SL(2,Z)-invariance and D-instanton contributions to the D^6R^4 interaction

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Abstract. The modular invariant coefficient of the D^6R^4 interaction in the low energy expansion of type IIB string theory has been conjectured to be a solution of an inhomogeneous Laplace eigenvalue equation, obtained by considering the toroidal compactification of two-loop Feynman diagrams of eleven-dimensional supergravity. In this paper we determine its exact $SL(2,\mathbb{Z})$-invariant solution $f(\Omega)$ as a function of the complex modulus, $\Omega = x + iy$, satisfying an appropriate moderate growth condition as $y \to \infty$ (the weak coupling limit). The solution is presented as a Fourier series with modes $\widehat{f}_n(y)e^{2\pi inx}$, where the mode coefficients, $\widehat{f}_n(y)$ are bilinear in K-Bessel functions. Invariance under $SL(2,\mathbb{Z})$ requires these modes to satisfy the nontrivial boundary condition $\widehat{f}_n(y) = O(y^{-2})$ for small y, which uniquely determines the solution. The large-y expansion of $f(\Omega)$ contains the known perturbative (power-behaved) terms, together with precisely-determined exponentially decreasing contributions that have the form expected of D-instantons, anti-D-instantons and D-instanton/anti-D-instanton pairs.

Contents

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1. Introduction

The low energy expansion of string theory has a rich dependence on the moduli, or scalar fields, that parameterize the coset space $G(\mathbb{R})/K(\mathbb{R})$, where G is the duality group and K its maximal compact subgroup. In this paper we will be concerned with the simplest nontrivial example, type IIB superstring theory in $D = 10$ space-time dimensions, in which $G = SL(2)$ and $K = SO(2)$. Duality invariance of the theory implies that the IIB scattering amplitudes should transform covariantly under the discrete arithmetic subgroup, $G(\mathbb{Z}) = SL(2,\mathbb{Z})$. This implies that the coefficients of the terms at any order in the low energy expansion of the amplitude are modular functions, which restricts their dependence on the moduli.

Terms of sufficiently low dimension in the effective action preserve a fraction of the 32 supercharges, i.e., they are BPS interactions. Such interactions have particularly simple moduli-dependent coefficients. The lowest-order terms that contribute to the four-particle amplitude, beyond the Einstein–Hilbert action, are the $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS interactions of order R^4 and D^4R^4, where R^4 denotes four powers of the Riemann curvature tensor with the sixteen indices contracted by a standard sixteen-index tensor [1, appendix 9.A] that will not concern us here. These interactions have coefficients given by non-holomorphic Eisenstein series, $E_{\frac{3}{2}}(\Omega)$ and $E_{\frac{5}{2}}(\Omega)$, respectively (we refer to (2.3) for a definition of these series). Here $\Omega = x + iy$ is the complex modulus and $y^{-1} = g_B$ is the type IIB string coupling.

It is the coefficient of the next term, the $\frac{1}{8}$-BPS interaction D^6R^4, that is the subject of this paper. This interaction enters into the type IIB string frame low energy effective action in the form

$$\ell_s^4 \int d^{10}x \sqrt{-\det G^{(10)}}\, y^{-1}\, f(\Omega)\, D^6 R^4\,, \tag{1.1}$$

where $G^{(10)}$ is the ten-dimensional string frame metric, ℓ_s is the string length scale and we have suppressed an overall numerical coefficient. The factor of y^{-1} cancels when $G^{(10)}$ is rescaled in a manner that converts the expression to the Einstein frame, in which $SL(2,\mathbb{Z})$ duality should be manifest. The coefficient $f(\Omega)$ is a modular function that was conjectured in [2] to be the solution of an inhomogeneous Laplace eigenvalue equation[1]

$$(\Delta_\Omega - 12)\, f(\Omega) \;\;=\;\; -\Big(2\,\zeta(3)\, E_{\frac{3}{2}}(\Omega)\Big)^2\,, \tag{1.2}$$

where $\Delta_\Omega = y^2\,\left(\partial_x^2 + \partial_y^2\right)$. The basis of this conjecture was an implementation of the duality that relates M-theory compactified on a torus to type IIB string theory compactified on a circle [5–7]. More precisely, the procedure used in [2] was to evaluate the terms of order $D^6\,R^4$ in two-loop four-graviton supergravity amplitude compactified on a two-torus to nine dimensions. The

[1] In reference [2], $f(\Omega)$ was denoted by $\mathcal{E}_{\frac{1}{2},\frac{1}{2}}$; it has also been denoted by $\mathcal{E}_{(0,1)}$ in earlier papers on this subject, such as [3] and [4].

complex structure of the torus, Ω, translates into the complex coupling constant of the type IIB string theory while the torus volume, $\mathcal{V}$, is proportional to an inverse power of the radius of the string theory circle, r_B.

However, the analysis of the solution of (1.2) in [2] was incomplete in several respects. Although the power-behaved terms in the large-y expansion of $f(\Omega)$ were determined in [2], a general analysis of the solution including the non power-behaved parts of the solution was missing. The objective of this paper is to develop such an analysis. Furthermore, since (1.2) is the simplest of the more general inhomogeneous eigenvalue equations that arise at higher orders in the low energy expansion [8], such an analysis should be of more general significance.

The layout of this paper is as follows. The detailed solution of (1.2) is given in section 2. Our procedure is to consider the inhomogeneous second order differential equations satisfied by the mode coefficients of the Fourier series

$$f(\Omega) \quad = \quad \sum_n \widehat{f}_n(y)\, e^{2\pi i n x}\,. \tag{1.3}$$

This requires the imposition of appropriate boundary conditions on $\widehat{f}_n(y)$ at $y \to \infty$ and $y \to 0$. The $y \to \infty$ condition (the weak coupling limit) is determined by the moderate growth condition[2] requiring that

$$f(\Omega) \quad = \quad O(y^3)\,, \tag{1.4}$$

which corresponds to tree-level behaviour of the D^6R^4 interaction in string perturbation theory. The $y \to 0$ condition (the strong coupling limit), which is much less obvious, requires

$$\widehat{f}_n(y) \quad = \quad O(y^{-2})\,. \tag{1.5}$$

We will see that this condition follows from a subtle relation between the weak coupling limit condition and $SL(2,\mathbb{Z})$ invariance. These boundary conditions pin down the solution completely (with no arbitrary undetermined coefficients) and we are able to determine the exact solution for $\widehat{f}_n(y)$ for all n.

The solution has the form

$$\begin{aligned}\widehat{f}_n(y) \quad = \quad & \delta_{n,0}\,\tilde{f}(y) \; + \; \alpha_n\,\sqrt{y}\,K_{\frac{7}{2}}(2\pi|n|y) \\ & + \sum_{\substack{n_1+n_2\,=\,n \\ (n_1,n_2)\,\neq\,(0,0)}} \; \sum_{i,j\,=\,0,1} M^{ij}_{n_1,n_2}(\pi|n|y)\,K_i(2\pi|n_1|y)\,K_j(2\pi|n_2|y)\,,\end{aligned} \tag{1.6}$$

where α_n are constants and $M^{ij}_{n_1,n_2}(z)$ are quadratic polynomials in z and $1/z$. The K-Bessel functions must be replaced by an appropriate limit when either n, n_1, or n_2 vanishes; see section 2.2 for complete details. The first term in (1.6), which contributes only to the $n = 0$ mode, has the form

[2]In the present context, this condition means that for any $y_0 > 0$ there exists some constant $C > 0$ such that $|f(x+iy)| \leq Cy^3$ for all $y \geq y_0$.

$\tilde{f}(y) = a_0 y^3 + a_1 y + a_2 y^{-1}$, which corresponds to the sum of perturbative contributions up to genus-two. One additional power behaved term arises in (1.6) from the $n = 0$ contribution $\lim_{n\to 0} \alpha_n \sqrt{y}\, K_{\frac{7}{2}}(2\pi|n|y) = \beta y^{-3}$, where β is a constant. This corresponds to a genus-three contribution in string perturbation theory. The coefficients a_0, a_1, a_2, and β were determined by somewhat different means in [2].

The large-y behaviour of the K-Bessel functions in (1.6) gives a rich spectrum of exponentially decreasing terms that may be interpreted as D-instanton effects in string theory[3]. It is particularly notable that there are instanton/anti-instanton terms in the large-y expansion. For example, the zero mode, $\widehat{f}_0$, contains a sum of an infinite series of exponentially suppressed terms of the form $\sum_{n_1=1}^{\infty} c_{n_1}\, e^{-4\pi|n_1|y}$, where the coefficients c_{n_1} are easily deduced from the large-y limit of (1.6) as we will also describe in section 2.

In section 3 we will discuss how the information in the solution of (1.2) makes contact with string theory. In particular, the small coupling (equivalently, large-y) expansion of the solution obtained in section 2 contains a rich array of instanton and anti-instanton contributions. One of the main new observations in this paper is that these conspire to ensure that the strong coupling ($y \to 0$) limits of the Fourier modes satisfy the appropriate small-y boundary condition. This appears somewhat analogous to the manner in which instanton effects conspire to ensure the absence of a singularity in three-dimensional $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in the work of Seiberg–Witten [9].

For completeness, we will present several alternative procedures for determining the solution to (1.2) in three appendices. In appendix A, we will make the $SL(2,\mathbb{Z})$ properties of (1.2) explicit by expressing the solution as a series of the form

$$f(\Omega) \quad = \quad \frac{2\,\zeta(3)^2}{3}\, E_3(\Omega) \ + \ \sum_{\gamma\,\in\,\mathcal{S}} (\det\gamma)^{-3}\, F(\gamma\Omega)\,, \tag{1.7}$$

with $\mathcal{S} = \{\pm 1\}\backslash\{ \left(\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \cap GL^+(2,\mathbb{R}) |\, \gcd(m_1,n_1) = \gcd(m_2,n_2) = 1\}$ (which is the set of 2×2 matrices with integer entries and co-prime rows modulo an overall sign). The function $F(\Omega)$ depends only on the ratio of the real and imaginary parts of Ω, and satisfies a second order inhomogeneous ordinary differential equation given in (A.5). The convergence of the sum over the images of $F(\Omega)$ under $SL(2,\mathbb{Z})$ transformations is obtained only if one imposes suitable boundary conditions in the limits $x/y \to 0$ and

[3]The terminology is motivated by the fact that large-y behaviour proportional to $e^{2\pi i(n_1+n_2)y} e^{-2\pi(|n_1|+|n_2|)y}$ is characteristic of contributions of D-instantons and anti-D-instantons, although the precise form of such contributions has not been obtained by explicit D-instanton calculations.

$x/y \to \infty$.[4] The Fourier modes of the $SL(2,\mathbb{Z})$-invariant expression (1.7) are considered in appendix A.2, where we give an alternative expression of the Fourier modes $\widehat{f}_n(y)$ of $f(\Omega)$ in terms of integrals. We have not succeeded in directly computing those integrals, but their values are of course determined by the analysis of section 2. Furthermore, the convergence properties of these integrals again leads to the $y \to 0$ boundary condition that was deduced by general arguments in section 2.

In appendix B we will describe how the solution may be obtained in a manner suggested by Schmid's work on automorphic distributions of Eisenstein series [10,11]. This gives yet another formula for $\widehat{f}_n(y)$ in lemma B.12. In appendix C we will comment on the solution using the Rölcke-Selberg spectral expansion. This leads to a complete solution of (1.2), but one which seems to be very difficult to use in practice (at least for the nonzero Fourier modes) since it involves properties of unknown cusp forms.

2. Fourier modes of the inhomogeneous Laplace equation

2.1. **Fourier modes and boundary conditions.** We will now consider (1.2) in terms of the Fourier modes of both sides. We write the Fourier expansion of the solution as

$$f(x+iy) \quad = \quad \sum_{n\,\in\,\mathbb{Z}} \widehat{f}_n(y)\, e^{2\pi i n x} \tag{2.1}$$

and the Fourier expansion of the source term as

$$S(x+iy) \quad = \quad -4\,\zeta(3)^2\, E_{\frac{3}{2}}(x+iy)^2 \quad = \quad \sum_{n\,\in\,\mathbb{Z}} S_n(y)\, e^{2\pi i n x}\,. \tag{2.2}$$

The latter are determined by the standard Fourier expansion of the non-holomorphic Eisenstein series,

$$E_s(x+iy) \quad = \quad \frac{1}{2\zeta(2s)} \sum_{(c,d)\,\neq\,(0,0)} \frac{y^s}{|c(x+iy)+d|^{2s}} \quad = \quad \sum_{n\in\mathbb{Z}} \mathcal{F}_{n,s}(y)\, e^{2\pi i n x}\,, \tag{2.3}$$

where the zero mode consists of two power behaved terms,

$$\mathcal{F}_{0,s}(y) \quad = \quad y^s \; + \; \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)}\, y^{1-s}\,, \tag{2.4}$$

and the non-zero modes are proportional to K-Bessel functions,

$$\mathcal{F}_{n,s}(y) \quad = \quad \frac{2\,\pi^s}{\Gamma(s)\zeta(2s)}\, |n|^{s-\frac{1}{2}}\, \sigma_{1-2s}(|n|) \sqrt{y}\, K_{s-\frac{1}{2}}(2\pi|n|y)\,, \quad n \neq 0 \tag{2.5}$$

(see [12, §1.6]).

[4] We are grateful to Don Zagier for describing the solution satisfying the appropriate boundary conditions, as well as for discussions concerning the relevance of this solution.

Since the Laplace operator Δ_Ω commutes with all group translations, the differential equation (1.2) can be equivalently stated as the simultaneous differential equations

$$(y^2\,\partial_y^2 - 12 - 4\pi^2 n^2 y^2)\,\widehat{f}_n(y) \;=\; S_n(y)\,,\quad n\in\mathbb{Z}\,, \tag{2.6}$$

for each Fourier mode of (2.1). We will determine the solution for each value of n in the form

$$\widehat{f}_n(y) \;=\; \widehat{f}_n^P(y) \;+\; \widehat{f}_n^H(y)\,, \tag{2.7}$$

where $\widehat{f}_n^P(y)$ is a particular solution to the equation and $\widehat{f}_n^H(y)$ is a solution of the homogenous equation which is chosen in order that the solution $\widehat{f}_n(y)$ satisfies appropriate boundary conditions.

We now need to consider these boundary conditions. The large y (meaning weak string coupling) growth condition (1.4) on $f(x+iy)$ carries over to each fourier coefficient $\widehat{f}_n(y)$, thus

$$\widehat{f}_n(y) \;=\; O(y^3)\;\text{ for large }\,y\,. \tag{2.8}$$

In fact modes with $n \neq 0$ will be shown to decay like a constant times $y^{e_n}\exp(-2\pi|n|y)$ in this limit, with values of e_n that will be discussed later.

In addition to this boundary condition on each $\widehat{f}_n(y)$ for large y, there is also a condition for small y which is in fact a consequence of (2.8) together with the $SL(2,\mathbb{Z})$-invariance of $f(\Omega)$. It is given by the following lemma.

Lemma 2.9. *If $h(x+iy)$ is an $SL(2,\mathbb{Z})$-invariant function on the upper half plane satisfying the large-y growth condition $h(x+iy) = O(y^s)$ for some $s>1$, then each Fourier mode $\widehat{h}_n(y) = \int_0^1 h(x+iy)e^{-2\pi i n x}dx$ of h satisfies the bound $\widehat{h}_n(y) = O(y^{1-s})$ for small y. In particular, the small-y boundary condition for any mode number n is*

$$\widehat{f}_n(y) \;=\; O(y^{-2})\,. \tag{2.10}$$

Proof. Note the inequality $E_s(x+iy) \geq y^s$ for $s>1$, which comes from dropping all terms with $c \neq 0$ in the definition (2.3). By assumption, the large-y bound states that there is a constant $C>0$ such that $|h(x+iy)| \leq Cy^s$ for any $x+iy$ in $\mathcal{F}$, the standard fundamental domain for $SL(2,\mathbb{Z})$. It follows that $|h(x+iy)| \leq CE_s(x+iy)$ in $\mathcal{F}$, and hence, by automorphy, everywhere in the upper-half plane. This, together with the fact that $E_s(x+iy) > 0$, implies

$$|\widehat{h}_n(y)| \;\leq\; C\int_0^1 E_s(x+iy)\,dx \;=\; C\left(y^s + \frac{\sqrt{\pi}\,\Gamma(s-\frac12)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}y^{1-s}\right). \tag{2.11}$$

Therefore $\widehat{h}_n(y) = O(y^{1-s})$ as $y\to 0$. In the particular case $h=f$ and $s=3$, the bound (1.4) then implies (2.10). □

The conditions (2.8) and (2.10) specify a unique solution to (2.6). To be explicit, we observe that the solution space of the corresponding homogeneous differential equation

$$(y^2\,\partial_y^2 - 12 - 4\pi^2 n^2 y^2)\widehat{f}_n^H \quad = \quad 0 \tag{2.12}$$

consists of the two-dimensional space

$$\begin{aligned} &\{\widehat{f}_n^H \;=\; a\sqrt{y}\, I_{\frac{7}{2}}(2\pi|n|y) \,+\, b\sqrt{y}\, K_{\frac{7}{2}}(2\pi|n|y) \,|\, a,b\in\mathbb{C}\}\,, \quad n\,\neq\,0\,,\\ \text{or}\quad &\{\widehat{f}_0^H \;=\; ay^4 \,+\, by^{-3}\,|\, a,b\in\mathbb{C}\}\,, \qquad\qquad n\,=\,0\,, \end{aligned} \tag{2.13}$$

where the modified Bessel functions of the third kind are defined by

$$K_{\frac{7}{2}}(y) \quad = \quad \sqrt{\tfrac{\pi}{2y}}\,P(y)\,e^{-y} \tag{2.14}$$

$$\text{and}\quad I_{\frac{7}{2}}(y) \quad = \quad \tfrac{1}{\sqrt{2\pi y}}\,\big(P(-y)\,e^{y} + P(y)\,e^{-y}\big)\,, \tag{2.15}$$

with $P(y) \,=\, 1 + \frac{6}{y} + \frac{15}{y^2} + \frac{15}{y^3}$. The unique expression (2.7) that satisfies the boundary conditions in the two dimensional solution space to (2.6), for $n \neq 0$, can be deduced by noting the following asymptotic behaviour of Bessel functions. In the $y \to \infty$ limit the relevant functions behave as

$$\begin{aligned} \sqrt{y}\,K_{7/2}(2\pi|n|y) \quad &= \quad \frac{e^{-2\pi|n|y}}{2|n|^{1/2}}\,(1+O(\tfrac{1}{y}))\\ \text{and}\quad \sqrt{y}\,I_{7/2}(2\pi|n|y) \quad &= \quad \frac{e^{2\pi|n|y}}{2\pi|n|^{1/2}}\,(1+O(\tfrac{1}{y^2}))\,, \end{aligned} \tag{2.16}$$

so only the $K_{7/2}$ solution satisfies the boundary condition, which means that $a = 0$ in (2.13). The coefficient b of the solution to the inhomogeneous equation is then determined by noting the $y \to 0$ asymptotics

$$\sqrt{y}\,K_{7/2}(2\pi|n|y) \quad = \quad \frac{15}{16|n|^{\frac{7}{2}}\pi^3 y^3} \quad - \quad \frac{3}{8|n|^{\frac{3}{2}}\pi y} \;+\; O(y) \tag{2.17}$$

and imposing the condition (2.10) for small y, which requires the y^{-3} term in (2.17) to cancel with a similar term in the particular solution $\widehat{f}_n^P(y)$. The situation for $n = 0$ is of course simpler and again has $a = 0$, and b determined by asymptotics at the origin.

In order to analyze the particular solutions of (2.6) we need first to discuss the Fourier modes of the source term, which can be conveniently broken into a sum of products of Fourier modes of the nonholomorphic Eisenstein series given in (2.4) and (2.5),

$$S_n(y) \quad = \quad \sum_{\substack{n_1,n_2\,\in\,\mathbb{Z}\\ n_1+n_2\,=\,n}} s_{n_1,n_2}(y)\,. \tag{2.18}$$

The s_{n_1,n_2} are naturally divided into the following classes:

- When $n_1 = n_2 = 0$,

$$s_{0,0}(y) \;=\; -\,(2\,\zeta(3)\,y^{\frac{3}{2}} \,+\, 4\,\zeta(2)\,y^{-\frac{1}{2}})^2\,. \tag{2.19}$$

- When either $n_1 = 0$ and $n_2 = n \neq 0$ or $n_2 = 0$ and $n_1 = n \neq 0$,

$$s_{n,0}(y) \;=\; s_{0,n}(y) \;=\; -\,8\,\pi\,(2\,\zeta(3)\,y^2 \,+\, 4\,\zeta(2))\,\frac{\sigma_2(|n|)}{|n|}\,K_1(2\pi|n|y)\,, \tag{2.20}$$

  where $\sigma_2(|n|) = \sum_{k|n} k^2$, the sum being over positive divisors.
- When $n_1 \neq 0$ and $n_2 \neq 0$,

$$s_{n_1,n_2}(y) \;=\; -\,64\,\pi^2\,y\,\frac{\sigma_2(|n_1|)\,\sigma_2(|n_2|)}{|n_1 n_2|}\,K_1(2\pi|n_1|y)\,K_1(2\pi|n_2|y)\,. \tag{2.21}$$

In parallel with (2.18), it will be useful to express $\widehat{f}_n(y)$ as the sum

$$\widehat{f}_n(y) \;=\; \sum_{n_1+n_2\,=\,n} \widehat{f}_{n_1,n_2}(y)\,, \tag{2.22}$$

where

$$(y^2\,\partial_y^2 - 12 - 4\pi^2(n_1+n_2)^2y^2)\,\widehat{f}_{n_1,n_2}(y) \;=\; s_{n_1,n_2}(y)\,. \tag{2.23}$$

The space of solutions to this equation is again two dimensional and obviously shares the same homogeneous solutions given in (2.13) with $n = n_1 + n_2$. There is an obvious ambiguity breaking apart (2.6) into a sum of differential equations (2.23): a homogeneous solution could be simultaneously added to one $\widehat{f}_{n_1,n_2}$ and subtracted from another $\widehat{f}_{n_1',n_2'}$, where $n_1' + n_2' = n_1 + n_2$, without affecting the overall sum (2.22). To avoid this ambiguity, we shall insist that each $\widehat{f}_{n_1,n_2}$ satisfies the same growth conditions as $\widehat{f}_n(y)$,

$$\begin{aligned} \widehat{f}_{n_1,n_2}(y) \;&=\; O(y^3) \quad \text{for } y \text{ large}\,, \\ \widehat{f}_{n_1,n_2}(y) \;&=\; O(y^{-2}) \quad \text{for } y \text{ small}\,. \end{aligned} \tag{2.24}$$

As before, such solutions are unique and have the form

$$\widehat{f}_{n_1,n_2}(y) \;=\; \widehat{f}^{P}_{n_1,n_2}(y) \,+ \alpha_{n_1,n_2}\,\sqrt{y}\,K_{7/2}(2\pi|n_1+n_2|y)\,, \tag{2.25}$$

for any values of n_1 and n_2, where $\widehat{f}^{P}_{n_1,n_2}$ is a particular solution satisfying the large-y bound $O(y^3)$ and α_{n_1,n_2} is the coefficient of the homogeneous solution, which will be determined by the small-y boundary condition $\widehat{f}_{n_1,n_2}(y) = O(y^{-2})$.

We will now determine the explicit solutions for various choices of the integers (n_1, n_2). These give rise to the following sectors:

(i) $n_1 = n_2 = 0$;
(ii) $n_1 = 0, n_2 \neq 0$ or $n_1 \neq 0, n_2 = 0$;
(iii) $n_1 n_2 > 0$;
(iv) $n_1 n_2 < 0$;
(v) $n_1, n_2 \neq 0$ and $n = n_1 + n_2 = 0$.

The last case is a special case of (iv) but merits separate discussion.

2.2. Solutions of the equations in distinct sectors of n_1 and n_2.

(i) $n_1 = n_2 = 0$

In this case the source term, $s_{0,0}(y)$, is given by the power behaved terms in (2.19) and it is easy to see that the solution to (2.23) is

$$\widehat{f}^P_{0,0}(y) \quad = \quad \frac{2\,\zeta(3)^2}{3}\,y^3 \; + \; \frac{4\,\zeta(2)\,\zeta(3)}{3}\,y \; + \; \frac{4\,\zeta(4)}{y}\,. \tag{2.26}$$

Furthermore, $\alpha_{0,0} = 0$ and $\widehat{f}_{0,0}(y) = \widehat{f}^P_{0,0}(y)$.

The complete zero mode, $\widehat{f}_0(y)$ is given by the sum of $\hat{f}_{0,0}(y)$ and the terms of the form $\widehat{f}_{n_1,-n_1}(y)$ that arise in case (v), and will be discussed in section 2.3.

(ii) $n_1 = 0, n_2 \neq 0$ or $n_1 \neq 0, n_2 = 0$

It is easy to verify by substitution that (2.23) with source term (2.20) has a particular solution given by

$$\begin{aligned}\widehat{f}^P_{n,0}(y) \quad &= \quad \widehat{f}^P_{0,n}(y) \\ = \quad \frac{8\,\sigma_2(|n|)}{9\,\pi\,|n|^3} &\times \Big(q^0_{n,0}(\pi|n|y)K_0(2\pi|n|y) + q^1_{n,0}(\pi|n|y)K_1(2\pi|n|y)\Big),\end{aligned} \tag{2.27}$$

where the coefficients are given by

$$q^0_{n,0}(z) \quad = \quad \frac{1}{z}\,\left(90\zeta(3) - n^2\pi^4 + 9z^2\zeta(3)\right) \tag{2.28}$$

$$\text{and} \quad q^1_{n,0}(z) \quad = \quad \frac{1}{z^2}\,\left(90\zeta(3) - n^2\pi^4 + 54z^2\zeta(3)\right). \tag{2.29}$$

Note that the expression (2.27) respects the symmetries

$$\widehat{f}_{n_1,n_2}(y) \quad = \quad \widehat{f}_{-n_1,-n_2}(y) \quad = \quad \widehat{f}_{n_2,n_1}(y)\,. \tag{2.30}$$

Since $\widehat{f}^P_{n,0}(y) \sim -\frac{4\sigma_2(|n|)(n^2\pi^4-90\zeta(3))}{9n^6\pi^4}\,y^{-3}$ as $y \to 0$, the coefficient $\alpha_{n,0}$ of the second term in (2.25) must be taken to be

$$\alpha_{n,0} \quad = \quad \alpha_{0,n} \quad = \quad \frac{64\,\sigma_2(|n|)\,(n^2\pi^4 - 90\,\zeta(3))}{135\,|n|^{\frac{5}{2}}\,\pi} \tag{2.31}$$

in order that complete solution satisfies the boundary condition (2.24) at the origin.

Thus the full solution (2.25) given by

$$\begin{aligned}\widehat{f}_{n,0}(y) &= \widehat{f}_{0,n}(y) = \\ &\frac{8\,\sigma_2(|n|)}{9\,\pi|n|^3}\times\Big(q^0_{n,0}(\pi|n|y)K_0(2\pi|n|y)+q^1_{n,0}(\pi|n|y)K_1(2\pi|n|y)\Big)\\ &\qquad+\frac{64\,\sigma_2(|n|)\,(n^2\pi^4-90\,\zeta(3))}{135\,|n|^{\frac52}\,\pi}\sqrt{y}\,K_{\frac72}(2\pi|n|y) \qquad (2.32)\end{aligned}$$

behaves as

$$\widehat{f}_{n,0}(y) = \widehat{f}_{0,n}(y) = \frac{4\,\pi^2\,\sigma_2(|n|)}{15\,n^2\,y} + O(1) \qquad (2.33)$$

in the $y\to 0$ limit. In the large-y limit the solution behaves as

$$\widehat{f}_{n,0}(y) = \widehat{f}_{0,n}(y) = e^{-2\pi|n|y}\times\Big(4\,\sigma_2(|n|)\,|n|^{-5/2}\,\zeta(3)\,y^{\frac12}+O(1)\Big)\,, \qquad (2.34)$$

where the exponential suppression has a form characteristic of a charge-n D-instanton and the other factors are associated with the instanton measure. This will be commented upon further in section 3.

(iii) $n_1n_2>0$ and **(iv)** $n_1n_2<0$ with $n_1+n_2\neq 0$

Let $\mathrm{sgn}(x)$ denote the sign function and $H(x)=\frac{1+\mathrm{sgn}(x)}{2}$ denote the Heavyside function. It is easy to check that an explicit particular solution to (2.23) with source given by (2.21) is given by the bilinear sum in K_0 and K_1 Bessel functions

$$\begin{aligned}\widehat{f}^P_{n_1,n_2}(y) &= \\ &\frac{32\,\pi\,\sigma_2(|n_1|)\,\sigma_2(|n_2|)}{3\,|n_1n_2|\,|n_1+n_2|^5}\sum_{i,j=0,1} q^{i,j}_{n_1,n_2}(\pi|n_1+n_2|y)\,K_i(2\pi|n_1|y)\,K_j(2\pi|n_2|y),\end{aligned} \qquad (2.35)$$

where the matrix coefficients are given by the expressions

$$q^{0,0}_{n_1,n_2}(z)=\mathrm{sgn}(n_1n_2)\,(-4z\,n_1n_2\left(n_1^2+n_2^2-6n_1n_2\right)-\tfrac{30}{z}\,n_1n_2(n_1-n_2)^2), \qquad (2.36)$$

$$\begin{aligned}q^{0,1}_{n_1,n_2}(z) &= (H(n_1n_2)+H(-n_1n_2)\,\mathrm{sgn}(n_1)\,\mathrm{sgn}(n_1+n_2))\\ &\times(-n_1\left(13n_1^2n_2-65n_1n_2^2+n_1^3+19n_2^3\right)+\tfrac{30}{z^2}\,n_1n_2^2(n_1-n_2))\,, \qquad (2.37)\end{aligned}$$

$$\begin{aligned}q^{1,0}_{n_1,n_2}(z) &= (H(n_1n_2)+H(-n_1n_2)\,\mathrm{sgn}(n_2)\,\mathrm{sgn}(n_1+n_2))\\ &\times(-n_2\left(13n_2^2n_1-65n_2n_1^2+n_2^3+19n_1^3\right)+\tfrac{30}{z^2}\,n_2n_1^2(n_2-n_1))\,, \qquad (2.38)\end{aligned}$$

and

$$q^{1,1}_{n_1,n_2}(z) = -4z\, n_1 n_2 \left(n_1^2+n_2^2-6n_1n_2\right) - \frac{14n_1^3n_2 - 94n_1^2n_2^2 + 14n_1n_2^3 + n_1^4 + n_2^4}{z}\,. \quad (2.39)$$

Imposing the small-y boundary condition on $\widehat{f}_{n_1,n_2}(y)$ in (2.25) requires

$$\alpha_{n_1,n_2} = \mathrm{sgn}(n_1+n_2)\frac{128\,\pi\,\sigma_2(|n_1|)\,\sigma_2(|n_2|)}{45\,n_1^2\,n_2^2\,|n_1+n_2|^{\frac{7}{2}}}\Big(n_1^5+n_2^5+15n_1^4n_2+15n_1n_2^4 - 80n_1^3n_2^2 - 80n_1^2n_2^3 + 60n_1^2n_2^2(n_1-n_2)\log(|\tfrac{n_1}{n_2}|)\Big), \quad (2.40)$$

and the resulting $y \to 0$ behaviour of (2.25) is given by

$$\widehat{f}_{n_1,n_2}(y) = \frac{8\,\sigma_2(|n_1|)\,\sigma_2(|n_2|)}{5\,n_1^2\,n_2^2\,y} + O(1)\,. \quad (2.41)$$

In sector (iii), where $|n_1+n_2| = |n_1|+|n_2|$, the $y \to \infty$ behaviour of (2.25) has the instantonic form

$$\widehat{f}_{n_1,n_2}(y) = e^{-2\pi|n_1+n_2|y}\Big(\frac{\alpha_{n_1,n_2}}{2|n_1+n_2|^{\frac{1}{2}}} - \frac{64\,\pi^2\,\sigma_2(|n_1|)\,\sigma_2(|n_2|)}{3\,|n_1n_2|^{\frac{1}{2}}}\,\frac{n_1^2+n_2^2-6n_1n_2}{(n_1+n_2)^4} + O(y^{-1})\Big)\,. \quad (2.42)$$

In sector (iv), where $|n| = |n_1+n_2| < |n_1|+|n_2|$ a qualitatively new feature is that there are an infinite number of values of n_1 and n_2 having a fixed value of $n = n_1+n_2$. Because of this, the $y \to \infty$ limit is very different from the large-y limit for the $n_1n_2 > 0$ case in (2.42) since the particular solution contains terms that decrease exponentially relative to BPS D-instanton terms. Explicitly, when $n_1n_2 < 0$ the large-y behaviour is given by

$$\begin{aligned}\widehat{f}_{n_1,n_2}(y) &= \alpha_{n_1,n_2}\,\sqrt{y}\,K_{7/2}(2\pi|n_1+n_2|y) - e^{-2\pi(|n_1|+|n_2|)y}\left(\frac{\sigma_2(|n_1|)\,\sigma_2(|n_2|)}{|n_1n_2|^{\frac{5}{2}}\,y^2} + O(y^{-3})\right)\\ &= e^{-2\pi|n_1+n_2|y}\,\frac{\alpha_{n_1,n_2}}{2|n_1+n_2|^{\frac{1}{2}}}(1+O(y^{-1})) - e^{-2\pi(|n_1|+|n_2|)y}\left(\frac{\sigma_2(|n_1|)\,\sigma_2(|n_2|)}{|n_1n_2|^{\frac{5}{2}}\,y^2} + O(y^{-3})\right).\end{aligned} \quad (2.43)$$

The second term in either expression can be more exponentially damped than the first term as n_1 or n_2 increases with $n = n_1+n_2$ held fixed.

(v) $n_1, n_2 \neq 0$ with $n = n_1+n_2 = 0$

This is a special case of (iv) and the particular solution can now be obtained by carefully considering the limit $n_2 = -n_1+\epsilon$ with $\epsilon \to 0$ in (2.35).

Superficially, the presence of the $|n_1+n_2|^{-5}$ factor there suggests that this limit gives a badly divergent result. However, there are massive cancelations caused by properties of the K-Bessel functions and the resulting limit simplifies to be

$$\widehat{f}^P_{n_1,-n_1}(y) \;=\; \frac{32\,\pi\,\sigma_2(|n_1|)^2}{315\,|n_1|^3}\sum_{i,j=0,1} r^{i,j}(\pi|n_1|y)\,K_i(2\pi|n_1|y)\,K_j(2\pi|n_1|y) \tag{2.44}$$

where the coefficient matrix, r^{ij}, has components

$$\begin{aligned} r^{0,0}(z) &= z\left(-512z^4+48z^2-15\right)\\ r^{0,1}(z) &= r^{1,0}(z) \;=\; -\left(128z^4+12z^2+15\right)\\ r^{1,1}(z) &= z^{-1}\left(512z^6+16z^4+33z^2-15\right). \end{aligned} \tag{2.45}$$

The solution of the homogeneous equation solution can also be obtained by setting $n_2=-n_1+\epsilon$ in α_{n_1,n_2} and considering the limit $\epsilon\to 0$, which leads to

$$\lim_{n_2\to -n_1}\alpha_{n_1,n_2}\sqrt{y}\,K_{7/2}(2\pi|n_1+n_2|y) \;=\; \frac{8\,\sigma_2(|n_1|)^2}{21\,n_1^6\,\pi^2\,y^3}\,. \tag{2.46}$$

In order to verify that the full solution

$$\widehat{f}_{n_1,-n_1}(y) \;=\; \widehat{f}^P_{n_1,-n_1}(y) + \frac{8\,\sigma_2(|n_1|)^2}{21\,n_1^6\,\pi^2\,y^3} \tag{2.47}$$

satisfies the $y=0$ boundary condition, we also note that for small y

$$\widehat{f}^P_{n_1,-n_1}(y) \;=\; -\frac{8\,\sigma_2(|n_1|)^2}{21\,n_1^6\,\pi^2\,y^3} + \frac{8\,\sigma_2(|n_1|)^2}{5\,n_1^4\,y} + O(1)\,. \tag{2.48}$$

Therefore, it follows from (2.47) that at small y the full solution for the $(n_1,-n_1)$ mode is

$$\widehat{f}_{n_1,-n_1}(y) \;=\; \frac{8\,\sigma_2(|n_1|)^2}{5\,|n_1|^4\,y} + O(1)\,, \tag{2.49}$$

and at large y it is

$$\widehat{f}_{n_1,-n_1}(y) \;=\; \frac{8\,\sigma_2(|n_1|)^2}{21\,n_1^6\,\pi^2\,y^3} - e^{-4\pi|n_1|y}\Big(\frac{\sigma_2(|n_1|)^2}{|n_1|^5\,y^2}+O(y^{-3})\Big). \tag{2.50}$$

Note that the power behaved term proportional to $1/y^3$ was uncovered by a different method in [2] and is interpreted as a genus-three contribution to the amplitude in string theory perturbation theory. The exponentially decaying term is characteristic of the contribution of a charge-$(n_1,-n_1)$ D-instanton/anti D-instanton pair.

2.3. The complete expression for each Fourier mode, $\widehat{f}_n(y)$.

Having determined the expressions for $\widehat{f}_{n_1,n_2}(y)$ we shall now study the n-th mode $\widehat{f}_n(y)$, which we recall was given in (2.22) as the sum of $\widehat{f}_{n_1,n_2}(y)$ over n_1 and n_2 with $n_1+n_2=n$. We first note that by (2.26) and the explicit formulas for each $\widehat{f}_{n_1,n_2}$ given in section 2.2, the $SL(2,\mathbb{Z})$-invariant function $f(\Omega)-\frac{2\zeta(3)^2}{3}E_3(\Omega)$ is $O(y)$ for y large. Applying lemma 2.9, we conclude that its Fourier coefficients $\widehat{f}_n(y)-\frac{2\zeta(3)^2}{3}\mathcal{F}_{n,3}(y)$ obey the bound $O(y^{-\varepsilon})$ for any fixed positive real number $\varepsilon>0$. Using formulas (2.4) and (2.5), this gives the asymptotic statement

$$\widehat{f}_n(y) \;=\; \frac{945\,\zeta(3)^2\,\sigma_{-5}(|n|)}{4\,\pi^5}\frac{1}{y^2} \;+\; O(y^{-\varepsilon})\,, \tag{2.51}$$

again for any fixed $\varepsilon>0$. In the case $n=0$, $\sigma_{-5}(|n|)$ should be interpreted as $\zeta(5)$. The error term can be slightly improved using the Kronecker limit formula, though this will not be important for our purposes. Note that even though each term in (2.22) satisfies the small-y bound $O(y^{-1})$, their aggregate sum diverges like y^{-2} in (2.51).

The constant term: The $n=0$ mode is given by

$$\widehat{f}_0(y) \;=\; \widehat{f}_{0,0}(y) \;+\; \sum_{n_1\neq 0}\widehat{f}_{n_1,-n_1}(y)\,. \tag{2.52}$$

The sum of the second term in (2.47) over all nonzero integers n_1 is

$$\frac{16}{21\,\pi^2\,y^3}\sum_{m>0}\frac{\sigma_2(m)^2}{m^6} \;=\; \frac{16}{21\,\pi^2\,y^3}\,\frac{\zeta(6)\,\zeta(4)^2\,\zeta(2)}{\zeta(8)} \;=\; \frac{4\,\zeta(6)}{27\,y^3}\,, \tag{2.53}$$

where we have used the Ramanujan identity

$$\sum_{m=1}^{\infty}\frac{\sigma_t(m)\,\sigma_{t'}(m)}{m^r} \;=\; \frac{\zeta(r)\,\zeta(r-t)\,\zeta(r-t')\,\zeta(r-t-t')}{\zeta(2r-t-t')}\,. \tag{2.54}$$

As a result of this and (2.26), we can write the complete solution for the zero mode as

$$\widehat{f}_0(y) \;=\; \frac{2\,\zeta(3)^2}{3}\,y^3 \;+\; \frac{4\,\zeta(2)\,\zeta(3)}{3}\,y \;+\; \frac{4\,\zeta(4)}{y} \;+\; \frac{4\,\zeta(6)}{27\,y^3} \;+\; \sum_{n\neq 0}\widehat{f}^{P}_{n,-n}(y)\,, \tag{2.55}$$

where the expression for $\widehat{f}^{P}_{n,-n}(y)$ is given in (2.44) and is exponentially suppressed as $y\to\infty$. The behaviour as $y\to 0$ is more subtle since the sum in (2.22) does not commute with the small-y limit, and was given above in (2.51). A finer asymptotic expansion can be obtained using Mellin transform methods.

The non-zero Fourier modes: Modes with $n \neq 0$ get contributions from the sectors labelled (ii), (iii) and (iv), so that,

$$\widehat{f}_n(y) \;=\; \widehat{f}_{n,0}(y) \;+\; \widehat{f}_{0,n}(y) \;+\; \sum_{n_1=1}^{n-1} \widehat{f}_{n_1,n-n_1}(y) \;+\; 2 \sum_{n_1 \geq n+1} \widehat{f}_{n_1,n-n_1}(y)\,. \tag{2.56}$$

It is first important to verify that the last sum is convergent. This involves an estimate of the behaviour of its terms as $|n_1| \to \infty$, which arises in case (iv). The $K_i(2\pi|n_1|y)K_j(2\pi|n-n_1|y)$ terms in the n_1 sum (coming from (2.35) are exponentially suppressed as $|n_1|$ gets large. Furthermore, for fixed n, an analysis of formula (2.40) shows that $\alpha_{n_1,n-n_1} = O(n_1^{-6})$ as $n_1 \to \infty$. Thus the terms coming from the homogeneous solutions $\alpha_{n_1,n_2}\sqrt{y}K_{7/2}(2\pi|n_1 + n_2|y)$ also converge because the sum $\sum_{n_1=-\infty}^{\infty} \alpha_{n_1,n-n_1}$ is finite.

The leading behaviour in the weak coupling limit $y \to \infty$ has the form

$$\widehat{f}_n(y) \;=\; e^{-2\pi|n|y}\left(8\,\frac{\sigma_2(|n|)}{|n|^{5/2}}\,\zeta(3)\,y^{1/2} + O(1)\right), \tag{2.57}$$

which is dominated by the behaviour of $\widehat{f}_{n,0}$ and $\widehat{f}_{0,n}$. The behaviour for small y was given in (2.51). It is also possible to study these asymptotics using the explicit formulas for $\widehat{f}_{n_1,n_2}$ given in section 2.2, or from an analysis of (A.44) (which gives an alternative description of the terms in (2.56)). See also formula (B.13), which gives yet another formula for $\widehat{f}_n(y)$.

3. Discussion and connections with string theory

The motivation for considering the differential equation (1.2) from [2] was based on considering the compactification of the two-loop Feynman diagrams of the four-graviton amplitude of eleven-dimensional supergravity on T^2, in the zero-volume limit, $\mathcal{V} \to 0$. The first non-leading term in the low-energy expansion of this amplitude was argued in [2] to give the effective type IIB string theory interaction $f(\Omega)\,D^6R^4$, with $f(\Omega)$ satisfying (1.2). In this paper we have determined the exact solution for all the Fourier modes $\widehat{f}_n(y)$ from (1.3).

The zero mode $\widehat{f}_0(y)$ (2.55) possesses four terms that are power behaved in y that were originally discussed in detail in [2]. The coefficients of these powers are rational numbers multiplying products of zeta values. The values of these coefficients should agree with explicit perturbative string theory calculations up to genus three. The genus zero and genus one string results were known to agree at the time of publication of [2]. The genus-two contribution has been related [13] to the integral of an invariant introduced in [14, 15], which has also recently been evaluated [16] and agrees with the genus-two term (the y^{-1} term in (2.55)). The genus-three part (the y^{-3} contribution in (2.55)) agrees precisely with the prediction for that term in the type IIA theory, that arises from the expansion of the one-loop eleven-dimensional supergravity amplitude compactified on a circle [17]. Furthermore, a recent genus-three string theory calculation [18] also precisely reproduces this y^{-3} contribution.

In solving for the modes $\widehat{f}_n(y)$, it was important to understand the nature of the boundary conditions at $y = \infty$ and $y = 0$. Although the condition at large y (the weak coupling regime) is simply that no term can be more singular than y^3, which is the power corresponding to tree-level perturbation theory, the condition at $y = 0$ is more subtle. We showed in lemma 2.9 that the necessary condition is that $\widehat{f}_n(y) = O(y^{-2})$ in the limit $y \to 0$, which follows as consequence of $SL(2,\mathbb{Z})$ invariance together with the $y \to \infty$ bound, $\widehat{f}_n(y) = O(y^3)$. This is a highly non-trivial condition, in that it implies that the infinite series of terms that manifests itself as a series of exponentially decreasing D-instanton and anti D-instanton contributions at large y, simultaneously conspires to cancel a singular term in $\widehat{f}_n(y)$ at small y. This bears some similarity to the behaviour of the metric on the Coulomb branch of three-dimensional $\mathcal{N} = 4$ supersymmetric $SU(2)$ Yang–Mills theory with no flavour fields in Seiberg-Witten theory [9] (see also [19]). In that case, the expansion of the moduli space metric at large values of the Higgs field also gets contributions from an infinite series of exponentially suppressed terms [20], but the solution can be uniquely determined by requiring the Coulomb branch metric to be non-singular at the origin[5].

The expressions for the Fourier modes contain detailed information concerning the instanton-like contributions that decrease exponentially at large y. Such terms that have the form expected of contributions arising from D-instantons, anti D-instantons and D-instanton/anti D-instanton pairs. This is explicit in the large-y limits given in (2.44) for the terms contributing to $\widehat{f}_0(y)$ and in (2.34), (2.42) and (2.43) for the terms contributing to $\widehat{f}_n(y)$. In particular, (2.50) shows that the constant term, $\widehat{f}_0(y)$, has an infinite series of exponentially decreasing terms in the large-y limit, which have exponential factors $e^{-4\pi|n|y}$ that have the form which would arise from a D-instanton/anti D-instanton pair with charges n and $-n$. Furthermore, the measure contains the square of the divisor sum $\sigma_2(|n|)$,

$$e^{-4\pi|n|y}\,\frac{\sigma_2(|n|)^2}{|n|^5}\,\frac{1}{y^2}\,. \tag{3.1}$$

Since the measure for a single charge-n D-instanton contains a single power of a divisor sum, this is another indication that terms of this form in $\widehat{f}_0(y)$ might be identified with D-instanton/anti D-instanton pairs. Such instanton/anti-instanton terms should break all supersymmetries, giving rise to extra fermionic zero modes. Soaking these up should ought to account

[5] It has been suggested that the series of exponentially suppressed terms might be interpreted as instanton/anti-instanton contributions [21]. However, the identification of the radial coordinate in the Atiyah-Hitchin metric with the corresponding scalar vacuum expectation value in the explicit semi-classical solution is ambiguous. Owing to the high degree of supersymmetry in our case, it is not possible to redefine the modular parameter Ω without losing $SL(2,\mathbb{Z})$ invariance, so this ambiguity is not present.

for the fact that they are suppressed by the factor of $1/y^2$ in (3.1), although we have not determined such factors in the measure from an explicit D-instanton calculation.

The exponentially suppressed terms that contribute to $\widehat{f}_{n,0}$ and $\widehat{f}_{0,n}$ with $n \neq 0$ might be interpreted as contributions of single charge-n D-instantons or charge-n anti D-instantons with a measure that can be read off from (2.34),

$$e^{-2\pi|n|y} \left(\frac{4\sigma_2(|n|)}{|n|^{5/2}} \, \zeta(3) \, y^{1/2} + O(1) \right), \tag{3.2}$$

which has a factor of $y^{5/2}$ relative to (3.1). Likewise, the large-y contribution to $\widehat{f}_n(y)$ with $n = n_1 + n_2$ and $\mathrm{sign}(n_1) = \mathrm{sign}(n_2)$, obtained in (2.42) has the form

$$e^{-2\pi|n_1+n_2|y} \, \sigma_2(|n_1|) \, \sigma_2(|n_2|) \times (\text{function of } n_1, n_2) \,, \tag{3.3}$$

which has a power of y^0.

It would be desirable to understand the particular powers of y in the prefactors of (3.1), (3.2) and (3.3) in terms of the zero modes associated with supersymmetry breaking, but we have not understood this in a systematic manner.

In any case, given the non-standard application of M-theory/string theory duality that motivated (1.2), we would like to determine whether this equation accurately describes the coefficient of the D^6R^4 interaction beyond the checks outlined above. Further motivation for this equation and its generalisation to higher-rank duality groups was obtained in [22–24] in considering properties of the low energy effective action of type II string theory in lower dimensions obtained by toroidal compactification to dimension D. In these cases the coefficient of the D-dimensional D^6R^4 interaction, $f^{(D)}$, is a function of the moduli associated with the $E_{11-D}(\mathbb{Z})$ duality group[6]. Equation (1.2) then generalises to an inhomogeneous Laplace eigenvalue equation [3]

$$\left(\Delta^{(D)} - \frac{6(14-D)(D-6)}{D-2} \right) f^{(D)} = -\left(\mathcal{E}^{(D)}_{\frac{3}{2}} \right)^2 + 120\,\zeta(3)\,\delta_{D-6,0} \,, \tag{3.4}$$

where $\Delta^{(D)}$is the laplacian on the homogeneous space and $\mathcal{E}^{(D)}_s$ is the maximal parabolic Langlands Eisenstein series attached to the parabolic associated with the first node of the Dynkin diagram (which is the coefficient of the R^4 interaction in D dimensions). The constant terms in various parabolic subgroups were analysed to a certain extent for the cases with $D \geq 6$ in [22,23] and for $D = 3$ in [3], and agreed with expectations based on perturbative string theory calculations. This has also been extended to the cases of $D = 1$ and 2 in [25]. The analysis of the non-zero Fourier modes presents new challenges that extends the considerations of [4], which considered the maximal parabolic Langlands Eisenstein series that arise as coefficients of the R^4 and D^4R^4 interactions. The four dimensional version of (3.4) has

[6]Recall that the duality groups of rank ≤ 8 are specific real split forms of $SL(2,\mathbb{Z})$, $SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$, $SL(5,\mathbb{Z})$, $Spin(5,5,\mathbb{Z})$, $E_6(\mathbb{Z})$, $E_7(\mathbb{Z})$, $E_8(\mathbb{Z})$.

also received support from consideration of the soft scalar limits of $\mathcal{N}=8$ supergravity amplitudes in four dimensions [26] .

Since the natural region of validity of perturbative supergravity is $\mathcal{V} \gg \ell_{11}^2$ it is not obvious why the M-theory argument that leads to $f(\Omega)$ should be a good approximation to the exact answer. However, in common with analogous duality arguments for BPS quantities, the fact that the D^6R^4 interaction is $\frac{1}{8}$-BPS seems to justify what would otherwise be an outrageous continuation in $\mathcal{V}$. In considering higher order interactions in the low energy expansion there is no reason, based on our current understanding, for expecting such a continuation from large to small $\mathcal{V}$ to be valid. Nevertheless, it might be of interest to analyze the structure of the compactified Feynman diagrams of eleven-dimensional supergravity further, if only to find inspiration for the possible mathematical structure of higher order terms. The paper [8] contains a detailed discussion of higher order corrections to the low energy expansion, that arise by expanding the two-loop four-graviton amplitude of eleven-dimensional supergravity to higher orders beyond the D^6R^4 interaction studied in this paper. This does not yield any contributions that survive the $r_B \to \infty$ limit to $D=10$ dimensions, but does give contributions that may be useful at finite values of r_B (i.e., in the $D=9$ type IIB theory). Even though the analysis in [8] is not the complete story, the equations that emerge from the higher order expansion of the two-loop amplitude suggest that (1.2) is a specially simple example of a more general set of equations for the higher-order coefficients.

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Appendices: Other methods to solve the inhomogeneous Laplace equation (1.2)

In the following three appendices we will briefly describe other approaches to constructing solutions to (1.2). None of them is completely satisfactory, though each reveals different information about the solution; further variants of these might be useful in studying the D^6R^4 coefficient in lower dimensions (equivalently, for higher rank groups).

Appendix A. The solution as a Poincaré Series

The procedure used to determine the solutions for $\widehat{f}_n(y)$ in section 2 has the drawback that the modularity properties of the complete solution $f(\Omega)$ in (1.3) are obscured in the mode-by-mode analysis. In particular, the values of the coefficients α_{n_1,n_2} in (2.25) were determined only by invoking a boundary condition at $y=0$. This appendix presents a solution for $f(\Omega)$ in terms of a Poincaré series that makes both the $SL(2,\mathbb{Z})$-invariance and growth conditions of each $\widehat{f}_{n_1,n_2}(y)$ manifest. The results of this section should be viewed as complementary to those of section 2, where the modes were expressed very explicitly in terms of products of K-Bessel functions (whereas here they will be given in terms of integrals).

A.1. **Sum over translates of one-dimensional solution.** Squaring the definition of $E_{3/2}$ given in (2.3), we write

$$\begin{aligned} E_{\frac{3}{2}}(\Omega)^2 &= \sum_{\gamma_1,\gamma_2\,\in\,\Gamma_\infty\backslash\Gamma} \mathrm{Im}\,(\gamma_1\Omega)^{3/2}\,\mathrm{Im}\,(\gamma_2\Omega)^{3/2} \\ &= E_3(\Omega) + \sum_{\gamma_1\neq\gamma_2\,\in\,\Gamma_\infty\backslash\Gamma} \mathrm{Im}\,(\gamma_1\Omega)^{3/2}\,\mathrm{Im}\,(\gamma_2\Omega)^{3/2} \\ &= E_3(\Omega) + \sum_{\gamma\,\in\,\mathcal{S}} (\det\gamma)^{-3}\,\mathcal{T}(\gamma\Omega)\,, \end{aligned} \tag{A.1}$$

where $\Gamma = SL(2,\mathbb{Z})$, $\Gamma_\infty = \{\left(\begin{smallmatrix}\pm1 & \star\\ 0 & \pm1\end{smallmatrix}\right)\in SL(2,\mathbb{Z})\}$, $\mathcal{S} = \{\pm1\}\backslash\{\left(\begin{smallmatrix}m_1 & n_1\\ m_2 & n_2\end{smallmatrix}\right)\in M_2(\mathbb{Z})\cap GL^+(2,\mathbb{R})|\,\gcd(m_1,n_1)=\gcd(m_2,n_2)=1\}$, and

$$\mathcal{T}(\Omega) \;=\; \mathcal{T}(x+iy) \;:=\; \sigma(\tfrac{x}{y})\,,\quad \sigma(u) \;=\; (u^2+1)^{-3/2}. \tag{A.2}$$

Indeed, this can easily be seen using the calculations

$$\frac{\mathrm{Re}\;\gamma\Omega}{\mathrm{Im}\;\gamma\Omega} \;=\; \frac{n_1n_2+m_2n_1x+m_1n_2x+m_1m_2(x^2+y^2)}{y\,\det\gamma} \tag{A.3}$$

and

$$\begin{aligned} &\big((n_1n_2+m_2n_1x+m_1n_2x+m_1m_2(x^2+y^2))^2+(y\det\gamma)^2\big)^{-3/2} \\ &\qquad\qquad = \;\big(|m_1z+n_1|^2|m_2z+n_2|^2\big)^{-3/2}. \end{aligned} \tag{A.4}$$

We shall use the fact that $\mathcal{T}(x+iy)$ only depends on the single parameter $\frac{x}{y}$ to construct a solution to (1.2) from (A.1).

Consider the differential equation

$$\left(\frac{d}{du}\left((1+u^2)\,\frac{d}{du}\right)-12\right)h(u) \;=\; -\,\sigma(u)\,, \tag{A.5}$$

where the differential operator on the lefthand side corresponds to $\Delta_\Omega - 12$ acting on functions of the ratio $u=\frac{x}{y}$.

Lemma A.6. *The function*[7]

$$h(u) \;=\; \frac{7+44u^2+40u^4}{3\sqrt{1+u^2}} \;-\; \frac{16}{3\pi}\left(\frac{4}{3}+5u^2+u(3+5u^2)\tan^{-1}(u)\right) \quad \text{(A.7)}$$

is the unique smooth, even function satisfying both (A.5) and the decay condition $h(u) \sim \frac{1}{6|u|^3}$ as $u \to \pm\infty$. It furthermore extends to a holomorphic function on $\mathbb{C} - \{iv \,|\, |v| \geq 1\}$, with jump discontinuities along these branch cuts given by

$$\lim_{u\to 0^+} h(u+iv) \;-\; \lim_{u\to 0^-} h(u+iv) \;=\; \begin{cases} \frac{2i}{3}\left(8v\left(5v^2-3\right) - \frac{40v^4-44v^2+7}{\sqrt{v^2-1}}\right), & v>1 \\ \frac{2i}{3}\left(8v\left(5v^2-3\right) + \frac{40v^4-44v^2+7}{\sqrt{v^2-1}}\right), & v<-1. \end{cases} \quad \text{(A.8)}$$

With $h(u)$ as defined in (A.7), define

$$F(\Omega) \;=\; F(x+iy) \;=\; 4\,\zeta(3)^2\,h(\tfrac{x}{y}) \quad \text{(A.9)}$$

so that $(\Delta - 12)F = -4\zeta(3)^2\mathcal{T}$. Appealing to the expression (A.1), the general solution to (1.2) among automorphic functions having polynomial growth has the form

$$f(\Omega) \;=\; \tfrac{2\zeta(3)^2}{3}\,E_3(\Omega) + \sum_{\gamma\in\mathcal{S}}(\det\gamma)^{-3}F(\gamma\Omega) + \alpha\,E_4(\Omega)\,,\;\; \alpha \in \mathbb{C}\,. \quad \text{(A.10)}$$

Note that h is bounded by a constant multiple of σ, and so F is bounded by a constant multiple of $\mathcal{T}$. Thus the absolute convergence of the sum (A.10) follows from that of (A.1). This argument furthermore shows that the γ-sum is $O(y^3)$ as $y \to \infty$; since $E_3(\Omega)$ satisfies the same bound, this implies the coefficient

$$\alpha \;=\; 0 \quad \text{(A.11)}$$

in order for (1.4) to hold. Thus (A.10) simplifies to

$$f(\Omega) \;=\; \tfrac{2\,\zeta(3)^2}{3}\,E_3(\Omega) \;+\; \sum_{\gamma\in\mathcal{S}}(\det\gamma)^{-3}F(\gamma\Omega) \quad \text{(A.12)}$$

as a result.

Consider the function

$$\phi\left(\left(\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right),\Omega\right) \;:=\; |m_1n_2 - n_1m_2|^{-3}F\left(\tfrac{m_1\Omega+n_1}{m_2\Omega+n_2}\right). \quad \text{(A.13)}$$

[7] The first term of the solution (A.7) (which solves (A.5)) was essentially obtained in [2], but in that reference the second term (which solves the homogeneous version of (A.5)) was instead a function of the two variables x and y rather than only of their ratio. The combination of terms in (A.7), which was pointed out to us by Don Zagier, has asymptotic behaviour that guarantees convergence of the sums that arise in (A.10), whereas the expression used in [2] leads to a divergent result.

For Ω fixed, it has the well-defined limit $\frac{2\zeta(3)^2}{3}\frac{y^3}{|m_2\Omega+n_2|^6}$ along the singular set $m_1n_2-n_1m_2=0$, which we shall take as its defining value there. Thus (A.12) can be rewritten as

$$f(\Omega) \;=\; \sum_{(m_1,n_1),(m_2,n_2)\,\in\,(\mathbb{Z}\times\mathbb{Z})'/\pm} \phi\left(\left(\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2\end{smallmatrix}\right),\Omega\right), \tag{A.14}$$

where

$$(\mathbb{Z}\times\mathbb{Z})'/\pm \;=\; \{(0,1)\}\sqcup\{(c,d)\,|\,c>0,\gcd(c,d)=1\}\,. \tag{A.15}$$

We shall later derive expressions for the Fourier modes of (A.12) in terms of the Fourier transform of h,

$$\begin{aligned}\widehat{h}(r) \;&=\; \int_{\mathbb{R}} h(u)\,e^{-2\pi i r u}\,du \\ &=\; 2\left(\frac{10}{\pi^2r^2}+1\right)K_0(2\pi|r|) \;+\; \frac{4\left(3\pi^2r^2+5\right)}{\pi^3|r|^3}K_1(2\pi|r|) \\ &\qquad -\;\frac{32}{3\pi\sqrt{|r|}}K_{\frac{7}{2}}(2\pi|r|)\,,\;\; r\,\neq\,0\,, \\ \widehat{h}(0) \;&=\; \frac{1}{6}\,.\end{aligned} \tag{A.16}$$

This computation may be performed in a variety of ways: shifting the contour of integration and wrapping around the the branch cuts, using properties (A.8); explicitly computing the Fourier transform $h(u)(1+u^2)^{-s}$ for Re s large, and then analytically continuing to $s=0$; or taking the Fourier transform of the differential equation (A.5), and explicitly solving the resulting differential equation subject to the constraints that $\hat{h}$ is continuous and has rapid decay.

A.2. **Fourier coefficients via a term-by-term analysis.** Returning to the expression (A.14), we break the sum into pieces defined by

$$f(\Omega) \;=\; \Sigma^{0,0}(\Omega) \;+\; \Sigma^{0,1}(\Omega) \;+\; \Sigma^{1,0}(\Omega) \;+\; \Sigma^{1,1}(\Omega)\,, \tag{A.17}$$

where

$$\Sigma^{0,0}(\Omega) \;:=\; \phi\left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1\end{smallmatrix}\right),\Omega\right) \;=\; \frac{2\,\zeta(3)^2}{3}\,y^3\,, \tag{A.18}$$

$$\Sigma^{0,1}(\Omega) \;:=\; \sum_{m_2\,=\,1}^{\infty}\;\sum_{\gcd(n_2,m_2)\,=\,1} \phi\left(\left(\begin{smallmatrix} 0 & 1 \\ m_2 & n_2\end{smallmatrix}\right),\Omega\right), \tag{A.19}$$

$$\Sigma^{1,0}(\Omega) \;:=\; \sum_{m_1\,=\,1}^{\infty}\;\sum_{\gcd(n_1,m_1)\,=\,1} \phi\left(\left(\begin{smallmatrix} m_1 & n_1 \\ 0 & 1\end{smallmatrix}\right),\Omega\right), \tag{A.20}$$

and

$$\Sigma^{1,1}(\Omega) \;:=\; \sum_{m_1\,=\,1}^{\infty}\;\sum_{\gcd(n_1,m_1)\,=\,1}\;\sum_{m_2\,=\,1}^{\infty}\;\sum_{\gcd(n_2,m_2)\,=\,1} \phi\left(\left(\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2\end{smallmatrix}\right),\Omega\right). \tag{A.21}$$

The contribution of these terms to the Fourier modes, $\widehat{f}_n(y)$, will be described in the rest of this appendix. In sections A.2.1 and A.2.2 we will derive Fourier expansions

$$\Sigma^{0,1}(\Omega) \quad = \quad \Sigma^{1,0}(\Omega) \quad = \quad \sum_{n\,\in\,\mathbb{Z}} e^{2\pi i n x}\,\widehat{\Sigma}^{0,1}_n(y)\,, \tag{A.22}$$

and

$$\Sigma^{1,1}(\Omega) \quad = \quad \sum_{n_1\,\in\,\mathbb{Z}}\sum_{n_2\,\in\,\mathbb{Z}} e^{2\pi i(n_1+n_2)x}\,\widehat{\Sigma}^{1,1}_{n_1,n_2}(y)\,, \tag{A.23}$$

respectively. The Fourier modes $\widehat{\Sigma}^{0,1}_n(y)$ and $\widehat{\Sigma}^{1,1}_n(y)$ are related to the $\widehat{f}_{n_1,n_2}$ of section 2.1 by the formulas

$$\begin{aligned} \widehat{f}_{0,0}(y) \quad &= \quad \Sigma^{0,0}(\Omega) \;+\; 2\,\widehat{\Sigma}^{0,1}_0(y) \;+\; \widehat{\Sigma}^{1,1}_{0,0}(y)\,, \\ \widehat{f}_{n,0}(y) \quad &= \quad \widehat{f}_{0,n}(y) \quad = \quad \widehat{\Sigma}^{0,1}_n(y) \;+\; \widehat{\Sigma}^{1,1}_{n,0}(y)\,, \quad n \neq 0\,, \\ \text{and} \quad \widehat{f}_{n_1,n_2}(y) \quad &= \quad \widehat{\Sigma}^{1,1}_{n_1,n_2}(y)\,, \quad n_1, n_2 \neq 0\,. \end{aligned} \tag{A.24}$$

Writing (2.26) as $\widehat{f}_{0,0}(y) = \widehat{f}^{(1)}_{0,0}(y) + \widehat{f}^{(2)}_{0,0}(y) + \widehat{f}^{(3)}_{0,0}(y)$, where $\widehat{f}^{(r)}_{0,0}(y)$ is proportional to y^{5-2r}, we see that

$$\widehat{f}^{(1)}_{0,0}(y) \quad = \quad \Sigma^{0,0}(\Omega) \quad = \quad \frac{2\,\zeta(3)^2}{3}\,y^3\,. \tag{A.25}$$

A.2.1. *Poisson summation for* $\Sigma^{0,1}$ *and* $\Sigma^{1,0}$. We will now apply Poisson summation to (A.19) to put it in the form (A.25). First, reindexing the sum shows

$$\begin{aligned} \Sigma^{0,1}(\Omega) \quad &= \quad \sum_{m_2\,=\,1}^{\infty}\;\sum_{n_2\,\in\,(\mathbb{Z}/m_2\mathbb{Z})^*}\;\sum_{r\,\in\,\mathbb{Z}} \phi\left(\left(\begin{smallmatrix} 0 & 1 \\ m_2 & n_2+rm_2 \end{smallmatrix}\right), \Omega\right) \\ &= \quad \sum_{m_2\,=\,1}^{\infty}\;\sum_{n_2\,\in\,(\mathbb{Z}/m_2\mathbb{Z})^*}\;\sum_{r\in\mathbb{Z}} \frac{1}{m_2^3}\,F((m_2x+n_2+rm_2+im_2y)^{-1}) \\ &= \quad \sum_{m_2\,=\,1}^{\infty}\;\sum_{n_2\,\in\,(\mathbb{Z}/m_2\mathbb{Z})^*}\;\sum_{r\in\mathbb{Z}} \frac{4\,\zeta(3)^2}{m_2^3}\,h\left(\frac{m_2x+n_2+rm_2}{m_2y}\right). \end{aligned} \tag{A.26}$$

The sum over n_2 ranges over all residue classes of integers modulo m_2 that are coprime to m_2. Applying Poisson summation to the inner sum,

$$\begin{aligned} \sum_{r\in\mathbb{Z}} h\left(\frac{m_2x+n_2+rm_2}{m_2y}\right) \quad &= \quad \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} e^{-2\pi i n r}\,h\left(\frac{m_2x+n_2+rm_2}{m_2y}\right)dr \\ &= \quad \sum_{n\in\mathbb{Z}} e^{2\pi i n \frac{m_2x+n_2}{m_2}}\,y\,\widehat{h}(ny)\,, \end{aligned} \tag{A.27}$$

we obtain

$$\Sigma^{0,1}(\Omega) \quad = \quad \sum_{m_2\,=\,1}^{\infty} \frac{4\,\zeta(3)^2}{m_2^3} \sum_{n_2\,\in\,(\mathbb{Z}/m_2\mathbb{Z})^*}\;\sum_{n\,\in\,\mathbb{Z}} e^{2\pi i n(x+\frac{n_2}{m_2})}\,y\,\widehat{h}(ny)\,. \tag{A.28}$$

The n_2-sum can be written in terms of the *Ramanujan sum*

$$c_{m_2}(n) \quad = \sum_{\substack{n_2=1 \\ (m_2,n_2)=1}}^{m_2} e^{2\pi i \frac{n_2}{m_2} n}\,, \tag{A.29}$$

which satisfies the identity

$$\sum_{m_2=1}^{\infty} \frac{c_{m_2}(n)}{m_2^r} \quad = \quad \frac{\sigma_{1-r}(|n|)}{\zeta(r)}\,. \tag{A.30}$$

When applied in the $r=3$ case to (A.28), this results in the expression

$$\Sigma^{0,1}(\Omega) \quad = \quad \Sigma^{1,0}(\Omega) \quad = \quad 4\,\zeta(3) \sum_{n\,\in\,\mathbb{Z}} e^{2\pi i n x}\,\sigma_{-2}(|n|)\,y\,\widehat{h}(ny)\,, \tag{A.31}$$

which has the form (A.25). Formula (A.30) also applies when $n=0$ provided we use the convention that $\sigma_{-2}(0)=\zeta(2)$. Since $\widehat{h}(0)=1/6$ by (A.16), it follows that the $n=0$ term in (A.31) gives the total contribution

$$\widehat{\Sigma}_0^{0,1}(y) \;+\; \widehat{\Sigma}_0^{1,0}(y) \quad = \quad \tfrac{4}{3}\,\zeta(2)\,\zeta(3)\,y \quad = \quad \widehat{f}_{0,0}^{(2)} \tag{A.32}$$

to $\widehat{f}_{0,0}(y)$, which is the second term on the right hand side of (2.26). Furthermore,

$$\widehat{\Sigma}_n^{0,1}(y) \quad = \quad 4\,\zeta(3)\,\sigma_{-2}(|n|)\,y\,\widehat{h}(ny) \qquad \text{for} \quad n\neq 0\,. \tag{A.33}$$

A.2.2. *Poisson summation for* $\Sigma^{1,1}$. In order to produce the Fourier series (A.23), we perform a double Poisson summation on the definition of $\Sigma_{1,1}$ in (A.21) to obtain the formula

$$\begin{aligned}
&\Sigma^{1,1}(\Omega) \\
&\quad = \sum_{m_1\,=\,1}^{\infty} \sum_{n_1\,\in\,(\mathbb{Z}/m_1\mathbb{Z})^*} \sum_{r_1\,\in\,\mathbb{Z}} \sum_{m_2\,=\,1}^{\infty} \sum_{n_2\,\in\,(\mathbb{Z}/m_2\mathbb{Z})^*} \sum_{r_2\,\in\,\mathbb{Z}} \phi\left(\left(\begin{smallmatrix} m_1 & n_1+r_1m_1 \\ m_2 & n_2+r_2m_2 \end{smallmatrix}\right), \Omega\right) \\
&= \sum_{m_1,m_2\,=\,1}^{\infty} \sum_{\substack{n_1\,\in\,(\mathbb{Z}/m_1\mathbb{Z})^* \\ n_2\,\in\,(\mathbb{Z}/m_2\mathbb{Z})^* \\ \hat{n}_1,\hat{n}_2\,\in\,\mathbb{Z}}} \int_{\mathbb{R}^2} \phi\left(\left(\begin{smallmatrix} m_1 & n_1+r_1m_1 \\ m_2 & n_2+r_2m_2 \end{smallmatrix}\right), \Omega\right) e^{-2\pi i(\hat{n}_1 r_1+\hat{n}_2 r_2)}\,dr_1\,dr_2\,.
\end{aligned} \tag{A.34}$$

The integral is given by

$$\int_{\mathbb{R}^2} F\left(\tfrac{m_1(x+iy)+(n_1+r_1m_1)}{m_2(x+iy)+(n_2+r_2m_2)}\right) \frac{e^{-2\pi i(\hat{n}_1 r_1+\hat{n}_2 r_2)}}{|m_1(n_2+r_2m_2)-m_2(n_1+r_1m_1)|^3}\,dr_1\,dr_2\,. \tag{A.35}$$

With the change of variables $r_1 \mapsto r_1 - x - \frac{n_1}{m_1}$, $r_2 \mapsto r_2 - x - \frac{n_2}{m_2}$ this integral becomes

$$
\begin{aligned}
& e^{2\pi i\left((\hat{n}_1+\hat{n}_2)x+\hat{n}_1\frac{n_1}{m_1}+\hat{n}_2\frac{n_2}{m_2}\right)} \\
& \qquad \times \int_{\mathbb{R}^2} |m_1 m_2(r_2-r_1)|^{-3}\, F(\tfrac{m_1(r_1+iy)}{m_2(r_2+iy)})\, e^{-2\pi i(\hat{n}_1 r_1+\hat{n}_2 r_2)}\, dr_1\, dr_2 \\
& \qquad\qquad = \quad e^{2\pi i\left((\hat{n}_1+\hat{n}_2)x+\hat{n}_1\frac{n_1}{m_1}+\hat{n}_2\frac{n_2}{m_2}\right)}\, y^{-1}\, |m_1 m_2|^{-3} \\
& \qquad\qquad\quad \times \int_{\mathbb{R}^2} |r_2-r_1|^{-3}\, F(\tfrac{r_1+i}{r_2+i})\, e^{-2\pi i(\hat{n}_1 y r_1+\hat{n}_2 y r_2)}\, dr_1\, dr_2\,, \qquad (A.36)
\end{aligned}
$$

after changing variables $r_1 \mapsto yr_1$ and $r_2 \mapsto yr_2$. After applying (A.30) twice, we can write (A.34) in the form (A.23) with mode coefficients given by

$$\widehat{\Sigma}^{1,1}_{\hat{n}_1,\hat{n}_2}(y) \quad = \quad \frac{4}{y}\,\sigma_{-2}(|\hat{n}_1|)\,\sigma_{-2}(|\hat{n}_2|)\,\mathcal{I}(\hat{n}_1,\hat{n}_2;y) \qquad (A.37)$$

where

$$\mathcal{I}(\hat{n}_1,\hat{n}_2;y) \quad = \quad \frac{1}{4\,\zeta(3)^2}\int_{\mathbb{R}^2} \frac{F(\frac{r_1+i}{r_2+i})}{|r_2-r_1|^3}\, e^{-2\pi i(\hat{n}_1 y r_1+\hat{n}_2 y r_2)}\, dr_1\, dr_2\,. \qquad (A.38)$$

Using

$$F(\tfrac{r_1+i}{r_2+i}) \quad = \quad 4\,\zeta(3)^2\, h(\tfrac{r_1 r_2+1}{r_2-r_1})\,, \qquad (A.39)$$

this can be rewritten as

$$\mathcal{I}(\hat{n}_1,\hat{n}_2;y) \quad := \quad \int_{\mathbb{R}^2} \frac{h(\frac{r_1 r_2+1}{r_2-r_1})}{|r_2-r_1|^3}\, e^{-2\pi i(\hat{n}_1 y r_1+\hat{n}_2 y r_2)}\, dr_1\, dr_2\,. \qquad (A.40)$$

Lemma A.41. *The integral (A.40) is absolutely convergent. Consequently, $\mathcal{I}(\hat{n}_1,\hat{n}_2;y)$ is bounded in y and the Fourier modes (A.37) are $O(y^{-1})$.*

Proof. Since the exponential has modulus 1 and $h \geq 0$, it suffices to show the convergence of $\mathcal{I}(0,0;y)$. Recalling that $h(u)$ is bounded by a constant multiple of $\sigma(u) = (u^2+1)^{-3/2}$, this reduces to the convergence of the integral

$$
\begin{aligned}
& \int_{\mathbb{R}^2} |r_2-r_1|^{-3}\,(1+(\tfrac{r_1 r_2+1}{r_2-r_1})^2)^{-3/2}\, dr_1\, dr_2 \quad = \\
& \qquad\qquad\qquad \int_{\mathbb{R}^2} (1+r_1^2)^{-3/2}\,(1+r_2^2)^{-3/2}\, dr_1\, dr_2\,, \qquad (A.42)
\end{aligned}
$$

which is of course finite. □

In fact, we know from (A.37) and (2.26) that

$$\widehat{\Sigma}^{1,1}_{0,0}(y) \quad = \quad \frac{4}{y}\,\zeta(2)^2\,\mathcal{I}(0,0;y) \quad = \quad \widehat{f}^{(3)}_{0,0}(y) \quad = \quad \frac{4\,\zeta(4)}{y}\,, \qquad (A.43)$$

so that $\mathcal{I}(0,0;y) = 2/5$ (this can also be verified by numerical integration).

Equations (A.32)-(A.33) show that $\widehat{\Sigma}^{0,1}_n(y) = O(y)$ for small y, and so (A.24) and lemma A.41 imply that the modes $\widehat{f}_{n_1,n_2}$ are at most $O(1/y)$

for small values of y. Nevertheless, the $y \to 0$ limit of this expression is more singular than $1/y$ because the n-th Fourier coefficient $\widehat{f}_n(y)$ includes the sum

$$\sum_{n_1 \neq 0,n} \widehat{f}_{n_1,n-n_1}(y) \quad = \quad \frac{4}{y} \sum_{n_1 \neq 0,n} \sigma_{-2}(|n_1|)\, \sigma_{-2}(|n-n_1|)\, \mathcal{I}(n_1, n-n_1; y)\,. \tag{A.44}$$

Indeed, in (2.51) the $y \to 0$ behaviour of this expression was shown to be proportional to a constant multiple of $1/y^2$.

Appendix B. Poincaré series and Eisenstein automorphic distributions

We now return to the sum over γ in (A.12) ,

$$\sum_{\gamma \in \mathcal{S}} (\det \gamma)^{-3} F(\gamma\Omega)\,, \tag{B.1}$$

where we recall that $\mathcal{S} = \{\pm 1\}\backslash\{\left(\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right) \in M_2(\mathbb{Z}) \cap GL^+(2,\mathbb{R}) |\, \gcd(m_1,n_1) = \gcd(m_2,n_2) = 1\}$. We begin with some comments about the structure of the set $\mathcal{S}$. Suppose $[m\, n] \in \mathbb{Z}^2$ is a vector with $d = \gcd(m,n)$. Then d divides $[m\, n]\gamma$ for any integral 2×2 matrix γ. Consequently, if $\gamma \in SL(2,\mathbb{Z})$ then $\gcd([m\, n]) = \gcd([m\, n]\gamma)$. If γ has the form $\gamma = \begin{pmatrix} n_2 & b \\ -m_2 & d \end{pmatrix} \in SL(2,\mathbb{Z})$ and $\left(\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right) \in \mathcal{S}$, then $\left(\begin{smallmatrix} m_1 & n_1 \\ m_2 & n_2 \end{smallmatrix}\right)\gamma = \left(\begin{smallmatrix} p_1 & p_2 \\ 0 & 1 \end{smallmatrix}\right)$ for some relatively prime integers $p_1, p_2 \in \mathbb{Z}$, where $p_1 = m_1 n_2 - n_1 m_2 > 0$. Thus

$$\mathcal{S} \quad = \quad \{\left(\begin{smallmatrix} p_1 & p_2 \\ 0 & 1 \end{smallmatrix}\right)\gamma \,|\, p_1 > 0, p_2 \in (\mathbb{Z}/p_1\mathbb{Z})^*, \gamma \in \Gamma\} \tag{B.2}$$

parameterizes elements of $\mathcal{S}$ via $\Gamma = PSL(2,\mathbb{Z})$.

In light of (B.2), (B.1) becomes

$$\sum_{\gamma \in \mathcal{S}} (\det \gamma)^{-3} F(\gamma\Omega) \quad = \quad \sum_{p_1=1}^{\infty} \sum_{p_2 \in (\mathbb{Z}/p_1\mathbb{Z})^*} \sum_{\gamma \in PSL(2,\mathbb{Z})} p_1^{-3}\, F\left(\left(\begin{smallmatrix} p_1 & p_2 \\ 0 & 1 \end{smallmatrix}\right)\gamma\Omega\right). \tag{B.3}$$

By virtue of its definition in (A.9), $F\left(\left(\begin{smallmatrix} p_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\Omega\right) = F(p_1\Omega) = F(\Omega)$ and so (B.3) can be written as

$$\begin{aligned} &\sum_{p_1=1}^{\infty} \sum_{p_2 \in (\mathbb{Z}/p_1\mathbb{Z})^*} \sum_{\gamma \in PSL(2,\mathbb{Z})} p_1^{-3}\, F\left(\begin{pmatrix} 1 & p_2/p_1 \\ 0 & 1 \end{pmatrix}\gamma\Omega\right) \\ &\qquad = \sum_{\frac{p_2}{p_1} \in \mathbb{Q}} p_1^{-3} \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} F(\tfrac{p_2}{p_1} + \gamma\Omega) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \Phi(\gamma\Omega)\,, \end{aligned} \tag{B.4}$$

where

$$\Phi(\Omega) \quad = \quad \sum_{\frac{p_2}{p_1} \in \mathbb{Q}} p_1^{-3} F(\tfrac{p_2}{p_1} + \Omega)\,. \tag{B.5}$$

Thus (B.4) writes the sum (B.1) as a sum of left translates of Φ over $\Gamma_\infty\backslash\Gamma$, the type of the sum that the terminology "Poincaré series" is traditionally reserved for.

A standard double coset decomposition for $\Gamma_\infty\backslash\Gamma/\Gamma_\infty$ and application of Poisson summation (see [27,28]) to the last expression in (B.4) gives

$$\begin{aligned}\sum_{\gamma\in\mathcal{S}}(\det\gamma)^{-3}F(\gamma\Omega) \quad &= \quad \Phi(\Omega)\ + \\ &\sum_{c=1}^{\infty}\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*}\sum_{n\in\mathbb{Z}} e^{2\pi i n(x+d/c)}\int_{\mathbb{R}} e^{-2\pi i n r}\,\Phi(\tfrac{a}{c}-\tfrac{1}{c^2(r+iy)})\,dr\,. \qquad \text{(B.6)}\end{aligned}$$

To compute this integral, we use the following Fourier expansion of Φ:

Lemma B.7. *In terms of the function h from (A.9),*

$$\Phi(x+iy) \quad = \quad 4\,\zeta(3)\sum_{n\in\mathbb{Z}}\sigma_{-2}(|n|)\,e^{2\pi i n x}\,y\,\widehat{h}(ny)\,, \qquad \text{(B.8)}$$

where $\hat{h}(\cdot)$ was computed in (A.16) and $\sigma_{-2}(|n|)=\sum_{d|n}d^{-2}$ is to be interpreted as $\zeta(2)$ when $n=0$.

Proof. Writing the rationals in (B.5) as an integer plus a rational in the interval $[0,1)$ we have that

$$\begin{aligned}\Phi(x+iy) \quad &= \quad \sum_{p_1=1}^{\infty}p_1^{-3}\sum_{p_2\in(\mathbb{Z}/p_1\mathbb{Z})^*}\sum_{n\in\mathbb{Z}}F(\tfrac{p_2}{p_1}+x+n+iy) \\ &= \quad \sum_{p_1=1}^{\infty}p_1^{-3}\sum_{p_2\in(\mathbb{Z}/p_1\mathbb{Z})^*}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}F(\tfrac{p_2}{p_1}+x+u+iy)\,e^{-2\pi i n u}\,du \\ &= \quad \sum_{p_1=1}^{\infty}p_1^{-3}\sum_{p_2\in(\mathbb{Z}/p_1\mathbb{Z})^*}\sum_{n\in\mathbb{Z}}e^{2\pi i n(x+\frac{p_2}{p_1})}\int_{\mathbb{R}}F(u+iy)\,e^{-2\pi i n u}\,du \\ &= \quad \sum_{n\in\mathbb{Z}}e^{2\pi i n x}\left(\sum_{p_1=1}^{\infty}p_1^{-3}\sum_{p_2\in(\mathbb{Z}/p_1\mathbb{Z})^*}e^{2\pi i n p_2/p_1}\right) \\ &\qquad\qquad \times\ 4\,\zeta(3)^2\int_{\mathbb{R}}h(\tfrac{u}{y})\,e^{-2\pi i n u}\,du\end{aligned} \qquad \text{(B.9)}$$

after applying Poisson summation and (A.9). The lemma now follows from (A.30). ☐

After inserting (B.8), the integral in (B.6) becomes

$$4\,\zeta(3)\int_{\mathbb{R}}e^{-2\pi i n r}\sum_{m\in\mathbb{Z}}\sigma_{-2}(|m|)\,e^{2\pi i\frac{ma}{c}-2\pi i\frac{mr}{c^2(r^2+y^2)}}\,\frac{y}{c^2(r^2+y^2)}\,\hat{h}(\tfrac{my}{c^2(r^2+y^2)})\,dr\,. \qquad \text{(B.10)}$$

In terms of the Kloosterman sum $S(a,b;c) := \sum_{x\in(\mathbb{Z}/c\mathbb{Z})^*} e((ax+bx^{-1})/c)$, the second term on the righthand side of (B.6) equals

$$4\,\zeta(3)\sum_{c=1}^{\infty}\sum_{m,n\,\in\,\mathbb{Z}} e^{2\pi inx}\,\sigma_{-2}(|m|)\,S(m,n;c)$$
$$\times\int_{\mathbb{R}} e^{-2\pi i\left(nr+\frac{mr}{c^2(r^2+y^2)}\right)}\,\frac{y}{c^2(r^2+y^2)}\,\hat{h}\big(\tfrac{my}{c^2(r^2+y^2)}\big)\,dr\,. \quad \text{(B.11)}$$

The Fourier coefficient of the Fourier mode $x\mapsto e(nx)$ of (B.1) can thus be read off from this and (B.6); combining with (A.12) proves the following

Lemma B.12. *The Fourier modes $\widehat{f}_n(y)$ from (1.3) are given as the sums*

$$\widehat{f}_n(y) \;=\; \tfrac{2\,\zeta(3)^2}{3}\mathcal{F}_{n,3}(y) \;+\; 4\,\zeta(3)\,\sigma_{-2}(|n|)\,y\,\widehat{h}(ny) \;+$$
$$4\zeta(3)\sum_{\substack{c\,>\,0\\ m\,\in\,\mathbb{Z}}}\sigma_{-2}(|m|)S(m,n;c)\int_{\mathbb{R}} e^{-2\pi ir\left(n+\frac{m}{c^2(r^2+y^2)}\right)}\,\frac{y}{c^2(r^2+y^2)}\,\hat{h}\left(\tfrac{my}{c^2(r^2+y^2)}\right)dr, \quad \text{(B.13)}$$

where $\mathcal{F}_{n,s}(y)$ is defined in (2.4)-(2.5).

Remark: The function $\Phi(x+iy)$ can be interpreted in terms of Schmid's automorphic Eisenstein distribution as

$$\Phi(x+iy) \;=\; 4\,\zeta(3)\int_{\mathbb{R}}\tau_2(u+x)\,h(\tfrac{u}{y})\,du\,, \quad \text{(B.14)}$$

where

$$\tau_\nu(u) \;=\; \sum_{\substack{p,q\,\in\,\mathbb{Z}\\ q>0}} q^{-\nu-1}\,\delta_{u=p/q} \;=\; \sum_{n\,\in\,\mathbb{Z}}\sigma_{-\nu}(|n|)\,e^{2\pi inu} \quad \text{(B.15)}$$

is the automorphic distribution corresponding to the Eisenstein series $E_{\nu+1}$ (see [11, §4]). It can be alternatively be thought of as a distributional limit of values of $E_{\nu+1}(x+iy)$ as $y\to 0$ [10]. Integrals such as (B.14) represent the embedding of a vectors (playing the role of h) in the line model of a principal series representation of $SL(2,\mathbb{R})$, into spaces of automorphic functions. Thus Φ can itself be naturally viewed as an automorphic function. Furthermore, identity (B.8) is an immediate consequence of (B.15).

Appendix C. Solution as expressed via spectral theory

We shall now present the spectral expansion of the solution f to (1.2). Though we shall not directly link it to the expression (B.13), there is in fact a famous connection between Kloosterman sums and the spectral theory of automorphic forms (see, for example, [27, 29]).

A significant complication here is that the source term $-(2\zeta(3)E_{\frac{3}{2}})^2$ in (1.2) is not integrable over the automorphic quotient, hence a divergent contribution must be subtracted from it. However, it is striking that the corresponding source term *is* integrable for D_5, E_6, E_7, and E_8 [3,4].

Consider $L^2(\Gamma\backslash\mathbb{H})$, with the inner product

$$\langle f_1, f_2\rangle \;=\; \int_{\Gamma\backslash\mathbb{H}} f_1(x+iy)\,\overline{f_2(x+iy)}\,\frac{dxdy}{y^2}\,. \tag{C.1}$$

The Rölcke-Selberg spectral expansion theorem states that any function $H\in L^2(\Gamma\backslash\mathbb{H})$ can be expanded as

$$H(z) \;=\; \sum_{j=0}^{\infty}\langle H,\phi_j\rangle\,\phi_j(\Omega) \;+\; \frac{1}{4\pi i}\int_{\mathrm{Re}\ s=1/2}\langle H,E_s\rangle E_s(\Omega)\,ds\,, \tag{C.2}$$

where the ϕ_j are an orthonormal basis for the discrete spectrum satisfying $(\Delta+\lambda_j)\phi_j=0$ for some eigenvalue $\lambda_j\geq 0$ (ϕ_0 is the constant $\sqrt{\frac{3}{\pi}}$, whereas the ϕ_j for $j\geq 1$ are Maass cusp forms).

Consider the inhomogeneous differential equation

$$(\Delta-12)F \;=\; -E_{3/2}^2\,,\quad F \text{ automorphic under } \Gamma\,, \tag{C.3}$$

which differs from (1.2) by the constant multiple $4\zeta(3)^2$. The polynomially-bounded solutions to its homogeneous analog

$$(\Delta-12)F \;=\; 0 \tag{C.4}$$

which are automorphic under Γ are all scalar multiples of the Eisenstein series E_4, which we recall grows like y^4 as $y\to\infty$. Because of the growth condition (1.4), the solution f to (1.2) is $4\zeta(3)^2$ times the unique Γ-automorphic solution $F(x+iy)$ to (C.3) which grows by at most $O(y^3)$ in the cusp.

For large values of y, the Eisenstein series $E_s(x+iy)$ is asymptotic to its constant term $\mathcal{F}_{0,s}(y)=y^s+c(2s-1)y^{1-s}$ given in (2.4), where $c(s)=\frac{\xi(s)}{\xi(s+1)}$ and $\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Thus the righthand side of (C.3) is asymptotic to $-y^3-2c(2)y-c(2)^2y^{-1}$ as $y\to\infty$. The automorphic function $\hat{E}_1$, defined as the constant term in the Laurent expansion

$$E_s \;=\; \tfrac{3}{\pi(s-1)} \;+\; \hat{E}_1 \;+\; (s-1)\,\hat{\hat{E}}_1 \;+\; O((s-1)^2) \tag{C.5}$$

of E_s at $s=1$, is asymptotic to y as $y\to\infty$. Therefore

$$H \;:=\; -E_{3/2}^2 \;+\; E_3 \;+\; 2\,c(2)\,\hat{E}_1 \;=\; O(\log(y)) \tag{C.6}$$

and is in particular in $L^2(\Gamma\backslash\mathbb{H})$. Applying the spectral expansion (C.2), we see that

$$(\Delta-12)^{-1}H = -\sum_{j=0}^{\infty}\frac{\langle H,\phi_j\rangle}{\lambda_j+12}\,\phi_j(\Omega)+\frac{1}{4\pi i}\int_{\mathrm{Re}\ s=1/2}\frac{\langle H,E_s\rangle}{s(s-1)-12}E_s(\Omega)\,ds\,. \tag{C.7}$$

Since H is square integrable and since the coefficients on the righthand side of (C.7) are each smaller than their respective counterparts in (C.2), Parseval's theorem shows that $(\Delta-12)^{-1}H$ is also in $L^2(\Gamma\backslash\mathbb{H})$.

Applying Δ to both sides of (C.5) and comparing constant terms in s at $s = 1$ results in the differential equation $\Delta\hat{E}_1 = \frac{3}{\pi}$. Since $\Delta E_3 = 6E_3$, we conclude

$$\begin{aligned} F \;&=\; \tfrac{1}{6}E_3(\Omega) + 2c(2)(\tfrac{1}{12}\hat{E}_1(\Omega) + \tfrac{1}{48\pi}) \;-\; \sum_{j=0}^{\infty} \frac{\langle H, \phi_j\rangle}{\lambda_j + 12}\,\phi_j(\Omega) \\ &\quad +\; \frac{1}{4\pi i}\int_{\mathrm{Re}\ s=1/2} \frac{\langle H, E_s\rangle}{s(s-1)-12} E_s(\Omega)\, ds \qquad (C.8) \end{aligned}$$

is the unique solution to (C.3) which is $O(y^3)$ for large y.

The solution (C.8) can be explicated by computing the inner products $\langle H, \phi_j\rangle$ and $\langle H, E_s\rangle$. For $j \geq 1$ the former are more complicated and can be computed in terms the L-functions of the Maass forms ϕ_j using the Rankin-Selberg unfolding method. However, since the Maass forms themselves are quite mysterious, this is of dubious direct utility. The Maass forms are characterized by having zero constant term. This indicates that the nonconstant Fourier modes of the solution to (1.2) are difficult to directly compute using the spectral expansion.

At the same time, the inner products $\langle H, \phi_0\rangle$ and $\langle H, E_s\rangle$ can be computed very explicitly, and together give an alternative derivation of the constant Fourier mode (2.55) of the solution to (1.2). The rest of this appendix indicates how these computations are carried out.

Let $\mathcal{F}_C := \{x + iy | x \in [-\frac{1}{2}, \frac{1}{2}], x^2 + y^2 \geq 1, y \leq C\}$ denote the points in the standard fundamental domain $\mathcal{F}$ for $SL(2,\mathbb{Z})$ having imaginary part bounded by C. Let Λ^C be the truncation operator on automorphic functions which subtracts the constant term at points in the Γ-translates of $\mathcal{F} - \mathcal{F}_C$:

$$(\Lambda^C\phi)(x+iy) \;=\; \begin{cases} \phi(x+iy) - \int_0^1 \phi(u+iy)du, & y \in \mathcal{F} - \mathcal{F}_C\,, \\ \phi(x+iy), & y \in \mathcal{F}_C \end{cases} \qquad (C.9)$$

(this formula defines $\Lambda^C\phi$ on the fundamental domain $\mathcal{F}$; its value elsewhere is determined by automorphy).

The *Maass-Selberg relations* state that

$$\begin{aligned} \langle \Lambda^C E_{s_1}, E_{s_2}\rangle \;&=\; \frac{C^{s_1+\overline{s_2}-1}}{s_1+\overline{s_2}-1} + c(2\overline{s_2}-1)\frac{C^{s_1-\overline{s_2}}}{s_1-\overline{s_2}} + c(2s_1-1)\frac{C^{\overline{s_2}-s_1}}{\overline{s_2}-s_1} \\ &\quad + c(2s_1-1)c(2\overline{s_2}-1)\frac{C^{1-s_1-\overline{s_2}}}{1-s_1-\overline{s_2}}\,. \qquad (C.10) \end{aligned}$$

Since $\Lambda^C E_{s_1}$ decays rapidly in the cusp, this inner product differs from $\int_{\mathcal{F}_C} E_{s_1}\overline{E_{s_2}}\,\frac{dxdy}{y^2}$ by an additive term of size $o(1)$ as $C \to \infty$.

Since H is square-integrable its inner product with $E_{1/2+it}$ converges, though the inner products of its three constituent terms in (C.6) do not.

Write

$$\begin{aligned}\langle H, E_{1/2+it}\rangle \quad &= \quad -\lim_{C\to\infty}\int_{\mathcal{F}_C} E_{\frac{3}{2}}^2(\Omega)E_{\frac{1}{2}-it}(\Omega)\,\tfrac{dx\,dy}{y^2}\;+ \\ \lim_{C\to\infty}\int_{\mathcal{F}_C} E_3(\Omega)E_{\frac{1}{2}-it}(\Omega)\,\tfrac{dx\,dy}{y^2} &\;+\; 2\,c(2)\lim_{C\to\infty}\int_{\mathcal{F}_C}\hat{E}_1(\Omega)E_{\frac{1}{2}-it}(\Omega)\,\tfrac{dx\,dy}{y^2}\,.\end{aligned} \tag{C.11}$$

Since the compensating $o(1)$ terms disappear in the $C \to \infty$ limit, $E_{1/2-it}$ can be replaced with $\Lambda^C E_{1/2-it}$ in each of the above integrals. The second integral on the righthand side can then be directly handled by (C.10), while the third integral requires simply taking the constant term in the Laurent expansion of both sides at $s_2 = 1$.

However, the integral involving $E_{\frac{3}{2}}(z)^2 E_s(z)$ is more subtle since standard regularization techniques (such as Zagier's method [30]) do not directly apply. This is because if we unfold the fastest growing series, $E_{\frac{3}{2}}$, there still remains the product $E_{\frac{3}{2}}E_{\frac{1}{2}-it}$, which has size on the order of y^2. Zagier's method first truncates $E_{\frac{3}{2}}$ by subtracting a term of size $y^{3/2}$, and what remains decays only like $y^{-1/2}$; this is not enough to get integrability. Instead we play a similar game as in (C.6) by using other Eisenstein series which match those growth rates, and rewrite the first integral on the righthand side of (C.11) as

$$\begin{aligned}\int_{\mathcal{F}_C} E_{3/2}^2(\Omega)E_{\frac{1}{2}-it}(\Omega)\,\tfrac{dxdy}{y^2} \quad &= \\ \int_{\mathcal{F}_C} E_{3/2}(\Omega)[E_{3/2}(\Omega)E_{\frac{1}{2}-it}(\Omega) &- E_{2-it}(\Omega) - \phi(\tfrac{1}{2}-it)E_{2+it}(\Omega)]\,\tfrac{dxdy}{y^2} \\ + \int_{\mathcal{F}_C} E_{\frac{3}{2}}(\Omega)\,E_{2-it}(\Omega)\,\tfrac{dxdy}{y^2} \;&+\; \phi(\tfrac{1}{2}-it)\int_{\mathcal{F}_C} E_{3/2}(\Omega)\,E_{2+it}(\Omega)\,\tfrac{dxdy}{y^2}\,.\end{aligned} \tag{C.12}$$

The advantage of this rearrangement is that the bracketed expression on the righthand side grows at most like $O(\log y)$, and so Zagier's truncation applies. The last two integrals can be estimated using the Maass-Selberg relations (C.10).

Finally, the inner product $\langle H, \phi_0\rangle$ can be also computed using these techniques (using the fact it the residue of E_s at $s = 1$ is a constant function), though an important simplification occurs because ϕ_0 is constant: namely, the truncated integral of $E_{3/2}^2\phi_0 = \frac{\sqrt{3}}{\pi}E_{3/2}E_{3/2}$ can be computed using the Maass-Selberg relations (C.10).

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