Proof of a modular relation between 1-, 2- and 3-loop Feynman diagrams on a torus

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Abstract

The coefficients of the higher-derivative terms in the low energy expansion of genus-one graviton scattering amplitudes are determined by integrating sums of non-holomorphic modular functions over the complex structure modulus of a torus. In the case of the four-graviton amplitude, each of these modular functions is a multiple sum associated with a Feynman diagram for a free massless scalar field on the torus. The lines in each diagram join pairs of vertex insertion points and the number of lines defines its weight w, which corresponds to its order in the low energy expansion. Previous results concerning the low energy expansion of the genus-one four-graviton amplitude led to a number of conjectured relations between modular functions of a given w, but different numbers of loops $\leq w-1$. In this paper we shall prove the simplest of these conjectured relations, namely the one that arises at weight $w=4$ and expresses the three-loop modular function D_4 in terms of modular functions with one and two loops. As a byproduct, we prove three intriguing new holomorphic modular identities.

1 Introduction and summary of results

In an earlier paper [1] we elucidated certain properties of the non-holomorphic modular functions that enter into the low energy expansion of the genus-one four-graviton scattering amplitude in string perturbation theory.[1] These non-holomorphic modular functions arise from vacuum Feynman diagrams for a massless scalar field on a torus of fixed modulus τ with marked points at the positions of the four vertex operators. The lines in a diagram correspond to Green functions (i.e., propagators) joining pairs of these points. The *weight* w is the number of scalar lines in the diagram; it governs the order at which the diagram contributes to the low energy expansion. The number of loops of a diagram will be denoted L. Expressing a Feynman diagram in terms of the discrete momenta on the torus gives a representation of its value in terms of a multiple sum over $2L$ independent integers that are generalizations of the standard non-holomorphic Eisenstein series (for which $L = 1$)[2]. There are contributions from diagrams of weight w with different numbers of loops L, subject to the constraint $L \leq w - 1$. A key to the progress made in elucidating the properties of these modular functions in [1] was understanding the structure that emerges by considering families of modular functions with a fixed number of loops L.

• For $L = 1$ (which is the lowest non-trivial value for L due to momentum conservation on the torus) and weight w, the modular function is unique (up to a constant normalization factor) and given by the non-holomorphic Eisenstein series E_w, defined by,

$$E_w(\tau, \bar\tau) = \sum_{(m,n)\neq(0,0)} \left(\frac{\tau_2}{\pi\, |m\tau + n|^2} \right)^w \tag{1.1}$$

The sum is over integers $m, n \in \mathbb{Z}$ which parametrize the discrete momenta on the torus; the real and imaginary parts of τ are respectively $\tau_1, \tau_2 \in \mathbb{R}$; and the factor of $1/\pi^w$ has been included for convenience. The Eisenstein series satisfies the Laplace-eigenvalue equation,

$$\Delta E_w = w(w - 1)E_w \tag{1.2}$$

where the Laplace-Beltrami operator Δ on the upper half plane is given by $\Delta = 4\tau_2^2\, \partial_\tau \partial_{\bar\tau}$.

• For $L = 2$, the most general vacuum Feynman diagram of weight w is given by a

[1] Non-holomorphic modular functions are functions of the complex structure parameter τ of the torus and its complex conjugate $\bar\tau$ which are invariant under the canonical action of $SL(2, \mathbb{Z})$ on τ and $\bar\tau$ by Möbius transformations. For paedagigical introductions to the subject, one may consult [2] [3], [4], while a historical perspective is presented in [5]. Since we shall deal with both holomorphic and non-holomorphic functions of τ in this paper, we shall indicate the dependence on both τ and $\bar\tau$ for non-holomorphic objects throughout, which is a different notation from that adopted in [1].

[2] These bear some resemblance to multiple Kronecker–Eisenstein series of the type discussed in [6].

multiple sum of the form,

$$C_{a_1,a_2,a_3}(\tau,\bar\tau) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\,\delta_{n,0} \prod_{r=1}^{3} \left(\frac{\tau_2}{\pi\,|m_r\tau+n_r|^2} \right)^{a_r} \tag{1.3}$$

The sum is over integers $m_r, n_r \in \mathbb{Z}$ for $r = 1,2,3$, while $m = m_1 + m_2 + m_3$ and $n = n_1 + n_2 + n_3$ are constrained to vanish by the Kronecker symbols. The parameters a_1, a_2, a_3 are positive integers subject to $w = a_1 + a_2 + a_3$. A rich structure of the space of all $L = 2$ modular functions of arbitrary given weight w was exhibited in [1]. This was achieved by showing that the functions C_{a_1,a_2,a_3} satisfy a system of Laplace eigenvalue equations whose inhomogeneous parts are quadratic polynomials in Eisenstein series, and whose eigenvalues and eigenspaces are governed by the representation theory of a hidden $SO(2,1)$. The simplest examples may be exhibited for the lowest weights $w = 3, 4$ where we have,

$$\begin{aligned} \Delta C_{1,1,1} &= 6\,E_3 \\ (\Delta-2)C_{2,1,1} &= 9\,E_4 - E_2^2 \end{aligned} \tag{1.4}$$

The first equation may be integrated and the integration constant may be determined by matching the asymptotic behavior near the cusp $\tau_2 \to \infty$ to give $C_{1,1,1} = E_3 + \zeta(3)$, a result that had been obtained earlier by Zagier by direct summation of the series [7]. The second equation admits no such simple integration, but its significance will become clear shortly.

• For $L \geq 3$, the situation is more complicated and considerably less well-understood. There is no longer a single formula (such as for $C_{a,b,c}$ for $L = 2$) to evaluate all diagrams, since more than a single diagram topology contributes when $L \geq 3$. Moreover, there is no systematic way known to derive equations of the Laplace eigenvalue type for the corresponding multiple sums. Therefore, the methods used to expose the structure at two loops appear of little use for higher loop diagrams. It has been possible, however, to formulate certain conjectured relations between the weight four and weight five modular functions.

The simplest of these conjectures was for the modular function D_4 characterized by three loops, $L = 3$, and weight $w = 4$, and given by the following sum,

$$D_4(\tau,\bar\tau) = \sum_{(m_r,n_r)\neq 0} \delta_{m,0}\,\delta_{n,0} \prod_{r=1}^{4} \left(\frac{\tau_2}{\pi\,|m_r\tau+n_r|^2} \right) \tag{1.5}$$

The sum is over integers $m_r, n_r \in \mathbb{Z}$ with $r = 1,2,3,4$, subject to the vanishing of $m = m_1 + m_2 + m_3 + m_4$ and $n = n_1 + n_2 + n_3 + n_4$ as enforced by the Kronecker δ-symbols.

The purpose of the present paper is to provide an analytical proof of the conjecture for D_4, which we shall henceforth refer to as the Theorem for D_4. This involves the following polynomial combination of modular functions,

$$F(\tau,\bar\tau) = D_4(\tau,\bar\tau) - 24C_{2,1,1}(\tau,\bar\tau) - 3E_2(\tau,\bar\tau)^2 + 18E_4(\tau,\bar\tau) \tag{1.6}$$

By construction, F is a modular function of weight 4 and involves contributions with one, two, and three loops. The precise statement is as follows.

Theorem: *The modular function $D_4(\tau, \bar\tau)$ satisfies the relation*

$$F(\tau, \bar\tau) = 0 \tag{1.7}$$

The evidence for the validity of (1.7) presented in [1] was based on an analysis of the behavior of $F(\tau, \bar\tau)$ near the cusp $\tau_2 \to \infty$. The expansion of $F(\tau, \bar\tau)$ in powers of q and $\bar q$, whose coefficients are powers of τ_2, was verified to vanish to lowest and first orders in q and $\bar q$. This gave us compelling evidence but, of course, did not constitute an analytic proof of (1.7). To prove the Theorem, we shall first prove the following auxiliary Lemma.

Lemma:

1. *The modular function $F(\tau, \bar\tau)$, defined in (1.6), admits a decomposition in inverse powers of τ_2, with a finite number of terms,*

$$F(\tau, \bar\tau) = \sum_{k=0}^{7} (\pi \tau_2)^{4-k} \mathcal{F}_k(q, \bar q) \tag{1.8}$$

   where the coefficients $\mathcal{F}_k(q, \bar q)$ are entire functions of $q = e^{2\pi i \tau}$ and $\bar q = e^{-2\pi i \bar\tau}$;

2. The coefficients $\mathcal{F}_k$ vanish for $k = 0, 1, 2, 3, 7$;

3. The coefficients $\mathcal{F}_k$ for $k = 4, 5, 6$ are harmonic functions of q and $\bar q$, and may be expressed in terms of holomorphic functions $\varphi_k(q)$ of q,

$$\mathcal{F}_k(q, \bar q) = \varphi_k(q) + \overline{\varphi_k(q)} \tag{1.9}$$

   where $\varphi_k(q)$ obeys the conjugation properties,

$$\overline{\varphi_k(q)} = \varphi_k(\bar q) \qquad\qquad \varphi_k(1/q) = (-)^k \varphi_k(q) \tag{1.10}$$

   The first relation in (1.10) is complex conjugation and implies that the Taylor series of $\varphi_k(q)$ in powers of q has real coefficients.

The proof of the Lemma will proceed by direct summation over the integers n_r in the multiple sum that defines D_4 in (1.5), $C_{2,1,1}$ in (1.3), and E_2, E_4 in (1.1). This method generalizes the calculation of Zagier for the function $C_{1,1,1}$. The proof of items 1 and 2 will be relatively straightforward, but the proof of item 3 will require extensive algebraic manipulations, most of which will be summarized in Appendix B.

The proof of the Theorem itself will proceed by showing that the holomorphic functions $\varphi_k(q)$ for $k = 4, 5, 6$ are modular forms of weight $0, -2$, and -4 respectively, and therefore must vanish (φ_4 must be constant by this argument, but will vanish in view of its asymptotic behavior near the cusp).

Holomorphic Corollaries:

The vanishing of the holomorphic coefficients $\varphi_k(q)$ for $k = 4, 5, 6$ leads to three holomorphic identities given in (B.7), (B.22) and (B.32). While their vanishing is a consequence of the Theorem, no direct analytical proof of the identities is known to us. The simplest of these identities results from splitting $\varphi_6(q) = \varphi_6^{(1)}(q) + \varphi_6^{(2)}(q)$ and then proving by combinatorial rearrangements that $\varphi_6^{(1)} = 0$, which leaves the non-trivial identity $\varphi_6^{(2)}(q) = 0$, namely,

$$\begin{aligned} \frac{15}{8}\zeta(6) &= \sum_{m_1+m_2\neq 0}{}' \frac{3}{16m_1m_2(m_1+m_2)^4} \frac{(1+q^{m_1})(1+q^{m_2})}{(1-q^{m_1})(1-q^{m_2})} \\ &\quad + \sum_{m_1,m_2}{}' \frac{3}{16m_1^3m_2^3} \frac{(1+q^{m_1})(1+q^{m_2})}{(1-q^{m_1})(1-q^{m_2})} - \sum_{m}{}' \frac{9}{4m^6} \frac{q^m}{(1-q^m)^2} \end{aligned} \tag{1.11}$$

The sum extends over integers $m_1, m_2 \in \mathbb{Z}$ which do not vanish (as indicated by the prime). A direct combinatorial proof of this identity is not known to us, but we have confirmed its validity order by order in an expansion in q around $q = 0$ using MAPLE to order $\mathcal{O}(q^{400})$.

Our proof of the D_4 conjecture of [1] provides significant encouragement that the conjectures advanced in [1] for the three non-trivial weight $w = 5$ modular functions, namely $D_5, D_{3,1,1}$ and $D_{2,2,1}$ may be proven by the same methods, even if the algebraic manipulations involved will be even more arduous. One would hope that with more insight a simpler proof will emerge, which would facilitate the proofs for $D_5, D_{3,1,1}$ and $D_{2,2,1}$ and the generalizations to higher weight. We shall report on such developments in future work.

1.1 Organization

The remainder of this paper is organized as follows.

In section 2 we provide a brief review of the role of non-holomorphic modular functions in the low energy expansion of genus-one string perturbation theory. In section 3, we provide the proof of the Lemma, supported by results derived in Appendices A and B. In section 4, we provide the proof of the Theorem by combining the results of the Lemma and further results on the structure of $F(\tau, \bar\tau)$ derived in Appendix C. In section 5, we spell out the various components of the Holomorphic Corollaries, and exhibit the holomorphic identities which arise as spin-offs of the proof of the Theorem. Finally, in section 6 we summarize our results, and discuss the outlook of our work.

2 String theory origin of the modular functions

We will here give a brief overview of the motivation for considering non-holomorphic modular functions from the physical perspective of superstring perturbation theory. This section is not essential for the remainder of the paper, whose purpose is rather to give a purely mathematical proof of the Theorem stated in the introduction. The starting point is the full genus-one four-graviton amplitude $\mathcal{A}_1$ in Type II closed superstring perturbation theory. It is given by an integral over the moduli space of genus-one Riemann surfaces of a partial amplitude $\mathcal{B}_1$ which is defined at fixed modulus. It is this partial amplitude $\mathcal{B}_1$ that will be of direct interest in this paper, and we shall start by reviewing its structure. For completeness, we shall also include a brief discussion of the structure of the full amplitude $\mathcal{A}_1$.

2.1 The partial amplitude $\mathcal{B}_1$ at fixed modulus

We consider a torus with modulus $\tau = \tau_1 + i\tau_2$ where $\tau_1, \tau_2 \in \mathbb{R}$ and $\tau_2 > 0$. The partial amplitude $\mathcal{B}_1$ is a family of non-holomorphic modular functions $\mathcal{B}_1(s,t,u|\tau,\bar\tau)$ which may be defined in terms of an exponential of the scalar Green function G on the torus,

$$\mathcal{B}_1(s,t,u|\tau,\bar\tau) = \prod_{i=1}^4 \int_\Sigma \frac{d^2 z_i}{\tau_2} \exp\left\{ \sum_{1\leq j<k\leq 4} s_{jk}\, G(z_j - z_k|\tau,\bar\tau) \right\} \tag{2.1}$$

The integral is over four copies of the Riemann surface Σ of modulus τ. The parameters $s_{12} = s_{34} = s$, $s_{23} = s_{14} = t$, and $s_{13} = s_{24} = u$ are dimensionless Lorentz invariants $s_{ij} = -\alpha' k_i \cdot k_j/2$ of the momenta k_i of the four gravitons labelled by $i,j = 1,2,3,4$. They obey the relation $s+t+u=0$ in view of overall momentum conservation for massless states. We exhibit all three parameters s,t,u – despite their interdependence – because $\mathcal{B}_1(s,t,u|\tau,\bar\tau)$ is in fact a symmetric function of s,t,u due to Bose symmetry of the gravitons.

The scalar Green function $G(z|\tau,\bar\tau)$ on the torus satisfies $\partial_z \partial_{\bar z} G(z|\tau,\bar\tau) = \pi\delta^{(2)}(z) - \pi/\tau_2$. In view of the relation $s+t+u=0$, the Green function G may be shifted by an arbitrary z-independent quantity without affecting the integrand in (2.1), or of $\mathcal{B}_1$ itself. We use this symmetry to impose the normalization condition $\int_\Sigma d^2 z\, G(z|\tau,\bar\tau) = 0$ on G, and express the resulting Green function as a Fourier sum over the integers $m,n \in \mathbb{Z}$,

$$G(z|\tau,\bar\tau) = \sum_{(m,n)\neq(0,0)} \mathcal{G}(m,n|\tau,\bar\tau)\, e^{2\pi i(m\alpha - n\beta)} \tag{2.2}$$

where $\alpha, \beta \in \mathbb{R}$ parametrize z by $z = \alpha + \beta\tau$. The Fourier modes $\mathcal{G}(m,n|\tau,\bar\tau)$ are given by,

$$\mathcal{G}(m,n|\tau,\bar\tau) = \frac{\tau_2}{\pi|m\tau+n|^2} \tag{2.3}$$

and we shall set $\mathcal{G}(0,0|\tau,\bar\tau) = 0$ by convention. The integers m,n label the two-dimensional momenta of the scalar field on the torus, the zero mode being excluded. For fixed τ, $G(z|\tau,\bar\tau)$ is regular in z, except for a logarithmic singularity at the origin where $G(z|\tau,\bar\tau) \sim -\ln|z|^2$.

2.2 The low energy expansion

For fixed τ, the singularities of $\mathcal{B}_1$ as a function of s,t,u are simple and double poles located at positive integer values of s,t,u. Low energy corresponds to $|s|$, $|t|$, $|u| \ll 1$, which is a region where $\mathcal{B}_1$ is analytic and admits a Taylor series expansion in s,t,u with finite radius of convergence. This expansion was investigated in [1], following less complete work in [8,9]. The low energy expansion may be obtained by Taylor expanding the exponential of (2.1) in its argument, and denoting the total degree of homogeneity in the variables s,t,u by the weight w,

$$\mathcal{B}_1(s,t,u|\tau,\bar\tau) = \sum_{w=0}^{\infty} \frac{1}{w!} \prod_{i=1}^{4} \int_\Sigma \frac{d^2 z_i}{\tau_2} \left(\sum_{1\leq j<k\leq 4} s_{jk}\, G(z_j - z_k|\tau,\bar\tau) \right)^w \tag{2.4}$$

The coefficients in this expansion are modular functions of τ. They can be represented by sums of vacuum Feynman diagrams in which the four vertices labelled by $i = 1,2,3,4$ represent the integration points z_i, and in which ℓ_{jk} lines represent the Green function of (2.2) joining the vertices j and k. The weight of the diagram is given by,

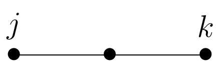
$$w = \sum_{1\leq j<k\leq 4}^{6} \ell_{jk} \tag{2.5}$$

Carrying out the integrations over the vertex positions z_i will then produce the multiple sums that were described in the introduction. A systematic discussion of the combinatorial notation for general Feynman diagrams encountered in the low energy expansion of $\mathcal{B}_1$ was presented in [1] following [8, 9], but will not be needed here. Instead, we shall content ourselves with writing down the Feynman diagrams that correspond to the limited number of functions needed in this paper.

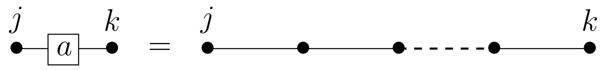
We denote the Green function $G(z_j - z_k|\tau,\bar\tau)$ by a line between the vertices i and j,

$G(z_j - z_k|\tau,\bar\tau)$ =


We shall represent an integrated vertex by an unlabeled dot. For example, the integrated string of two Green functions is denoted as follows,

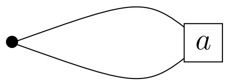
$\int_\Sigma \frac{d^2 z_i}{\tau_2} G(z_j - z_i|\tau,\bar\tau)\, G(z_i - z_k|\tau,\bar\tau)$ =


For an integrated string with a Green functions, we shall use the following notation,



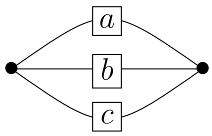
Given this notation, it is straightforward to express the multiple sums encountered in the introduction in terms of Feynman diagrams. For $L = 1$, we have,

$$E_a =$$



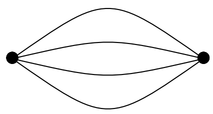
where the vertex on the left side of the diagram is to be integrated, as is consistent with the notation of the unlabelled dot. For $L = 2$, we have,

$$C_{a,b,c} =$$



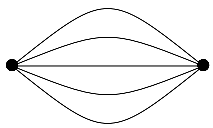
The lowest weight example with $L = 3$ is the modular function D_4 of weight $w = 4$, which is the function of central interest in this paper and is associated with the following diagram,

$$D_4 =$$

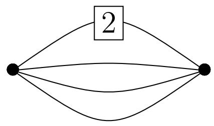


Finally, we also list the modular functions of weight 5 that were also the subject of conjectures in [1] that will be described later,

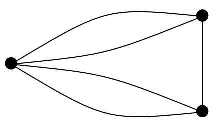
$$D_5 =$$



$$D_{3,1,1} =$$



$$D_{2,2,1} =$$



The modular function D_5 has $L = 4$, while $D_{3,1,1}$ and $D_{2,2,1}$ have $L = 3$. The expressions for these modular functions in terms of multiple sums are easily obtained from the diagrams. For example, for $D_{2,2,1}$, we have,

$$D_{2,2,1} = \sum_{(m,n)}{}' \frac{\tau_2}{\pi\,|m+n\tau|^2} \prod_{i=1}^{2} \left(\sum_{(m_i,n_i)}{}' \frac{\tau_2^2}{\pi^2 |m_i + n_i\tau|^2\, |m + m_i + (n+n_i)\tau|^2} \right) \tag{2.6}$$

where the prime superscripts on the sums indicate that the zero mode is to be omitted.

2.3 The conjectured relations at weight 5

In addition to the conjectured relation for D_4 stated in the Theorem (1.7) of the introduction, and which is to be proven in this paper, a number of further relations for weight five modular functions were conjectured in [1]. These comprise the following relations,

$$\begin{aligned} 0 &= D_5 - 60\, C_{3,1,1} - 10\, E_2\, C_{1,1,1} + 48\, E_5 - 16\, \zeta(5) \\ 0 &= 40\, D_{3,1,1} - 300\, C_{3,1,1} - 120\, E_2\, E_3 + 276\, E_5 - 7\, \zeta(5) \\ 0 &= 10\, D_{2,2,1} - 20\, C_{3,1,1} + 4\, E_5 - 3\, \zeta(5) \end{aligned} \tag{2.7}$$

Here, $\zeta(w)$ denotes the Riemann ζ-function, to which we assign weight w. Each equation in (2.7) relates a weight five D-function to a weight five polynomial in modular functions of lower loop number, i.e. lower depth. We expect that the conjectures of (2.7) may be proven as well by the methods used in this paper. Further relations should be expected to proliferate at higher weights w and it would be fascinating to understand their complete structure.

2.4 The full amplitude as an integral over moduli

The full genus-one four-graviton amplitude in Type II superstring theories, $\mathcal{A}_1$, is obtained by integrating the partial amplitude $\mathcal{B}_1$ over the moduli space $\mathcal{M}_1$ of genus-one Riemann surfaces [10],

$$\mathcal{A}_1 = \kappa^2\, \mathcal{R}^4 \int_{\mathcal{M}_1} d\mu_1\, \mathcal{B}_1(s,t,u|\tau,\bar\tau) \tag{2.8}$$

The symbol $\mathcal{R}^4$ denotes four powers of the linearized Riemann curvature tensor, and the normalization κ^2 is proportional to Newton's constant in ten-dimensional space-time. The integral is over a fundamental domain $\mathcal{M}_1$ of the modular group $SL(2,\mathbb{Z})$ acting on the upper half plane, and $d\mu_1 = d\tau_1\, d\tau_2/\tau_2^2$ is the volume form of the Poincaré metric.

The integral of $\mathcal{B}_1$ over $\mathcal{M}_1$ is absolutely convergent only for purely imaginary s,t,u. Constructing and evaluating $\mathcal{A}_1$ beyond this region requires analytic continuation. This analytic continuation was shown to exist, to be computable, and to produce dependences on s,t,u which are no longer analytic near the origin $s=t=u=0$ in [11]. The physical origin for this non-analytic behavior is well-known and well-understood, as it results from the propagation of massless string states in closed loops.

The emergence of non-analyticities at low energy means that the expansion in powers of s,t,u in (2.4) and the integration of $\mathcal{B}_1$ over moduli space $\mathcal{M}_1$ in (2.8) cannot be interchanged, since doing so would produce divergent integrals. To evaluate the low energy expansion, one may first extract the exact non-analytic behavior to a given order in s,t,u, and then evaluate the remaining finite part which is polynomial in s,t,u. Alternatively, space-time may be partially compactified on a flat torus T^d, and the non-analytic part may be canceled when comparing the low energy contributions at different moduli of T^d. We refer the reader to [1] for detailed discussions of both approaches.

3 Proof of the Lemma

In this section, we shall provide a proof of the Lemma given in (1.8)-(1.10).

The strategy for proving part 1 of the Lemma is to show that each term contributing to F in (1.6), namely $D_4, C_{2,1,1}, E_4$, and E_2^2, admits by itself an expansion of the form (1.8). To show this, and to compute the contribution to $\mathcal{F}_k$ from each term, we will perform the summation over the integers n_r (but not the integers m_r), using the fundamental formula,

$$\sum_{n\in\mathbb{Z}} \frac{1}{z+n} = -i\pi \, \frac{1+e^{2\pi iz}}{1-e^{2\pi iz}} \tag{3.1}$$

This formula, and other formulas involving higher powers of $1/(z+n)$ that are derived from it by taking successive derivatives in z, are discussed Appendix A. Inspection of the results of these summations will prove part 1 of the Lemma, and will provide explicit formulas for the coefficients $\mathcal{F}_k$. The proof of part 2 of the Lemma is an easy application of the explicit formulas for $\mathcal{F}_k$ derived in part 1.

The strategy for proving part 3 of the Lemma is to use extensive algebraic simplifications and rearrangements of the coefficients $\mathcal{F}_k$ to prove that all non-harmonic contributions cancel. These calculations are considerably facilitated by the use of MAPLE.

To avoid a proliferation of factors of π and τ_2 while carrying out the calculation of $\mathcal{F}_k$ described above, it will be convenient to extract from F a common factor of τ_2^4/π^4, by defining the following reduced functions,

$$F = \frac{\tau_2^4}{\pi^4}\,\mathcal{F} \qquad D_4 = \frac{\tau_2^4}{\pi^4}\,\mathcal{D} \qquad C_{2,1,1} = \frac{\tau_2^4}{\pi^4}\,\mathcal{C} \qquad E_s = \frac{\tau_2^s}{\pi^s}\,\mathcal{E}_s \tag{3.2}$$

in terms of which the decomposition of the Lemma takes the form,

$$\mathcal{F}(\tau,\bar\tau) = \mathcal{D}(\tau,\bar\tau) - 24\,\mathcal{C}(\tau,\bar\tau) - 3\,\mathcal{E}_2(\tau,\bar\tau)^2 + 18\,\mathcal{E}_4(\tau,\bar\tau) = \sum_{k=0}^{7} \frac{\pi^8}{(\pi\tau_2)^k}\,\mathcal{F}_k(q,\bar q) \tag{3.3}$$

where $q = e^{2\pi i\tau}$ and $\mathcal{F}_k(q,\bar q)$ is an entire function of q and $\bar q$.

3.1 Expansion of $\mathcal{E}_2$ and $\mathcal{E}_4$

We will now write the expression for the Eisenstein series in a form that has the structure of the power series in τ_2 on the right-hand side of (3.3), where the coefficients are functions of q and $\bar q$. We will later derive expressions for $\mathcal{D}$ and $\mathcal{C}$ in the same format. This representation of $\mathcal{E}_s$ can be deduced from the defining relation for E_s in (1.1). To carry out the sum over n, it will be convenient to split the sum over $(m,n) \neq (0,0)$ into a contribution from $m=0$, in which case the sum over n must exclude $n=0$, and the contribution from $m \neq 0$, in which case the sum over n runs over all integers. One obtains the following decomposition,

$$\mathcal{E}_s(\tau,\bar\tau) = \sum_{n\neq 0} \frac{1}{|n|^{2s}} + \sum_{m\neq 0} \sum_{n\in\mathbb{Z}} \frac{1}{|m\tau+n|^{2s}} \tag{3.4}$$

The first term on the right side of (3.4) equals $2\zeta(2s)$, and provides the leading behavior near the cusp $\tau_2 \to \infty$ for $\mathrm{Re}\,(s) > 1$. We shall evaluate the second term on the right side for the special values $s = 2, 4$ needed here.

To compute the sum over n for $\mathcal{E}_2$, we decompose the argument of the sum, which is a rational function in $z = m\tau$ for $m \neq 0$, into partial fractions, and express the result using the notation $z = x + iy$ for $x, y \in \mathbb{R}$,

$$\frac{1}{|z+n|^4} = -\frac{1}{4y^2\,(z+n)^2} - \frac{1}{4y^2\,(\bar z+n)^2} + \frac{i}{4y^3\,(z+n)} - \frac{i}{4y^3\,(\bar z+n)} \tag{3.5}$$

The summations over n may now be performed by using (3.1) and the first formula in (A.2). The result is,

$$\mathcal{E}_2 = \frac{\pi^4}{45} + \sum_{m\neq 0} \left(\frac{\pi}{4m^3\tau_2^3} \frac{1+q^m}{1-q^m} + \frac{\pi^2}{m^2\tau_2^2} \frac{q^m}{(1-q^m)^2} + \text{c.c.} \right) \tag{3.6}$$

Here and below, the notation c.c. denotes the addition of the complex conjugate of the entire preceding expression. The first term on the right-hand side of (3.6) accounts for the contribution from $m = 0$, which has been evaluated, noting that $2\zeta(4) = \pi^4/45$.

The calculation for $\mathcal{E}_4$ proceeds by decomposition into partial fractions as well, and requires the use of (A.1), as well as all three formulas in (A.2). The result is as follows,

$$\begin{aligned}
\mathcal{E}_4 \;=\; & \frac{\pi^8}{4725} + \sum_{m\neq 0} \left(\frac{5\pi}{32m^7\tau_2^7} \frac{1+q^m}{1-q^m} + \frac{5\pi^2}{8m^6\tau_2^6} \frac{q^m}{(1-q^m)^2} \right. \\
& \left. + \frac{\pi^3}{2m^5\tau_2^5} \frac{q^m(1+q^m)}{(1-q^m)^3} + \frac{\pi^4}{6m^4\tau_2^4} \frac{q^m + 4q^{2m} + q^{3m}}{(1-q^m)^4} + \text{c.c.} \right)
\end{aligned} \tag{3.7}$$

The first term is the $m = 0$ contribution, which is equal to $2\zeta(8) = \pi^8/4725$.

The expressions (3.6) and (3.7) for $\mathcal{E}_2$ and $\mathcal{E}_4$ manifestly have the structure of the representation of $\mathcal{F}_k$ in terms of a power series in $1/\tau_2$ on the right-hand side of (3.3). In fact, the contributions to $\mathcal{F}_k$ from $\mathcal{E}_4$ are manifestly harmonic, and thus satisfy the property of $\mathcal{F}_k$ in part 3 of the Lemma.

3.2 Expansion of $\mathcal{C}$

Obtaining an expansion of $\mathcal{C}$ in a form that is analogous to the expansions of $\mathcal{E}_2, \mathcal{E}_4$ is more involved. We start with the expression for $\mathcal{C}$ as a multiple sum, which may be obtained from (1.3),

$$\mathcal{C}(\tau,\bar\tau) = \sum_{(m_r,n_r)\neq(0,0)} \frac{\delta_{m,0}\delta_{n,0}}{|m_1\tau+n_1|^4\,|m_2\tau+n_2|^2\,|m_3\tau+n_3|^2} \tag{3.8}$$

To carry out the summations over the integers n_r, we must again partition the contributions according to the vanishing pattern of the integers m_r. Note that the summand is symmetric under the permutation of the indices $2,3$, but there is no such symmetry with index 1. As a result, $\mathcal{C}$ may be decomposed as follows,

$$\mathcal{C}(\tau,\bar\tau) = \mathcal{C}^{(3)} + \sum_{m_1\neq 0}\big(2\,\mathcal{C}^{(2)}(m_1\tau)+\mathcal{C}^{(1)}(m_1\tau)\big) + \sum_{m_r\neq 0}\delta_{m,0}\,\mathcal{C}^{(0)}(m_1\tau,m_2\tau,m_3\tau) \tag{3.9}$$

where the combinatorial coefficients take into account the symmetries of the various multiple sums. The partial contributions are given by,

$$\begin{aligned}
\mathcal{C}^{(3)} &= \sum_{n_r\neq 0}\frac{\delta_{n,0}}{n_1^4\,n_2^2\,n_3^2}\\
\mathcal{C}^{(2)}(z_1) &= \sum_{n_2\neq 0}\sum_{n_1,n_3}\frac{\delta_{n,0}}{n_2^2\,|z_1+n_1|^4\,|-z_1+n_3|^2}\\
\mathcal{C}^{(1)}(z_1) &= \sum_{n_2\neq 0}\sum_{n_1,n_3}\frac{\delta_{n,0}}{n_2^4\,|z_1+n_1|^2\,|-z_1+n_3|^2}\\
\mathcal{C}^{(0)}(z_1,z_2,z_3) &= \sum_{n_r}\frac{\delta_{n,0}}{|z_1+n_1|^4\,|z_2+n_2|^2\,|z_3+n_3|^2}
\end{aligned} \tag{3.10}$$

Conservation of m_r imposes the condition $z_1+z_2+z_3=0$ with $z_r\neq 0$ in $\mathcal{C}^{(0)}$. We have retained the dependence on all three variables z_r, as the associated manifest permutation symmetry in their indices will be convenient for later purposes.

The calculation of the constant $\mathcal{C}^{(3)}$ is presented in Appendix A, and we find,

$$\mathcal{C}^{(3)} = \frac{2\pi^8}{14175} \tag{3.11}$$

The evaluation of the remaining $\mathcal{C}$-functions proceeds in analogy with the evaluation of $\mathcal{E}_2$ given earlier, and has been performed using MAPLE. In expressing the results, we shall use the notation $q_r = e^{2\pi i z_r}$ and $y_r = \text{Im}\,(z_r)$. The results are as follows,

$$\begin{aligned}
\mathcal{C}^{(2)}(z_1) &= \left(\frac{\pi^2}{16y_1^6} + \frac{\pi^4}{12y_1^4}\right)\left(\frac{q_1}{(1-q_1)^2} + \frac{\bar{q}_1}{(1-\bar{q}_1)^2}\right) - \frac{\pi^2}{4y_1^6}\frac{1+q_1\bar{q}_1}{|1-q_1|^2} \\
&\quad + \left(\frac{5\pi}{32y_1^7} + \frac{\pi^3}{8y_1^5}\right)\frac{1-q_1\bar{q}_1}{|1-q_1|^2} - \frac{\pi^3}{8y_1^5}\frac{(q_1+\bar{q}_1)(1-q_1\bar{q}_1)}{|1-q_1|^4}
\end{aligned} \tag{3.12}$$

and

$$\mathcal{C}^{(1)}(z_1) = \left(\frac{\pi^5}{90\,y_1^3} - \frac{\pi^3}{24\,y_1^5} - \frac{\pi}{32y_1^7}\right)\frac{1-q_1\bar{q}_1}{|1-q_1|^2} + \frac{\pi^2}{16\,y_1^6}\frac{1+q_1\bar{q}_1}{|1-q_1|^2} \tag{3.13}$$

Finally, the most involved part is $\mathcal{C}^{(0)}$, which is given as follows,

$$\begin{aligned}
\mathcal{C}^{(0)}(z_1,z_2,z_3) &= \frac{\pi^4}{4\,y_1^2\,y_2\,y_3}\frac{1+q_1}{(1-q_1)^2(1-q_2)(1-q_3)} \\
&\quad + \frac{\pi^3}{8\,y_1^3\,y_2^2}\frac{q_1(1+\bar{q}_2)}{(1-q_1)^2(1-\bar{q}_2)} + \frac{\pi^3(3y_1+2y_2)}{8\,y_1^3\,y_2\,y_3^2}\frac{q_1(1+q_2)}{(1-q_1)^2(1-q_2)} \\
&\quad - \frac{\pi^2(6y_1^2+8y_1y_2+3y_2^2)}{64\,y_1^4\,y_2\,y_3^3}\frac{(1+q_1)(1+q_2)}{(1-q_1)(1-q_2)} - \frac{\pi^2(y_1-3y_2)}{64\,y_1^4\,y_2^3}\frac{(1+q_1)(1+\bar{q}_2)}{(1-q_1)(1-\bar{q}_2)} \\
&\quad + \frac{\pi^2}{64\,y_2^3\,y_3^3}\frac{q_2+\bar{q}_3}{(1-q_2)(1-\bar{q}_3)} + (2\leftrightarrow 3) + \text{c.c.}
\end{aligned} \tag{3.14}$$

The notation $(2 \leftrightarrow 3)$ denotes the addition of the contribution obtained by interchanging indices 2 and 3 of the entire expression, while c.c. denotes the addition of the complex conjugate of the entire expression. After substituting $z_r = m_r\tau$ and $q_r = q^{m_r}$ for $r = 1,2,3$ into $\mathcal{C}^{(2)}$, $\mathcal{C}^{(1)}$, and $\mathcal{C}^{(0)}$, we see that the contribution of $\mathcal{C}$ to $\mathcal{F}$ has the same form as the expansion in powers of $1/\tau_2$ on the right-hand side of (3.3), thereby proving part 1 of the Lemma for $\mathcal{C}$.

3.3 Expanding $\mathcal{D}$

The expansion of $\mathcal{D}$ proceeds in analogy with the expansions used above for $\mathcal{E}_2, \mathcal{E}_4$, and $\mathcal{C}$. We start with the expression for $\mathcal{D}$ as a multiple sum, which may be derived from (1.5),

$$\mathcal{D}(\tau,\bar{\tau}) = \sum_{(m_r,n_r)\neq(0,0)} \delta_{m,0}\,\delta_{n,0}\prod_{r=1}^{4}\frac{1}{|m_r\tau+n_r|^2} \tag{3.15}$$

where $m = m_1+m_2+m_3+m_4$ and $n = n_1+n_2+n_3+n_4$. To carry out the summations over the integers n_r, we partition the contributions according to the vanishing pattern of the integers m_r. As a result, $\mathcal{D}$ may be decomposed as follows,

$$\begin{aligned}\mathcal{D}(\tau,\bar\tau) &= \mathcal{D}^{(4)} + \sum_{m_1\neq 0} 6\,\mathcal{D}^{(2)}(m_1\tau) + \sum_{m_2,m_3,m_4\neq 0} 4\,\delta_{m,0}\,\mathcal{D}^{(1)}(m_2\tau,m_3\tau,m_4\tau) \\ &\quad + \sum_{m_1,m_2,m_3,m_4\neq 0} \delta_{m,0}\,\mathcal{D}^{(0)}(m_1\tau,m_2\tau,m_3\tau,m_4\tau) \end{aligned} \tag{3.16}$$

By a slight abuse of notation, m will stand for the sum of all summation variables m_r, whether that number of variables is three as in the third term on the right side, or four as in the fourth term. The reduced contributions are given by,

$$\begin{aligned}\mathcal{D}^{(4)} &= \sum_{n_r\neq 0} \frac{\delta_{n,0}}{n_1^2\,n_2^2\,n_3^2\,n_4^2} \\ \mathcal{D}^{(2)}(z_1) &= \sum_{n_3,n_4\neq 0}\ \sum_{n_1,n_2\in\mathbb{Z}} \frac{\delta_{n,0}}{n_3^2\,n_4^2\,|z_1+n_1|^2\,|-z_1+n_2|^2} \\ \mathcal{D}^{(1)}(z_2,z_3,z_4) &= \sum_{n_1\neq 0}\ \sum_{n_2,n_3,n_4\in\mathbb{Z}} \frac{\delta_{n,0}}{n_1^2} \prod_{r=2}^{4} \frac{1}{(z_r+n_r)(\bar z_r+n_r)} \\ \mathcal{D}^{(0)}(z_1,z_2,z_3,z_4) &= \sum_{n_r\in\mathbb{Z}} \delta_{n,0} \prod_{r=1}^{4} \frac{1}{(z_r+n_r)(\bar z_r+n_r)} \end{aligned} \tag{3.17}$$

In each case we impose the m_r conservation condition, $\sum_r z_r = 0$.

- The contribution from the constant $\mathcal{D}^{(4)}$ is evaluated in Appendix A, and we find,

$$\mathcal{D}^{(4)} = \frac{\pi^8}{945} \tag{3.18}$$

There is no $\mathcal{D}^{(3)}$ contribution because imposing the vanishing of three of the integers m_r implies the vanishing of the fourth one as well. The remaining $\mathcal{D}$-functions are as follows.

- The contributions to $\mathcal{D}^{(2)}$ partitions into two parts: one from $n_1+n_2=-n_3-n_4\neq 0$, and the other from $n_1+n_2=-n_3-n_4=0$. Collecting both gives,

$$\begin{aligned}\mathcal{D}^{(2)}(z_1) &= \left(\frac{\pi^5}{18y_1^3}+\frac{\pi^3}{3y_1^5}+\frac{3\pi}{16y_1^7}\right)\frac{1-q_1\bar q_1}{|1-q_1|^2} \\ &\quad -\left(\frac{\pi^4}{6y_1^4}+\frac{3\pi^2}{8y_1^6}\right)\frac{1+q_1\bar q_1}{|1-q_1|^2}+\frac{\pi^6}{45y_1^2}\left(\frac{q_1}{(1-q_1)^2}+\frac{\bar q_1}{(1-\bar q_1)^2}\right)\end{aligned} \tag{3.19}$$

- The contribution $\mathcal{D}^{(1)}(z_1,z_3,z_4)$ may be expressed as follows,

$$\mathcal{D}^{(1)}(z_2,z_3,z_4) = \Phi(z_2,z_3)+\Phi(z_3,z_4)+\Phi(z_4,z_2) \tag{3.20}$$

where we have defined the function,

$$
\begin{aligned}
\Phi(z_r, z_s) &= \frac{\pi^4}{12 y_r^2\, y_s^2} \,\mathrm{Re}\left(\frac{1+q_r}{1-q_r}\right) \mathrm{Re}\left(\frac{1+q_s}{1-q_s}\right) \\
&\quad - \frac{\pi^3}{4 y_r\, y_s\, y_t^3} \,\mathrm{Re}\left(\frac{q_r q_s + \bar q_t}{(1-q_r)(1-q_s)(1-\bar q_t)}\right) \\
&\quad + \frac{\pi^2}{8 y_r y_s y_t^4} \,\mathrm{Re}\left(\frac{q_r q_s - \bar q_t}{(1-q_r)(1-q_s)(1-\bar q_t)}\right)
\end{aligned}
\tag{3.21}
$$

with $t \in \{2,3,4\}$ and $t \notin \{r,s\}$, so that $y_r + y_s + y_t = 0$ and $q_r q_s q_t = 1$.

- The contribution from $\mathcal{D}^{(0)}$ must be partitioned into the part for which all pairs satisfy $z_r + z_s \neq 0$ for $r \neq s$, and the part for which we have $z_1 + z_2 = z_3 + z_4 = 0$ or permutations thereof. The results may be collected as follows.

$$
\begin{aligned}
\mathcal{D}^{(0)}(z_1, z_2, z_3, z_4) &= \frac{\pi^4}{y_1 y_2 y_3 y_4} \, \frac{1}{(1-q_1)(1-q_2)(1-q_3)(1-q_4)} \\
&\quad - \frac{\pi^3 (1-\delta_{y_1+y_2,0})}{4 y_1 y_2 y_3 y_4 (y_1+y_2)} \, \frac{(q_1 q_2 - \bar q_3 \bar q_4)}{(1-q_1)(1-q_2)(1-\bar q_3)(1-\bar q_4)} + \ 2 \text{ perms} \\
&\quad + \frac{\pi^4 \delta_{y_1+y_2,0}}{y_1^2 y_3^2} \, \frac{q_1 \bar q_3}{(1-q_1)^2 (1-\bar q_3)^2} + \ 2 \text{ perms} \\
&\quad - \frac{\pi^3}{4 y_1^2 y_2 y_3 y_4} \, \frac{(q_1 - \bar q_2 \bar q_3 \bar q_4)}{(1-q_1)(1-\bar q_2)(1-\bar q_3)(1-\bar q_4)} + 3 \text{ perms} \\
&\quad + \text{complex conjugate}
\end{aligned}
\tag{3.22}
$$

The terms listed on the second and third lines correspond to the partition (12|34); the two permutations to be added correspond to the partitions (13|24) and (14|23). The term listed on the fourth line corresponds to the partition (1|234); the three permutations to be added correspond to the partitions (2|134), (3|124), and (4|123). The complex conjugate, which interchanges q_r with $\bar q_r$ and leaves y_r invariant, of the entire expression is to be added.

3.4 Summary of contributions to $\mathcal{F}_k$

A summary of the contributions of the various non-vanishing components of $\mathcal{D}$, $\mathcal{C}$, $\mathcal{E}_2^2$ and $\mathcal{E}_4$ to $\mathcal{F}$, is presented in table 1.

Substituting $z_r = m_r \tau$ and $q_r = q^{m_r}$ into the expressions (3.19), (3.20) and (3.22), results in expansions of the functions $\mathcal{D}^{(2)}$, $\mathcal{D}^{(1)}$, and $\mathcal{D}^{(0)}$ in the form of power series in $1/\tau_2$. We therefore see that the contribution of $\mathcal{D}$ to $\mathcal{F}$ is again expressed as a power series in $1/\tau_2$ of the same form as the right-hand side of (3.3). This completes the proof of part 1 of the Lemma, since now all contributions, namely $\mathcal{D}$, $\mathcal{C}$, $\mathcal{E}_2^2$ and $\mathcal{E}_4$ have been proven to have the form given in (3.3).

$\mathcal{D}^{(0)}$				τ_2^{-4}	τ_2^{-5}		
$\mathcal{D}^{(1)}$				τ_2^{-4}	τ_2^{-5}	τ_2^{-6}	
$\mathcal{D}^{(2)}$		τ_2^{-2}	τ_2^{-3}	τ_2^{-4}	τ_2^{-5}	τ_2^{-6}	τ_2^{-7}
$\mathcal{D}^{(4)}$	τ_2^{0}						
$\mathcal{C}^{(0)}$				τ_2^{-4}	τ_2^{-5}	τ_2^{-6}	
$\mathcal{C}^{(1)}$			τ_2^{-3}		τ_2^{-5}	τ_2^{-6}	τ_2^{-7}
$\mathcal{C}^{(2)}$					τ_2^{-5}	τ_2^{-6}	τ_2^{-7}
$\mathcal{C}^{(3)}$	τ_2^{0}						
$\mathcal{E}_2^2$	τ_2^{0}	τ_2^{-2}	τ_2^{-3}	τ_2^{-4}	τ_2^{-5}	τ_2^{-6}	
$\mathcal{E}_4$	τ_2^{0}			τ_2^{-4}	τ_2^{-5}	τ_2^{-6}	τ_2^{-7}

Table 1: The non-vanishing powers of τ_2 in the expansion of the contributions to $\mathcal{F}$.

3.5 Vanishing of the contributions $\mathcal{F}_k$ with $k = 0, 1, 2, 3, 7$

By inspection of the above results, one readily shows the vanishing of the coefficients $\mathcal{F}_k$ for $k = 0, 1, 2, 3, 7$. The arguments are as follows.

- The cancellation of $\mathcal{F}_0$ follows by checking the pure power terms, as was done in [1], since none of these terms depend on q or $\bar{q}$. The result of [1] is double checked by adding the contributions computed to this order, which come from the terms listed in the first column of table 1,

$$\mathcal{D}^{(4)} - 24\mathcal{C}^{(3)} - 3 \times 4 \times \zeta(4)^2 + 18 \times 2 \times \zeta(8) = 0 \tag{3.23}$$

- The cancellation of $\mathcal{F}_1$ results from the observation that no terms of order τ_2^{-1} arise in any of the contributions to $\mathcal{F}$.
- The cancellation of $\mathcal{F}_2$ results from the combination of just two contributions, namely $\mathcal{D}^{(2)}$ and $\mathcal{E}_2^2$. They both have the same functional dependence on q and $\bar{q}$,

$$\sum_{m \neq 0} \frac{1}{m^2} \left(\frac{q^m}{(1-q^m)^2} + \frac{\bar{q}^m}{(1-\bar{q}^m)^2} \right) \tag{3.24}$$

  Upon properly including the combinatorial factors, we find that their sum cancels.
- The cancellation of $\mathcal{F}_3$ results from combining the three terms of order τ_2^{-3} in $\mathcal{D}^{(2)}$, $\mathcal{C}^{(1)}$, and $\mathcal{E}_2^2$. They all have the same functional dependence on $q, \bar{q}$, given by,

$$\sum_{m \neq 0} \frac{1}{m^3} \frac{1 - q^m \bar{q}^m}{(1-q^m)(1-\bar{q}^m)} \tag{3.25}$$

The coefficients are as follows (including combinatorial factors),

$$6 \times \frac{1}{18} - 24 \times \frac{1}{90} - 3 \times 2 \times \frac{1}{45} \times \frac{1}{2} = 0 \tag{3.26}$$

- Finally, the cancellation of $\mathcal{F}_7$ results from combining the four terms of order τ_2^{-7}, namely in $\mathcal{D}^{(2)}$, $\mathcal{C}^{(1)}$, $\mathcal{C}^{(2)}$, and $\mathcal{E}_4$. These four contributions all have the same functional dependence on $q, \bar{q}$, given by,

$$\sum_{m \neq 0} \frac{1}{m^7} \frac{1 - q^m \bar{q}^m}{(1 - q^m)(1 - \bar{q}^m)} \tag{3.27}$$

The coefficients are as follows (including combinatorial factors),

$$6 \times \frac{3}{16} - 24 \times \left(-\frac{1}{32} \right) - 24 \times 2 \times \frac{5}{32} + 18 \times \frac{5}{16} = 0 \tag{3.28}$$

This cancellation completes the proof of part 2 of the Lemma that $\mathcal{F}_k = 0$ for $k = 0, 1, 2, 3, 7$.

3.6 Harmonic structure of $\mathcal{F}_k$ for $k = 4, 5, 6$

The analysis of the remaining terms in $\mathcal{F}$, namely, $\mathcal{F}_4$, $\mathcal{F}_5$ and $\mathcal{F}_6$ is considerably more complicated, and we relegate the detailed discussion to appendix B.

There, we will begin by collecting all the terms that contribute to $\mathcal{F}_k$ for $k = 4, 5, 6$. Some contributions to $\mathcal{F}_k$ are *manifestly harmonic* in view of the fact that they enter as the sum of a holomorphic function of q and its complex conjugate. Other contributions are not manifestly harmonic, and will be collected into a sub-contribution denoted by $\mathcal{F}_k^{\rm nh}$. Using extensive algebraic rearrangements of the sum of the terms in $\mathcal{F}_k^{\rm nh}$ it will be shown that all non-harmonic dependence in $\mathcal{F}_k^{\rm nh}$ in fact cancels, so that $\mathcal{F}_k^{\rm nh}$ also contributes harmonic terms, thereby proving that $\mathcal{F}_k$ is a purely real harmonic expression of the form,

$$\mathcal{F}_k(\tau, \bar{\tau}) = \varphi_k(q) + \varphi_k(\bar{q}) \tag{3.29}$$

The explicit form of $\varphi_k(q)$ will be presented in Appendix B. By inspection, we will show that the functions $\varphi_k(q)$ satisfy the conjugation properties of (1.10) of the Lemma given in the introduction. From the first equation of (1.10), we conclude that φ_k is a real function of q, whose Taylor series expansion in powers of q has real coefficients. The combination of these two properties implies that, when viewed as a function of τ, the function $\psi_k(\tau) = \varphi_k(q)$ has the following equivalent conjugation properties,

$$\psi_k(\tau) = \varphi_k(e^{2\pi i \tau}) \qquad \begin{aligned} \overline{\psi_k(\tau)} &= (-)^k \, \psi_k(\bar{\tau}) \\ \psi_k(-\tau) &= (-)^k \, \psi_k(\tau) \end{aligned} \tag{3.30}$$

Using these explicit formulas, and MAPLE based calculations, we have shown that $\varphi_k(q) = \mathcal{O}(q^{300})$. In the subsequent section, we shall produce a proof of the vanishing of φ_k to all orders in q by exploiting the modular invariance of $F(\tau, \bar{\tau})$.

4 Proof of the Theorem

The Lemma of the introduction, proven in the preceding section, strongly constrains the structure of the non-holomorphic function $F(\tau, \bar{\tau})$, introduced in (1.6). To prove the Theorem (1.7) we need to prove that $F = 0$. Our strategy will be to combine the strongly constrained structure of F, derived from explicit computations of $\mathcal{F}_k$ in the preceding section, with the modular invariance property of F.

4.1 Modular properties of $\psi_k(\tau)$

We begin by describing the strong structural constraint derived in the Lemma in a manner that will be suitable for investigating the modular properties of $\psi_k(\tau) = \varphi_k(q)$. To this end, we restore the normalizations of (3.2), and use the decomposition provided by the Lemma in (3.3) and (3.29) to arrive at the following expression,

$$\pi^4 F(\tau, \bar{\tau}) = H(\tau, y) + H(\bar{\tau}, y) \tag{4.1}$$

where we use the notation $y = \tau_2$ for later convenience. The function H may be expressed in terms of the holomorphic functions $\psi_k(\tau)$ with $k = 4, 5, 6$ by the relation,

$$H(\tau, y) = \psi_4(\tau) + \frac{1}{y}\psi_5(\tau) + \frac{1}{y^2}\psi_6(\tau) \tag{4.2}$$

It will generally be more convenient to express the modular properties on the form of H in terms of ψ_k, since then all dependence will be manifestly in terms of the modulus τ. The conjugation properties of (1.10) for $\varphi_k(q)$ and (3.30) for $\psi_k(\tau)$ guarantee that H satisfies the following conjugation properties,

$$\begin{aligned} \overline{H(\tau, y)} &= H(\bar{\tau}, y) \\ H(-\tau, -y) &= H(\tau, y) \end{aligned} \tag{4.3}$$

We shall sometimes refer to the functions H as *almost holomorphic*.

Next, we analyze the constraints on $\varphi_k(q) = \psi_k(\tau)$ imposed by the modular invariance of F. Since $\varphi_k(q)$ admits a q-expansion near the cusp, it is an entire function of q and hence invariant under the modular transformation $T : \tau \to \tau + 1$, which implies the periodicity,

$$\begin{aligned} \psi_k(\tau + 1) &= \psi_k(\tau) \\ H(\tau + 1, y) &= H(\tau, y) \end{aligned} \tag{4.4}$$

Modular invariance of F under the transformation $S : \tau \to -1/\tau$ requires,

$$F(\tau, \bar{\tau}) = F(-1/\tau, -1/\bar{\tau}) \tag{4.5}$$

Making use of the identities

$$\frac{|\tau|^2}{\tau_2} = \frac{\tau^2}{\tau_2} - 2i\tau = \frac{\bar\tau^2}{\tau_2} + 2i\bar\tau \tag{4.6}$$

allows us to express $F(-1/\tau, -1/\bar\tau)$ as the sum of an *almost holomorphic function* $H^S(\tau, y)$ and its complex conjugate $H^S(\bar\tau, y)$,

$$\pi^4 F(-1/\tau, -1/\bar\tau) = H^S(\tau, y) + H^S(\bar\tau, y) \tag{4.7}$$

where H^S may be chosen as follows (the decomposition is not unique, as may be seen by adding an imaginary constant to H^S),

$$\begin{aligned} H^S(\tau, y) &= \psi_4(-1/\tau) - 2i\tau\psi_5(-1/\tau) - 4\tau^2\psi_6(-1/\tau) \\ &\quad + \frac{1}{y}\Big(\tau^2\psi_5(-1/\tau) - 4i\tau^3\psi_6(-1/\tau)\Big) + \frac{1}{y^2}\tau^4\psi_6(-1/\tau) \end{aligned} \tag{4.8}$$

It is clear by inspection that H^S satisfies the same conjugation relations as H does in (4.3). By eliminating F between (4.1), (4.5), and (4.7), we obtain a relation which expresses the modular S-invariance of F in terms of the functions H and H^S,

$$H(\tau, y) + H(\bar\tau, y) = H^S(\tau, y) + H^S(\bar\tau, y) \tag{4.9}$$

An alternative representation of the same formula is given by

$$K(\tau, y) = -K(\bar\tau, y) \tag{4.10}$$

where we have defined the function K by,

$$K(\tau, y) = H(\tau, y) - H^S(\tau, y) \tag{4.11}$$

In view of relation (4.10), K is purely imaginary for all τ and y, while in view of the conjugation relations of (4.3) for H and H^S, the following conjugation relations hold,

$$\begin{aligned} \overline{K(\tau, y)} &= K(\bar\tau, y) \\ K(-\tau, -y) &= K(\tau, y) \end{aligned} \tag{4.12}$$

Furthermore, from the expressions for H and H^S in terms of $\psi_k(\tau)$ and $\psi_k(-1/\tau)$, we see that $K(\tau, y)$ has an expansion in powers of y just as H and H^S do,

$$K(\tau, y) = K_4(\tau) + \frac{1}{y}K_5(\tau) + \frac{1}{y^2}K_6(\tau) \tag{4.13}$$

where $K_k(\tau)$ are holomorphic functions of τ.

In appendix C, we show that the general form of K in (4.13), subject to the condition that it be purely imaginary, and satisfies the conjugation properties of (4.12), implies[3] that K is restricted to the following form, for a real and as yet undetermined constant A,

$$K(\tau, y) = A + \frac{i}{y} A\tau \tag{4.14}$$

The relation between H, H^S and K depends on y and τ. These may be viewed as independent variables, as follows from writing $y^2 K(\tau, y)$ as a power series in τ and $\bar{\tau}$. Consequently, the various powers in y may be identified by comparing (4.11) and (4.14),

$$\begin{aligned}
\psi_4(\tau) - \psi_4(-1/\tau) + 2i\tau\psi_5(-1/\tau) + 4\tau^2\psi_6(-1/\tau) &= A \\
\psi_5(\tau) - \tau^2\psi_5(-1/\tau) + 4i\tau^3\psi_6(-1/\tau) &= iA\tau \\
\psi_6(\tau) - \tau^4\psi_6(-1/\tau) &= 0
\end{aligned} \tag{4.15}$$

These equations must hold for all τ.

4.2 Proving the vanishing of A, and ψ_4, ψ_5, ψ_6

The last equation in (4.15) implies that $\psi_6(\tau)$ is actually a modular form in τ of weight -4. Since we have already shown that $\psi_k(\tau) = \varphi_k(q)$ is a holomorphic function of τ, we deduce that $\psi_6(\tau) = 0$, as no holomorphic modular forms of weight -4 exist. To make progress on the remaining equations in (4.15), we need further information on A. We shall show next that further use of the periodicity $\psi_k(\tau + 1) = \psi_k(\tau)$ implies $A = 0$.

To do so, we consider the transformation $\tau \to -1/(\tau - 1)$ by the element $T^{-1}S \in SL(2,\mathbb{Z})$. It has a fixed point at $\tau = 1/2 + i\sqrt{3}/2$, which is one of the orbifold points of the canonical fundamental domain of $SL(2,\mathbb{Z})$ in the upper half-plane. Having already determined that $\psi_6 = 0$, we now apply the transformation $T^{-1}S$ to the second equation in (4.15), use periodicity of ψ_k under T, and multiply by $(\tau - 1)^2$ throughout. This results in,

$$(\tau - 1)^2\, \psi_5(-1/(\tau - 1)) - \psi_5(\tau) = -iA\,(\tau - 1) \tag{4.16}$$

Applying the transformation $\tau \to -1/(\tau - 1)$ in this equation gives,

$$\tau^2\, \psi_5(-1/\tau) - (\tau - 1)^2\, \psi_5(-1/(\tau - 1)) = iA\,\tau(\tau - 1) \tag{4.17}$$

Adding the second equation of (4.15) to (4.16) and (4.17) leads to the following condition, $iA(1 - \tau + \tau^2) = 0$. Since this relation has to hold for all τ, we deduce that $A = 0$. The second equation in (4.15) (with $\psi_6(\tau) = 0$) now simplifies to,

$$\psi_5(\tau) = \tau^2\psi_5(-1/\tau) \tag{4.18}$$

[3]We are very grateful to Stephen Miller for suggesting this procedure for constraining K.

This condition means that $\psi_5(\tau)$ is a modular form of weight -2; since it must also be holomorphic, it must vanish, $\psi_5(\tau)=0$. Finally we see that the first equation in (4.15) with $A=\psi_6=\psi_5=0$ implies that $\psi_4(\tau)$ is a holomorphic modular form of weight 0, which must be constant. This constant must vanish since the asymptotic behaviour near the cusp at $\tau_2=\infty$ has no constant term, and so we have $\psi_4=0$.

Since the functions $\psi_k(\tau)=\varphi_k(q)$ vanish for $k=4,5,6$, it follows from (3.29) that $\mathcal{F}_k$ vanishes for those values of k. Together with the results of section 3.5 for $k=0,1,2,3,7$, we conclude that $\mathcal{F}_k$ vanishes for $k=0,1,\cdots,7$. Therefore, in view of (1.8), we have, finally, proven the Theorem (1.7) of the introduction.

5 Holomorphic corollaries

In section 4, we proved that the holomorphic functions $\varphi_4(q), \varphi_5(q)$, and $\varphi_6(q)$ are modular forms of respective weights $0, -2$, and -4. Therefore, using the known asymptotic behavior of φ_4 near the cusp, it follows that φ_4, φ_5 and φ_6 must individually vanish. These results were obtained by exploiting the holomorphicity of φ_4, φ_5 and φ_6, and the vanishing of $\mathcal{F}_k$ for $k=0,1,2,3,7$ established in section 3.5, along with the modular invariance of F.

In addition, we obtained *explicit expressions* for $\varphi_4, \varphi_5, \varphi_6$ respectively in (B.7), (B.22), and (B.32), which, in view of the above result, manifestly satisfy the corollaries:

- They are modular forms of weights $0, -2, -4$ respectively;
- They therefore vanish as functions of q.

Neither of these properties is manifest from the explicit expressions for $\varphi_4, \varphi_5, \varphi_6$ in (B.7), (B.22), and (B.32), and we have not succeeded in proving either of these properties directly from the explicit expressions of $\varphi_4, \varphi_5, \varphi_6$ by analytical combinatorial methods.

Remarkably, the identity $\varphi_6(q)=0$ may be split up into two simpler sums,

$$\varphi_6(q) = \varphi_6^{(1)}(q) + \varphi_6^{(2)}(q) \tag{5.1}$$

where

$$\varphi_6^{(1)}(q) = \frac{3}{4}\zeta(6) + \sum_{m_1+m_2\neq 0}^{\prime} \frac{9}{8m_1^2 m_2^2 (m_1+m_2)^2} \frac{(1+q^{m_1})(1+q^{m_2})}{(1-q^{m_1})(1-q^{m_2})} \tag{5.2}$$

and

$$\begin{aligned} \varphi_6^{(2)}(q) \;\; = \;\; & -\frac{15}{8}\zeta(6) - \sum_{m}^{\prime} \frac{9}{4m^6} \frac{q^m}{(1-q^m)^2} + \sum_{m_1,m_2}^{\prime} \frac{3}{16 m_1^3 m_2^3} \frac{(1+q^{m_1})(1+q^{m_2})}{(1-q^{m_1})(1-q^{m_2})} \\ & + \sum_{m_1+m_2\neq 0}^{\prime} \frac{3}{16 m_1 m_2 (m_1+m_2)^4} \frac{(1+q^{m_1})(1+q^{m_2})}{(1-q^{m_1})(1-q^{m_2})} \end{aligned} \tag{5.3}$$

each of which vanishes separately. We have verified the validity of these identities using MAPLE up to order $\mathcal{O}(q^{400})$. In addition, the identity $\varphi_6^{(1)}(q) = 0$ may be proven by simple combinatorial arguments. Since the proof of the Theorem has provided a proof for $\varphi_6(q) = 0$, it follows that the identity $\varphi_6^{(2)}(q) = 0$ is also proven.

To prove $\varphi_6^{(1)}(q) = 0$, we evaluate the sum,

$$\phi_6 = \frac{2}{3}\zeta(6) + \sum_{m_1+m_2\neq 0}{}' \frac{f(q^{m_1})\, f(q^{m_2})}{m_1^2 m_2^2 (m_1+m_2)^2} \tag{5.4}$$

where $f(x) = (1+x)/(1-x)$. Using the symmetry property $f(1/x) = -f(x)$ of the function $f(x)$, we recast the sum as follows,

$$\phi_6 = \frac{2}{3}\zeta(6) + 2 \sum_{m_1,m_2\geq 1} \frac{f(q^{m_1})f(q^{m_2}) - f(q^{m_1})f(q^{m_1+m_2}) - f(q^{m_2})f(q^{m_1+m_2})}{m_1^2 m_2^2 (m_1+m_2)^2} \tag{5.5}$$

Using the algebraic identity $f(x)f(y) - f(x)f(xy) - f(y)f(xy) = -1$, the numerator in the sum equals -1, and the summand is therefore independent of q,

$$\phi_6 = \frac{2}{3}\zeta(6) - 2 \sum_{m_1,m_2\geq 1} \frac{1}{m_1^2 m_2^2 (m_1+m_2)^2} \tag{5.6}$$

Repeated use of the partial fraction identity for positive integers $a, b > 0$,

$$\frac{1}{m^a n^b} = \sum_{r=b}^{a+b-1} \frac{\binom{r-1}{b-1}}{(m_1+m_2)^r m_1^{a+b-r}} + \sum_{r=a}^{a+b-1} \frac{\binom{r-1}{a-1}}{(m_1+m_2)^r m_2^{a+b-r}} \tag{5.7}$$

leads to the following evaluation of the general sums,

$$\sum_{m_1,m_2\geq 1} \frac{1}{m_1^a m_2^b (m_1+m_2)^c} = \sum_{\substack{r+s=a+b\\ r,s>0}} \left(\binom{r-1}{a-1} \zeta(c+r,s) + \binom{r-1}{b-1} \zeta(c+r,s) \right) \tag{5.8}$$

where the multi-zeta function $\zeta(r_1, r_2)$ is defined by,

$$\zeta(r_1,r_2) = \sum_{0<n_2<n_1} \frac{1}{n_1^{r_1} n_2^{r_2}} \tag{5.9}$$

Therefore we find that,

$$\phi_6 = \frac{2}{3}\zeta(6) - 4\zeta(4,2) - 8\zeta(5,1) = 0 \tag{5.10}$$

It is easy to show that there is no analogous way of partitioning the sum of the six terms in the explicit equation $\varphi_4(q) = 0$ into separate identities, as we did for $\varphi_6(q)$. To see this, one can simply truncate the q-expansion of each term to the first six non-trivial orders, and show that no linear combination other than their total sum can vanish. Thus, the identity $\varphi_4(q) = 0$ cannot be reduced further. The holds true for $\varphi_5(q) = 0$.

6 Discussion

The proof of the Theorem confirms the conjectured relationship that expresses D_4 as a polynomial in functions of lower depth, namely, the functions $C_{2,1,1}$, E_4 and E_2. The conjecture made in [1] was based on the analysis of the first two lowest powers of $q = e^{2\pi i\tau}$ in the expansion of these modular functions near the cusp $\tau_2 \to \infty$. Each of these terms was accompanied by a Laurent polynomial in τ_2 so the conjecture was based on matching a number of leading and sub-leading coefficients. Analogous conjectures were made, based on similar asymptotic analysis, that related each of the weight $w = 5$ functions, D_5, $D_{3,1,1}$, and $D_{2,2,1}$, to polynomials in modular functions of lower depth. Although the proof of the D_4 conjecture suggests the validity of the weight 5 conjectures in (2.7), the methods used in this paper may be too cumbersome to be applied systematically to these cases.

In a separate paper we will present an alternative formulation of the Feynman graphs in which the modular functions are expressed in terms of single-valued multiple elliptical polylogs based on generalized Bloch–Wigner polylogarithms discussed in [12]. It seems likely that this approach will lead to a more general analysis of the properties of the modular functions that arise in the low energy expansion and we are hopeful this will lead to an understanding of higher-weight terms and possibly to the complete one-loop amplitude. Separately, there has been some progress in expressing one-loop amplitudes in terms of multiple elliptical polylogs [13] of the type discussed, for example, in [14] and it would be interesting to discover the relationship of these to the closed string expansion under discussion in this paper.

Finally, a natural generalization of the questions addressed here and in [1] is to the case of genus two and higher. In fact, the study of the modular properties of the low energy expansion for the two-loop four-graviton superstring amplitude in [15,16] was a direct motivation for the investigations in [1]. It would be fascinating to understand further the modular structure of two-loop partial amplitudes, and the possibility that modular relations, such as the ones proven here, emerge also at two loops.

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A Calculations of the q expansions

In this section, we shall provide various details of the proof of the Lemma, part 1.

A.1 Basic summation formulas

In the course of the calculations we make repeated use of the summation formula,

$$\sum_{n\in\mathbb{Z}} \frac{1}{z+n} = -i\pi\,\frac{1+q}{1-q} \tag{A.1}$$

where we shall use the convenient abbreviation $q = e^{2\pi i z}$. The sum is only conditionally convergent and, following Eisenstein, should be understood as defined by the limit as $N \to \infty$ of the cutoff sum with $-N \leq n \leq N$. The identity (A.1) is evident from the equality of the its residues at the poles located at $z = -n$ for $n \in \mathbb{Z}$. Further identities that will be of use here follow by differentiation of (A.1) with respect to z, and we also have,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^2} &= -4\pi^2\,\frac{q}{(1-q)^2} \\
\sum_{n\in\mathbb{Z}} \frac{1}{(n+z)^3} &= 4i\pi^3\,\frac{q+q^2}{(1-q)^3} \\
\sum_{n\in\mathbb{Z}} \frac{1}{(n+z)^4} &= \frac{8\pi^4}{3}\,\frac{q+4q^2+q^3}{(1-q)^4}
\end{aligned} \tag{A.2}$$

and so on. The general formula may be expressed as follows,

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^{k+1}} = -i\pi\delta_{k,0} + \frac{(-2\pi i)^{k+1}}{\Gamma(k+1)} \sum_{\ell=1}^{\infty} \ell^k q^\ell \tag{A.3}$$

There are obvious variants of these formulas that will be needed as well when the summation over n excludes the value $n = 0$, and we have for example,

$$\sum_{n\in\mathbb{Z}}{}' \frac{1}{z+n} = -i\pi\,\frac{1+q}{1-q} - \frac{1}{z} \tag{A.4}$$

where the prime superscript instructs us to omit the value $n = 0$ from the sum. Formulas analogous to (A.2) and (A.3) may be derived again by successive differentiation.

A.2 Calculations of the constants $\mathcal{C}^{(3)}$ and $\mathcal{D}^{(4)}$

To compute $\mathcal{C}^{(3)}$, defined in (3.10), we eliminate $n_3 = -n_1 - n_2$, so that,

$$\mathcal{C}^{(3)} = \sum_{n_1 \neq 0} \sum_{n_2 \neq 0, -n_1} \frac{1}{n_1^4 \, n_2^2 \, (n_1 + n_2)^2} \tag{A.5}$$

The summation over n_2, for $n_1 \neq 0$, gives,

$$\sum_{n_2 \neq 0, -n_1} \frac{1}{n_2^2 \, (n_1 + n_2)^2} = \frac{2\pi^2}{3n_1^2} - \frac{6}{n_1^4} \tag{A.6}$$

The remaining summation over n_1 gives,

$$\mathcal{C}^{(3)} = \sum_{n_1 \neq 0} \left(\frac{2\pi^2}{3n_1^6} - \frac{6}{n_1^8} \right) = \frac{4\pi^2}{3} \zeta(6) - 12\zeta(8) \tag{A.7}$$

which readily leads to the result of (3.11).

To compute $\mathcal{D}^{(4)}$, defined in (3.17), we eliminate $n_4 = -n_1 - n_2 - n_3$, and partition the contributions according to whether $n_1 + n_2 = 0$ or not,

$$\mathcal{D}^{(4)} = \sum_{n_1 \neq 0} \sum_{n_3 \neq 0} \frac{1}{n_1^4 n_3^4} + \sum_{n_1 \neq 0} \sum_{n_2 \neq 0, -n_1} \sum_{n_3 \neq 0, -n_1 - n_2} \frac{1}{n_1^2 n_2^2 n_3^2 (n_1 + n_2 + n_3)^2} \tag{A.8}$$

The first term readily evaluates to $4\zeta(4)^2$, while the sum over n_3 may be carried out in the second term with the help of (A.6), and we find,

$$\mathcal{D}^{(4)} = 4\zeta(4)^2 + \sum_{n_1 \neq 0} \sum_{n_2 \neq 0, -n_1} \frac{1}{n_1^2 \, n_2^2} \left(\frac{2\pi^2}{3(n_1 + n_2)^2} - \frac{6}{(n_1 + n_2)^4} \right) \tag{A.9}$$

To sum over n_2, we make use again of (A.6) for the first term in the parentheses, and of the following summation formula for the second term in the parentheses,

$$\sum_{n_2 \neq 0, -n_1} \frac{1}{n_2^2 (n_1 + n_2)^4} = \frac{\pi^4}{45 n_1^2} + \frac{4\pi^2}{3n_1^4} - \frac{15}{n_1^6} \tag{A.10}$$

The sums over n_1 may be performed in terms of ordinary ζ-values, and we find,

$$\mathcal{D}^{(4)} = 4\zeta(4)^2 + \frac{28\pi^4}{45} \zeta(4) - 24\pi^2 \zeta(6) + 18\zeta(8) = \frac{\pi^8}{945} \tag{A.11}$$

which is the leading asymptotic term in the expansion of $\mathcal{D}$ in (3.18).

B Harmonicity of $\mathcal{F}_4$, $\mathcal{F}_5$, and $\mathcal{F}_6$.

In this appendix, we shall collect and simplify the contributions to the coefficients $\mathcal{F}_k$ in (4.2) for $k = 4, 5, 6$, arising from $\mathcal{D}$, $\mathcal{C}$, $\mathcal{E}_2^2$ and $\mathcal{E}_4$. We shall show that all contributions combine into a purely harmonic result. We shall calculate the functions $\varphi_k(q)$, and prove their conjugation properties, by inspection. The analysis of these contributions is considerably more complicated than the analysis for $k = 0, 1, 2, 3, 7$ considered in the body of the text.

B.1 Vanishing of non-harmonic terms in $\mathcal{F}_4$

We begin by collecting all contributions to $\mathcal{F}_4$ which are not *manifestly harmonic*, and denote the result by $\mathcal{F}_4^{\rm nh}$. It is given as follows,[4]

$$\begin{aligned}
\mathcal{F}_4^{\rm nh} \;\; = \;\; & -{\sum_m}' \frac{1}{m^4} \frac{1+q^m\bar q^m}{|1-q^m|^2} + {\sum_{m_1+m_2\neq 0}}' \frac{1}{m_1^2 m_2^2} \,{\rm Re}\left(\frac{1+q^{m_1}}{1-q^{m_1}}\right) {\rm Re}\left(\frac{1+q^{m_2}}{1-q^{m_2}}\right) \\
& + {\sum_{m_1+m_2\neq 0}}' \frac{6}{m_1^2 m_2^2} \frac{q^{m_1}\bar q^{m_2}}{(1-q^{m_1})^2(1-\bar q^{m_2})^2} + {\sum_m}' \frac{6}{m^4} \frac{q^m \bar q^m}{|1-q^m|^4} \\
& - {\sum_{m_1,m_2}}' \frac{6}{m_1^2 m_2^2} \frac{q^{m_1}\bar q^{m_2}}{(1-q^{m_1})^2(1-\bar q^{m_2})^2} \qquad \text{(B.1)}
\end{aligned}$$

The first term arises from $\mathcal{D}^{(2)}$, including the combinatorial factor of 6; the second term arises from $\mathcal{D}^{(1)}$, including the combinatorial factor of 4, and a factor of 3 to account for the sum over three equal terms in $\mathcal{D}^{(1)}$; the third term arises from the non-generic case with a single independent pair vanishing in $\mathcal{D}^{(0)}$, including the combinatorial factor of 3; the fourth term arises from the non-generic case with a two independent pairs vanishing in $\mathcal{D}^{(0)}$, including the combinatorial factor of 3; and the fifth term arises from $-3\mathcal{E}_2^2$.

- The last three terms in (B.1) manifestly cancel one another.
- To show the absence of non-harmonic terms in the first two terms, we use the identity,

$$ {\sum_m}' \frac{1}{m^2} \,{\rm Re}\left(\frac{1+q^m}{1-q^m}\right) = 0 \qquad \text{(B.2)}$$

which follows from the fact that the summand is odd in m. Using (B.2), the second term of (B.1) is expressed as follows,

$$ {\sum_{m_1+m_2\neq 0}}' \frac{1}{m_1^2 m_2^2} \,{\rm Re}\left(\frac{1+q^{m_1}}{1-q^{m_1}}\right) {\rm Re}\left(\frac{1+q^{m_2}}{1-q^{m_2}}\right) = {\sum_m}' \frac{1}{m^4} \left[{\rm Re}\left(\frac{1+q^m}{1-q^m}\right)\right]^2 \qquad \text{(B.3)}$$

[4]Here, and in the following formulas, the prime superscript on the summation symbol indicates that the term with $m_r = 0$ is omitted from the sum.

which may be decomposed into harmonic and non-harmonic sums as follows,

$$\sum_m{}' \frac{1}{m^4} \left[\mathrm{Re} \left(\frac{1+q^m}{1-q^m} \right) \right]^2 = \sum_m{}' \frac{1}{4m^4} \left(\frac{(1+q^m)^2}{(1-q^m)^2} + \frac{(1+\bar q^m)^2}{(1-\bar q^m)^2} + 2 \frac{|1+q^m|^2}{|1-q^m|^2} \right) \tag{B.4}$$

• The first term of (B.1) can be usefully re-expressed with the help of the following rearrangement of the numerator,

$$1 + q^m \bar q^m = \frac{1}{2} |1-q^m|^2 + \frac{1}{2} |1+q^m|^2 \tag{B.5}$$

so that the sum becomes,

$$- \sum_m{}' \frac{1}{m^4} \frac{1+q^m \bar q^m}{|1-q^m|^2} = - \sum_m{}' \frac{1}{2m^4} - \sum_m{}' \frac{1}{2m^4} \frac{|1+q^m|^2}{|1-q^m|^2} \tag{B.6}$$

The second term on the right side of (B.6) cancels the last term in the sum on the right side of (B.4). The remaining terms, namely the first term on the right side of (B.6), and the first two terms in the sum on the right side of (B.4) are manifestly harmonic, and will need to be retained to compute φ_4. Therefore $\mathcal{F}_4$ is harmonic.

B.2 Calculation and properties of $\varphi_4(q)$

Collecting all harmonic contributions to $\mathcal{F}_4$ is most easily done by regrouping the terms that constitute $\varphi_4(q)$, and we find,

$$\begin{aligned} \varphi_4(q) \ = \ & \sum_{m_1,m_2,m_3,m_4}{}' \frac{\delta_{m,0}}{m_1 m_2 m_3 m_4} \frac{1}{(1-q^{m_1})(1-q^{m_2})(1-q^{m_3})(1-q^{m_4})} \\ & - \sum_{m_1,m_2,m_3}{}' \frac{12\, \delta_{m,0}}{m_1^2 m_2 m_3} \frac{1+q^{m_1}}{(1-q^{m_1})^2(1-q^{m_2})(1-q^{m_3})} \\ & - \sum_{m_1,m_2}{}' \frac{3}{m_1^2 m_2^2} \frac{q^{m_1} q^{m_2}}{(1-q^{m_1})^2(1-q^{m_2})^2} + \sum_m{}' \frac{18}{m^4} \frac{q^{2m}}{(1-q^m)^4} \end{aligned} \tag{B.7}$$

The first term on the right side arises from $\mathcal{D}^{(0)}$; the second term from $\mathcal{C}^{(0)}$; the third term from $-3\mathcal{E}_2^2$; and the fourth term arises from combining the contributions from $18\mathcal{E}_4$, $-48\mathcal{C}^{(2)}$, the harmonic terms in (B.4), and the constant term in (B.6).

It may be readily verified, by inspection term by term of (B.7), that $\varphi_4(q)$ obeys the conjugation properties of (1.10) for $k=4$. In particular, its Taylor series in powers of q has real coefficients.

B.3 Vanishing of non-harmonic terms in $\mathcal{F}_5$

We begin by collecting all contributions to $\mathcal{F}_5$ which are *not manifestly harmonic*, and denote the result by $\mathcal{F}_5^{\rm nh}$. Its expression may be organized as follows,

$$\begin{aligned}
\mathcal{F}_5^{\rm nh} \ &= \ {\sum_m}' \frac{2}{m^5} \frac{1-q^m\bar q^m}{|1-q^m|^2} + f_5^{(0)} + f_5^{(1)} + f_5^{(2)} - {\sum_{m_1,m_2}}' \frac{3}{m_1^3 m_2^2} \,{\rm Re} \left(\frac{(1+q^{m_1})\,\bar q^{m_2}}{(1-q^{m_1})(1-\bar q^{m_2})^2} \right) \\
& \quad + {\sum_m}' \frac{6}{m^5} \frac{(q^m+\bar q^m)(1-q^m\bar q^m)}{|1-q^m|^4} - {\sum_m}' \frac{6}{m^5} \frac{1-q^m\bar q^m}{|1-q^m|^2} + {\sum_m}' \frac{1}{m^5} \frac{1-q^m\bar q^m}{|1-q^m|^2} \\
& \quad - {\sum_{m_1+m_2\neq 0}}' \frac{12}{m_1^3 m_2^2} \,{\rm Re} \left(\frac{q^{m_1}(1+\bar q^{m_2})}{(1-q^{m_1})^2(1-\bar q^{m_2})} \right)
\end{aligned} \tag{B.8}$$

where we have used the following abbreviations,

$$\begin{aligned}
f_5^{(0)} \ &= \ - {\sum_{m_r}}' \frac{3\,\delta_{m,0}}{m_1 m_2 m_3^3} \,{\rm Re} \left(\frac{q^{m_1+m_2} + \bar q^{m_3}}{(1-q^{m_1})(1-q^{m_2})(1-\bar q^{m_3})} \right) \\
f_5^{(1)} \ &= \ - {\sum_{m_r}}' \frac{3\,\delta_{m,0}(1-\delta_{m_1+m_2,0})}{2 m_1 m_2 m_3 m_4 (m_1+m_2)} \,{\rm Re} \left(\frac{q^{m_1+m_2} - \bar q^{m_3+m_4}}{(1-q^{m_1})(1-q^{m_2})(1-\bar q^{m_3})(1-\bar q^{m_4})} \right) \\
f_5^{(2)} \ &= \ - {\sum_{m_r}}' \frac{2\,\delta_{m,0}}{m_1^2 m_2 m_3 m_4} \,{\rm Re} \left(\frac{q^{m_1} - \bar q^{m_2+m_3+m_4}}{(1-q^{m_1})(1-\bar q^{m_2})(1-\bar q^{m_3})(1-\bar q^{m_4})} \right)
\end{aligned} \tag{B.9}$$

The first term in (B.8) arises from $6\mathcal{D}_4^{(2)}$, the second from $4\mathcal{D}_4^{(1)}$, the third and fourth from $\mathcal{D}_4^{(0)}$, the fifth from $-3\mathcal{E}_2^2$, the sixth and seventh from $-48\mathcal{C}^{(2)}$, the eighth from $-24\mathcal{C}^{(1)}$, and the ninth from $-24\mathcal{C}^{(0)}$. We have the following simplifications.

- The first term on the second line of (B.8) cancels the term on the third line, using (B.2).
- To simplify $f_5^{(0)}$, we use the following identity for the numerator in the sum,

$$q^{m_1+m_2} + \bar q^{m_3} = \frac{1}{4} \sum_{\sigma,\sigma'=\pm 1} \sigma' (1+\sigma\, q^{m_1})(1+\sigma\sigma'\, q^{m_2})(1+\sigma'\,\bar q^{m_3}) \tag{B.10}$$

as a result of which we obtain,

$$\begin{aligned}
f_5^{(0)} \ &= \ \hat f_5^{(0)} - {\sum_{m_r}}' \frac{3\,\delta_{m,0}}{4 m_1 m_2 m_3^3} \,{\rm Re} \left(\frac{1+\bar q^{m_3}}{1-\bar q^{m_3}} \right) + {\sum_{m_r}}' \frac{3\delta_{m,0}}{2 m_1 m_2 m_3^3} \,{\rm Re} \left(\frac{1+q^{m_1}}{1-q^{m_1}} \right) \\
\hat f_5^{(0)} \ &= \ - {\sum_{m_r}}' \frac{3\,\delta_{m,0}}{4 m_1 m_2 m_3^3} \,{\rm Re} \left(\frac{(1+q^{m_1})(1+q^{m_2})(1+\bar q^{m_3})}{(1-q^{m_1})(1-q^{m_2})(1-\bar q^{m_3})} \right)
\end{aligned} \tag{B.11}$$

• To simplify $f_5^{(1)}$ we use the following decomposition formula,

$$q^{m_1+m_2} - \bar q^{m_3+m_4} = +\frac{1}{4}\sum_{\sigma=\pm 1}(1+q^{m_1+m_2})(1-\sigma\,\bar q^{m_3})(1+\sigma\,\bar q^{m_4}) \\ -\frac{1}{4}\sum_{\sigma=\pm 1}(1+\sigma\,q^{m_1})(1-\sigma\,q^{m_2})(1+\bar q^{m_3+m_4}) \qquad \text{(B.12)}$$

Symmetry of the sum in $f_5^{(1)}$ under permutations $m_1 \leftrightarrow m_2$ and $m_3 \leftrightarrow m_4$, as well as under complex conjugation combined with the reversal of signs of all m_r, implies that all four terms above produce equal contributions to $f_5^{(1)}$. As a result, the simplified $f_5^{(1)}$ is given as follows,

$$f_5^{(1)} = -\sum_{m_r}{}' \frac{3\,\delta_{m,0}(1-\delta_{m_1+m_2,0})}{2m_1m_2m_3m_4(m_1+m_2)}\,\mathrm{Re}\left(\frac{(1+q^{m_1+m_2})(1+\bar q^{m_3})}{(1-q^{m_1})(1-q^{m_2})(1-\bar q^{m_3})}\right) \qquad \text{(B.13)}$$

Further simplification is achieved by using the following decomposition,

$$1+q^{m_1+m_2} = \frac{1}{2}(1-q^{m_1})(1-q^{m_2}) + \frac{1}{2}(1+q^{m_1})(1+q^{m_2}) \qquad \text{(B.14)}$$

as a result of which we obtain,

$$f_5^{(1)} = \hat f_5^{(1)} - \sum_{m_r}{}' \frac{3\,\delta_{m,0}(1-\delta_{m_1+m_2,0})}{4m_1m_2m_3m_4(m_1+m_2)}\,\mathrm{Re}\left(\frac{1+\bar q^{m_3}}{1-\bar q^{m_3}}\right) \qquad \text{(B.15)}$$

$$\hat f_5^{(1)} = -\sum_{m_r}{}' \frac{3\,\delta_{m,0}(1-\delta_{m_1+m_2,0})}{4m_1m_2m_3m_4(m_1+m_2)}\,\mathrm{Re}\left(\frac{(1+q^{m_1})(1+q^{m_2})(1+\bar q^{m_3})}{(1-q^{m_1})(1-q^{m_2})(1-\bar q^{m_3})}\right)$$

Note that the second term of the first of these equations is harmonic.

• To simplify $f_5^{(2)}$, we use the following decomposition formula,

$$q^{m_1} - \bar q^{m_2+m_3+m_4} = \frac{1}{2}\sum_{\sigma=\pm 1}(1+\sigma\,q^{m_1})(1-\sigma\,\bar q^{m_2+m_3+m_4}) \qquad \text{(B.16)}$$

The contribution from the $\sigma=-1$ term above is harmonic, while the contribution from the $\sigma=+1$ term may be further simplified by using the following decomposition formula,

$$1-\bar q^{m_2+m_3+m_4} = \frac{1}{4}\sum_{\sigma,\sigma'=\pm 1}(1+\sigma\,\bar q^{m_2})(1+\sigma'\,\bar q^{m_3})(1-\sigma\sigma'\,\bar q^{m_4}) \qquad \text{(B.17)}$$

Hence we find the simplified formula,

$$f_5^{(2)} = \hat f_5^{(2)} + \sum_{m_r}{}' \frac{\delta_{m,0}}{m_1^2m_2m_3m_4}\,\mathrm{Re}\left(\frac{1+\bar q^{m_2+m_3+m_4}}{(1-\bar q^{m_2})(1-\bar q^{m_3})(1-\bar q^{m_4})}\right)$$

$$
\begin{aligned}
& - \sum_{m_r}{}' \frac{\delta_{m,0}}{4m_1^2 m_2 m_3 m_4} \, \mathrm{Re} \left(\frac{1+q^{m_1}}{1-q^{m_1}} \right) \\
\hat{f}_5^{(2)} \quad = \quad & - \sum_{m_r}{}' \frac{3\,\delta_{m,0}}{4m_1 m_2 m_3^2 m_4} \, \mathrm{Re} \left(\frac{(1+q^{m_1})(1+q^{m_2})(1+\bar{q}^{m_3})}{(1-q^{m_1})(1-q^{m_2})(1-\bar{q}^{m_3})} \right)
\end{aligned}
\tag{B.18}
$$

where in writing $\hat{f}_5^{(2)}$ we have taken the liberty of permuting the indices m_1 and m_3 and taking the complex conjugate of the expression under the reality sign, for later convenience. Note that the second and third terms of the first of these equations are harmonic.

• To combine $\hat{f}_5^{(1)}$ and $\hat{f}_5^{(2)}$ we make use of the following identity,

$$
-\frac{3\,\delta_{m,0}(1-\delta_{m_1+m_2,0})}{4m_1 m_2 m_3 m_4 (m_1+m_2)} - \frac{3\,\delta_{m,0}\,(1-\delta_{m_1+m_2,0})}{4m_1 m_2 m_3^2 m_4} = \frac{3\,\delta_{m,0}(1-\delta_{m_1+m_2,0})}{4m_1 m_2 m_3^2 (m_1+m_2)}
\tag{B.19}
$$

We may carry out the sum over m_4 explicitly, and express the result in the following form,

$$
\begin{aligned}
\hat{f}_5^{(1)} + \hat{f}_5^{(2)} \quad &= \quad \hat{f}_5^{(3)} + \sum_{m_1,m_3}{}' \frac{3}{4m_1^2 m_3^3} \, \mathrm{Re} \left(\frac{(1+q^{m_1})^2(1+\bar{q}^{m_3})}{(1-q^{m_1})^2(1-\bar{q}^{m_3})} \right) \\
\hat{f}_5^{(3)} \quad &= \quad \sum_{m_r}{}' \frac{3\,\delta_{m,0}}{4m_1 m_2 m_3^3} \, \mathrm{Re} \left(\frac{(1+q^{m_1})(1+q^{m_2})(1+\bar{q}^{m_3})}{(1-q^{m_1})(1-q^{m_2})(1-\bar{q}^{m_3})} \right)
\end{aligned}
\tag{B.20}
$$

In expressing $\hat{f}_5^{(3)}$ on the second line above, we have carried out the sum over m_4 to eliminate $\delta_{m,0}$, which imposes the condition $m_1 + m_2 + m_3 \neq 0$ on the remaining sum.

• It is clear by inspection that $\hat{f}_5^{(0)} + \hat{f}_5^{(3)} = 0$. The remaining contributions to $\mathcal{F}_5$ which are not yet manifestly harmonic arise from the first, fifth, seventh, and eighth terms in (B.8) (the sixth and ninth terms cancel one another, as shown earlier), as well as from the second term on the right side of the first equation in (B.20). Assembling those contributions gives,

$$
\begin{aligned}
& \sum_{m_1,m_2}{}' \frac{3}{4m_1^2 m_2^3} \, \mathrm{Re} \left(\frac{(1+q^{m_1})^2(1+\bar{q}^{m_2})}{(1-q^{m_1})^2(1-\bar{q}^{m_2})} \right) \\
& - \sum_{m_1,m_2}{}' \frac{3}{m_1^2 m_2^3} \, \mathrm{Re} \left(\frac{q^{m_1}\,(1+\bar{q}^{m_2})}{(1-q^{m_1})^2(1-\bar{q}^{m_2})} \right) - \sum_{m}{}' \frac{3}{m^5} \, \frac{(1-q^m \bar{q}^m)}{(1-q^m)(1-\bar{q}^m)}
\end{aligned}
\tag{B.21}
$$

it is easy to see that the sum is in fact harmonic as well.

In summary, we have shown that $\mathcal{F}_5$ is purely harmonic.

B.4 Calculation and properties of $\varphi_5(q)$

Collecting all harmonic contributions to $\mathcal{F}_5$ is most easily done by regrouping the terms that constitute $\varphi_5(q)$, and we find,

$$\begin{aligned}
\varphi_5(q) \;=\; & \sum_{m_1,m_2,m_3,m_4}{}^{'} \frac{\delta_{m,0}}{2m_1^2 m_2 m_3 m_4} \frac{1+q^{m_2+m_3+m_4}}{(1-q^{m_2})(1-q^{m_3})(1-q^{m_4})} \\
& - \sum_{m_1,m_2,m_3}{}^{'} \frac{(18m_1+12m_2)\delta_{m,0}}{m_1^3 m_2 m_3^2} \frac{q^{m_1}(1+q^{m_2})}{(1-q^{m_1})^2(1-q^{m_2})} + \sum_{m_1}{}^{'} \frac{9}{m_1^5} \frac{q^{m_1}(1+q^{m_1})}{(1-q^{m_1})^3} \\
& - \sum_{m_1,m_2}{}^{'} \frac{3}{2m_1^3 m_2^2} \frac{(1+q^{m_1})\, q^{m_2}}{(1-q^{m_1})(1-q^{m_2})^2} + \sum_{m_1}{}^{'} \phi(m_1) \frac{1+q^{m_1}}{1-q^{m_1}}
\end{aligned} \tag{B.22}$$

The coefficient $\phi(m_1)$ in the last term is given by the following multiple sums,

$$\begin{aligned}
\phi(m_1) \;=\; & -\frac{3}{2m_1^5} + \sum_{m_2}{}^{'} \frac{3}{8m_1^3 m_2^2} - \sum_{m_2,m_3}{}^{'} \frac{3\,\delta_{m,0}}{8m_1^3 m_2 m_3} + \sum_{m_2,m_3}{}^{'} \frac{3\,\delta_{m,0}}{4m_1 m_2 m_3^3} \\
& - \sum_{m_2,m_3,m_4}{}^{'} \frac{\delta_{m,0}}{8m_1^2 m_2 m_3 m_4} - \sum_{m_2,m_3,m_4}{}^{'} \frac{3\,\delta_{m,0}(1-\delta_{m_3+m_4,0})}{8m_1 m_2 m_3 m_4 (m_3+m_4)}
\end{aligned} \tag{B.23}$$

It may be readily verified, by inspection of (B.22), that $\varphi_5(q)$ obeys the conjugation properties of (1.10) for $k=5$. In particular, its Taylor series in powers of q has only real coefficients.

B.5 Vanishing of non-harmonic terms in $\mathcal{F}_6$

We begin by collecting all contributions to $\mathcal{F}_6$ which are *not manifestly harmonic*, and denote the result by $\mathcal{F}_5^{\rm nh}$. Its expression may be organized as follows,

$$\begin{aligned}
\mathcal{F}_6^{\rm nh} \;=\; & \sum_{m}{}^{'} \frac{33}{4m^6} \frac{1+q^m\bar q^m}{(1-q^m)(1-\bar q^m)} \\
& + \sum_{m_r}{}^{'} \frac{3\,\delta_{m,0}}{2m_1 m_2 m_3^4} \,{\rm Re}\left(\frac{q^{m_1+m_2}-\bar q^{m_3}}{(1-q^{m_1})(1-q^{m_2})(1-\bar q^{m_3})} \right) \\
& - \sum_{m_1,m_2}{}^{'} \frac{3}{8m_1^3 m_2^3} \,{\rm Re}\left(\frac{(1+q^{m_1})(1+\bar q^{m_2})}{(1-q^{m_1})(1-\bar q^{m_2})} \right) \\
& - \sum_{m_1+m_2\neq 0}{}^{'} \frac{3}{2m_1^3 m_2^3} \,{\rm Re}\left(\frac{q^{m_1}+\bar q^{m_2}}{(1-q^{m_1})(1-\bar q^{m_2})} \right) \\
& + \sum_{m_1+m_2\neq 0}{}^{'} \frac{3(m_1-3m_2)}{2m_1^4 m_2^3} \,{\rm Re}\left(\frac{(1+q^{m_1})(1+\bar q^{m_2})}{(1-q^{m_1})(1-\bar q^{m_2})} \right)
\end{aligned} \tag{B.24}$$

The first line arises from combining the contributions from $6\mathcal{D}_4^{(2)}$, $-48\mathcal{C}^{(2)}$, and $-24\mathcal{C}^{(1)}$; the second line arises from $4\mathcal{D}^{(1)}$, the third from $-3\mathcal{E}_2^2$; the fourth from the fourth line in $-24\mathcal{C}^{(0)}$; and the fifth from the last two lines in $-24\mathcal{C}^{(0)}$ (which contribute equally).

• The sums in the second and fourth lines may be simplified by using the following identities for their numerator,

$$\begin{aligned} q^{m_1+m_2} - \bar{q}^{m_3} &= \frac{1}{4} \sum_{\sigma,\sigma'=\pm1} \sigma'(1+\sigma\, q^{m_1})(1+\sigma\sigma'\, q^{m_2})(1-\sigma'\, \bar{q}^{m_3}) \\ q^{m_1} + \bar{q}^{m_2} &= \frac{1}{2}(1+q^{m_1})(1+\bar{q}^{m_2}) - \frac{1}{2}(1-q^{m_1})(1-\bar{q}^{m_2}) \end{aligned} \tag{B.25}$$

The contribution with $\sigma' = +1$ in the first line is harmonic, as is the contribution from the second term in the second line above. The remaining not-manifestly-harmonic contributions are as follows,

$$-\frac{3}{8} f_6 - {\sum_m}' \frac{9}{8m^6} \frac{(1+q^m)(1+\bar{q}^m)}{(1-q^m)(1-\bar{q}^m)} + {\sum_{m_1,m_2}}' \frac{3}{8m_1^3 m_2^3} \,\mathrm{Re}\left(\frac{(1+q^{m_1})(1+\bar{q}^{m_2})}{(1-q^{m_1})(1-\bar{q}^{m_2})} \right) \tag{B.26}$$

where we have introduced the notation,

$$f_6 = {\sum_{m_r}}' \frac{\delta_{m,0}}{m_1 m_2 m_3^4} \frac{(1+q^{m_1})(1+\bar{q}^{m_3})}{(1-q^{m_1})(1-\bar{q}^{m_3})} + {\sum_{m_r}}' \frac{\delta_{m,0}}{m_1 m_2 m_3^4} \frac{(1+q^{m_3})(1+\bar{q}^{m_1})}{(1-q^{m_3})(1-\bar{q}^{m_1})} \tag{B.27}$$

In the second sum, we relabel the indices by permuting m_1 and m_3, which makes the dependence on q and $\bar{q}$ the same in both terms, and gives,

$$f_6 = {\sum_{m_1,m_2,m_3}}' \left(\frac{\delta_{m,0}}{m_1 m_2 m_3^4} + \frac{\delta_{m,0}}{m_1^4 m_2 m_3} \right) \frac{(1+q^{m_1})(1+\bar{q}^{m_3})}{(1-q^{m_1})(1-\bar{q}^{m_3})} \tag{B.28}$$

We now use the following rearrangement formula,

$$\frac{\delta_{m,0}}{m_1 m_2 m_3^4} + \frac{\delta_{m,0}}{m_1^4 m_2 m_3} = -\frac{\delta_{m,0}}{m_1^4 m_3^4} \left(m_1^2 - m_1 m_3 + m_3^2 \right) \tag{B.29}$$

As a result, f_6 becomes,

$$f_6 = {\sum_{m_1,m_2,m_3}}' \left(\frac{\delta_{m,0}}{m_1^3 m_3^3} - \frac{2\,\delta_{m,0}}{m_1^2 m_3^4} \right) \mathrm{Re}\left(\frac{(1+q^{m_1})(1+\bar{q}^{m_3})}{(1-q^{m_1})(1-\bar{q}^{m_3})} \right) \tag{B.30}$$

Using (B.2) in the sum for the last term in the parentheses, we obtain,

$$f_6 = {\sum_{m_1,m_3}}' \frac{1}{m_1^3 m_3^3} \,\mathrm{Re}\left(\frac{(1+q^{m_1})(1+\bar{q}^{m_3})}{(1-q^{m_1})(1-\bar{q}^{m_3})} \right) - {\sum_m}' \frac{3}{m^6} \frac{(1+q^m)(1+\bar{q}^m)}{(1-q^m)(1-\bar{q}^m)} \tag{B.31}$$

The expression (B.26) is seen to vanish by substituting the expression for f_6 derived in (B.31).

This concludes the proof that $\mathcal{F}_6$ is purely harmonic.

B.6 Calculation and properties of $\varphi_6(q)$

Collecting all harmonic contributions to $\mathcal{F}_6$ is most easily done by regrouping the terms that constitute $\varphi_6(q)$, and we find,

$$\begin{aligned}\varphi_6 \quad = \quad & -\sum_{m}{}' \frac{9}{16m^6}\frac{(1+q^m)^2}{(1-q^m)^2} + \sum_{m_1,m_2}{}' \frac{3}{16m_1^3m_2^3}\frac{(1+q^{m_1})(1+q^{m_2})}{(1-q^{m_1})(1-q^{m_2})} \\ & + \sum_{m_1+m_2\neq 0}{}' \frac{9}{8m_1^2m_2^2(m_1+m_2)^2}\frac{(1+q^{m_1})(1+q^{m_2})}{(1-q^{m_1})(1-q^{m_2})} \\ & + \sum_{m_1+m_2\neq 0}{}' \frac{3}{16m_1m_2(m_1+m_2)^4}\frac{(1+q^{m_1})(1+q^{m_2})}{(1-q^{m_1})(1-q^{m_2})} \end{aligned} \qquad \text{(B.32)}$$

It may be readily verified, by inspection of (B.32), that $\varphi_6(q)$ obeys the conjugation properties of (1.10) for $k=6$. In particular, its Taylor series in powers of q has only real coefficients.

C Details of the analysis of $K(\tau, y)$

In this Appendix,[5] we shall show that $K(\tau, y)$, which has been defined in (4.11) and is of the form (4.13) with $K_4(\tau), K_5(\tau), K_6(\tau)$ holomorphic in τ, and which is purely imaginary for all τ and y, must be of the form given in (4.14), with K_4, K_5, K_6 polynomials in τ of degree one, two, and three respectively. This is a crucial auxiliary result, which is used in section 4 to prove the Theorem of this paper.

Note that $K(\tau, y)$ must vanish along the imaginary axis, as on the one hand it must be purely imaginary, yet on the other hand, its building blocks H and H^S are real there. If $K(\tau, y)$ had been a holomorphic function, then its vanishing along the imaginary axis would have implied its vanishing everywhere. However, $K(\tau, y)$ is only *almost holomorphic*, so the situation is more subtle, as we will see.

We will Taylor expand $y^2K(\tau, y)$ around an arbitrary point on the imaginary axis in the upper half τ-plane. It will be convenient to choose that point to be $\tau = i$, and to expand in the variable $\tilde\tau$ (or in the variables $\tilde x, \tilde y$), defined by,

$$\tau = i + \tilde\tau \qquad\qquad \tau = \tilde x + i\tilde y, \quad \tilde x, \tilde y \in \mathbb{R} \qquad \text{(C.1)}$$

The expansion of the holomorphic functions $\psi_k(\tau)$ around $\tau = i$ is best organized in terms of their expansion in terms of (positive powers of) q, and we have,

$$\psi_k(\tau) = \sum_{n=0}^{\infty} d_{k,n}\, e^{2\pi i n\tilde\tau} \qquad \text{(C.2)}$$

[5]The analysis in this appendix was suggested by Stephen Miller.

where the coefficients $d_{k,n}$ are real. (Displacing the expansion point to another point on the imaginary axis will maintain reality of the coefficients $d_{k,n}$, but change their values.)

We will now use the fact that the S transformation $\tau \to -1/\tau$ leaves the point $\tau = i$ invariant, and takes the following form on $\tilde{\tau}$,

$$i + \tilde{\tau} \to -\frac{1}{i + \tilde{\tau}} = \frac{i}{1 - i\tilde{\tau}} = i \sum_{n=0}^{\infty} i^n \tilde{\tau}^n \tag{C.3}$$

to write the expansion of $K(\tau, y) \equiv H(\tau, y) - H^S(\tau, y)$ in powers of $\tilde{x}$ and $\tilde{y}$,

$$y^2 K(\tau, y) = \sum_{r=0}^{2} \sum_{n=0}^{\infty} \tilde{y}^r \, c_{n,r} \, i^n \, (\tilde{x} + i\tilde{y})^n \tag{C.4}$$

where the coefficients $c_{n,r}$ are again real constants. Expanding the n-th power gives,

$$y^2 K(\tau, y) = \sum_{r=0}^{2} \sum_{n=0}^{\infty} \sum_{l=0}^{n} c_{n,r} \binom{n}{l} (-1)^l \, i^{n-l} \, \tilde{x}^{n-l} \, \tilde{y}^{l+r} \tag{C.5}$$

setting $n = a + l \geq 0$ and $l = b - r \geq 0$ we have

$$y^2 K(\tau, y) = \sum_{a=0}^{\infty} \sum_{r=0}^{2} \sum_{b=r}^{\infty} c_{a+b-r,r} \begin{pmatrix} a+b-r \\ b-r \end{pmatrix} (-1)^{b-r} \, i^a \, \tilde{x}^a \, \tilde{y}^b \tag{C.6}$$

Since $K(\tau, y)$ must be imaginary for all $\tilde{x}$ and $\tilde{y}$, we conclude that all contributions with even index a must vanish identically as a function of $\tilde{y}$. The terms with $b = 0$ and $b = 1$ are special in that they receive contributions only from $r = 0$ and $r = 0, 1$ respectively. The conditions for $b = 0, 1$, and all even a, are as follows,

$$c_{a,0} = 0 \qquad\qquad c_{a,1} = (a+1) c_{a+1,0} \tag{C.7}$$

The conditions for $b \geq 2$, and all even a, are as follows,

$$c_{a+b,0} \begin{pmatrix} a+b \\ b \end{pmatrix} - c_{a+b-1,1} \begin{pmatrix} a+b-1 \\ b-1 \end{pmatrix} + c_{a+b-2,2} \begin{pmatrix} a+b-2 \\ b-2 \end{pmatrix} = 0 \tag{C.8}$$

The equations for $0 \leq b \leq 5$ may be readily solved: they set all coefficients $c_{n,r}$ to zero, except for the following set,

$$c_{1,0} \qquad c_{3,0} \qquad c_{1,1} \qquad c_{2,1} = 3c_{3,0} \qquad c_{1,2} = 2c_{3,0} \tag{C.9}$$

whose values are left undetermined by the equations. The equations for $b \geq 6$ do not impose any further conditions on the coefficients $c_{n,r}$. So we find that consistency with modular invariance places a very restrictive condition on $K(\tau, y)$.

Substituting this result into the expansion (C.6) shows that $K(\tau,y)$ is given in terms of a very small number of terms with odd powers of $\tilde{x}$,

$$y^2K(\tau,y)=i\tilde{x}\Big(c_{1,0}+c_{1,1}\,\tilde{y}+c_{3,0}\,(\tilde{x}^2+\tilde{y}^2)\Big) \tag{C.10}$$

Since the expression only has a finite number of powers of $\tilde{x}+i\tilde{y}=\tilde{\tau}$ we can return to the original τ coordinate by replacing $\tilde{\tau}\to\tau-i$, $\tilde{y}\to y-1$, which results in

$$K(\tau,y)=A+2i\,B\,\tau+\frac{1}{y}(C+i\,A\,\tau-3\,B\,\tau^2)+\frac{1}{y^2}(i\,C\,\tau-i\,B\,\tau^3) \tag{C.11}$$

where the coefficients $A=c_{11}+2\,c_{30}$, $B=c_{3,0}$, $C=c_{1,0}-c_{1,1}-c_{3,0}$ are all real.

Finally, the conjugation property $\psi_k(-\tau)=(-)^k\psi_k(\tau)$ of (3.30) implies that K is an even function of its arguments, $K(-\tau,-y)=K(\tau,y)$. Imposing this further condition on K implies that $B=C=0$, so that we are left with,

$$K(\tau,y)=A+\frac{i}{y}A\tau=iA\,\frac{x}{y} \tag{C.12}$$

which is manifestly imaginary, since A is real.

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