Deformations of complex structures on Riemann surfaces and integrable structures of Whitham type hierarchies

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Abstract

We obtain variational formulas for holomorphic objects on Riemann surfaces with respect to arbitrary local coordinates on the moduli space of complex structures. These formulas are written in terms of a canonical object on the moduli space which corresponds to the pairing between the space of quadratic differentials and the tangent space to the moduli space. This canonical object satisfies certain commutation relations which appear to be the same as the ones that emerged in the integrability theory of Whitham type hierarchies. Driven by this observation, we develop the theory of Whitham type hierarchies integrable by hydrodynamic reductions as a theory of certain differential-geometric objects. As an application we prove that the universal Whitham hierarchy is integrable by hydrodynamic reductions.

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Contents

1 Introduction

Various calculations with holomorphic objects on a Riemann surface can be done efficiently by using the Fay identity [1, 2]. It seems that the Fay identity contains all information about identities between the Riemann theta function, normalized holomorphic differentials, the prime form and their derivatives with respect to coordinates on a Riemann surface and on its Jacobian. On the other hand, these holomorphic objects also depend on moduli of complex structures and one needs to be able to compute variations with respect to these moduli. Such formulas were obtained by Rauch [3]. He represented a Riemann surface as a ramified covering of $\mathbb{C}P^1$ and computed variations of holomorphic objects in terms of branch points of this covering. Rauch formulas have proven its usefulness and efficiency in various contexts [4, 5, 6]. It is desirable however to have universal variational formulas which are independent of a particular representation of a Riemann surface and work for arbitrary coordinates on the moduli space. This problem can be approached as follows. It is known that the space of quadratic holomorphic differentials on a Riemann surface $\mathcal{E}$ is dual to the tangent space of the moduli space M_g of complex structures at the point corresponding to $\mathcal{E}$ (see [7] for a general theory of deformations of complex structures and [8] for the Serre duality theorem). Let $v_1, ..., v_{3g-3}$ be local coordinates on M_g, let $\frac{\partial}{\partial v_1}, ..., \frac{\partial}{\partial v_{3g-3}}$ be the corresponding basis in the tangent space and $g_1(p)dp^2, ..., g_{3g-3}(p)dp^2$ be the dual basis in the space of quadratic differentials. The object

$$G(p)dp^2 = \sum_{i=1}^{3g-3} g_i(p)dp^2 \frac{\partial}{\partial v_i}$$

does not depend on any choice of coordinates. Moreover, let $M_{g,n}$ be the moduli space of Riemann surfaces with n punctures $u_1, ..., u_n$. Here we can vary both the complex structure of $\mathcal{E}$ and points $u_1, ..., u_n$ in $\mathcal{E}$. A basis in the tangent space looks like $\frac{\partial}{\partial u_1}, ..., \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_1}, ..., \frac{\partial}{\partial v_{3g-3}}$ and the corresponding object is

$$\hat{G}(p)dp^2 = \sum_{i=1}^{n} F(p, u_i)dp^2 \frac{\partial}{\partial u_i} + \sum_{j=1}^{3g-3} g_j(p)dp^2 \frac{\partial}{\partial v_j}$$

where $F(p, u)$ has a pole of order one at $p = u$ and is holomorphic outside the diagonal. The residue of $F(p, u)$ at $p = u$ is a constant and without loss of generality we assume $F(p, u) = \frac{1}{p-u} + O(1)$. Roughly speaking, $F(p, u)$ should be a quadratic differential with respect to p and a vector field with respect to u. However, the transformation law for this object is more complicated. Indeed, if we change coordinates by $p = \mu(\tilde{p}, v_1, ..., v_{3g-3})$, $u_i = \mu(\tilde{u}_i, v_1, ..., v_{3g-3})$ (and do not change $v_1, ..., v_{3g-3}$), then $\hat{G}(p)dp^2$ should transform as a vector field with respect to $u_1, ..., u_n, v_1, ..., v_{3g-3}$ (and as a quadratic differential with respect to p). This leads to the following transformation law for $F(p, u)$:

$$\tilde{F}(\tilde{p}, \tilde{u}) = \frac{\mu'(\tilde{p})^2}{\mu'(\tilde{u})} \Big(F(\mu(\tilde{p}), \mu(\tilde{u})) - G(\mu(\tilde{p}))(\mu(\tilde{u})) \Big)$$

with the same coefficient at F as if it were a quadratic differential with respect to the first argument and a vector field with respect to the second one but with an additional term depending on G.

The first main result of the paper is the set of explicit formulas for the action of $G(p)$ on various holomorphic objects on the Riemann surface $\mathcal{E}$ such as prime form, holomorphic 1-forms, period matrix, see (2.13), (2.14), (2.15).

We have also obtained commutation relations for $G(p)$:

$$[G(p_1), G(p_2)] =$$

$$= F(p_2, p_1)G'(p_1) - F(p_1, p_2)G'(p_2) + 2F(p_2, p_1)_{p_1}G(p_1) - 2F(p_1, p_2)_{p_2}G(p_2). \tag{1.1}$$

The same relations hold for $\hat{G}(p)$. One can check that these relations are invariant with respect to an arbitrary change of coordinates by virtue of transformation laws of G and F.

Our main motivation for these studies came from attempts to understand better integrable structures of the so-called Whitham type hierarchies [9, 10, 11]. Recall that a Whitham type hierarchy is defined as compatibility conditions of the following system of PDEs:

$$\frac{\partial \psi}{\partial t_i} = h_i(z, u_1, ..., u_n),\ i = 1, ..., N. \tag{1.2}$$

Here $\psi, u_1, ..., u_n$ are functions of times $t_1, ..., t_N$ and z is a parameter. The system (1.2) is understood as a parametric way of defining $N-1$ relations between partial derivatives $\frac{\partial \psi}{\partial t_i}$, $i = 1, ..., N$ obtained by excluding z from these equations. Functions h_i are called potentials of this Whitham type hierarchy.

An important class of such hierarchies associated with the moduli space of Riemann surfaces of genus g with n punctures (the so-called universal Whitham hierarchy) was constructed and studied in [9, 12]. The universal Whitham hierarchy is important in the theory of Frobenius manifolds [13], matrix models and other areas of mathematics. Note that the set of times in the universal Whitham hierarchy coincides with a set of meromorphic differentials on a Riemann surface (holomorphic outside punctures), and that the potentials $h_i(z)$ are integrals of these differentials.

A natural question is in which sense a Whitham type hierarchy is integrable. In this paper we concentrate on an approach to integrability theory of such systems based on the so-called hydrodynamic reductions [14, 15, 16]. In this approach a quasi-linear system is called integrable if it possesses a large family of hydrodynamic reductions. This family of hydrodynamic reductions must be parametrized by solutions of another system of PDEs called Gibbons-Tsarev system [17, 18, 19]. Therefore, Gibbons-Tsarev systems play a crucial role in this integrability theory. See [19] and references therein for the definition and examples of Gibbons-Tsarev systems.

The second main result of the paper is a description of integrable structures that appeared in the hydrodynamic reduction method as a certain differential-geometric structure. We call it a GT structure.

By definition, a GT structure is defined locally by a family of vector fields $g(p)$ and a function $f(p_1, p_2)$ satisfying relations similar to (1.1), see Section 3 for precise definitions. It becomes transparent from these definitions that a natural GT structure exists on the moduli space $M_{g,n}$ and is represented by the objects $\hat{G}(p)$ and $F(p, u)$ described above.

Given a GT structure one could ask how to find all corresponding integrable Whitham type hierarchies. It turns out that in order to classify all possible integrable hierarchies with given GT structure one needs to find all functions $\lambda(p_1, p_2)$ satisfying the functional equation

$$g(p_1)(\lambda(p_2, p_3)) = \lambda(p_1, p_3)\lambda(p_2, p_1)_{p_1} - \lambda(p_2, p_3)f(p_1, p_2)_{p_2} -$$

$$-f(p_1, p_2)\lambda(p_2, p_3)_{p_2} - f(p_1, p_3)\lambda(p_2, p_3)_{p_3}.$$

Moreover, to find all potentials $h(p)$ of a given hierarchy one needs to solve another functional equation

$$g(p_1)(h(p_2)) = \lambda(p_1, p_2)h'(p_1) - f(p_1, p_2)h'(p_2).$$

It is natural to ask if the universal Whitham hierarchy is integrable by hydrodynamic reductions. One could expect that the corresponding GT structure is given by $\hat{G}(p)$ and $F(p_1, p_2)$ and needs to find a function $\lambda(p_1, p_2)$ which gives the universal Whitham hierarchy.

The third main result of this paper is a proof that the universal Whitham hierarchy is indeed integrable in all genera by hydrodynamic reductions. We give the corresponding function $\lambda(p_1, p_2)$ and the precise form of potentials.

Let us describe the content of the paper. In Section 2 we recall main definitions and notations of holomorphic objects on a Riemann surface, construct our main object $G(p)$ and compute its action on holomorphic objects. We compute commutation relations for $G(p)$ as well. We give some examples and explain how the Rauch formulas are connected with ours. In Section 3 we introduce GT structures and develop a theory of these structures. In particular, we explain how to construct new GT structures from a given one and how to construct potentials if we are given a function $\lambda(p_1, p_2)$ defining our hierarchy. We also explain a relation between GT structures and Lie algebroids of a certain type. In Section 4 we recall the definition and basic properties of Whitham type hierarchies. In Section 5 we recall definition of Gibbon-Tsarev systems and prove that there exists a one-to-one correspondence between Gibbons-Tsarev systems and GT structures. In Section 6 we discuss the definition of integrability of Whitham type hierarchies in our framework. We explain that the definition based on hydrodynamic reductions and Gibbons-Tsarev systems is equivalent to ours. We refer to [19] for a full discussion of integrability of Whitham type hierarchies based on hydrodynamic reductions and Gibbons-Tsarev systems. It was not possible to make this paper self-contained and repeat this discussion here without essential increasing of the length of the present paper. In Section 7 we recall the definition of the universal Whitham hierarchy and prove that this hierarchy is integrable by hydrodynamic reductions.

2 Holomorphic objects on Riemann surfaces and deformations of complex structures

Let $\mathcal{E} = \mathbb{D}/\Gamma$ be a compact Riemann surface of genus $g > 1$, $\mathbb{D} \subset \mathbb{C}$ its universal covering and $\Gamma = \pi_1(\mathcal{E})$. Denote by $a_\alpha, b_\alpha,\ \alpha = 1, ..., g$ a canonical basis in the homology group $H_1(\mathcal{E}, \mathbb{Z})$. Let us choose a coordinate in $\mathbb{D}$ and use the same symbols for holomorphic objects on $\mathcal{E}$ and their lifting on $\mathbb{D}$. We will also use the same symbol for a point in $\mathcal{E}$, its lifting in $\mathbb{D}$ and its coordinate. Let $\omega_\alpha(z)dz$ be the basis of holomorphic 1-forms on $\mathcal{E}$ normalized by $\int_{a_\alpha} \omega_\beta dz = \delta_{\alpha\beta}$. Choose a basepoint z_0 and define the Abel map $q_\alpha(z) = \int_{z_0}^{z} \omega_\alpha(z)dz$. Note that $\omega_\alpha = q'_\alpha$. Denote the prime form[1] by $E(x,y)(dx)^{-1/2}(dy)^{-1/2}$. Let $B_{\alpha\beta} = \int_{b_\alpha} \omega_\beta dz$ be the matrix of b-periods. Details on holomorphic objects on Riemann surfaces are given in [1, 2, 5]. Recall that

$$E(v,u) = -E(u,v), \qquad E(u,v) = u - v - \frac{1}{12}S(u)(u-v)^3 + O((u-v)^4), \tag{2.3}$$

where $S(p)$ is the Bergman projective connection on $\mathcal{E}$. Note that $E(u,v)$ is multivalued. If u or v is moved by a_α, it remains invariant. If u moves by b_α to $\bar{u}$ or v moves by b_α to $\bar{v}$, then

$$E(\bar{u},v) = E(u,v)\exp\Big(-\pi i B_{\alpha\alpha} + 2\pi i(q_\alpha(v) - q_\alpha(u))\Big), \tag{2.4}$$

$$E(u,\bar{v}) = E(u,v)\exp\Big(-\pi i B_{\alpha\alpha} - 2\pi i(q_\alpha(v) - q_\alpha(u))\Big).$$

Let $W(u,v) = (\ln(E(u,v))_{uv}$ be the Bergman kernel. Recall that

$$\int_{a_i} W(u,v)du = 0, \quad \int_{b_\alpha} W(u,v)du = 2\pi i\omega_\alpha(v), \quad \int_{b_\alpha}\int_{b_\beta} W(u,v)dudv = 2\pi i B_{\alpha\beta}. \tag{2.5}$$

Recall a description of the tangent space to the moduli space M_g of Riemann surfaces at the point corresponding to $\mathcal{E}$ [20, 21]. Let $p \in \mathcal{E}$ be the center of a small disc $D \subset \mathcal{E}$. Let L be the Lie algebra of holomorphic vector fields on $D \setminus \{p\}$ and L_p, L_{out} be subalgebras of L consisting of vector fields holomorphic at p and holomorphic on $\mathcal{E} \setminus \{p\}$ correspondingly. It is known that the tangent space to the moduli space M_g is isomorphic to the quotient $L/(L_p \oplus L_{out})$. Let $M_{g,1}$ be the moduli space of Riemann surfaces with a puncture at $u \in \mathcal{E}$. The tangent space to $M_{g,1}$ is isomorphic to the quotient $L/(L_p \oplus L_{out,u})$ where $L_{out,u} \subset L_{out}$ consists of vector fields with zero at u. Let us construct vector spaces dual to these tangent spaces using the Serre duality theorem [8]. There exists a non degenerate pairing between the space L and the space Q of quadratic differentials holomorphic on $D \setminus \{p\}$. This pairing is given by $(v,q) = \mathrm{Res}_p(vq)$. The space dual to the tangent space of M_g is equal to $(L_p \oplus L_{out})^\perp \subset Q$ and consists of quadratic differentials holomorphic on $\mathcal{E}$. Similarly, the space dual to the tangent space of $M_{g,1}$ is equal

[1]In this paper we represent differential-geometric objects as functions with prescribed transformation laws with respect to an arbitrary change of coordinates. For example if $x = \mu(\tilde{x}),\ y = \mu(\tilde{y})$, then the prime form transforms as $\tilde{E}(\tilde{x},\tilde{y}) = \mu'(\tilde{x})^{-1/2}\mu'(\tilde{y})^{-1/2}E(\mu(\tilde{x}),\mu(\tilde{y}))$.

to $(L_p \oplus L_{out,u})^{\perp} \subset Q$ and consists of quadratic differentials holomorphic on $\mathcal{E} \setminus \{u\}$ with pole of order less or equal to one at u. More generally, the space dual to the tangent space of $M_{g,n}$ of the moduli space of Riemann surfaces with punctures at $u_1, ..., u_n$ consists of quadratic differentials holomorphic on $\mathcal{E} \setminus \{u_1, ..., u_n\}$ with poles of order less or equal to one at $u_1, ..., u_n$.

Let $v_1, ..., v_{3g-3}$ be local coordinates on moduli space M_g. Let $\frac{\partial}{\partial v_1}, ..., \frac{\partial}{\partial v_{3g-3}}$ be the corresponding basis in the tangent space and $g_1(p)dp^2, ..., g_{3g-3}(p)dp^2$ be the dual basis in the space of quadratic differentials. The object[2]

$$G(p)dp^2 = \sum_{i=1}^{3g-3} g_i(p)dp^2 \frac{\partial}{\partial v_i}$$

does not depend on the choice of coordinates. A similar construction for $M_{g,n}$ gives the object

$$\hat{G}(p)dp^2 = \sum_{i=1}^{n} F(p, u_i)dp^2 \frac{\partial}{\partial u_i} + \sum_{j=1}^{3g-3} g_j(p)dp^2 \frac{\partial}{\partial v_j}$$

where $u_1, ..., u_n$ are coordinates of n points in $\mathcal{E}$ and

$$F(p_1, p_2) = \frac{1}{p_1 - p_2} + O(1). \tag{2.6}$$

Proposition 2.1. Under an arbitrary change of coordinates of the form

$$p = \mu(\tilde{p}, v_1, ..., v_{3g-3}),\ \ u_i = \mu(\tilde{u}_i, v_1, ..., v_{3g-3}) \tag{2.7}$$

the objects $G(p)$, $F(p_1, p_2)$ obey the following transformation rules

$$\tilde{G}(\tilde{p}) = \mu'(\tilde{p})^2 G(\mu(\tilde{p})), \tag{2.8}$$

$$\tilde{F}(\tilde{p}_1, \tilde{p}_2) = \frac{\mu'(\tilde{p}_1)^2}{\mu'(\tilde{p}_2)} \Big(F(\mu(\tilde{p}_1), \mu(\tilde{p}_2)) - G(\mu(\tilde{p}_1))(\mu(\tilde{p}_2)) \Big) \tag{2.9}$$

Proof. The relation (2.8) means that $G(p)$ is a quadratic differential in p (with values in vector fields in $v_1, ..., v_{3g-3}$). In order to obtain (2.9) we perform an arbitrary change of coordinates of the form $p = \mu(\tilde{p}, v_1, ..., v_{3g-3})$, $u_i = \mu(\tilde{u}_i, v_1, ..., v_{3g-3})$, $v_j = \tilde{v}_j$ and require that the object $\hat{G}(p)$ transforms as a vector field in $u_1, ..., u_n, v_1, ..., v_{3g-3}$. The relation (2.9) is a consequence of this requirement. □

Proposition 2.2. The following identities hold

$$[G(p_1), G(p_2)] = F(p_2, p_1)G'(p_1) - F(p_1, p_2)G'(p_2)+ \tag{2.10}$$

[2]Note that the functions G, g_i, F etc. depend also on $v_1, ..., v_{3g-3}$. We will often omit these arguments in order to simplify formulas.

$$+2F(p_2,p_1)_{p_1}G(p_1)-2F(p_1,p_2)_{p_2}G(p_2),$$

$$[\hat G(p_1),\hat G(p_2)]=F(p_2,p_1)\hat G'(p_1)-F(p_1,p_2)\hat G'(p_2)+ \qquad (2.11)$$
$$+2F(p_2,p_1)_{p_1}\hat G(p_1)-2F(p_1,p_2)_{p_2}\hat G(p_2),$$

$$G(p_2)(F(p_1,p_3))-G(p_1)(F(p_2,p_3))=F(p_1,p_2)F(p_2,p_3)_{p_2}-F(p_2,p_1)F(p_1,p_3)_{p_1}+ \qquad (2.12)$$
$$+F(p_1,p_3)F(p_2,p_3)_{p_3}-F(p_2,p_3)F(p_1,p_3)_{p_3}+2F(p_2,p_3)F(p_1,p_2)_{p_2}-2F(p_1,p_3)F(p_2,p_1)_{p_1},$$

$$\frac{G(p_1)(E(p_2,p_3))}{E(p_2,p_3)}=\frac{1}{2}F(p_1,p_2)_{p_2}+\frac{1}{2}F(p_1,p_3)_{p_3}- \qquad (2.13)$$
$$-F(p_1,p_2)\frac{E(p_2,p_3)_{p_2}}{E(p_2,p_3)}-F(p_1,p_3)\frac{E(p_2,p_3)_{p_3}}{E(p_2,p_3)}-\frac{1}{2}\Big(\frac{E(p_1,p_2)_{p_1}}{E(p_1,p_2)}-\frac{E(p_1,p_3)_{p_1}}{E(p_1,p_3)}\Big)^2,$$

$$G(p_1)\Big(\int_{p_2}^{p_3}\omega_i\Big)=F(p_1,p_2)\omega_i(p_2)-F(p_1,p_3)\omega_i(p_3)- \qquad (2.14)$$
$$-\frac{E(p_1,p_2)_{p_1}}{E(p_1,p_2)}\omega_i(p_1)+\frac{E(p_1,p_3)_{p_1}}{E(p_1,p_3)}\omega_i(p_1),$$

$$G(p)(B_{jk})=2\pi i\omega_j(p)\omega_k(p). \qquad (2.15)$$

Proof. Notice that (2.11) is a formal consequence of (2.10) and (2.12) (see Proposition 3.1).

Consider the difference of the l.h.s. and the r.h.s. of each of (2.10), (2.12), (2.13), (2.14). Expanding these expressions on each diagonal $p_i=p_j,\ i\neq j$ and using (2.3) and (2.6) one can check that each of these expressions is holomorphic on all diagonals. Making an arbitrary change of coordinates of the form $p_i=\mu(\tilde p_i,v_1,...,v_{3g-3}),\ i=1,2,3$ one can check that all these differences are transformed as tensor fields in p_1,p_2,p_3. In particular, the difference between the l.h.s. and the r.h.s. of (2.12) is a holomorphic quadratic differential in p_1,p_2 and holomorphic vector field in p_3. This proves (2.12) because any holomorphic vector field vanishes. Similarly, the differences between the l.h.s. and the r.h.s. of (2.13), (2.14) are holomorphic quadratic differentials in p_1 and holomorphic functions in p_2,p_3. Moreover, these functions vanish on the diagonal $p_2=p_3$. This would prove (2.13), (2.14) (any holomorphic function is a constant) provided that we prove that the differences between the l.h.s. and the r.h.s. are single valued.

Taking the second derivative of the equation (2.13) we get

$$G(p_1)(W(p_2,p_3))= \qquad (2.16)$$

$$= -\Big(F(p_1,p_2)\frac{E(p_2,p_3)_{p_2}}{E(p_2,p_3)} + F(p_1,p_3)\frac{E(p_2,p_3)_{p_3}}{E(p_2,p_3)} - \frac{E(p_1,p_2)_{p_1}E(p_1,p_3)_{p_1}}{E(p_1,p_2)E(p_1,p_3)}\Big)_{p_2p_3}$$

where $(W(p_2,p_3) = (\ln(E(p_2,p_3))_{p_2p_3}$ is the Bergman kernel. Let us prove this identity. Let $\Delta(p_1,p_2,p_3)$ be the difference of the l.h.s. and the r.h.s. of (2.16). It is a quadratic differential in p_1 and 1-form in both p_2,p_3. Using transformation properties (2.4) we see that $\Delta(p_1,p_2,p_3)$ is single valued. Therefore, $\Delta(p_1,p_2,p_3) = \sum_{\alpha,\beta=1}^{g} r_{\alpha\beta}(p_1)\omega_\alpha(p_2)\omega_\beta(p_3)$ where $r_{\alpha\beta}(p_1)$ are some holomorphic quadratic differentials. Computing $\int_{a_\alpha}\int_{a_\beta}\Delta(p_1,p_2,p_3)dp_2dp_3$ we obtain $r_{\alpha\beta}(p_1) = 0$ which proves (2.16). Computing $\int_{b_\alpha}\int_{b_\beta} dp_2dp_3$ of the l.h.s. and the r.h.s. of (2.16) and using (2.5) and (2.4) we obtain (2.15). The difference between the l.h.s. and the r.h.s. of (2.14) is single valued by virtue of (2.15). This proves (2.14). Equation (2.13) is proven in a similar way. Note that the difference between the l.h.s. and the r.h.s. of (2.13) is single valued by virtue of (2.14). Equation (2.10) is proven by applying its l.h.s. and the r.h.s. to B_{jk}. For example, on the l.h.s. we have $G(p_1)(G(p_2)(B_{jk})) - G(p_2)(G(p_1)(B_{jk}))$. Computing by virtue of (2.15), (2.14) we prove (2.10). □

Remark 2.1. Recall that the Riemann theta-function is defined by

$$\theta(z_1,...,z_g) = \sum_{\mathbf{m}\in\mathbb{Z}^g} \exp(2\pi i\mathbf{m}\cdot\mathbf{z} + \pi i\mathbf{m}\mathbf{B}\mathbf{m}^t).$$

Here we use bold symbols for the corresponding vectors: $\mathbf{m} = (m_1,...,m_g)$, $\mathbf{z} = (z_1,...,z_g)$, $\mathbf{m}\cdot\mathbf{z} = m_1z_1 + ... + m_gz_g$, and $\mathbf{B}$ is the period matrix. We have

$$G(p)(\theta(z_1,...,z_g)) = \sum_{\alpha,\beta=1}^{g} \frac{\partial\theta(z_1,...,z_g)}{\partial B_{\alpha\beta}} G(p)(B_{\alpha\beta}) = \frac{1}{2}\sum_{\alpha,\beta=1}^{g} \frac{\partial^2\theta(z_1,...,z_g)}{\partial z_\alpha\partial z_\beta}\omega_\alpha(p)\omega_\beta(p)$$

where we used heat equation for θ and (2.15).

Remark 2.2. Expanding (2.13) on diagonal $p_2 = p_3$ we obtain

$$G(p_1)(S(p_2)) + F(p_1,p_2)_{p_2^3} + 2S(p_2)F(p_1,p_2)_{p_2} + S(p_2)_{p_2}F(p_1,p_2) - 6W(p_1,p_2)^2 = 0.$$

Example 2.1. Let $g = 2$. Represent $\mathcal{E}$ as a 2-fold covering of $\mathbb{C}P^1$. Let x be an affine coordinate in $\mathbb{C}P^1$ and let branch points of the covering be at $x = 0,1,\infty,a,b,c$. The curve $\mathcal{E}$ is given by $y^2 = x(x-1)(x-a)(x-b)(x-c)$. One can check that

$$G(p) = \frac{1}{2p(p-1)}\Big(\frac{a(a-1)}{p-a}\frac{\partial}{\partial a} + \frac{b(b-1)}{p-b}\frac{\partial}{\partial b} + \frac{c(c-1)}{p-c}\frac{\partial}{\partial c}\Big),$$

$$F(p_1,p_2) = \frac{(p_1-a)(p_1-b)(p_1-c)p_2(p_2-1) + q_1q_2}{2(p_1-p_2)p_1(p_1-1)(p_1-a)(p_1-b)(p_1-c)}$$

where p,p_1,p_2 are affine coordinates in $\mathbb{C}P^1$ and $q_i^2 = p_i(p_i-1)(p_i-a)(p_i-b)(p_i-c)$.

Let us compare our variational formulas with Rauch ones. The equation (2.15) reads

$$\frac{1}{2p(p-1)}\Big(\frac{a(a-1)}{p-a}\frac{\partial}{\partial a} + \frac{b(b-1)}{p-b}\frac{\partial}{\partial b} + \frac{c(c-1)}{p-c}\frac{\partial}{\partial c}\Big)(B_{jk})dp^2 = 2\pi i\omega_j(p)dp\cdot\omega_k(p)dp. \quad (2.17)$$

Let τ be a local coordinate near branch point a, we have $p = a + \tau^2$, $dp = 2\tau d\tau$. Expanding the l.h.s. of (2.17) we get $(2\frac{\partial B_{jk}}{\partial a} + O(\tau))d\tau^2$. Therefore $\frac{\partial B_{jk}}{\partial a} = \pi i \frac{\omega_j(p)dp}{d\tau}|_{p=a}\frac{\omega_k(p)dp}{d\tau}|_{p=a}$ and we arrive at a Rauch formula.

In general, if $\mathcal{E}$ is represented as a branched covering of $\mathbb{C}P^1$ ramified at $a_k \in \mathbb{C}P^1$ with ramification indexes r_k, $k = 1, 2, ...$, then $G(p)dp^2 = \Big(\frac{1}{r_k(p-a_k)}\frac{\partial}{\partial a_k} + o((p-a_k)^{-1})\Big)dp^2$ and Rauch formulas can be derived from ours in a similar way.

Example 2.2. Let us choose $B_{j_1,k_1}, ..., B_{j_{3g-3},k_{3g-3}}$ as local coordinates in M_g. Applying (2.15) to B_{j_l,k_l} we get

$$G(p) = 2\pi i \sum_{l=1}^{3g-3} \omega_{j_l}(p)\omega_{k_l}(p)\frac{\partial}{\partial B_{j_l,k_l}}.$$

Applying again (2.15) to an arbitrary B_{jk} we obtain quadratic relations between normalized differentials. Namely, if $S(B_{11}, B_{12}, ..., B_{gg}) = 0$ is a relation between the entries of the period matrix (there are $\frac{g(g+1)}{2} - 3g + 3$ functionally independent ones), then

$$\sum_{j,k=1}^{g} \frac{\partial S}{\partial B_{jk}}\omega_j(p)\omega_k(p) = 0.$$

See [22] for a similar formula for quadratic relations between normalized differentials.

3 GT structures

Based on the identities (2.10), (2.12) we want to introduce a general differential-geometric structure: a family of vector fields $g(p)$ and a function $f(p_1, p_2)$ satisfying the same relations. We will see later that this structure is equivalent to an integrability structure of Whitham type hierarchies, the so-called Gibbons-Tsarev system.

Let $g(p) = \sum_{i=1}^{m} g_i(p, v_1, ...v_m)\frac{\partial}{\partial v_i}$ be a family of vector fields parameterized by p and $f(p_1, p_2, v_1, ..., v_m)$ be a function.

Definition 3.1. A local GT structure is a family $g(p)$ and a function $f(p_1, p_2)$ satisfying the following relations:

$$[g(p_1), g(p_2)] = f(p_2, p_1)g'(p_1) - f(p_1, p_2)g'(p_2) + 2f(p_2, p_1)_{p_1}g(p_1) - 2f(p_1, p_2)_{p_2}g(p_2), \quad (3.18)$$

$$g(p_2)(f(p_1, p_3)) - g(p_1)(f(p_2, p_3)) = f(p_1, p_2)f(p_2, p_3)_{p_2} - f(p_2, p_1)f(p_1, p_3)_{p_1} + \quad (3.19)$$

$$+f(p_1, p_3)f(p_2, p_3)_{p_3} - f(p_2, p_3)f(p_1, p_3)_{p_3} + 2f(p_2, p_3)f(p_1, p_2)_{p_2} - 2f(p_1, p_3)f(p_2, p_1)_{p_1},$$

$$f(p_1,p_2)=\frac{1}{p_1-p_2}+O(1). \tag{3.20}$$

Here and in the sequel we often omit additional arguments $v_1,...v_m$, indexes stand for partial derivatives and $g'(p)=\frac{\partial g(p,v_1,...,v_m)}{\partial p}$.

Given a GT structure we can construct new GT structures in different ways.

Proposition 3.1. Let $g(p)$, $f(p_1,p_2)$ satisfy relations (3.18), (3.19) and

$$\hat{g}(p)=f(p,u_1)\frac{\partial}{\partial u_1}+...+f(p,u_n)\frac{\partial}{\partial u_n}+g(p). \tag{3.21}$$

Then $\hat{g}(p)$, $f(p_1,p_2)$ also satisfy relations (3.18), (3.19).

Proof. Equation (3.18) is verified by direct computation for $n=1$ and through induction by n for $n>1$. Equation (3.19) remains the same because $f(p_1,p_2)$ does not depend on $u_1,...,u_n$. $\square$

We say that a GT structure given by $\hat{g}(p)$, $f(p_1,p_2)$ is obtained from a GT structure $g(p)$, $f(p_1,p_2)$ by adding n points $u_1,...,u_n$. This procedure corresponds to a regular fields extension of a Gibbons-Tsarev system [19].

Proposition 3.2. Let $g(p)$, $f(p_1,p_2)$ satisfy relations (3.18), (3.19) and

$$\hat{g}^{(n_1,...,n_k)}(p)= \tag{3.22}$$

$$=\sum_{\substack{1\le j\le k,\\ 0\le i_{j,1},...,i_{j,n_j},\\ i_{j,1}+...+i_{j,n_j}\le n_j}}\frac{(i_{j,1}+2i_{j,2}+...+n_j i_{j,n_j})!}{i_{j,1}!...i_{j,n_j}!\;1!^{i_{j,1}}...n_j!^{i_{j,n_j}}}\frac{\partial^{i_{j,1}+...+i_{j,n_j}}f(p,u_{j,0})}{\partial u_{j,0}^{i_{j,1}+...+i_{j,n_j}}}\frac{\partial}{\partial u_{j,i_{j,1}+...+i_{j,n_j}}}+g(p).$$

Then $\hat{g}^{(n_1,...,n_k)}(p)$, $f(p_1,p_2)$ also satisfy relations (3.18), (3.19).

Proof. Let us start with the following local GT structure

$$\hat{g}(p)=\sum_{\substack{1\le j\le k,\\ 0\le l\le n_j}}f(p,v_{j,l})\frac{\partial}{\partial v_{j,l}}+g(p). \tag{3.23}$$

We make the following change of coordinates

$$v_{j,0}=u_{j,0},$$
$$v_{j,1}=u_{j,0}+\epsilon u_{j,1},$$
$$v_{j,2}=u_{j,0}+2\epsilon u_{j,1}+\epsilon^2 u_{j,2}, \tag{3.24}$$

$$.............$$

$$v_{j,n_j}=u_{j,0}+n_j\epsilon u_{j,1}+\frac{n_j(n_j-1)}{2}\epsilon^2 u_{j,2}+...+\epsilon^{n_j}u_{j,n_j}.$$

In new coordinates we have

$$\hat{g}(p)=\sum_{1\le j\le k}\Big(f(p,u_{j,0})\frac{\partial}{\partial u_{j,0}}+\frac{1}{\epsilon}(f(p,u_{j,0}+\epsilon u_{j,1})-f(p,u_{j,0}))\frac{\partial}{\partial u_{j,1}}+$$

$$+\frac{1}{\epsilon^2}(f(p,u_{j,0}+2\epsilon u_{u,1}+\epsilon^2 u_{j,2})-2f(p,u_{j,0}+\epsilon u_{j,1})+f(p,u_{j,0}))\frac{\partial}{\partial u_{j,2}}+...$$

$$+\frac{1}{\epsilon^{n_j}}(f(p,u_{j,0}+n_j\epsilon u_{j,1}+...+\epsilon^{n_j}v_{j,n_j})-...+(-1)^{n_j}f(p,u_{j,0}))\frac{\partial}{\partial u_{j,n_j}})\Big)+g(p).$$

Taking the limit $\epsilon\to 0$ we obtain (3.22). □

We say that the GT structure (3.22) is obtained from the GT structure (3.23) by colliding points $v_{j,0},v_{j,1},...,v_{j,n_j}$ for each j.

Remark 3.1. Equation (3.19) is equivalent to Jacobi identity for (3.18) provided that vector fields $g(p_1),\ g(p_2),\ g(p_3),\ g'(p_1),\ g'(p_2),\ g'(p_3)$ are linearly independent for generic $p_1,\ p_2,\ p_3$.

Remark 3.2. A local GT structure can be regarded as a certain Lie algebroid. Let

$$g(p)=e_2+(p-z)e_3+(p-z)^2e_4+....$$

In other words, let $e_{i+2}=i!\ g^{(i)}(z)$. Let $e_1=\frac{\partial}{\partial z}$ and

$$f(p_1,p_2)=\frac{1}{p_1-p_2}+\sum_{i,j=0}^{\infty}f_{i,j}(z)\ (p_1-z)^i(p_2-z)^j.$$

Then we have $[e_1,e_i]=(i-1)e_{i+1}$ and equation (3.18) is equivalent to

$$[e_i,e_j]=(j-i)e_{i+j}+\sum_{r=0}^{i-1}(i+r-1)f_{j-2,r}e_{i-r+1}-\sum_{r=0}^{j-1}(j+r-1)f_{i-2,r}e_{j-r+1}.$$

In particular, if $f(p_1,p_2)=\frac{1}{p_1-p_2}$, then we get $[e_i,e_j]=(j-i)e_{i+j}$ for $e_1,e_2,....$ Note that (3.19) always holds for $f(p_1,p_2)=\frac{1}{p_1-p_2}$. Therefore, a local GT structure can be regarded as a certain deformation of a Lie algebra with basis $e_1,e_2,...$ and bracket $[e_i,e_j]=(j-i)e_{i+j}$ in the class of Lie algebroids.

Given a local GT structure one wants to classify all Whitham type hierarchies that are integrable by hydrodynamic reductions and that correspond to a given Gibbons-Tsarev system. It turns out that in order to do this one needs to find all functions $\lambda(p_1,p_2,v_1,...,v_m)$ satisfying a certain condition. This can be formalized in the following way:

Definition 3.2. An enhanced local GT structure is a family of vector fields $g(p)$, a function $f(p_1, p_2)$ and an additional function $\lambda(p_1, p_2, v_1, ..., v_m)$ satisfying the relations (3.18), (3.19), (3.20) and

$$g(p_1)(\lambda(p_2, p_3)) = \lambda(p_1, p_3)\lambda(p_2, p_1)_{p_1} - \lambda(p_2, p_3)f(p_1, p_2)_{p_2} - \\ -f(p_1, p_2)\lambda(p_2, p_3)_{p_2} - f(p_1, p_3)\lambda(p_2, p_3)_{p_3}, \tag{3.25}$$

$$\lambda(p_1, p_2) = \frac{1}{p_1 - p_2} + O(1).$$

Given an enhanced local GT structure one wants to find a vector space of all potentials of the corresponding Whitham type hierarchy. In all known examples these spaces are spaces of solutions of linear systems of PDEs. However, in the general case we can define this vector space as a space of solutions of a linear functional equation.

Definition 3.3. Given an enhanced local GT structure we define the corresponding vector space of potentials as the space of solutions of the following functional equation for a function $h(p, v_1, ..., v_m)$:

$$g(p_1)(h(p_2)) = \lambda(p_1, p_2)h'(p_1) - f(p_1, p_2)h'(p_2). \tag{3.26}$$

Note that expanding (3.26) near diagonal $p_2 = p_1$ we obtain for $h(p, v_1, ..., v_m)$ a system of linear PDEs equivalent to (3.26).

The following procedure gives a standard way to obtain solutions of (3.26):

Proposition 3.3. Let γ be a path in $\mathbb{C}$ such that $\int_\gamma \frac{\partial(\lambda(t,p_2)f(p_1,t))}{\partial t}dt = 0$. Then

$$h(p) = \int_\gamma \lambda(t, p)dt$$

is a solution of (3.26).

Proof. Substitute this expression for $h(p)$ into (3.26) and use (3.25). Direct computation shows that the difference between the r.h.s and the l.h.s. of (3.26) is $\int_\gamma \frac{\partial(\lambda(t,p_2)f(p_1,t))}{\partial t}dt$. □

Let us promote local GT structures to differential-geometric ones.

Proposition 3.4. Relations (3.18), (3.19), (3.20) are invariant with respect to arbitrary transformations of the form

$$p_i = \mu(\tilde{p}_i, v_1, ..., v_m), \qquad \tilde{g}(\tilde{p}) = \mu'(\tilde{p})^2 g(\mu(\tilde{p})), \\ \tilde{f}(\tilde{p}_1, \tilde{p}_2) = \frac{\mu'(\tilde{p}_1)^2}{\mu'(\tilde{p}_2)}\Big(f(\mu(\tilde{p}_1), \mu(\tilde{p}_2)) - g(\mu(\tilde{p}_1))(\mu(\tilde{p}_2))\Big). \tag{3.27}$$

Let $\pi: M \to B$ be a bundle with m dimensional fiber F and one dimensional base B.

Definition 3.4. A GT structure on π is a local GT structure on each trivialization for each $U \subset B$ such that for different trivializations these local GT structures are connected by (3.27). Here $v_1, ..., v_m$ stands for coordinates on F and p is a coordinate on B.

Proposition 3.5. Relations (3.25) are invariant with respect to an arbitrary transformations of the form (3.27) provided that λ is transformed as

$$\tilde{\lambda}(\tilde{p}_1, \tilde{p}_2) = \mu'(\tilde{p}_1)\lambda(\mu(\tilde{p}_1), \mu(\tilde{p}_2)) \tag{3.28}$$

Definition 3.5. An enhanced GT structure on π is an enhanced local GT structure on each trivialization for each $U \subset B$ such that for different trivializations these enhanced local GT structures are connected by (3.27), (3.28).

Example 3.1. It is clear from (2.10), (2.12) that $g(p) = G(p)$, $f(p_1, p_2) = F(p_1, p_2)$ is a GT structure on the bundle $M_{g,1} \to M_g$.

Similar GT structures exist for $g = 0, 1$. In the case $g = 0$ we consider the moduli space $M_{0,n+3}$ of complex structures on $\mathbb{C}P^1$ with punctures in $n+3$ points. We fix 3 points at $0, 1, \infty$ and move other points. The formulas for the corresponding GT structures read

$$f(p_1, p_2) = \frac{p_2(p_2 - 1)}{(p_1 - p_2)p_1(p_1 - 1)}, \quad g(p) = \sum_{i=1}^{n} \frac{u_i(u_i - 1)}{(p - u_i)p(p - 1)} \frac{\partial}{\partial u_i}. \tag{3.29}$$

In the case $g = 1$ we consider the moduli space $M_{1,n+1}$ of complex structures on an elliptic curve with punctures in $n + 1$ points. We fix one point at 0 and move other points. We also deform the complex structure on our elliptic curve. The space of complex structures is one dimensional in this case. We use the modular parameter τ with $\mathrm{Im}\tau > 0$ as a coordinate on the moduli space of elliptic curves. The formulas for the corresponding GT structures read

$$f(p_1, p_2) = \rho(p_1 - p_2, \tau) - \rho(p_1), \quad g(p) = 2\pi i \frac{\partial}{\partial \tau} + \sum_{j=1}^{n} (\rho(p - u_j, \tau) - \rho(p, \tau)) \frac{\partial}{\partial u_j} \tag{3.30}$$

where $\rho(p, \tau) = \frac{\partial}{\partial p} \ln(\theta(p, \tau))$ and $\theta(p, \tau) = \sum_{k \in \mathbb{Z}} (-1)^k e^{2\pi i (kp + \frac{k(k-1)}{2}\tau)}$.

Remark 3.3. In these GT structures we can also collide points and obtain new GT structures. Moreover, in the case $g = 0$ (resp. $g = 1$) we can collide points with $0, 1, \infty$ (resp. with 0) by doing a substitution similar to (3.24). In the case $g = 0$ we can also make an arbitrary fractional linear transformation with constant coefficients sending $0, 1, \infty$ to a, b, c and collide some of a, b, c.

Remark 3.4. Consider an enhanced local GT structure with $g(p)$ given by (3.23). Colliding points $v_{j,0}, v_{j,1}, ..., v_{j,n_j}$ by substitution (3.24) and taking the limit $\epsilon \to 0$ we can do the same substitution and limit in the function λ and obtain a new enhanced local GT structure.

4 Whitham type hierarchies

Given a set of independent variables $t_1, ..., t_N$ called times, a set of dependent variables $v_1, ..., v_m$ called fields and a set of functions $h_i(z, v_1, ..., v_m)$, $i = 1, ..., N$ called potentials we define a Whitham type hierarchy as compatibility conditions of the following system of PDEs:

$$\frac{\partial \psi}{\partial t_i} = h_i(z, v_1, ..., v_m), \ i = 1, ..., N. \tag{4.31}$$

Here $\psi, v_1, ..., v_m$ are functions of times $t_1, ..., t_N$ and z is a parameter. The system (4.31) is understood as a parametric way of defining $N-1$ relations between partial derivatives $\frac{\partial \psi}{\partial t_i}$, $i = 1, ..., N$ obtained by eliminating z from these equations. Let us assume that the system (4.31) is compatible. Compatibility conditions can be written as

$$\sum_{l=1}^{m} \Big(\Big(\frac{\partial h_i}{\partial z} \frac{\partial h_j}{\partial v_l} - \frac{\partial h_j}{\partial z} \frac{\partial h_i}{\partial v_l} \Big) \frac{\partial v_l}{\partial t_k} + \Big(\frac{\partial h_j}{\partial z} \frac{\partial h_k}{\partial v_l} - \frac{\partial h_k}{\partial z} \frac{\partial h_j}{\partial v_l} \Big) \frac{\partial v_l}{\partial t_i} + \Big(\frac{\partial h_k}{\partial z} \frac{\partial h_i}{\partial v_l} - \frac{\partial h_i}{\partial z} \frac{\partial h_k}{\partial v_l} \Big) \frac{\partial v_l}{\partial t_j} \Big) = 0 \tag{4.32}$$

where $i, \ j, \ k = 1, ..., N$ are pairwise distinct. Let $V_{i,j,k}$ be the linear space of functions in z spanned by $\frac{\partial h_i}{\partial z} \frac{\partial h_j}{\partial v_l} - \frac{\partial h_j}{\partial z} \frac{\partial h_i}{\partial v_l}$, $\frac{\partial h_j}{\partial z} \frac{\partial h_k}{\partial v_l} - \frac{\partial h_k}{\partial z} \frac{\partial h_j}{\partial v_l}$, $\frac{\partial h_k}{\partial z} \frac{\partial h_i}{\partial v_l} - \frac{\partial h_i}{\partial z} \frac{\partial h_k}{\partial v_l}$, $l = 1, ..., m$.

Proposition 4.1. Let $V_{i,j,k}$ be finite dimensional and $\dim V_{i,j,k} = D$. Then (4.32) is equivalent to a hydrodynamic type system of D linearly independent equations of the form

$$\sum_{l=1}^{m} \Big(a_{rl}(v_1, ..., v_m) \frac{\partial v_l}{\partial t_i} + b_{rl}(v_1, ..., v_m) \frac{\partial v_l}{\partial t_j} + c_{rl}(v_1, ..., v_m) \frac{\partial v_l}{\partial t_k} \Big) = 0, \ r = 1, ..., D. \tag{4.33}$$

Proof. Let $\{S_1(z), ..., S_D(z)\}$ be a basis in $V_{i,j,k}$ and
$\frac{\partial h_i}{\partial z} \frac{\partial h_j}{\partial v_l} - \frac{\partial h_j}{\partial z} \frac{\partial h_i}{\partial v_l} = \sum_{r=1}^{D} c_{rl} S_r$, $\frac{\partial h_j}{\partial z} \frac{\partial h_k}{\partial v_l} - \frac{\partial h_k}{\partial z} \frac{\partial h_j}{\partial v_l} = \sum_{r=1}^{D} a_{rl} S_r$, $\frac{\partial h_k}{\partial z} \frac{\partial h_i}{\partial v_l} - \frac{\partial h_i}{\partial z} \frac{\partial h_k}{\partial v_l} = \sum_{r=1}^{D} b_{rl} S_r$.

Substituting these expressions into (4.32) and equating to zero coefficients at $S_1, ..., S_D$ we obtain (4.33). □

Remark 4.1. In the theory of integrable systems of hydrodynamic type the system (4.31) is often referred to as a pseudo-potential representation of the system (4.33).

Remark 4.2. In all known examples of integrable Whitham type hierarchies we have $n \leq D \leq 2n - 1$. Therefore, this inequality can be regarded as a criterion of integrability. However, in this paper we explore another criterion of integrability given by the so-called hydrodynamic reduction method.

5 Gibbons–Tsarev systems

Gibbons-Tsarev systems are the main ingredient of the approach to integrability of Whitham type hierarchies and, more generally, to integrability of quasi-linear systems of the form (4.33) based on hydrodynamic reductions. In this approach hydrodynamic reductions of a given hierarchy are parameterized by solutions of a Gibbons-Tsarev system. In this Section we explain a connection between Gibbons-Tsarev systems and GT structures.

Let $p_1,..,p_M,\ v_1,...,v_m$ be functions of auxiliary variables $r_1,...,r_M$ and $\partial_i=\frac{\partial}{\partial r_i}$.

Definition 5.1. A Gibbons–Tsarev system is a compatible system of partial differential equations of the form.

$$\begin{gathered}\partial_i p_j=f(p_i,p_j,v_1,\dots,v_m)\partial_i v_1,\quad i\neq j,\quad i,j=1,\dots,M,\\ \partial_i v_j=g_j(p_i,v_1,\dots,v_m)\partial_i v_1,\quad j=2,\dots,m,\quad i=1,\dots,M,\\ \partial_i\partial_j v_1=q(p_i,p_j,v_1,\dots,v_m)\partial_i v_1\partial_j v_1,\quad i\neq j,\quad i,j=1,\dots,M.\end{gathered}\tag{5.34}$$

Remark 5.1. It follows from the compatibility assumption that the space of solutions of a Gibbons-Tsarev system is locally parameterized by $2M$ functions in one variable. Note that $f,\ g_i,\ q$ do not depend on M and therefore M can be arbitrary large for a given Gibbons-Tsarev system.

We say that a Gibbons-Tsarev system is non-degenerate if $f(p_1,p_2,v_1,...,v_m)$ has a pole of order one on the diagonal $p_2=p_1$. Assume in the sequel that all Gibbons-Tsarev systems are non-degenerate.

Proposition 5.1. There exists a one-to-one correspondence between non-degenerate Gibbons–Tsarev systems and local GT structures.

Proof. Redefining f,g_i from (5.34) we write a Gibbons-Tsarev system in the form

$$\begin{gathered}\partial_i p_j=\frac{f(p_i,p_j,v_1,\dots,v_m)}{g_1(p_i,v_1,...,v_m)}\partial_i v_1,\quad i\neq j,\quad i,j=1,\dots,M,\\ \frac{\partial_i v_1}{g_1(p_i,v_1,\dots,v_m)}=\frac{\partial_i v_j}{g_j(p_i,v_1,\dots,v_m)},\quad j=2,\dots,m,\quad i=1,\dots,M,\\ \partial_i\partial_j v_1=q(p_i,p_j,v_1,\dots,v_m)\partial_i v_1\partial_j v_1,\quad i\neq j,\quad i,j=1,\dots,M\end{gathered}\tag{5.35}$$

where $f(p_1,p_2)=\frac{1}{p_1-p_2}+O(1)$. Indeed, $\frac{1}{g_1(p_i)}$ is the residue of $f(p_i,p_j)$ from (5.34) at $p_j=p_i$. Write

$$g(p)=\sum_{i=1}^m g_i(p,v_1,...,v_m)\frac{\partial}{\partial v_i}.$$

Compatibility of the system (5.35) implies $\partial_1\partial_2\phi(p_3,v_1,...,v_m)=\partial_2\partial_1\phi(p_3,v_1,...,v_m)$ for an arbitrary function ϕ. This can be written as

$$\Big(f(p_1,p_2)\frac{\partial}{\partial p_2}+f(p_1,p_3)\frac{\partial}{\partial p_3}+g(p_2)\Big)\Big((f(p_2,p_3)\frac{\partial}{\partial p_3}+g(p_2))\phi(p_3)\cdot\frac{\partial_2 u_1}{g_1(p_2)}\Big)\cdot\frac{\partial_1 u_1}{g_1(p_1)}=$$

$$\Big(f(p_2,p_1)\frac{\partial}{\partial p_1}+f(p_2,p_3)\frac{\partial}{\partial p_3}+g(p_1)\Big)\Big((f(p_1,p_3)\frac{\partial}{\partial p_3}+g(p_1))\phi(p_3)\cdot\frac{\partial_1 u_1}{g_1(p_1)}\Big)\cdot\frac{\partial_2 u_1}{g_1(p_2)}.$$

Expanding this equation and equating coefficients at ϕ and ϕ_{p_3} we get

$$f(p_1,p_2)f(p_2,p_3)_{p_2}-f(p_2,p_1)f(p_1,p_3)_{p_1}+f(p_1,p_3)f(p_2,p_3)_{p_3}-f(p_2,p_3)f(f(p_1,p_3)_{p_3}+$$
$$+g(p_1)(f(p_2,p_3))-g(p_2)(f(p_1,p_3))+$$
$$f(p_2,p_3)\Big(g_1(p_1)\frac{\partial_1\partial_2u_1}{\partial_1u_1\partial_2u_1}-\frac{1}{g_1(p_2)}\Big(f(p_1,p_2)\frac{\partial}{\partial p_2}+g(p_1)\Big)(g_1(p_2))\Big)-$$
$$f(p_1,p_3)\Big(g_1(p_2)\frac{\partial_1\partial_2u_1}{\partial_1u_1\partial_2u_1}-\frac{1}{g_1(p_1)}\Big(f(p_2,p_1)\frac{\partial}{\partial p_1}+g(p_2)\Big)(g_1(p_1))\Big)=0,$$

$$f(p_1,p_2)g'(p_2)-f(p_2,p_1)g'(p_1)+[g(p_1),g(p_2)]+$$
$$\Big(g_1(p_1)\frac{\partial_1\partial_2u_1}{\partial_1u_1\partial_2u_1}-\frac{1}{g_1(p_2)}\Big(f(p_1,p_2)\frac{\partial}{\partial p_2}+g(p_1)\Big)(g_1(p_2))\Big)g(p_2)-$$
$$\Big(g_1(p_2)\frac{\partial_1\partial_2u_1}{\partial_1u_1\partial_2u_1}-\frac{1}{g_1(p_1)}\Big(f(p_2,p_1)\frac{\partial}{\partial p_1}+g(p_2)\Big)(g_1(p_1))\Big)g(p_1)=0.$$

Expanding the first of these equations near the diagonal $p_2=p_3$ and noting that

$$f(p_1,p_2)f(p_2,p_3)_{p_2}+f(p_1,p_3)f(p_2,p_3)_{p_3}-f(p_2,p_3)f(p_1,p_3)_{p_3}=-\frac{2f(p_1,p_2)_{p_2}}{p_2-p_3}+O(1)$$

we obtain

$$g_1(p_1)\frac{\partial_1\partial_2u_1}{\partial_1u_1\partial_2u_1}-\frac{1}{g_1(p_2)}\Big(f(p_1,p_2)\frac{\partial}{\partial p_2}+g(p_1)\Big)(g_1(p_2))=2f(p_1,p_2)_{p_2}.$$

Substituting this into our equations we arrive at relations (3.19), (3.18) for a local GT structure.

One can check that all these steps are invertible and any local GT structure with relations (3.18), (3.19) gives a Gibbons-Tsarev system (5.35) with

$$\partial_1\partial_2u_1=\Big(\frac{2f(p_1,p_2)_{p_2}}{g_1(p_1)}+\frac{1}{g_1(p_1)g_1(p_2)}\Big(f(p_1,p_2)\frac{\partial}{\partial p_2}+g(p_1)\Big)(g_1(p_2))\Big)\partial_1u_1\partial_2u_1.$$

□

6 Integrability of Whitham type hierarchies

In this Section we explain a relation between integrable Whitham type hierarchies and enhanced GT structures.

Proposition 6.1. A Whitham type hierarchy with potentials $h_i(p,v_1,...,v_m),\ i=1,...,N$ is integrable by hydrodynamic reductions if and only if there exists a Gibbons-Tsarev system (5.35) such that

$$h_j'(p_1)\partial_1(h_i(p_2))=h_i'(p_1)\partial_1(h_j(p_2)),\quad i,j=1,...,N \tag{6.36}$$

by virtue of (5.35).

Proof. The equation (6.36) can be written as

$$f(p_1,p_2)=\frac{\sum_{k=1}^m\Big(h_i'(p_1)h_j(p_2)_{v_k}-h_j'(p_1)h_i(p_2)_{v_k}\Big)g_k(p_1)}{h_j'(p_1)h_i'(p_2)-h_j'(p_2)h_i'(p_1)} \tag{6.37}$$

and, therefore, coincides with the formula (77) from [19]. It is proven in [19] that the equation (6.37) is equivalent to the integrability of a given Whitham type hierarchy. □

Proposition 6.2. There exists a one-to-one correspondence between integrable Whitham type hierarchies and enhanced local GT structures. Under this correspondence the space of potentials of a Whitham type hierarchy coincides with the space of solutions of the linear system (3.26).

Proof. Write (6.36) as $\frac{\partial_1(h_i(p_2))}{h_i'(p_1)}=\frac{\partial_1(h_j(p_2))}{h_j'(p_1)}$. By executing ∂_1 in numerators we get

$$\frac{f(p_1,p_2)h_i'(p_2)+g(p_1)(h_i(p_2))}{h_i'(p_1)}=\frac{f(p_1,p_2)h_j'(p_2)+g(p_1)(h_j(p_2))}{h_j'(p_1)}.$$

Let $\lambda(p_1,p_2)=\frac{f(p_1,p_2)h_i'(p_2)+g(p_1)(h_i(p_2))}{h_i'(p_1)}$, this function does not depend on i. Therefore, we get

$$g(p_1)(h_i(p_2))=\lambda(p_1,p_2)h_i'(p_1)-f(p_1,p_2)h_i'(p_2)$$

which coincides with (3.26). Applying the relation (3.18) to $h_i(p_3)$ we can write

$$g(p_1)g(p_2)(h_i(p_3))-g(p_2)g(p_1)(h_i(p_3))=$$

$$f(p_2,p_1)b'(p_1)(h_i(p_3))-f(p_1,p_2)b'(p_2)(h_i(p_3))+2f(p_2,p_1)_{p_1}b(p_1)(h_i(p_3))-2f(p_1,p_2)_{p_2}b(p_2)(h_i(p_3)).$$

Computing the l.h.s. and the r.h.s. of this relation by virtue of (3.26) we obtain (3.25). □

7 The universal Whitham hierarchy

In this Section we use notations introduced in Section 2, including $G(p)$ and $F(p_1,p_2)$.

According to [9] the universal Whitham hierarchy is given by potentials obtained by integration of meromorphic differentials on a Riemann surface. We are going to construct such an hierarchy explicitly[3] and prove that it is integrable by hydrodynamic reductions.

Proposition 7.1. Fix constants $s_1,...,s_m$ such that $s_1+...+s_m=1$ (the simplest possibility is $m=1$ and $s_1=1$). The following formulas define an enhanced GT structure:

$$g(p)=\sum_{j=1}^n F(p,u_j)\frac{\partial}{\partial u_j}+\sum_{k=1}^m F(p,w_j)\frac{\partial}{\partial w_k}+G(p),\quad f(p_1,p_2)=F(p_1,p_2), \tag{7.38}$$

[3] We need to choose constants of integrations carefully in order to obtain an integrable hierarchy.

$$\lambda(p_1,p_2)=\frac{E(p_1,p_2)_{p_1}}{E(p_1,p_2)}-\sum_{k=1}^{m}s_k\frac{E(p_1,w_k)_{p_1}}{E(p_1,w_k)}.$$

Moreover, the following functions belong to the space of potentials of this enhanced GT structure:

$$h_j(p)-h_1(p),\ j=2,...,n,\quad q_\alpha(p)-\sum_{k=1}^{m}s_kq_\alpha(w_k),\ \alpha=1,...,g$$

where

$$h_j(p)=\ln(E(p,u_j))-\sum_{k=1}^{m}s_k\ln(E(u_j,w_k)). \tag{7.39}$$

Proof. We need to prove identities (3.25) and (3.26) for given $\lambda(p_1,p_2)$ and potentials. This can be done by straightforward computation using identities from Proposition 2.2. The simplest way is to start from identity (3.26) for $h_j(p)-h_1(p)$ and check it using identity (2.13). It is clear from the proof of the Proposition 6.2 that (3.25) is a consequence of (3.26). It follows from Proposition 3.3 that $\frac{1}{2\pi i}\int_{b_\alpha}\lambda(t,p)dt$ are also potentials of our hierarchy. Computing these integrals by virtue of (2.4) we conclude that the functions $q_\alpha(p)-\sum_{k=1}^{m}s_kq_\alpha(w_k),\ \alpha=1,...,g$ belong to the space of potentials. □

Proposition 7.2. The universal Whitham hierarchy is integrable by hydrodynamic reductions.

Proof. It is clear that the vector space spanned by derivatives with pespect to p of potentials described in the previous Proposition coincides with the space of meromorphic differentials on $\mathcal{E}$ holomorphic outside $u_1,...,u_n$ and with poles of order less or equal to one in these points. Therefore, we obtain a part of the universal Whitham hierarchy. In order to obtain the full hierarchy we apply the procedure of colliding point, see Proposition 3.2 and Remark 3.4. This proves the Proposition in the case $g>1$.

In the case $g=0$ we define an enhanced GT structure by (3.29) and set $\lambda(p_1,p_2)=\frac{1}{p_1-p_2}$. The space of potentials contains the functions $h_j(p)-h_1(p),\ j=2,...,n+2$ where $h_j(p)=\ln(p-u_j),\ j=1,...,n,\ h_{n+1}(p)=\ln(p)$ and $h_{n+2}(p)=\ln(p-1)$. This gives a part of the universal Whitham hierarchy corresponding to meromorphic differentials on $\mathbb{C}P^1$ with poles of order less or equal to one in $u_1,...,u_n,0,1$. To obtain the full hierarchy we collide these points by a procedure similar to the one in the proof of Proposition 3.2, see also Remarks 3.3 and 3.4.

In the case $g=1$ we define an enhanced GT structure by (3.30) and set

$$\lambda(p_1,p_2)=\rho(p_1-p_2,\tau)-\rho(p_1)-2\pi i.$$

The space of potentials contains $p-\tau$ and the functions $h_j(p)-h_1(p),\ j=2,...,n$ where $h_j(p)=\ln(\theta(p-u_j,\tau))-\ln(\theta(u_j,\tau))$. This gives a part of the universal Whitham hierarchy corresponding to meromorphic differentials on $\mathcal{E}$ with poles of order less or equal to one in $u_1,...,u_n$. To obtain the full hierarchy we collide some of these points by a procedure similar to one in the proof of Proposition 3.2, see also Remark 3.4. □

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