresponding to the boundary segments and not only to the arcs.
2. Our treatment of self-folded triangles is different; in the triangulation quiver there is a loop at each vertex corresponding to the inner side of a self-folded triangle.
3. We do not delete 2-cycles that arise in the quiver (e.g. when there are precisely two arcs incident to a puncture).

These differences allow to attach triangulation quivers to marked surfaces that do not admit adjacency quivers, such as a monogon, a triangle or a sphere with three punctures, see Figure 1. On the other hand, there are situations where the triangulation quiver and the adjacency quiver of a triangulation coincide.

Lemma 2.6. *The triangulation quiver equals the adjacency quiver for any triangulation of a closed surface (i.e. with empty boundary) with at least three arcs incident to each puncture.*

The condition in the lemma was called (T3) in our work [20]. In particular, we get the following corollary.

Corollary 2.7. *For a closed surface with exactly one puncture, the triangulation quiver and the adjacency quiver associated to any triangulation coincide.*

2.2. **Brauer graph algebras and triangulation algebras.**

Definition 2.8. Let (Q, f) be a ribbon quiver. A function $\nu \colon \alpha \mapsto \nu_\alpha$ on Q_1 is called *g-invariant* if $\nu_{g(\alpha)} = \nu_\alpha$ for any arrow α.

A g-invariant function can thus be regarded as a function on the nodes of the associated ribbon graph.

Let (Q, f) be a ribbon quiver. For an arrow $\alpha \in Q_1$, set

$$\begin{aligned} n_\alpha &= \min\{n > 0 \,:\, g^n(\alpha) = \alpha\} \\ \omega_\alpha &= \alpha \cdot g(\alpha) \cdot \ldots \cdot g^{n_\alpha - 1}(\alpha) \\ \omega'_\alpha &= \alpha \cdot g(\alpha) \cdot \ldots \cdot g^{n_\alpha - 2}(\alpha) \end{aligned}$$

The function $\alpha \mapsto n_\alpha$ is obviously g-invariant, telling the length of the g-cycle ω_α starting at α. The path ω'_α is "almost" cycle; when $n_\alpha = 1$ the arrow α is a loop at some vertex i and ω'_α is understood to be the path of length zero starting at i.

Let K be a field. For a quiver Q, denote by KQ its path algebra over K and by $\widehat{KQ}$ the completed path algebra. The elements of KQ are finite K-linear combinations of paths in Q whereas those of $\widehat{KQ}$ are possibly infinite such combinations.

Definition 2.9. Let (Q, f) be a ribbon quiver, and let $m \colon Q_1 \to \mathbb{Z}_{>0}$ and $c \colon Q_1 \to K^\times$ be g-invariant functions of *multiplicities* and *scalars*, respectively. The *graph algebra* associated to these data is defined by

$$\Gamma(Q, f, m, c) = KQ / \langle \alpha \cdot f(\alpha) \,,\, c_\alpha \omega_\alpha^{m_\alpha} - c_{\bar{\alpha}} \omega_{\bar{\alpha}}^{m_{\bar{\alpha}}} \rangle_{\alpha \in Q_1}.$$

In other words, the graph algebra is the Brauer graph algebra [16] associated to the corresponding ribbon graph. In particular, it is special biserial and hence of tame representation type.

Definition 2.10. Let (Q, f) be a triangulation quiver and let $m\colon Q_1 \to \mathbb{Z}_{>0}$ and $c\colon Q_1 \to K^\times$ be g-invariant functions of *multiplicities* and *scalars*, respectively.

We say that m is *admissible* if $m_\alpha n_\alpha \geq 3$ for every arrow $\alpha \in Q_1$. In this case we define the *triangulation algebra* associated to these data as a quotient of the completed path algebra of Q by the closure of an ideal generated by suitable commutativity relations:

$$\Lambda(Q, f, m, c) = \widehat{KQ}/\overline{\langle \bar{\alpha} \cdot f(\bar{\alpha}) - c_\alpha \omega_\alpha^{m_\alpha - 1} \cdot \omega'_\alpha \rangle_{\alpha \in Q_1}}$$

Since the path $\omega_\alpha^{m_\alpha-1} \cdot \omega'_\alpha$ is of length $m_\alpha n_\alpha - 1$, the definition of a triangulation algebra makes sense also when $m_\alpha n_\alpha = 2$, but then the corresponding arrow could be eliminated from Q complicating somewhat the remaining relations. The admissibility condition ensures that the generating relations lie in the square of the ideal generated by all arrows of Q so no arrows have to be deleted.

Example 2.11. We identify some algebras in the literature as triangulation algebras. In the first three examples, we use the presentation as quiver with relations given in Theorem 1.1(a).

1. The triangulation algebras of the triangulation quiver corresponding to a monogon are algebras of quaternion type with one vertex (notation III.1(e) in [7]).
2. The triangulation algebras of the triangulation quiver corresponding to a punctured monogon are algebras of quaternion type with two vertices (denoted $Q(2\mathcal{B})_1$ in [7]).
3. The triangulation algebras of the triangulation quivers corresponding to triangulations of a sphere with three punctures are algebras of quaternion type with three vertices (denoted $Q(3\mathcal{D})$ and $Q(3\mathcal{K})$ in [7]).
4. As shown in [20], the Jacobian algebra of the quiver with potential associated by Labardini-Fragoso [18] to a triangulation of a closed surface satisfying condition (T3) is the triangulation algebra of its adjacency quiver (which is a triangulation quiver in view of Lemma 2.6) with all multiplicities set to 1.

The reason for the exclusion of the two exceptional cases from Theorem 1.1 is explained by the next statement.

Proposition 2.12. *Let (Q, f) be a connected triangulation quiver and $m\colon Q_1 \to \mathbb{Z}_{>0}$ an admissible g-invariant function of multiplicities. Then the following conditions are equivalent:*

(a) *The ribbon graph of (Q, f) with multiplicities is one of the two shown in Figure 2, i.e. a punctured monogon with multiplicities $(3, 1)$ or a tetrahedron with all multiplicities equal to 1.*
(b) $m_\alpha n_\alpha = 3$ *for all* $\alpha \in Q_1$.
(c) $(m_\alpha n_\alpha)^{-1} + (m_{f(\alpha)} n_{f(\alpha)})^{-1} + (m_{f^2(\alpha)} n_{f^2(\alpha)})^{-1} = 1$ *for some* $\alpha \in Q_1$.

2.3. **Mutations of triangulation quivers.** Motivated by the relation between flips of triangulations and Fomin-Zelevinsky mutation of their adjacency quivers [10], we introduce a notion of mutation for triangulation quivers.

Definition 2.13. Let (Q, f) be a triangulation quiver and let k be a vertex of Q without loops. Denote by α, $\bar{\alpha}$ the two arrows that start at k and observe that our assumption

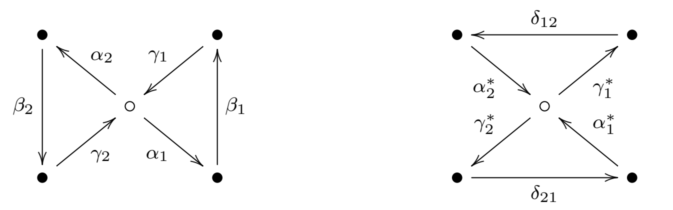

FIGURE 4. Mutation of triangulation quivers at the middle vertex $\circ$. Some of the other vertices may coincide, and only the arrows that change are shown.

on k implies that there are six distinct arrows

$$\alpha_1 = \alpha, \qquad \beta_1 = f(\alpha), \qquad \gamma_1 = f^2(\alpha), \qquad \alpha_2 = \bar{\alpha}, \qquad \beta_2 = f(\bar{\alpha}), \qquad \gamma_2 = f^2(\bar{\alpha})$$

which form two cycles of the permutation f.

The *mutation* of (Q, f) at k is the triangulation quiver (Q', f') obtained from (Q, f) by performing the following steps:

(1) Remove the two arrows β_1 and β_2;
(2) Replace the four arrows α_1, α_2, γ_1 and γ_2 with arrows in the opposite direction α_1^*, α_2^*, γ_1^* and γ_2^*;
(3) Add new arrows δ_{12} and δ_{21} with

$$s(\delta_{12}) = s(\gamma_1), \qquad t(\delta_{12}) = t(\alpha_2), \qquad s(\delta_{21}) = s(\gamma_2), \qquad t(\delta_{21}) = t(\alpha_1).$$

(4) Define the permutation f' on the new set of arrows Q'_1 by $f'(\varepsilon) = f(\varepsilon)$ if ε is an arrow of Q which has not been changed, and by

$$\begin{aligned} f'(\alpha_1^*) &= \gamma_2^*, & f'(\gamma_2^*) &= \delta_{21}, & f'(\delta_{21}) &= \alpha_1^* \\ f'(\alpha_2^*) &= \gamma_1^*, & f'(\gamma_1^*) &= \delta_{12}, & f'(\delta_{12}) &= \alpha_2^* \end{aligned}$$

for the other arrows.

At the level of the underlying quivers, this is very similar to Fomin-Zelevinsky mutation, but note that Q' may contain 2-cycles.

Lemma 2.14. *The triangulation quivers of two triangulations related by a flip at some arc are related by a mutation at the vertex corresponding to that arc.*

Proof. We need to verify that a vertex corresponding to a flippable arc cannot have loops. Indeed, for a loop α at some vertex k we have that either $f(\alpha) = \alpha$ or $g(\alpha) = \alpha$. In the former case k corresponds to a boundary segment, whereas in the latter case it corresponds to an arc which is the inner side of a self-folded triangle. $\square$

The permutation f' on Q'_1 defines the permutation g' by $g'(\alpha') = \overline{f'(\alpha')}$ for $\alpha' \in Q'_1$. Any g-invariant function ν gives rise to a g'-invariant function ν' on Q'_1 by setting $\nu'_\varepsilon = \nu_\varepsilon$ for the arrows in Q'_1 that are also in Q_1 and

$$\nu'_{\alpha_1^*} = \nu'_{\gamma_1^*} = \nu_{\beta_1}, \qquad \nu'_{\alpha_2^*} = \nu'_{\gamma_2^*} = \nu_{\beta_2}, \qquad \nu'_{\delta_{12}} = \nu_{\gamma_1}, \qquad \nu'_{\delta_{21}} = \nu_{\gamma_2}$$

for the other arrows. In particular, any two g-invariant functions $m\colon Q_1 \to \mathbb{Z}_{>0}$ and $c\colon Q_1 \to K^\times$ of multiplicities and scalars on (Q, f) give rise to g'-invariant functions of multiplicities $m'\colon Q'_1 \to \mathbb{Z}_{>0}$ and scalars $c'\colon Q'_1 \to K^\times$ on (Q', f').

For the rest of this section we fix a triangulation quiver (Q, f) and consider its mutation (Q', f') at some vertex k without loops.

Proposition 2.15. *The ribbon graphs of (Q, f) and (Q', f') are related by an elementary move in the sense of Kauer* [16]. *Hence the corresponding Brauer graph algebras $\Gamma(Q, f, m, c)$ and $\Gamma(Q', f', m', c')$ are derived equivalent for any choice of multiplicities and scalars.*

Let $p\colon Q_1 \to xK[[x]]$ be a g-invariant function whose values are power series without constant term. Consider the potential on Q defined by

$$W = \sum_{\alpha} \alpha \cdot f(\alpha) \cdot f^2(\alpha) - \sum_{\beta} p_\beta(\omega_\beta) \tag{2.1}$$

where the sums run over representatives α of f-cycles and β of g-cycles in Q_1. The function p gives rise to a g'-invariant function p' and hence to the potential on Q'

$$W' = \sum_{\alpha'} \alpha' \cdot f'(\alpha') \cdot f'^2(\alpha') - \sum_{\beta'} p'_{\beta'}(\omega_{\beta'})$$

where the sums run over representatives α' of f'-cycles and β' of g'-cycles in Q'_1.

The next proposition compares (Q', W') with the mutation of the quiver with potential (Q, W) at the vertex k as defined in [5].

Proposition 2.16. *Assume that there are no 2-cycles in Q passing through the vertex k. Then (Q', W') is right equivalent to the mutation of (Q, W) at k.*

In the notation of Definition 2.13, the condition in the proposition is equivalent to the conditions that $n_{\alpha_1} > 2$, $n_{\gamma_1} > 2$, $n_{\beta_1} > 1$ and $n_{\beta_2} > 1$.

Combining Proposition 2.16 with Corollary 2.7, we get:

Corollary 2.17. *Let Q be the adjacency quiver of a triangulation of a closed surface with exactly one puncture and view it as a triangulation quiver (Q, f). Then for any power series $p(x) \in xK[[x]]$ the potential*

$$-p(\omega) + \sum_{\alpha} \alpha \cdot f(\alpha) \cdot f^2(\alpha)$$

(where the sum runs over representatives α of f-cycles and ω is the cycle ω_β for some $\beta \in Q_1$) is non-degenerate. In particular, the set of power series

$$\{0\} \cup \{x^m \,:\, m \text{ is not divisible by the characteristic of } K\}$$

yields infinitely many non-degenerate potentials on Q whose Jacobian algebras are pairwise non-isomorphic.

Triangulation algebras are Jacobian algebras of quivers with potentials under some conditions on the characteristic of the ground field.

Lemma 2.18. *Let $m\colon Q_1 \to \mathbb{Z}_{>0}$ and $c\colon Q_1 \to K^\times$ be g-invariant functions and assume that all the multiplicities m_α are invertible over K. Then the triangulation algebra $\Lambda(Q, f, m, c)$ is the Jacobian algebra of (Q, W) where the potential W is of the form (2.1) for the g-invariant function $p\colon Q_1 \to xK[[x]]$ defined by $p_\alpha(x) = c_\alpha m_\alpha^{-1} x^{m_\alpha}$.*

By using the compatibility between mutations of quivers with potentials and mutations of cluster-tilting objects [3], noting that the vanishing condition needed in [3, Theorem 5.2] is always satisfied for symmetric (even for self-injective) algebras, Proposition 2.16 together with Theorem 1.5 imply the following derived equivalence. As we work with quivers with potentials, we have to impose some restrictions on the characteristic of the ground field.

Corollary 2.19. *Assume that there are no 2-cycles in Q passing through the vertex k. Let $m\colon Q_1 \to \mathbb{Z}_{>0}$ and $c\colon Q_1 \to K^\times$ be g-invariant functions of multiplicities and scalars, respectively. Let m' and c' be the corresponding g'-invariant functions on Q'_1. Assume that m is admissible and that moreover each of the numbers m_α is not divisible by the characteristic of K. Then the triangulation algebras $\Lambda(Q, f, m, c)$ and $\Lambda(Q', f', m', c')$ are derived equivalent.*

In fact, under the conditions of the corollary we have, in the notations of Theorem 1.5,

$$\operatorname{End}_{\operatorname{per}\Lambda} U^-_{P_k}(\Lambda) \simeq \Lambda' \simeq \operatorname{End}_{\operatorname{per}\Lambda} U^+_{P_k}(\Lambda)$$
$$\operatorname{End}_{\operatorname{per}\Gamma} U^-_{P_k}(\Gamma) \simeq \Gamma' \simeq \operatorname{End}_{\operatorname{per}\Gamma} U^+_{P_k}(\Gamma)$$

where

$$\Lambda = \Lambda(Q, f, m, c), \quad \Gamma = \Gamma(Q, f, m, c), \quad \Lambda' = \Lambda(Q', f', m', c'), \quad \Gamma' = \Gamma(Q', f', m', c')$$

and P_k denotes the indecomposable projective module corresponding to the vertex k over the appropriate algebra.

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Institut des Hautes Études Scientifiques, Le Bois Marie, 35, route de Chartres, 91440 Bures-sur-Yvette, France
E-mail address: `sefil@ihes.fr`