The Langlands-Shahidi method over function fields: the Ramanujan Conjecture and the Riemann Hypothesis for the unitary groups

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THE LANGLANDS-SHAHIDI METHOD OVER FUNCTION FIELDS: THE RAMANUJAN CONJECTURE AND THE RIEMANN HYPOTHESIS FOR THE UNITARY GROUPS

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Dedicated to Freydoon Shahidi

Résumé. On étudie la méthode de Langlands-Shahidi sur les corps de fonctions de caractéristique p. On preuve la fonctorialité de Langlands globale et locale des groupes unitaires vers groupes linéaires pour les représentations génériques. Supposant la conjecture de Shahidi pour les L-paquets modérés, on donne une extension de la définition des fonctions L et facteurs ε. Enfin, utilisant le travail de L. Lafforgue, on établit la conjecture de Ramanujan et on prouve que les fonctions L automorphes de Langlands-Shahidi satisfont l'hypothèse the Riemann.

Abstract. We study the Langlands-Shahidi method over a global field of characteristic p. We prove global and local Langlands functoriality from unitary groups to general linear groups for generic representations. Assuming Shahidi's tempered L-packet conjecture, we provide an extension of the definition of L-functions and ε-factors. Finally, thanks to the work of L. Lafforgue, we establish the Ramanujan conjecture and prove that Langlands-Shahidi automorphic L-functions satisfy the Riemann Hypothesis.

Introduction

We make the Langlands-Shahidi method available over function fields. The method was developed by Shahidi in the case of number fields over the course of several decades. Previously, the $\mathcal{LS}$ method in characteristic p was only well understood for the split classical groups.

Let $\mathbf{G}$ be a connected reductive group defined over a function field k. Let $\mathbf{P} = \mathbf{MN}$ be a maximal parabolic subgroup of $\mathbf{G}$ and let ${}^LG$ denote its Langlands dual group. The $\mathcal{LS}$ method allows us to study automorphic L-functions arising from the adjoint action $r = \oplus\, r_i$ of ${}^LM$ on ${}^L\mathfrak{n}$, where ${}^L\mathfrak{n}$ is the Lie algebra of the unipotent radical ${}^LN$ on the dual side. Let π be any globally generic cuspidal automorphic representation of $\mathbf{M}(\mathbb{A}_k)$. The Langlands-Shahidi method provides a definition of global L-functions and root numbers

$$L(s, \pi, r_i) \text{ and } \varepsilon(s, \pi, r_i),\ \ s \in \mathbb{C}.$$

Locally, we obtain a system of γ-factors, L-functions and ε-factors at every place v of k. Let $\psi = \otimes_v \psi_v = k\backslash\mathbb{A}_k \to \mathbb{C}^\times$ be a character, where $\mathbb{A}_k$ is the ring of adèles. Then we have

$$\gamma(s, \pi_v, r_{i.v}, \psi_v),\ \ L(s, \pi_v, r_{i,v}) \text{ and } \varepsilon(s, \pi_v, r_{i,v}, \psi_v).$$

The connection between the local and global theory can be seen via the global functional equation

$$L^S(s, \pi, r_i) = \prod_{v \in S} \gamma(s, \pi_v, r_{i,v}, \psi_v) L^S(1 - s, \tilde{\pi}, r_i).$$

The partial L-function being defined by

$$L^S(s, \pi, r_i) = \prod_{v \notin S} L(s, \pi_v, r_{i,v})$$

and local factors for tempered representations satisfy the following relation

$$\varepsilon(s, \pi_v, r_{i,v}, \psi_v) = \gamma(s, \pi_v, r_{i,v}, \psi_v) \frac{L(s, \pi_v, r_{i,v})}{L(1 - s, \tilde{\pi}_v, r_{i,v})}.$$

We begin in a purely local setting in sections 1 and 2, where we define local factors over any non-archimedean local field F of characteristic p via the Langlands-Shahidi local coefficient. We take π to be any generic representation of $\mathbf{M}(F)$ and ψ an additive character of F. The local coefficient

$$C_\psi(s, \pi, \tilde{w}_0)$$

is obtained via intertwining operators and the multiplicity one property for Whittaker models.

The rank one cases are addressed in Proposition 1.3, which shows compatibility of the Langlands-Shahidi local coefficient with the abelian γ-factors of Tate's thesis [60]. This result is essentially Propostion 3.2 of [41], which includes the case of $\mathrm{char}(F) = 2$. Let $\mathfrak{W} = \{\tilde{w}_\alpha, d\mu_\alpha\}_{\alpha \in \Delta}$ be a system of Weyl group element representatives $\tilde{w}_\alpha$ together with Haar measures, indexed by the simple roots Δ. Proposition 2.2 determines the behavior of the local coefficient as $\mathfrak{W}$ varies. When π is an unramified principal series representation, the local coefficient decomposes into product of rank one cases.

Globally, there is a connection to Langlands' theory of Eisenstein series over function fields [16, 46]. In order to be more precise, let π be a globally generic cuspidal representation of $\mathbf{M}(\mathbb{A})$. By choosing the appropriate test function in the space of π one is able to obtain a relation between the global and local Whittaker model, matching with the Casselman-Shalika formula at unramified places [6]. This enables us to prove in § 3 the crude functional equation involving the Langlands-Shahidi local coefficient and partial L-functions, Theorem 3.3. The connection between Eisenstein series and partial L-functions is stated in Corollary 3.4.

We then turn towards the main result of the Langlands-Shahidi method in § 4. Theorem 4.1 establishes the existence and uniqueness of a system of γ-factors, L-functions and root numbers. One ingredient in its proof is a very useful local to global result, Lemma 4.2, which allows us to lift any supercuspidal representation π_0 to a cuspidal automorphic representation π with controlled ramification at all other places; if π_0 is generic, then π is globally generic. Another ingredient is a recursive argument that is already present in Arthur's work, using endoscopic groups, in addition to Shahidi [53]; Lemma 4.4 allows us to produce individual functional equations for each r_i. We define local γ-factors recursively by means of the local coefficient and they connect to the global theory via the functional equation.

An inspiring list of axioms for γ-factors that uniquely characterize them can be found in [36]. Work on the uniqueness of Rankin-Selberg L-functions for general linear groups [21], led us to extend the characterization in a natural way to include L-functions and ε-factors beginning with the classical groups in [40]. We provide a simple proof of the local to global result for quasi-split reductive groups, originally written in [21] for GL_n and generalized in [14] as mentioned in § 4.2. Lemma 4.2 has allowed us to reduce the number of required axioms.

We conclude with the general treatment of the Langlands-Shahidi method over function fields in § 5, where we study their analytic properties and applications. Langlands-Shahidi autmorphic L-functions over function fields are rational, a property we prove based on Harder's rationality for Eisenstein series [16]. After twists by highly ramified characters, our automorphic L-functions in characteristic p become Laurent polynomials. Locally, a consequence of having a theory of L-functions leads towards reducibility results and Shahidi's applications to complementary series [53]. If we assume the Ramanujan conjecture, our automorphic L-functions are holomorphic for $\Re(s) > 1$; in § A.2 of the Appendix, we provide several examples when this property is true.

The second part of the article is devoted to the unitary groups. We extend the $\mathcal{LS}$ method to include the study of products of globally generic representations of two unitary groups. We prove stable Base Change for globally generic reprensentations of unitary groups. Globally, we base ourselves on previous work on the classical groups over function fields and we guide ourselves with the work of Kim and Krishnamurthy for the unitary groups in the case of number fields [28, 29].

Our approach is possible by combining the $\mathcal{LS}$ method and the Converse Theorem of Cogdell and Piatetski-Shapiro [7]. Over number fields, functoriality for the classical groups was established for globally generic representations by Cogdell, Kim, Piatetski-Shapiro and Shahidi [9]; the work of Arthur establishes the general case for not necessarily generic representations in [1]; and, Mok addresses the endoscopic classification for the unitary groups in [45].

Over function fields, work of V. Lafforgue addresses the Langlands correspondence from a connected reductive group to the Galois side [32]. The transfer σ of a cuspidal representation π of a connected reductive group, has the property that σ_v corresponds to π_v at every unramified place. A stong lift would require the local Langlands correspondence at every place, and not just unramified places. There is the ongoing work of A. Genestier and V. Lafforgue, who aim to prove the local Langlands correspondence in characteristic p. In contrast, in our approach we work purely with techniques from Automorphic Forms and Representation Theory of $\mathfrak{p}$-adic groups. The functorial lift is from globally generic cuspidal representations of a unitary group U_N to automorphic representations of $\mathrm{Res}\,\mathrm{GL}_N$. We note that once on the general linear group side, the work of L. Lafforgue [31] provides a one to one global correspondence with Galois representations. Locally, we prove the Langlands correspondence from generic representations of the quasi-split classical groups to admissible representations of a general linear group. Again, once on the general linear groups side, we have a one to one correspondence with Galois representations [37]. And, we reduce the general case to the generic case via the tempered L-packet conjecture. This is already a theorem for the split classical groups in characteristic p, thanks to the work of Ganapathy-Varma [15]. For two alternative approaches to the local Langlands correspondence for admissible representations of the quasi-split classical groups, see §§ 7 and 8 of [14].

Before studying L-functions for products of two unitary groups, we prepare in § 6 with the induction step of the $\mathcal{LS}$ method for the unitary groups. Namely, the case of Asai and twisted Asai L-functions studied in [20, 41]. We also retrieve from Theorem 4.1 the Rankin-Selberg product L-function of a unitary group and a general linear group. We have the main theorem on extended γ-factors, L-functions and root numbers, Theorem 7.3. Locally, we work with irreducible admissible representations in general. However, the proof is completed in § 10.2 under the assumption that the tempered L-packet conjecture holds to be true for the unitary groups in positive characteristic. As mentioned above, cases of this conjecture are already known. We then provide the list of axioms that uniquely characterize extended local factors. In addition, we list three important properties: the local functional equation; the global functional equation for completed L-functions; and, stability of γ-factors after twits by highly ramified characters. A proof of the latter property for all Langlands-Shahidi γ-factors in positive characteristic can be found in [14]; we use this to obtain a very useful stable form of local factors for the unitary groups after highly ramified twists.

In § 8, we establish stable Base Change for globally generic representations over function fields. In fact, we first produce a "weak" base change (agreeing with the local Base Change lift at every unramified place), before proving it is a "strong" Base Change (agreeing at every place) in §§ 9 and 10. Let K/k be a separable quadratic extension of function fields. Given a cuspidal automorphic representation π of a unitary group U_N, we construct a candidate admissible representation Π for the Base Change to $\mathrm{Res}\,\mathrm{GL}_N$. Namely, at every unramified place, let $\hat{A}_v$ be the semisimple conjugacy class of π_v in $\mathrm{GL}_N(\mathbb{C})$ obtained via the Satake parametrization. The Weil group $\mathcal{W}_{k_v}$ acts via the Galois group $\mathrm{Gal}(K_v/k_v) = \{1, \theta_v\}$. We have

$$\begin{array}{ccc} \pi_v \text{ of } \mathrm{U}_N(k_v) & \longrightarrow & \left\{(\hat{A}_v, w_{\theta,v})\right\} \text{ of } {}^L\mathrm{U}_N \\ \text{BC} \downarrow & & \downarrow \\ \Pi_v \text{ of } \mathrm{GL}_N(K_v) & \longleftarrow & \left\{(\hat{A}_v, \hat{A}_v, w_{\theta,v})\right\} \text{ of } {}^L\mathrm{Res}_{K_v/k_v}\mathrm{GL}_N \end{array}$$

where we use the fact that there is a natural bijection between $w_{\theta,v}$-conjugacy classes of ${}^L\mathrm{Res}_{K_v/k_v}\mathrm{GL}_N = \mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C}) \rtimes \mathcal{W}_{k_v}$ and conjugacy classes of ${}^L\mathrm{GL}_N = \mathrm{GL}_N(\mathbb{C}) \rtimes \mathcal{W}_{K_v}$ as in [42]. At ramified places we can basically take an arbitrary representation Π_v with the same central character as π_v, since we can locally incorporate stability under highly ramified twists.

For suitable twists by cuspidal automorphic representations τ of GL_m, we have that $L(s, \Pi \times \tau) = L(s, \pi \times \tau)$. The Converse Theorem requires that these L-functions be *nice*. Over a function field k with field of constants $\mathbb{F}_q$, this means they are rational on q^{-s} and satisfy the global functional equation. The required rationality property is Theorem 5.1 and the global functional equation is Property (xii) of § 4.4. An important property that allows us to work with ramified places is the stability of γ-factors after highly ramified twists. To summarize, we are then able to apply the Converse Theorem and establish the existence of a weak Base Change to $\mathrm{Res}\,\mathrm{GL}_N$, Theorem 8.6.

In § 9 we turn towards the local Langlands conjecture for the unitary groups, that is, local Base Change. Let U_N be a unitary group defined over a non-archimedean local field F of characteristic p and let E/F be the underlying quadratic extension. Let π be a generic representation of $\mathrm{U}_N(F)$. Then, Theorem 9.10 establishes the local transfer

$$\left\{ \begin{array}{c} \text{generic representations} \\ \pi \text{ of } \mathrm{U}_N(F) \end{array} \right\} \xrightarrow{\mathrm{BC}} \left\{ \begin{array}{c} \text{generic representations} \\ \Pi \text{ of } \mathrm{GL}_N(E) \end{array} \right\}$$

The local Base Change $\Pi = \mathrm{BC}(\pi)$ is known as stable base change. It is uniquely characterized by the property that it preserves local L-functions, γ-factors and root numbers, just as in the case of GL_N [19]. The proof is global in nature and we use the weak global Base Change of Theorem 8.6 to deduce existence for generic supercuspidals of $\mathrm{U}_N(F)$. We then go through the classification of representations of unitary groups. In particular, the construction of discrete series by Mœglin and Tadić [43] and the work on Muić on the standard module conjecture [48] play an important role. The Basic Assumption (BA) of [43] is part of Theorem 5.8, where we follow Shahidi for generic representations. In general, we verify (BA) in [14] without the generic assumption. In addition, the work of M. Tadić [59] on the classification of unitary representations of GL_m is very useful. Local Base Change in general is thus established recursively: Langlands' classification reduces to the tempered case; then, tempered representations are constructed via discrete series, which in turn are constructed via supercuspidals. In this article, we focus on generic representations, and we refer to §§ 7 and 8 of [14] for a discussion of local Base Change in general.

Let K/k be a separable quadratic extension of global function fields. The weak Base Change of Theorem 8.6 is proven to be the strong Base Change lift, i.e., it is compatible with local Base Change at every place v of k. More precisely, in Theorem 9.11, we have stable Base Change for globally generic representations of unitary groups:

$$\left\{ \begin{array}{c} \text{globally generic cupsidal} \\ \text{automorphic representations} \\ \pi \text{ of } \mathrm{U}_N(\mathbb{A}_k) \end{array} \right\} \xrightarrow{\mathrm{BC}} \left\{ \begin{array}{c} \text{automorphic representations} \\ \Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d \\ \text{of } \mathrm{GL}_N(\mathbb{A}_K) \end{array} \right\}$$

We use the analytic properties of automorphic L-functions over function fields of § 5 in order to write $\Pi = \mathrm{BC}(\pi)$ as an isobaric sum of cuspidal automorphic representations Π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_K)$. The approach of this article also applies to the classical groups over function fields and we refer to § A.3 for the proof of the strong functorial lift.

In § 10 we conclude our study of L-functions for the unitary groups. In the case of generic representations, both local and global, our treatment is entirely self contained using only methods of Automorphic Forms and $\mathfrak{p}$-adic Representation Theory. For representations that are not necessarily generic, we note in § 10.2 how to reduce the study of local L-functions, γ-factors and root numbers to the case of generic representations. This part is written under the assumption that Conjecture 10.3 is valid.

We conclude our treatise over function fields by transporting via Base Change two important problems from the unitary groups to GL_N. More precisely, we combine our results with those of L. Lafforgue [31] to prove the Ramanujan Conjecture and the Riemann Hypothesis for our automorphic L-functions.

In the Appendix, we take the opportunity to go back and remove the restriction of $p \neq 2$ that was present in our prior study of functoriality for the classical groups [39], and applications [40]. In § A.2 we prove an important holomorphy property of L-functions for the split classical groups as well as the unitary groups. In § A.3 we prove that the weak lift of [39] for the classical groups (resp. Theorem 8.6 for the unitary groups) agrees with the local Langlands functorial lift, thus completing the proof of the strong functorial lift.

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Contents

1. The Langlands-Shahidi local coefficient

In this section and the next we revisit the theory of the Langlands-Shahidi local coefficient [52]. Now in characteristic p, basing ourselves in [39, 41]. After some preliminaries, we normalize Haar measures and choose Weyl group element representatives for rank groups in § 1.3. The local coefficient is compatible with Tate's thesis in these cases. In § 2 we will turn towards the subtle issues that arise when gluing these pieces together.

1.1. **Local notation.** Throughout the article we let F denote a non-archimedean local field of characteristic p. The ring of integers is denoted by $\mathcal{O}_F$ and a fixed uniformizer by ϖ_F. Given a maximal Levi subgroup $\mathbf{M}$ of a quasi-split connected reductive group scheme $\mathbf{G}$, we let $\mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathbf{G})$ denote the class of triples (F, π, ψ) consisting of: a non-archimedean local field F, with $\mathrm{char}(F) = p$; a generic representation π of $M = \mathbf{M}(F)$; and, a smooth non-trivial additive character $\psi : F \to \mathbb{C}^\times$.

When $\mathbf{M}$ and $\mathbf{G}$ are clear from context, we will write $\mathscr{L}_{\mathrm{loc}}(p)$ for $\mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathbf{G})$. We say $(F, \pi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p)$ is supercuspidal (resp. discrete series, tempered, principal series) if π is a supercuspidal (resp. discrete series, tempered, principal series) representation.

Let us now fix the quasi-split connected reductive groups scheme $\mathbf{G}$. Let $\mathbf{B} = \mathbf{TU}$ be a fixed Borel subgroup of $\mathbf{G}$ with maximal torus $\mathbf{T}$ and unipotent radical $\mathbf{U}$. Parabolic subgroups $\mathbf{P}$ of $\mathbf{G}$ will be standard, i.e., $\mathbf{P} \supset \mathbf{B}$. We write $\mathbf{P} = \mathbf{MN}$, where $\mathbf{M}$ is the corresponding Levi subgroup and $\mathbf{N}$ its unipotent radical. Given an algebraic group $\mathbf{H}$, we let H denote its group of rational points, e.g., $H = \mathbf{H}(F)$.

Let Σ denote the roots of $\mathbf{G}$ with respect to the split component $\mathbf{T}_s$ of $\mathbf{T}$ and Δ the simple roots. Let Σ_r denote the reduced roots. The positive roots are denoted Σ^+ and the negative roots Σ^-, and similarly for Σ_r^+ and Σ_r^-. The fixed borel $\mathbf{B}$ corresponds to a pinning of the roots with simple roots Δ. Standard parabolic subgroups are then in correspondence with subsets $\theta \subset \Delta$; $\theta \leftrightarrow \mathbf{P}_\theta$. The opposite of a parabolic $\mathbf{P}_\theta$ and its unipotent radical $\mathbf{N}_\theta$ are denoted by $\mathbf{P}_\theta^-$ and $\mathbf{N}_\theta^-$, respectively.

Given the choice of Borel there is a Chevalley-Steinberg system. To each $\alpha \in \Sigma^+$ there is a subgroup $\mathbf{N}_\alpha$ of $\mathbf{U}$, stemming from the Bruhat-Tits theory of a not necessarily reduced root system. Given smooth characters $\psi_\alpha : N_\alpha / N_{2\alpha} \to \mathbb{C}^\times$, $\alpha \in \Delta$, we can construct a character of U via

$$\mathbf{U} \twoheadrightarrow \mathbf{U} / \prod_{\alpha \in \Sigma^+ - \Delta} \mathbf{N}_\alpha \cong \prod_{\alpha \in \Delta} \mathbf{N}_\alpha / \mathbf{N}_{2\alpha} \tag{1.1}$$

and taking $\psi = \prod_{\alpha \in \Delta} \psi_\alpha$.

The character ψ is called non-degenerate if each ψ_α is non-trivial. We often begin with a non-trivial smooth character $\psi : F \to \mathbb{C}^\times$. When this is the case, unless stated otherwise, it is understood that the character ψ of U is obtained from the additive character ψ of F by setting $\prod_{\alpha \in \Delta} \psi$ in (1.1).

Fix a non-degenerate character $\psi : U \to \mathbb{C}^\times$ and consider ψ as a one dimensional representation on U. Recall that an irreducible admissible representation π of G is called ψ-generic if there exists an embedding

$$\pi \hookrightarrow \mathrm{Ind}_U^G(\psi).$$

This is called a Whittaker model of π. More precisely, if V is the space of π then for every $v \in V$ there is a Whittaker functional $W_v : G \to \mathbb{C}$ with the property

$$W_v(u) = \psi(u) W_v(e), \text{ for } u \in U.$$

It is the multiplicity one result of Shalika [56] which states that the Whittaker model of a representation π is unique, if it exists. Hence, up to a constant, there is a unique functional

$$\lambda : V \to \mathbb{C}$$

satisfying

$$\lambda(\pi(u)v) = \psi(u)\lambda(v).$$

We have that

$$W_v(g) = \lambda(\pi(g)v), \text{ for } g \in G.$$

Given $\theta \subset \Delta$ let $\mathbf{P}_\theta = \mathbf{M}_\theta \mathbf{N}_\theta$ be the associated standard parabolic. Let $\mathbf{A}_\theta$ be the torus $(\cap_{\alpha \in \Delta} \ker(\alpha))^\circ$, so that $\mathbf{M}_\theta$ is the centralizer of $\mathbf{A}_\theta$ in $\mathbf{G}$. Let $X(\mathbf{M}_\theta)$ be the group of rational characters of $\mathbf{M}_\theta$, and let

$$\mathfrak{a}_{\theta,\mathbb{C}}^* = X(\mathbf{M}_\theta) \otimes \mathbb{C}.$$

There is the set of cocharacters $X^\vee(\mathbf{M}_\theta)$. And there is a pairing $\langle \cdot, \cdot \rangle : X(\mathbf{M}_\theta) \times X^\vee(\mathbf{M}_\theta) \to \mathbb{Z}$, which assigns a coroot $\alpha^\vee$ to every root α.

Let $X_{\mathrm{nr}}(\mathbf{M}_\theta)$ be the group of unramified characters of $\mathbf{M}_\theta$. It is a complex algebraic variety and we have $X_{\mathrm{nr}}(\mathbf{M}_\theta) \cong (\mathbb{C}^\times)^d$, with $d = \dim_{\mathbb{R}}(\mathfrak{a}_\theta^*)$. To see this, for every rational character $\chi \in X(\mathbf{M}_\theta)$ there is an unramified character $q^{\langle \chi, H_\theta(\cdot) \rangle} \in X_{\mathrm{nr}}(\mathbf{M}_\theta)$, where

$$q^{\langle \chi, H_\theta(m) \rangle} = |\chi(m)|_F \, .$$

This last relation can be extended to $\mathfrak{a}_{\theta,\mathbb{C}}^*$ by setting

$$q^{\langle s \otimes \chi, H_\theta(m) \rangle} = |\chi(m)|_F^s \, , \ s \in \mathbb{C}.$$

We thus have a surjection

$$\mathfrak{a}_{\theta,\mathbb{C}}^* \twoheadrightarrow X_{\mathrm{nr}}(\mathbf{M}_\theta). \tag{1.2}$$

Recall that, given a parabolic $\mathbf{P}_\theta$, the modulus character is given by

$$\delta_\theta(p) = q^{\langle \rho_\theta, H_\theta(m) \rangle}, \ p = mn \in P_\theta = MN,$$

where ρ_θ is half the sum of the positive roots in θ.

In [44] the variable appearing in the corresponding Eisenstein series ranges over the elements of $X_{\mathrm{nr}}(\mathbf{M}_\theta)$. Already in Tate's thesis [60], the variable ranges over the quasi-characters of GL_1. The surjection (1.2) allows one to use complex variables. In particular, our L-functions will be functions of a complex variable. For this we start by looking at a maximal parabolic subgroup $\mathbf{P} = \mathbf{MN}$ of $\mathbf{G}$. In this case, there is a simple root α such that $\mathbf{P} = \mathbf{P}_\theta$, where $\theta = \Delta - \{\alpha\}$. We fix a particular element $\tilde{\alpha} \in \mathfrak{a}_{\theta,\mathbb{C}}^*$ defined by

$$\tilde{\alpha} = \langle \rho_\theta, \alpha^\vee \rangle^{-1} \rho_\theta. \tag{1.3}$$

For general parabolics $\mathbf{P}_\theta$ we can reduce properties of L-functions to maximal parabolics via multiplicativity (Property (iv) of Theorem 4.1).

We make a few conventions concerning parabolic induction that we will use throughout the article. Let (π, V) be a smooth admissible representation of $M = M_\theta$ and let $\nu \in \mathfrak{a}_{\theta,\mathbb{C}}^*$. By parabolic induction, we mean normalized unitary induction

$$\mathrm{ind}_P^G(\pi),$$

where we extend the representation π to $P = MN$ by making it trivial on N. Also, whenever the parabolic subgroup $\mathbf{P}$ and ambient group $\mathbf{G}$ are clear from context, we will simply write

$$\mathrm{Ind}(\pi) = \mathrm{ind}_P^G(\pi).$$

We also incorporate twists by unramified characters. For any $\nu \in \mathfrak{a}_{\theta,\mathbb{C}}^*$, we let

$$\mathrm{I}(\nu,\pi) = \mathrm{ind}_{P_\theta}^G(q_F^{\langle \nu, H_\theta(\cdot)\rangle} \otimes \pi)$$

be the representation with corresponding space $\mathrm{V}(\nu,\pi)$. Finally, if $\mathbf{P}$ is maximal, we write

$$\mathrm{I}(s,\pi) = \mathrm{I}(s\tilde{\alpha},\pi), \ \ s \in \mathbb{C},$$

with $\tilde{\alpha}$ as in equation (1.3); its corresponding space is denoted by $\mathrm{V}(s,\pi)$. Furthermore, we write $\mathrm{I}(\pi)$ for $\mathrm{I}(0,\pi)$.

1.2. **The Langlands-Shahidi local coefficient.** Let W denote the Weyl group of Σ, which is generated by simple reflections $w_\alpha \in \Delta$. And, let W_θ denote the subgroup of W generated by w_α, $\alpha \in \theta$. We let

$$w_0 = w_l w_{l,\theta}, \tag{1.4}$$

where w_l and $w_{l,\theta}$ are the longest elements of W and W_θ, respectively. Choice of Weyl group element representatives in the normalizer $\mathrm{N}(T_s)$ will be addressed in section 2, in order to match with the semisimple rank one cases of § 1.3. For now, we fix a system of representatives $\mathfrak{W} = \{\tilde{w}_\alpha, d\mu_\alpha\}_{\alpha\in\Delta}$.

There is an intertwining operator

$$\mathrm{A}(\nu,\pi,\tilde{w}_0) : \mathrm{V}(\nu,\pi) \to \mathrm{V}(\tilde{w}_0(\nu), \tilde{w}_0(\pi)),$$

where $\tilde{w}_0(\pi)(x) = \pi(\tilde{w}_0^{-1} x \tilde{w}_0)$. Let $N_{w_0} = U \cap w_0 N_\theta^- w_0^{-1}$, then it is defined via the principal value integral

$$\mathrm{A}(\nu,\pi,\tilde{w}_0)f(g) = \int_{N_{w_0}} f(\tilde{w}_0^{-1} n g)\, dn.$$

With fixed ψ of U, let $\psi_{\tilde{w}_0}$ be the non-degenerate character on the unipotent radical $M_\theta \cap U$ of M_θ defined by

$$\psi_{\tilde{w}_0}(u) = \psi(\tilde{w}_0 u \tilde{w}_0^{-1}), \ \ u \in M_\theta \cap U. \tag{1.5}$$

This makes ψ and $\psi_{\tilde{w}_0}$ $\tilde{w}_0$-compatible.

Given an irreducible $\psi_{\tilde{w}_0}$-generic representation (π, V) of M_θ, Theorem 1.4 of [39] gives that $\mathrm{I}(\nu,\pi)$, $\nu \in \mathfrak{a}_{\theta,\mathbb{C}}^*$, is ψ-generic and establishes an explicit principal value integral for the resulting Whittaker functional

$$\lambda_\psi(\nu,\pi,\tilde{w}_0)f = \int_{N_{\theta'}} \lambda_{\psi_{\tilde{w}_0}}(f(\tilde{w}_0^{-1} n))\overline{\psi}(n)\, dn, \tag{1.6}$$

where $\theta' = w_0(\theta)$.

Definition 1.1. *For every $\psi_{\tilde{w}_0}$-generic $(F,\pi,\psi) \in \mathscr{L}_{\mathrm{loc}}(p)$, the Langlands-Shahidi local coefficient $C_\psi(s,\pi,\tilde{w}_0)$ is defined via the equation*

$$\lambda_\psi(s\tilde{\alpha},\pi,\tilde{w}_0) = C_\psi(s,\pi,\tilde{w}_0)\lambda_\psi(s\tilde{w}_0(\tilde{\alpha}),\tilde{w}_0(\pi),\tilde{w}_0)\mathrm{A}(s\tilde{\alpha},\pi,\tilde{w}_0), \tag{1.7}$$

where $s\tilde{\alpha} \in \mathfrak{a}_{\theta,\mathbb{C}}^*$ for every $s \in \mathbb{C}$.

Remark 1.2. *When it is clear from context, we identify $s \in \mathbb{C}$ with the element $s\tilde{\alpha} \in \mathfrak{a}_{\theta,\mathbb{C}}^*$. We thus write $\lambda_\psi(s,\pi,\tilde{w}_0)$ for $\lambda_\psi(s\tilde{\alpha},\pi,\tilde{w}_0)$ and $\mathrm{I}(s,\pi)$ for $\mathrm{I}(s\tilde{\alpha},\pi)$. Similarly, we identify s' with $s\tilde{w}_0(\tilde{\alpha}) \in \mathfrak{a}_{\theta',\mathbb{C}}^*$ and let $\pi' = \tilde{w}_0(\pi)$. Hence, we simply write $\lambda_\psi(s',\pi',\tilde{w}_0)$ instead of $\lambda_\psi(s\tilde{w}_0(\tilde{\alpha}),\tilde{w}_0(\pi),\tilde{w}_0)$.*

Let $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p)$ be $\psi_{\tilde{w}_0}$-generic. From Theorem 1.4 of [39] we know that $\lambda_\psi(s, \pi, \tilde{w}_0)$ is a polynomial in $\left\{q_F^s, q_F^{-s}\right\}$ for a test function $f_s \in {\rm I}(s, \pi)$. By Theorem 2.1 of [39], the Langlands-Shahidi local coefficient $C_\psi(s, \pi, \tilde{w}_0)$ is a rational function on q_F^{-s}, independent of the choice of test function.

1.3. **Rank one cases and compatibility with Tate's thesis.** Let F'/F be a separable extension of local fields. Let $\mathbf{G}$ be a connected quasi-split reductive group of rank one defined over F. The derived group is of the following form

$$\mathbf{G}_D = {\rm Res}_{F'/F}{\rm SL}_2 \text{ or } {\rm Res}_{F'/F}{\rm SU}_3.$$

Note that given a degree-2 finite étale algebra E over the field F', we consider the semisimple group ${\rm SU}_3$ given by the standard Hermitian form h for the unitary group in three variables as in § 4.4.5 of [11]. Given the Borel subgroup $\mathbf{B} = \mathbf{TU}$ of $\mathbf{G}$, the group $\mathbf{G}_D$ shares the same unipotent radical $\mathbf{U}$. The F rational points of the maximal torus $\mathbf{T}_D$ are given by

$$T_D = \left\{({\rm diag}(t, t^{-1}) \,|\, t \in F'^{\times}\right\},$$

in the former case, and by

$$T_D = \left\{({\rm diag}(z, \bar{z}z^{-1}, \bar{z}^{-1}) \,|\, z \in E^{\times}\right\}$$

in the latter case.

We now fix Weyl group element representatives and Haar measures. In these cases, Δ is a singleton $\{\alpha\}$, and we note that the root system of ${\rm SU}_3$ is not reduced. If $\mathbf{G}_D = {\rm Res}_{F'/F}{\rm SL}_2$, we set

$$\tilde{w}_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.8}$$

and, if $\mathbf{G}_D = {\rm Res}_{F'/F}{\rm SU}_3$, we set

$$\tilde{w}_\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.9}$$

Given a fixed non-trivial character $\psi : F \to \mathbb{C}^\times$, we then obtain a self dual Haar measure $d\mu_\psi$ of F, as in equation (1.1) of [41]. In particular, for ${\rm SL}_2$ we have the unipotent radical $\mathbf{N}_\alpha$, which is isomorphic to the the unique additive abelian group $\mathbf{G}_a$ of rank 1. Here, we fix the Haar measure $d\mu_\psi$ on $G_a \cong F$. Given a separable extension F'/F, we extend ψ to a character of F' via the trace, i.e., $\psi_{F'} = \psi \circ {\rm Tr}_{F'/F}$. We also have a self dual Haar measure $\mu_{\psi_{F'}}$ on $\mathbf{G}_a(F') \cong F'$.

Given a degree-2 finite étale algebra E over the field F, assume we are in the case $\mathbf{G}_D = {\rm SU}_3$. The unipotent radical is now $\mathbf{N} = \mathbf{N}_\alpha \mathbf{N}_{2\alpha}$, with $\mathbf{N}_\alpha$ and $\mathbf{N}_{2\alpha}$ the one parameter groups associated to the non-reduced positive roots. We use the trace to obtain a character $\psi_E : E \to \mathbb{C}^\times$ from ψ. We then fix measures $d\mu_\psi$ and $d\mu_{\psi_E}$, which are made precise in § 3 of [41] for N_α and $N_{2\alpha}$. We then extend to the case $\mathbf{G}_D = {\rm Res}_{F'/F}{\rm SU}_3$ by taking $\psi_{F'} = \psi \circ {\rm Tr}_{F'/F}$ and $\psi_E = \psi_{F'} \circ {\rm Tr}_{E/F'}$. Furthermore, we have corresponding self dual Haar measures $d\mu_{\psi_{F'}}$ and $d\mu_{\psi_E}$ for ${\rm Res}_{F'/F}\mathbf{N}_\alpha(F) \cong \mathbf{N}_\alpha(F')$ and ${\rm Res}_{F'/F}\mathbf{N}_{2\alpha}(F) \cong \mathbf{N}_{2\alpha}(E)$.

We also have the Langlands factors $\lambda(F'/F, \psi)$ defined in [34]. Let $\mathcal{W}_{F'}$ and $\mathcal{W}_F$ be the Weil groups of F' and F, respectively. Recall that if ρ is an n-dimensional semisimple smooth representation of $\mathcal{W}_{F'}$, then

$$\lambda(F'/F,\psi)^n = \frac{\varepsilon(s, \mathrm{Ind}_{\mathcal{W}_{F'}}^{\mathcal{W}_F}\rho, \psi)}{\varepsilon(s,\rho,\psi_{F'})}.$$

On the right hand side we have Galois ε-factors, see Chapter 7 of [5] for further properties. Given a degree-2 finite étale algebra E over F', the factor $\lambda(E/F',\psi)$ and the character $\eta_{E/F'}$ have the meaning of equation (1.8) of [41].

The next result addresses the compatibility of the Langlands-Shahidi local coefficient with the abelian γ-factors of Tate's thesis [60]. It is essentially Propostion 3.2 of [41], which includes the case of $\mathrm{char}(F) = 2$.

Proposition 1.3. *Let $(F,\pi,\psi) \in \mathscr{L}_{\mathrm{loc}}(p,\mathbf{T},\mathbf{G})$, where $\mathbf{G}$ is a quasi-split connected reductive group defined over F whose derived group $\mathbf{G}_D$ is either $\mathrm{Res}_{F'/F}\mathrm{SL}_2$ or $\mathrm{Res}_{F'/F}\mathrm{SU}_3$.*

(i) *If $\mathbf{G}_D = \mathrm{Res}_{F'/F}\mathrm{SL}_2$, let χ be the smooth character of T_D given by $\pi|_{T_D}$. Then*

$$C_{\psi_{F'}}(s,\pi,\tilde{w}_0) = \gamma(s,\chi,\psi_{F'}).$$

(ii) *If $\mathbf{G}_D = \mathrm{Res}_{F'/F}\mathrm{SU}_3$, χ and ν be the smooth characters of $E^\times$ and E^1, respectively, defined via the relation*

$$\pi|_{T_D}(\mathrm{diag}(t,z,\bar{t}^{-1})) = \chi(t)\nu(z).$$

Extend ν to a character of $E^\times$ via Hilbert's theorem 90. Then

$$C_{\psi_{F'}}(s,\pi,\tilde{w}_0) = \lambda(E/F',\overline{\psi}_{F'})\,\gamma_E(s,\chi\nu,\psi_E)\,\gamma(2s,\eta_{E/F'}\chi|_{F'^\times},\psi_{F'}).$$

The computations of [41] rely mostly on the unipotent group $\mathbf{U}$, which is independent of the group $\mathbf{G}$. However, there is a difference due to the variation of the maximal torus in the above proposition. For example, all smooth representations π of $\mathrm{SL}_2(F)$ are of the form $\pi(\mathrm{diag}(t,t^{-1})) = \chi(t)$ for a smooth character χ of $\mathrm{GL}_1(F)$; then $C_\psi(s,\pi,\tilde{w}_0) = \gamma(s,\chi,\psi)$. However, in the case of $\mathrm{GL}_2(F)$ we have $\pi(\mathrm{diga}(t_1,t_2)) = \chi_1(t_1)\chi_2(t_2)$ for smooth characters χ_1 and χ_2 of $\mathrm{GL}_1(F)$; then $C_\psi(s,\pi,\tilde{w}_0) = \gamma(s,\chi_1\chi_2^{-1},\psi)$. Note that the semisimple groups of rank one in the split case, ranging from adjoint type to simply connected, are PGL_2, GL_2 and SL_2. The cases $\mathrm{PGL}_2 \cong \mathrm{SO}_3$ and $\mathrm{SL}_2 = \mathrm{Sp}_2$ are also included in Propostion 3.2 of [*loc.cit.*]. Of particular interest to us in this article are $\mathrm{Res}_{E/F}\mathrm{GL}_2(F) \cong \mathrm{GL}_2(E)$, $\mathrm{U}_2(F)$ and $\mathrm{U}_3(F)$, which arise in connection with the quasi-split unitary groups.

Given an unramified character π of $T = \mathbf{T}(F)$, we have a parameter

$$\phi : \mathcal{W}_F \to {}^LT \tag{1.10}$$

corresponding to π. Let $\mathfrak{u}$ denote the Lie algebra of $\mathbf{U}$ and let r be the adjoint action of ${}^LT$ on ${}^L\mathfrak{u}$. Then r is irreducible if $\mathbf{G}_D = \mathrm{Res}_{F'/F}\mathrm{SL}_2$ and $r = r_1 \oplus r_2$ if $\mathrm{Res}_{F'/F}\mathrm{SU}_3$. As in [24], we normalize Langlands-Shahidi γ-factors in order to have equality with the corresponding Artin factors.

Definition 1.4. *Let $(F,\pi,\psi) \in \mathscr{L}_{\mathrm{loc}}(p,\mathbf{T},\mathbf{G})$ be as in Proposition 1.3:*

(i) *If $\mathbf{G}_D = \mathrm{Res}_{F'/F}\mathrm{SL}_2$, let*

$$\gamma(s,\pi,r,\psi) = \lambda(F'/F,\psi)\gamma(s,\chi,\psi_{F'}).$$

(ii) *If $\mathbf{G}_D = \mathrm{Res}_{F'/F}\mathrm{SU}_3$, let*

$$\gamma(s,\pi,r_1,\psi) = \lambda(E/F,\psi)\gamma(s,\chi\nu,\psi_E)$$

and

$$\gamma(s,\pi,r_2,\psi) = \lambda(F/F',\psi)\,\gamma(s,\eta_{E/F'}\chi|_{F'^\times},\psi_{F'}).$$

In this way, with ϕ as in (1.10), we have for each i:

$$\gamma(s,\pi,r_i,\psi) = \gamma(s,r_i \circ \phi,\psi).$$

The γ-factors on the right hand side are those defined by Deligne and Langlands [61]. We can then obtain corresponding L-functions and root numbers via γ-factors, see for example § 1 of [41].

2. Normalization of the local coefficient

We begin with Langlands lemma. This will help us choose a system of Weyl group element representatives in a way that the local factors agree with the rank one cases of the previous section. We refer to Shahidi's algorithmic proof of [52], for Lemma 2.1 below. In Proposition 2.2, we address the effect of varying the non-degenerate character on the Langlands-Shahidi local coefficient, in addition to changing the system of Weyl group element representatives and Haar measures. We then recall the multiplicativity property of the local coefficient, and we use this to connect between unramified principal series and rank one Proposition 1.3.

2.1. **Weyl group element representatives and Haar measures.** Recall that given two subsets θ and θ' of Δ are associate if $W(\theta,\theta') = \{w \in W | w(\theta) = \theta'\}$ is non-empty. Given $w \in W(\theta,\theta')$, define

$$N_w = U \cap wN_\theta^- w^{-1} \quad \overline{N}_w = w^{-1}N_w w.$$

The corresponding Lie algebras are denoted $\mathfrak{n}_w$ and $\overline{\mathfrak{n}}_w$.

Lemma 2.1. *Let $\theta,\theta' \subset \Delta$ are associate and let $w \in W(\theta,\theta')$. Then, there exists a family of subsets $\theta_1,\ldots,\theta_d \subset \Delta$ such that:*

(i) *We begin with $\theta_1 = \theta$ and end with $\theta_d = \theta'$.*
(ii) *For each j, $1 \le j \le d-1$, there exists a root $\alpha_j \in \Delta - \theta_j$ such that θ_{j+1} is the conjugate of θ_j in $\Omega_j = \{\alpha_j\} \cup \theta_j$.*
(iii) *Set $w_j = w_{j,\Omega_j} w_{l,\theta_j}$ in $W(\theta_j,\theta_{j+1})$ for $1 \le j \le d-1$, then $w = w_{d-1}\cdots w_1$.*
(iv) *If one sets $\dot{w}_1 = w$ and $\dot{w}_{j+1} = \dot{w}_j w_j^{-1}$ for $1 \le j \le d-1$, then $\dot{w}_d = 1$ and*

$$\overline{\mathfrak{n}}_{\dot{w}_j} = \overline{\mathfrak{n}}_{w_j} \oplus \mathrm{Ad}(w_j^{-1})\overline{\mathfrak{n}}_{\dot{w}_{j+1}}.$$

For each $\alpha \in \Delta$ there corresponds a group $\mathbf{G}_\alpha$ whose derived group is simply connected semisimple of rank one. We fix an embedding $\mathbf{G}_\alpha \hookrightarrow \mathbf{G}$. A Weyl group element representative $\tilde{w}_\alpha$ is chosen for each w_α and the Haar measure on the unipotent radical N_α are normalized as indicated in § 1.3. We take the corresponding measure on $N_{-\alpha}$ inside G_α. Fix

$$\mathfrak{W} = \{\tilde{w}_\alpha, d\mu_\alpha\}_{\alpha\in\Delta} \tag{2.1}$$

to be this system of Weyl group element representatives in the normalizer of $\mathrm{N}(T_s)$ together with fixed Haar measures on each N_α.

We can apply Lemma 2.1 by taking the Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ for the w_0 as in equation (1.4), i.e., we use $\theta = \emptyset$. In this way, we obtain a decomposition

$$w_0 = \prod_\beta w_{0,\beta}, \tag{2.2}$$

where β is seen as an index for the product ranging through $\beta \in \Sigma_r^+$. For each such $w_{0,\beta}$, there corresponds a simple reflection w_α for some $\alpha \in \Delta$.

From Langlands lemma and equation (2.2) we further obtain a decomposition of N in terms of N'_β, $\beta \in \Sigma_r^+$, where each N'_β corresponds to the unipotent group of N_α of $\mathbf{G}_\alpha$, for some $\alpha \in \Delta$. In this way, the measure on N is fixed by $\mathfrak{W}$ and we denote it by

$$dn = d\mu_N(n). \tag{2.3}$$

The decomposition of (2.2) is not unique. However, the choice $\mathfrak{W}$ of representatives determines a unique $\tilde{w}_0$.

We now summarize several facts known to the experts about the Langlands-Shahidi local coefficient in the following proposition.

Proposition 2.2. *Let $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p, \mathbf{M}, \mathbf{G})$. Let $\mathfrak{W}' = \{\tilde{w}'_\alpha, d\mu'_\alpha\}_{\alpha\in\Delta}$ be an arbitrary system of Weyl group element representatives and Haar measures. Let $\phi : U \to \mathbb{C}^\times$ be a non-degenerate character and assume that π is $\phi_{\tilde{w}'_0}$-generic. Let $\widetilde{\mathbf{G}}$ be a connected quasi-split reductive group defined over F, sharing the same derived group as $\mathbf{G}$, and with maximal torus $\widetilde{\mathbf{T}} = \mathbf{Z}_{\widetilde{\mathbf{G}}}\mathbf{T}$. Then, there exists an element $x \in \widetilde{T}$ such that the representation π_x, given by*

$$\pi_x(g) = \pi(x^{-1}gx)$$

is $\psi_{\tilde{w}_0}$-generic. And, there exists a constant $a_x(\phi, \mathfrak{W}')$ such that

$$C_\phi(s, \pi, \tilde{w}'_0) = a_x(\phi, \mathfrak{W}')C_\psi(s, \pi_x, \tilde{w}_0).$$

Let $(F, \pi_i, \psi) \in \mathscr{L}(p)$, $i = 1, 2$. If $\pi_1 \cong \pi_2$ and are both $\psi_{\tilde{w}_0}$-generic, then

$$C_\psi(s, \pi_1, \tilde{w}_0) = C_\psi(s, \pi_2, \tilde{w}_0).$$

Proof. The existence of a connected quasi-split reductive group $\widetilde{\mathbf{G}}$ of adjoint type, sharing the same derived group as $\mathbf{G}$, is due thanks to Proposition 5.4 of [55]. Its maximal torus is given by $\widetilde{\mathbf{T}} = \mathbf{Z}_{\widetilde{\mathbf{G}}}\mathbf{T}$. Let $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p)$, where (π, V) is $\phi_{\tilde{w}'_0}$-generic. Because $\widetilde{\mathbf{G}}$ is of adjoint type, the character ϕ lies on the same orbit as the fixed ψ. Thus, there indeed exists an $x \in \widetilde{T}$ such that π_x is $\psi_{\tilde{w}_0}$-generic.

The system $\mathfrak{W}'$ fixes a measure on N, which we denote by $dn' = d\mu'_N$. Uniqueness of Haar measures gives a constant $b \in \mathbb{C}$ such that $dn = b\, dn'$. Also, notice that we can extend π to a representation of $\widetilde{\mathbf{G}}$ which is trivial on $\mathbf{Z}_{\widetilde{G}}$. And, the restriction of π to T decomposes into irreducible constituents

$$\pi|_T = \oplus \tau_i.$$

Each τ_i, is one dimensional by Schur's lemma. Hence, for any element $y \in \widetilde{T}$ we have $\pi(y) \in \mathbb{C}$.

To work with the local coefficient, take $\varphi \in C_c^\infty(P_\theta w_0 B, V)$ and let $f = f_s = \mathcal{P}_s\varphi$ be as in Proposition 1.1 of [39]. Now, using the system $\mathfrak{W}'$ in the right hand side of the definition (1.7) for $C_\phi(s, \pi, \tilde{w}'_0)$, we have

$$\begin{aligned} &\lambda_\phi(s\tilde{w}'_0(\tilde{\alpha}), \tilde{w}'_0(\pi), \tilde{w}'_0)\mathrm{A}(s, \pi, \tilde{w}'_0)f \\ &\quad = \int_{N_{\overline{\theta}}} \int_{N_{w_0}} \lambda_{\phi_{\tilde{w}'_0}} \left(f(\tilde{w}_0'^{-1} n_1 \tilde{w}_0'^{-1} n_2) \right) \overline{\phi}(n_2)\, dn'_1\, dn'_2. \end{aligned}$$

Let $f_x(g) = f(x^{-1}gx)$ for $f \in \mathrm{I}(s,\pi)$, so that $f_x \in \mathrm{I}(s,\pi_x)$ is ψ-generic for the $\psi_{\tilde{w}_0}$-generic (π_x, V). Let c_x be the module for the automorphism $n \mapsto x^{-1}nx$. Then, after two changes of variables and an appropriate change in the domain of integration, the above integral is equal to

$$b^2c_x^2 \int_{N_{\overline{\theta}}} \int_{\tilde{w}_0' N_{w_0} \tilde{w}_0'^{-1}} \lambda_{\psi_{\tilde{w}_0}} \left(f_x(\tilde{w}_0'^{-2} n_1 n_2) \right) \overline{\psi}(n_2)\, dn_1\, dn_2,$$

letting $z = (\tilde{w}_0')^2$, $\tilde{w}_0' = \tilde{w}_0 t^{-1}$ and changing back the domain of integration, we obtain

$$\begin{aligned} b^2c_x^2c_z \int_{N_{\overline{\theta}}} \int_{N_{w_0}} & \lambda_{\psi_{\tilde{w}_0}} \left(f_x(t^2 \tilde{w}_0^{-1} n_1 \tilde{w}_0^{-1} n_2) \right) \overline{\psi}(n_2)\, dn_1\, dn_2 \\ &= b^2c_x^2c_z\pi(t^2)\lambda_{\psi_{\tilde{w}_0}}(s\tilde{w}_0(\tilde{\alpha}), \tilde{w}_0(\pi_x), \tilde{w}_0)\mathrm{A}(s,\pi,\tilde{w}_0)f_x. \end{aligned}$$

Now, working in a similar fashion with the left hand side of equation (1.7) we obtain

$$\lambda_\phi(s\tilde{\alpha}, \pi, \tilde{w}_0')f = bc_x \int_{N_{\overline{\theta}}} \lambda_{\psi_{\tilde{w}_0}} \left(f_x(x\tilde{w}_0'^{-1}x^{-1}\tilde{w}_0'\tilde{w}_0'^{-1}n) \right) \overline{\psi}(n)\, dn.$$

We can always find an $x \in \tilde{T}$ satisfying the above discussion and such that

$$d = x\tilde{w}_0'^{-1}x^{-1}\tilde{w}_0' \in T. \tag{2.4}$$

Alternatively, we can go to the separable closure, as in the discussion following Lemma 3.1 of [53], to obtain an element $x \in \mathbf{T}(F_s)$. To see this, we can reduce to rank one computations to produce the right x. In the case of SL_2, all additive characters are of the form ψ^a and we can take $t = \mathrm{diag}(a,1) \in \tilde{T}$ or $t = \mathrm{diag}(a^{1/2}, a^{-1/2}) \in \mathbf{T}(F_s)$. We then have

$$\begin{aligned} \lambda_\phi(s\tilde{\alpha}, \pi, \tilde{w}_0')f &= bc_x\pi(d) \int_{N_{\overline{\theta}}} \lambda_{\psi_{\tilde{w}_0}} \left(f_x(t\tilde{w}_0^{-1}n) \right) \overline{\psi}(n)\, dn \\ &= bc_x\pi(dt)\lambda_\psi(s,\pi,\tilde{w}_0)f_x. \end{aligned}$$

In this way, we finally arrive at the desired constant

$$a_x(\phi, \mathfrak{W}') = \frac{\pi(dt^{-1})}{bc_xc_z}.$$

To conclude, we notice that if $(F, \pi_i, \psi) \in \mathscr{L}(p)$, $i = 1, 2$, have $\pi_1 \cong \pi_2$ and are both $\psi_{\tilde{w}_0}$-generic, then Proposition 3.1 of [52] gives

$$C_\psi(s,\pi_1,\tilde{w}_0) = C_\psi(s,\pi_2,\tilde{w}_0).$$

□

2.2. **Multiplicativity of the local coefficient.** Shahidi's algorithm allows us to obtain a block based version of Langlands' lemma from the Corollary to Lemma 2.1.2 of [52]. We summarize the necessary results in this section. More precisely, let $\mathbf{P} = \mathbf{MN}$ be the maximal parabolic associated to the simple root $\alpha \in \Delta$, i.e., $\mathbf{P} = \mathbf{P}_\theta$ for $\theta = \Delta - \{\alpha\}$. We have the Weyl group element w_0 of equation (1.4). Consider a subset $\theta_0 \subset \theta$ and its corresponding parabolic subgroup $\mathbf{P}_{\theta_0}$ with maximal Levi $\mathbf{M}_{\theta_0}$ and unipotent radical $\mathbf{N}_{\theta_0}$.

Let $\Sigma(\theta_0)$ be the roots of $(\mathbf{P}_{\theta_0}, \mathbf{A}_{\theta_0})$. In order to be more precise, let $\Sigma^+(\mathbf{A}_{\theta_0}, \mathbf{M}_{\theta_0})$ be the positive roots with respect to the maximal split torus $\mathbf{A}_{\theta_0}$ in the center of

$\mathbf{M}_{\theta_0}$. We say that $\alpha, \beta \in \Sigma^+ - \Sigma^+(\mathbf{A}_{\theta_0}, \mathbf{M}_{\theta_0})$ are $\mathbf{A}_{\theta_0}$-equivalent if $\beta|_{A_{\theta_0}} = \alpha|_{A_{\theta_0}}$. Then

$$\Sigma^+(\theta_0) = \left(\Sigma^+ - \Sigma^+(\mathbf{A}_{\theta_0}, \mathbf{M}_{\theta_0})\right) / \sim .$$

Let $\Sigma_r^+(\theta_0)$ be the block reduced roots in $\Sigma^+(\theta_0)$. With the notation of Langlands' lemma, take $\theta_0' = w(\theta_0)$ and set

$$\Sigma_r(\theta_0, w) = \left\{[\beta] \in \Sigma_r^+(\theta_0) | w(\beta) \in \Sigma^-\right\} .$$

We have that

$$[\beta_i] = w_1^{-1} \cdots w_{j-1}^{-1}([\alpha_j]),\ 1 \leq j \leq n-1, \tag{2.5}$$

are all distinct in $\Sigma_r(\theta_0, w)$ and all $[\beta] \in \Sigma_r(\theta_0, w)$ are obtained in this way.

$$\overline{N}_{\dot{w}_j} = \mathrm{Ad}(w_j^{-1}) \overline{N}_{\dot{w}_{j+1}} \rtimes \overline{N}_{w_j}. \tag{2.6}$$

We now obtain a block based decomposition

$$w_0 = \prod_j w_{0,j}. \tag{2.7}$$

In addition, the unipotent group $\mathbf{N}_{w_0}$ decomposes into a product via successive applications of equation (2.6). Namely

$$\mathbf{N}_{w_0} = \prod_j \mathbf{N}_{0,j}. \tag{2.8}$$

where each $\mathbf{N}_{0,j}$ is a block unipotent subgroup of $\mathbf{G}$ corresponding to a finite subset Σ_j of $\Sigma_r(\theta_0, w_0)$. In this way, we can partition the block set of roots into a disjoint union

$$\Sigma_r(\theta_0, w_0) = \bigcup_i \Sigma_i. \tag{2.9}$$

Explicitly

$$\overline{\mathbf{N}}_{0,j} = \mathrm{Ad}(w_1^{-1} \cdots w_{j-1}^{-1}) \overline{\mathbf{N}}_{w_j}, \tag{2.10}$$

which gives a block unipotent $\mathbf{N}_j$ of $\mathbf{G}$. For each j we have an embedding

$$\mathbf{G}_j \hookrightarrow \mathbf{G}$$

of connected quasi-split reductive groups. Each constitutent of this decomposition of N_{w_0} is isomorphic to a block unipotent subgroup N_{w_j} of $\mathbf{M}_j$. The reductive group $\mathbf{M}_j$ has root system θ_j and is a maximal Levi subgroup of the reductive group $\mathbf{G}_j$ with root system Ω_j. We have $\mathbf{P}_j = \mathbf{M}_j \mathbf{N}_{w_j}$, a maximal parabolic subgroup of $\mathbf{G}_j$. Notice that each $\mathbf{M}_j$ corresponds to a simple root α_j of $\mathbf{G}_j$. We obtain from π a representation π_j of M_j. We let

$$\tilde{\alpha}_j = \left\langle \rho_{P_j}, \alpha_j^\vee \right\rangle^{-1} \rho_{P_j}.$$

We note that each w_j in equation (2.7) is of the form

$$w_j = w_{l,\mathbf{G}_j} w_{l,\mathbf{M}_j}. \tag{2.11}$$

Additionally, each w_j decomposes into a product of Weyl group elements corresponding to simple roots $\alpha \in \Delta$. While these decompositions are again not unique, the choice $\mathfrak{W}$ of representatives fixes a unique $\tilde{w}_j$, independently of the decomposition of w_j.

For each $[\beta] \in \Sigma_r^+(\theta_0, w_0)$, we let

$$i_{[\beta]} = \left\langle \tilde{\alpha}, \beta^\vee \right\rangle .$$

Since $\theta_0 \subset \theta$, we have that the values of $i_{[\beta]}$ range among the integer values

$$i = \langle \tilde{\alpha}, \gamma^\vee \rangle \,, \; 1 \leq i \leq m_r,$$

where $[\gamma] \in \Sigma_r^+(\theta, w_0)$. Let

$$a_j = \min \left\{ i_{[\beta]} | [\beta] \in \Sigma_j \right\},$$

where the Σ_i are as in (2.9). The following is Proposition 3.2.1 of [52].

Proposition 2.3. *Let $(F, \pi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathbf{G})$ and assume π is obtained via parabolic induction from a generic represetation π_0 of $\mathbf{M}_{\theta_0}$*

$$\pi \hookrightarrow \mathrm{ind}_{P_{\theta_0}}^{M_\theta} \pi_0.$$

Then, with the notation of Langlands' lemma, we have

$$C_\psi(s\tilde{\alpha}, \pi, \tilde{w}_0) = \prod_j C_\psi(a_j s \tilde{\alpha}_j, \pi_j, \tilde{w}_j).$$

2.3. **L-groups and the adjoint representation.** Given our connected reductive quasi-split group $\mathbf{G}$ over a non-archimedean local field or a global function field, it is also a group over its separable algebraic closure. Let $\mathcal{W}$ be the corresponding Weil group. The pinning of the roots determines a based root datum $\Psi_0 = (X^*, \Delta, X_*, \Delta^\vee)$. The dual root datum $\Psi_0^\vee = (X_*, \Delta^\vee, X^*, \Delta)$ determines the Chevalley group ${}^L G^\circ$ over $\mathbb{C}$. Then the L-group of $\mathbf{G}$ is the semidirect product

$${}^L G = {}^L G^\circ \rtimes \mathcal{W},$$

with details given in [3]. The base root datum $\Psi^\vee$ fixes a borel subgroup ${}^L B$, and we have all standard parabolic subgroups of the form ${}^L P = {}^L P^\circ \rtimes \mathcal{W}$. The Levi subgroup of ${}^L P$ is of the form ${}^L M = {}^L M^\circ \rtimes \mathcal{W}$, while the unipotent radical is given by ${}^L N = {}^L N^\circ$.

Let $r : {}^L M \to \mathrm{End}({}^L\mathfrak{n})$ be the adjoint representation of ${}^L M$ on the Lie algebra ${}^L\mathfrak{n}$ of ${}^L N$. It decomposes into irreducible components

$$r = \oplus_{i=1}^{m_r} r_i.$$

The r_i's are ordered according to nilpotency class. More specifically, consider the Eigenspaces of ${}^L M^\circ$ given by

$${}^L\mathfrak{n}_i = \left\{ X_{\beta^\vee} \in {}^L\mathfrak{n} | \, \langle \tilde{\alpha}, \beta \rangle = i \right\}, \; 1 \leq i \leq m_r.$$

Then each r_i is a representation of the complex vector space ${}^L\mathfrak{n}_i$.

2.4. **Unramified principal series and Artin L-functions.** Consider a triple $(F, \pi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathbf{G})$, where π has an Iwahori fixed vector. From Proposition 2.2, we can assume π is $\psi_{\tilde{w}_0}$-generic and $\tilde{w}_0$-compatible with ψ; the choice of Weyl group element representatives and Haar measures $\mathfrak{W}$ being fixed. With the multiplicativity of the local coefficient, we can proceed as in § 2 of [24] and § 3 of [53], and reduce the problem to the rank one cases of § 1.3. We now proceed to state the main result.

For every root $\beta \in \Sigma_r(\emptyset, w_0)$, we have as in § 2.1 a corresponding rank one group $\mathbf{G}_\alpha$. Let $\Sigma_r(w_0, \mathrm{SL}_2)$ denote the set consisting of $\alpha \in \Sigma_r(\emptyset, w_0)$ such that $\mathbf{G}_\alpha$ is as in case (i) of Proposition 1.3. Similarly, let $\Sigma_r(w_0, \mathrm{SU}_3)$ consist of $\alpha \in \Sigma_r(\emptyset, w_0)$ such that $\mathbf{G}_\alpha$ is as in the corresponding case (ii). Let

$$\lambda(\psi, w_0) = \prod_{\alpha \in \Sigma_r(w_0, \mathrm{SL}_2)} \lambda(F'_\alpha/F, \psi) \prod_{\alpha \in \Sigma_r(w_0, \mathrm{SU}_3)} \lambda(E_\alpha/F, \psi)^2 \lambda(F'_\alpha/F, \psi)^{-1}$$

Partitioning each of the sets Σ_i of equation (2.9) arising in this setting further by setting $\Sigma_i = \Sigma_i(\mathrm{SL}_2) \cup \Sigma_i(\mathrm{SU}_3)$ we can define $\lambda_i(\psi, w_0)$ appropriately, so that

$$\lambda(\psi, w_0) = \prod_i \lambda_i(\psi, w_0).$$

Proposition 2.4. *Let $(F, \pi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathbf{G})$ be such that π has an Iwahori fixed vector. Then*

$$C_\psi(s, \pi, \tilde{w}_0) = \lambda(\psi, w_0)^{-1} \prod_{i=1}^{m_r} \gamma(is, \pi, r_i, \psi).$$

Let $\phi : \mathcal{W}_F' \to {}^L M$ be the parameter of the Weil-Deligne group corresponding to π. Then

$$\prod_{i=1}^{m_r} \gamma(is, \pi, r_i, \psi) = \prod_{i=1}^{m_r} \gamma(is, r_i \circ \phi, \psi),$$

where on the right hand side we have the Artin γ-factors defined by Delinge and Langlands [60].

3. Partial L-functions and the local coefficient

The Langlands-Shahidi method studies L-functions arising from the adjoint representation $r : {}^L M \to \mathrm{End}({}^L \mathfrak{n})$. It decomposes into irreducible components r_i, $1 \leq i \leq m_r$, as in § 2.3. Locally, let $(F, \pi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p)$ be such that π has an Iwahori fixed vector. Then π corresponds to a conjugacy class $\{A_\pi \rtimes \sigma\}$ in ${}^L M$, where A_π is a semisimple element of ${}^L M^\circ$. Then

$$L(s, \pi, r_i) = \frac{1}{\det(I - r_i(A_\pi \rtimes \sigma) q_F^{-s})},$$

for $(F, \pi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p)$ unramified, with the notation of § 1.1.

3.1. **Global notation.** For the remainder of the article k will denote a global function field with field of constants $\mathbb{F}_q$. Let $\mathbb{A}_k$ denote its ring of adèles. Given an algebraic group $\mathbf{H}$ and a place v of k, we write q_v for q_{k_v} and H_v instead of $\mathbf{H}(k_v)$. Similarly with $\mathcal{O}_v$ and ϖ_v.

We globally fix a maximal compact open subgroup $\mathcal{K} = \prod_v \mathcal{K}_v$ of $\mathbf{G}(\mathbb{A}_k)$, where the $\mathcal{K}_v$ range through a fixed set of maximal compact open subgroups of G_v. Each $\mathcal{K}_v$ is special and $\mathcal{K}_v$ is hyperspecial at almost every place. In addition, we can choose each $\mathcal{K}_v$ to be compatible with the decomposition

$$\mathcal{K}_v = (N_v^- \cap \mathcal{K}_v)(M_v \cap \mathcal{K}_v)(N_v \cap \mathcal{K}_v),$$

for every standard parabolic subgroup $\mathbf{P} = \mathbf{MN}$. The group $M \cap \mathcal{K}$ is a maximal compact open subgroup of $M = \mathbf{M}(\mathbb{A}_k)$. Furthermore

$$G = P\mathcal{K}.$$

Given a finite set of places S of k, we let $G_S = \prod_{v \in S} G_v \prod_{v \notin S} \mathcal{K}_v$.

Let $\mathscr{L}_{\mathrm{glob}}(p, \mathbf{M}, \mathbf{G})$ be the class of quadruples (k, π, ψ, S) consisting of: k of characteristic p; a globally generic cuspidal automorphic representation $\pi = \otimes_v \pi_v$ of $\mathbf{M}(\mathbb{A}_k)$; a non-trivial character $\psi = \otimes_v \psi_v : k \backslash \mathbb{A}_k \to \mathbb{C}^\times$; and, a finite set of places S where k, π and ψ are unramified. We write $\mathscr{L}_{\mathrm{glob}}(p)$ when $\mathbf{M}$ and $\mathbf{G}$ are clear from context.

Given $(k,\pi,\psi,S) \in \mathscr{L}_{\rm glob}(p)$, we have partial L-functions

$$L^S(s,\pi,r_i) = \prod_{v \notin S} L(s,\pi_v,r_{i,v}).$$

They are absolutely convergent for $\Re(s) \gg 0$.

3.2. **Tamagawa measures.** Fix $(k,\pi,\psi,S) \in \mathscr{L}_{\rm glob}(p)$. From the discussion of § 2.1, the character $\psi = \otimes_v \psi_v$, gives a self-dual Haar measure $d\mu_v$ at every place v of k. We let

$$d\mu = \prod_v d\mu_v.$$

Notice that $d\mu_v(\mathcal{O}_v) = d\mu_v^\times(\mathcal{O}_v^\times) = 1$ for all $v \notin S$. Representatives of Weyl group elements are chosen using Langlands' lemma for $\mathbf{G}(k)$. This globally fixes the system

$$\mathfrak{W} = \left\{\tilde{w}_\alpha, d\mu_\alpha\right\}_{\alpha\in\Delta}. \tag{3.1}$$

As in § 2.1, this fixes the Haar measure on $\mathbf{N}(\mathbb{A}_k)$.

We obtain a character of $\mathbf{U}(\mathbb{A}_k)$ via the surjection (1.1) and the fixed character ψ. Given an arbitrary global non-degenerate character χ of $\mathbf{N}(\mathbb{A}_k)$, we obtain a global character $\chi_{\tilde{w}_0}$ of $\mathbf{N}_M(\mathbb{A}_k) = \mathbf{M}(\mathbb{A}_k) \cap \mathbf{N}(\mathbb{A}_k)$ via (1.5) which is $\tilde{w}_0$-compatible with χ. We note that the discussion of Appendix A of [9] is valid also for the case of function fields. In particular, Lemma A.1 of [*loc. cit.*] combined with Proposition 5.4 of [55] give Proposition 3.1 below, which allows us to address the variance of the globally generic character.

Proposition 3.1. *Let $(k,\pi,\psi,S) \in \mathscr{L}_{\rm glob}(p)$. There exists a connected quasi-split reductive group $\widetilde{\mathbf{G}}$ defined over k, sharing the same derived group as $\mathbf{G}$, and with maximal torus $\widetilde{\mathbf{T}} = \mathbf{Z}_{\widetilde{\mathbf{G}}}\mathbf{T}$. Then, there exists an element $x \in \widetilde{T}$ such that the representation π_x, given by*

$$\pi_x(g) = \pi(x^{-1}gx)$$

is $\psi_{\tilde{w}_0}$-generic. Furthermore, we have equality of partial L-functions

$$L^S(s,\pi,r_i) = L^S(s,\pi_x,r_i).$$

3.3. **Eisenstein series.** We build upon the discussion of § 5 of [39], which is written for split groups. Let $\phi : \mathbf{M}(k)\backslash\mathbf{M}(\mathbb{A}_k) \to \mathbb{C}$ be an automorphic form on the space of a cuspidal automorphic representation π of $\mathbf{M}(\mathbb{A}_k)$. Then ϕ extends to an automorphic function $\Phi : \mathbf{M}(k)\mathbf{U}(\mathbb{A}_k)\backslash\mathbf{G}(\mathbb{A}_k) \to \mathbb{C}$ as in § I.2.17 of [44]. For every $s \in \mathbb{C}$, set

$$\Phi_s = \Phi \cdot q^{\langle s\tilde{\alpha}+\rho_{\mathbf{P}}, H_{\mathbf{P}}(\cdot)\rangle}.$$

The function Φ_s is a member of the globally induced representation of G given by the restricted direct product

$$\mathrm{I}(s,\pi) = \otimes' \mathrm{I}(s,\pi_v).$$

The irreducible constituents of $\mathrm{I}(s,\pi)$ are automorphic representations $\Pi = \otimes' \Pi_v$ of G such that the representation π_v has $\mathcal{K}_v$-fixed vectors for almost all v. The restricted tensor product is taken with respect to functions $f_{v,s}^0$ that are fixed under the action of $\mathcal{K}_v$.

We use the notation of Remark 1.2, where $w_0 = w_l w_{l,\mathbf{M}}$. We have the global intertwining operator

$$\mathrm{M}(s,\pi,\tilde{w}_0) : \mathrm{I}(s,\pi) \to \mathrm{I}(s',\pi'),$$

defined by

$$\mathrm{M}(s,\pi,\tilde{w}_0)f(g) = \int_{N'} f(\tilde{w}_0^{-1}ng)dn,$$

for $f \in \mathrm{I}(s,\pi)$. It decomposes into a product of local intertwining operators

$$\mathrm{M}(s,\pi,\tilde{w}_0) = \otimes_v \mathrm{A}(s,\pi_v,\tilde{w}_0),$$

which are precisely those appearing in the definition of the Langlands-Shahidi local coefficient.

The following crucial result is found in [16] for everywhere unramified representations of split groups, the argument is generalized in [44, 46] and includes all the cases at hand.

Theorem 3.2 (Harder)**.** *The Eisenstein series*

$$E(s,\Phi,g,\mathbf{P}) = \sum_{\gamma\in\mathbf{P}(k)\backslash\mathbf{G}(k)} \Phi_s(\gamma g)$$

converges absolutely for $\Re(s) \gg 0$ *and has a meromorphic continuation to a rational function on* q^{-s}*. Furthermore*

$$\mathrm{M}(s,\pi) = \otimes_v \mathrm{A}(s,\pi_v,\tilde{w}_0)$$

is a rational operator in the variable q^{-s}*.*

We also have that the Fourier coefficient of the Eisenstein series $E(s,\Phi,g)$ is given by

$$E_\psi(s,\Phi,g,\mathbf{P}) = \int_{\mathbf{U}(K)\backslash\mathbf{U}(\mathbb{A}_k)} E(s,\Phi,ug)\overline{\psi}(u)\,du.$$

The Fourier coefficients are also rational functions on q^{-s}.

3.4. **The crude functional equation.** We now turn towards the link between the Langlands-Shahidi local coefficient and automorphic L-functions.

Theorem 3.3. *Let* $(k,\pi,\psi,S) \in \mathscr{L}_{\mathrm{glob}}(p)$ *be* $\psi_{\tilde{w}_0}$*-generic. Then*

$$\prod_{i=1}^{m_r} L^S(is,\pi,r_i) = \prod_{v\in S} C_\psi(s,\pi_v,\tilde{w}_0) \prod_{i=1}^{m_r} L^S(1-is,\tilde{\pi},r_i).$$

Proof. Since π is globally ψ-generic, by definition, there is a cusp form φ in the space of π such that

$$W_{M,\varphi}(m) = \int_{\mathbf{U}_M(K)\backslash\mathbf{U}_M(\mathbb{A}_k)} \varphi(um)\overline{\psi}(u)\,du \neq 0.$$

The function Φ defined above is such that the Eisenstein series $E(s,\Phi,g,P)$ satisfies

$$E_\psi(s,\Phi_s,g,P) = \prod_v \lambda_{\psi_v}(s,\pi_v)(\mathrm{I}(s,\pi_v)(g_v)f_{s,v}), \tag{3.2}$$

with $f_s \in \mathrm{V}(s,\pi)$, $f_{s,v} = f_{s,v}^\circ$ for all $v \notin S$. Here $E_\psi(s,\Phi_s,g,P)$ denotes the Fourier coefficient

$$E_\psi(s,\Phi_s,g,P) = \int_{\mathbf{U}(K)\backslash\mathbf{U}(\mathbb{A}_k)} E(s,\Phi_s,ug,P)\overline{\psi}(u)\,du.$$

The global intertwining operator $\mathrm{M}(s,\pi)$ is defined by

$$\mathrm{M}(s,\pi,\tilde{w}_0)f(g) = \int_{\mathbf{N}'(\mathbb{A}_k)} f(\tilde{w}_0^{-1}ng)\,dn,$$

where $f \in \mathrm{V}(s,\pi)$ and $\mathbf{N}'$ is the unipotent radical of the standard parabolic $\mathbf{P}'$ with Levi $\mathbf{M}' = w_0 \mathbf{M} w_0^{-1}$. It is the product of local intertwining operators

$$\mathrm{M}(s,\pi,\tilde{w}_0) = \prod_v \mathrm{A}(s,\pi_v,\tilde{w}_0).$$

It is a meromorphic operator, which is rational on q^{-s} (Proposition IV.1.12 of [44]).

We set $s' = s\tilde{w}_0(\tilde{\alpha})$ globally as well as locally, and we use the conventions of Remark 1.2. Now equation (3.2) gives

$$\begin{aligned} &E_\psi(s', \mathrm{M}(s,\pi,\tilde{w}_0)\Phi_s, g, P') \\ &\quad = \prod_v \lambda_{\psi_v}(s', w_0(\pi_v))(\mathrm{I}(s',\pi_v')(g_v)\mathrm{A}(s,\pi_v,\tilde{w}_0)f_{s,v}). \end{aligned}$$

Fourier coefficients of Eisenstein series satisfy the functional equation:

$$E_\psi(s', \mathrm{M}(s,\pi,\tilde{w}_0)\Phi_s, g, P') = E_\psi(s,\Phi_s,g,P).$$

And, equation (3.2) gives

$$E_\psi(s,\Phi_s,e,P) = \prod_v \lambda_{\psi_v}(s,\pi_v) f_{s,v}$$

$$E_\psi(s', \mathrm{M}(s,\pi,\tilde{w}_0)\Phi_s, e, P') = \prod_v \lambda_{\psi_v}(s',\pi_v')\mathrm{A}(s,\pi_v,\tilde{w}_0)f_{s,v}.$$

Then, the Casselman-Shalika formula for unramified quasi-split groups, Theorem 5.4 of [6], allows one to compute the Whittaker functional when π_v is unramified:

$$\lambda_{\psi_v}(s,\pi_v)f_{s,v}^0 = \prod_{i=1}^{m_r} L(1+is,\pi_v,r_{i,v})^{-1} f_{s,v}^0(e_v).$$

Also, for $v \notin S$, the intertwining operator gives a function $\mathrm{A}(s,\pi_v,w_0)f_{s,v}^0 \in \mathrm{I}(-s, w_0(\pi_v))$ satisfying

$$\mathrm{A}(s,\pi_v,\tilde{w}_0)f_{s,v}^0(e_v) = \prod_{i=1}^{m_r} \frac{L(is,\pi_v,r_{i,v})}{L(1+is,\pi_v,r_{i,v})} f_{s,v}^0(e_v).$$

This equation is established by means of the multiplicative property of the intertwining operator, which reduces the problem to semisimple rank one cases.

Finally, combining the last five equations together gives

$$\prod_{i=1}^{m_r} L^S(is,\pi,r_i) = \prod_{v\in S} \frac{\lambda_{\psi_v}(s,\pi_v)f_{s,v}}{\lambda_{\psi_v}(s',\pi_v')\mathrm{A}(s,\pi_v,\tilde{w}_0)f_{s,v}} \prod_{i=1}^{m_r} L^S(1-is,\tilde{\pi},r_i).$$

For every $v \in S$, equation (1.7) gives a local coefficient. Thus, we obtain the crude functional equation. □

The following useful corollary is a direct consequence of the proof of the theorem. It provides the connection between Eisenstein series and partial L-functions for globally generic representations.

Corollary 3.4. *Let* $(k,\pi,\psi,S) \in \mathscr{L}_{\mathrm{glob}}(p)$*, then*

$$E_\psi(s,\Phi,g,\mathbf{P}) = \prod_{v\in S} \lambda_{\psi_v}(s,\pi_v)\left(\mathrm{I}(s,\pi_v)(g_v)f_{s,v}\right) \prod_{i=1}^{m_r} L^S(1+is,\pi,r_i)^{-1},$$

for $g \in G_S$.

4. The Langlands-Shahidi method over function fields

4.1. **Main theorem.** We now come to the main result of the Langlands-Shahidi method over function fields. The corresponding result over number fields can be found in Theorem 3.5 of [53]. We note that compatibility with Artin factors for real groups is the subject of [51].

Theorem 4.1. *Let* $\mathbf{G}$ *be a connected quasi-split reductive group and* $\mathbf{M} = \mathbf{M}_\theta$ *a maximal Levi subgroup. Let* $r = \oplus r_i$ *be the adjoint action of* ${}^L M$ *on* ${}^L\mathfrak{n}$*. There exists a system of rational* γ*-factors,* L*-functions and* ε*-factors on* $\mathscr{L}_{\rm loc}(p)$*. They are uniquely determined by the following properties:*

(i) (Naturality). *Let* $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p)$*. Let* $\eta : F' \to F$ *be an isomorphism of non-archimedean local fields and let* $(F', \pi', \psi') \in \mathscr{L}_{loc}(p)$ *be the triple obtained via* η*. Then*

$$\gamma(s, \pi, r_i, \psi) = \gamma(s, \pi', r_i, \psi').$$

(ii) (Isomorphism). *Let* $(F, \pi_j, \psi) \in \mathscr{L}_{\rm loc}(p)$*,* $j = 1, 2$*. If* $\pi_1 \cong \pi_2$*, then*

$$\gamma(s, \pi_1, r_i, \psi) = \gamma(s, \pi_2, r_i, \psi).$$

(iii) (Compatibility with Artin factors). *Let* $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p)$ *be such that* π *has an Iwahori fixed vector. Let* $\sigma : \mathcal{W}'_F \to {}^L M$ *be the Langlands parameter corresponding to* π*. Then*

$$\gamma(s, \pi, r_i, \psi) = \gamma(s, r_i \circ \sigma, \psi).$$

(iv) (Multiplicativity). *Let* $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p, \mathbf{M}, \mathbf{G})$*be such that*

$$\pi \hookrightarrow {\rm ind}_{P_{\theta_0}}^M(\pi_0),$$

where π_0 *is a generic representation of* M_{θ_0}*, with* $\theta_0 \subset \theta$*. Suppose* $(F, \pi_j, \psi) \in \mathscr{L}_{\rm loc}(p, \mathbf{M}_j, \mathbf{G}_j)$*, where the* π_j *are those of Proposition 2.3. With* Σ_i *as in* (2.9) *we have*

$$\gamma(s, \pi, r_i, \psi) = \prod_{j \in \Sigma_i} \gamma(s, \pi_j, r_{i,j}, \psi).$$

(v) (Dependence on ψ). *Let* $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p)$*. For* $a \in F^\times$*, let* $\psi^a : F \to \mathbb{C}^\times$ *be the character given by* $\psi^a(x) = \psi(ax)$*. Then, there is an* h_i *such that*

$$\gamma(s, \pi, r_i, \psi^a) = \omega_\pi(a)^{h_i} \, |a|_F^{n_i(s - \frac{1}{2})} \cdot \gamma(s, \pi, r_i, \psi),$$

where $n_i = \dim {}^L\mathfrak{n}_i$*.*

(vi) (Functional equation). *Let* $(k, \pi, \psi, S) \in \mathscr{L}_{\rm glob}(p)$*. Then*

$$L^S(s, \pi, r_i) = \prod_{v \in S} \gamma(s, \pi, r_i, \psi) L^S(s, \tilde{\pi}, r_i).$$

(vii) (Tempered L-functions). *For* $(F, \pi, \psi) \in \mathscr{L}_{loc}(p)$ *tempered, let* $P_{\pi, r_i}(t)$ *be the unique polynomial with* $P_{\pi, r_i}(0) = 1$ *and such that* $P_{\pi, r_i}(q_F^{-s})$ *is the numerator of* $\gamma(s, \pi, r_i, \psi)$*. Then*

$$L(s, \pi, r_i) = \frac{1}{P_{\pi, r_i}(q_F^{-s})}.$$

is holomorphic and non-zero for $\Re(s) > 0$*.*

(viii) (Tempered ε-factors). *Let* $(F,\pi,\psi)\in\mathscr{L}_{\rm loc}(p)$ *be tempered, then*

$$\varepsilon(s,\pi,r_i,\psi)=\gamma(s,\pi,r_i,\psi)\frac{L(s,\pi,r_i)}{L(1-s,\tilde{\pi},r_i)}$$

is a monomial in q_F^{-s}*.*

(ix) (Twists by unramified characters). *Let* $(F,\pi,\psi)\in\mathscr{L}_{\rm loc}(p)$*, then*

$$L(s+s_0,\pi,r_i)=L(s,q_F^{\langle s_0\tilde{\alpha},H_\theta(\cdot)\rangle}\otimes\pi,r_i),$$
$$\varepsilon(s+s_0,\pi,r_i,\psi)=\varepsilon(s,q_F^{\langle s_0\tilde{\alpha},H_\theta(\cdot)\rangle}\otimes\pi,r_i,\psi).$$

(x) (Langlands' classification). *Let* $(F,\pi,\psi)\in\mathscr{L}_{\rm loc}(p,\mathbf{M},\mathbf{G})$*. Let* π_0 *be a tempered generic representation of* $M_0=M_{\theta_0}$ *and* χ *a character of* $A_0=A_{\theta_0}$ *which is in the Langlands' situation. Suppose* π *is the Langlands' quotient of the representation*

$$\xi=\mathrm{Ind}(\pi_{0,\chi}),$$

with $\pi_{0,\chi}=\pi_0\cdot\chi$*. Suppose* $(F,\pi_j,\psi)\in\mathscr{L}_{\rm loc}(p,\mathbf{M}_j,\mathbf{G}_j)$ *are quasi-tempered, where the* π_j *are obtained via Langlands' lemma and equation* (2.9)*. Then*

$$L(s,\pi,r_i)=\prod_{j\in\Sigma_i}L(s,\pi_j,r_{i,j}),$$
$$\varepsilon(s,\pi,r_i,\psi)=\prod_{j\in\Sigma_i}\varepsilon(s,\pi_j,r_{i,j},\psi).$$

4.2. **A local to global result and induction.** Lemma 4.2 below is the quasi-split reductive groups generalization of the local to global result of Henniart-Lomelí for GL_n over function fields [21]. A more general globalization theorem in characteristic p is now proved in collaboration with Gan [14]. Thanks to a mathematical discussion with Gan on the minimum number of places required, a simplified proof is included here for self-containment; Lemma 4.2 includes all of the cases at hand. This result can be seen as the function fields counterpart of Shahidi's number fields Propostion 5.1 of [53], which is in turn a subtle refinement of a result of Henniart and Vignéras [18, 62].

Lemma 4.2 (Henniart-Lomelí)**.** *Let* π_0 *be a supercuspidal unitary representation of* $\mathbf{G}(F)$*. There is a global function field* k *with a set of two places* $S=\{v_0,v_\infty\}$ *such that* $k_{v_0}\cong F$*. There exists a cuspidal automorphic representation* $\pi=\otimes_v\pi_v$ *of* $\mathbf{G}(\mathbb{A}_k)$ *satisfying the following properties:*

(i) $\pi_{v_0}\cong\pi_0$*;*
(ii) π_v *has an Iwahori fixed vector for* $v\notin S$*;*
(iii) π_{v_∞} *is a constituent of a tamely ramified principal series;*
(iv) *if* π_0 *is generic, then* π *is globally generic.*

Proof. We construct a function $f=\otimes_v f_v$ on $\mathbf{G}(\mathbb{A})$ in such a way that the Poincaré series

$$\mathrm{P}f(g)=\sum_{\gamma\in\mathbf{G}(k)}f(\gamma g)$$

is a cuspidal automorphic function.

Let $\mathbf{Z}$ be the center of $\mathbf{G}$. Note that the central character ω_π is unitary. It is possible to construct a global unitary character $\omega:\mathbf{Z}(k)\backslash\mathbf{Z}(\mathbb{A}_k)\to\mathbb{C}^\times$ such that: $\omega_{v_0}=\omega_{\pi_0}$; ω_{v_∞} is trivial on $Z^1_{v_\infty}$; and, ω_v is trivial on Z^0_v for every $v\notin\{v_0,v_\infty\}$.

Here, $Z^1_{v_\infty}$ is the pro-p unipotent radical mod $\mathfrak{p}_\infty$ of Z_{v_∞} and Z^0_v is the maximal compact open subgroup of Z_v.

We let $S' = \{v_0, v', v_\infty\}$, where v' is an auxiliary place of k. Outside of S' we consider the characteristic functions

$$f_v = \mathbb{1}_{\mathcal{K}_v},\ v \notin S'.$$

At the place v_0 of k, we let

$$f_{v_0}(g) = \langle \pi_0(g)x, y\rangle\ ,\ x, y \in V_{\pi_0},$$

be a matrix coefficient of π_0, where V_{π_0} is the space of π_0. The function f_{v_0} has compact support $\mathcal{C}_{v_0}$ modulo Z_{v_0}.

At the place v', consider the Iwahori subgroup $\mathcal{I}_{v'}$ of upper triangular matrices mod $\mathfrak{p}_{v'}$. At the place at infinity, we let $\mathcal{I}^1_{v_\infty}$ be the pro-p Iwahori subgroup of unipotent lower triangular matrices mod $\mathfrak{p}_{v_\infty}$. Let

$$f_{v'} = \mathbb{1}_{\mathcal{I}_{v'}} \text{ and } f_{v_\infty} = \mathbb{1}_{\mathcal{I}^1_{v_\infty}}.$$

Twist the global function $f = \otimes f_v$ by ω^{-1}, in order to be able to mod out by the center. Let $\mathbf{H} = \mathbf{G}/\mathbf{Z}$ and set

$$\mathcal{C} = \mathcal{C}_{v_0} \times \prod_{v \notin S'} \mathcal{K}_v \times \mathcal{I}_{v'} \times \mathcal{I}^1_{v_\infty}.$$

The projection $\mathcal{C}'$ of $\mathcal{C}$ on $\mathbf{H}(\mathbb{A}_k)$ is compact. Hence, $\mathcal{C}' \cap \mathbf{H}(k)$ is finite. Because of this, there exists a constant h, such that the height $\|g\| \leq h$ bounds the entries of $g = (g_{ij}) \in \mathbf{H}(k) \cap \mathcal{C}'$. See § I.2.2 of [44] for the height functions $\|g\|$ and $\|g\|_v$ for elements of $\mathbf{G}(\mathbb{A}_k)$ and G_v, respectively.

We now impose the further conditions on the choice of places v' and v_∞. These two places need to be such that the cardinality of their respective residue fields is larger than h. This is to ensure that the poles of the entries of all $g \in \mathbf{H}(k) \cap \mathcal{C}'$ are absorbed. Indeed, given a connected smooth projective curve X over $\mathbb{F}_q$, its set of points over the algebraic closure $\overline{\mathbb{F}}_q$ is infinite. Hence, it is always possible to find places v with residue field $\mathbb{F}_{q_v}$, with $q_v > h$.

We then have that

$$g_{ij} \in \mathcal{O}_{v'},\ i \leq j,$$

and the congruence relations

$$g_{ij} \equiv 0 \bmod \mathfrak{p}_{v'},\ i > j.$$

By incorporating the condition at infinity, we obtain similar relationships corresponding to the use of lower triangular matrices at v_∞. We can see that $\mathcal{C}' \cap \mathbf{H}(k) \in \mathbf{H}(\mathbb{F}_q)$. In fact, the pro-$p$ condition at v_∞ ensures that $\mathcal{C}' \cap \mathbf{H}(k) = \{I_n\}$. Now, we incorporate the twist by ω, and lift the function $f = \otimes f_v$ back to one of $\mathbf{G}(\mathbb{A}_k)$. We then have that

$$\mathrm{P}(f)(g) = f(g),\ \text{for } g \in \mathcal{C}.$$

We can now proceed as in p. 4033 of [21] to conclude that $\mathrm{P}f$ belongs to the space $L^2_0(\mathbf{G}, \omega)$ of cuspidal automorphic functions on $\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}_k)$ transforming via ω under $\mathbf{Z}(\mathbb{A}_k)$. The resulting cuspidal automorphic representation π of $\mathbf{G}(\mathbb{A}_k)$ is such that: $\pi_{v_0} \cong \pi_0$; $\pi_{v'}$ has a non-zero fixed vector under $\mathcal{I}_{v'}$; and, π_{v_∞} has a non-zero fixed vector under $\mathcal{I}^1_{v_\infty}$. The last three paragraphs of the proof of Theorem 3.1 of [*loc. cit.*] are general and can be used to establish property (iii) of our theorem.

Let $\psi : k\backslash\mathbb{A}_k \to \mathbb{C}^\times$ with ψ_v be an additive character which is unramified at every place. We extend ψ to a character of $\mathbf{U}(k)\backslash\mathbf{U}(\mathbb{A}_k)$ via equation (1.1). If π is ψ_{v_0}-generic, we can proceed as in Theorem 2.2 of [62] to show that

$$W_\psi(f)(g) = \int_{\mathbf{U}(k)\backslash\mathbf{U}(\mathbb{A}_k)} f(ng)\overline{\psi}(n)\,dn \neq 0.$$

Hence, the Whittaker model is globalized in this construction.

□

Remark 4.3. *The above proposition admits a modification. Property* (ii) *can be replaced by:* (ii)' π_v *is spherical for* $v \notin S$*; and, condition* (iii) *replaced by:* (iii)' π_{v_∞} *is a level zero supercuspidal representation in the sense of Morris* [47]*. If we begin with a level zero supercuspidal* π_0*, the above globalization produces a cuspidal automorphic representation* π *such that* π_v *arises from an unramified principal series at every* $v \neq v_0$*.*

The next result is present in Arthur's work, using endoscopy groups, in addition to Shahidi [53]. In particular, the discussion in § 4 of [*loc. cit.*] can be adapted to our situation. We here present a straight forward proof of the induction step.

Lemma 4.4. *Let* $(\mathbf{M}, \mathbf{G})$ *be a pair consisting of a connected quasi-split reductive group and* $\mathbf{M} = \mathbf{M}_\theta$ *a maximal Levi subgroup. Let* $r = \oplus_{i=1}^{m_r} r_i$ *be the adjoint action of* ${}^LM$ *on* ${}^L\mathfrak{n}$*. For each* $i > 1$*, there exists a pair* $(\mathbf{M}_i, \mathbf{G}_i)$ *such that the corresponding adjoint action of* ${}^LM_i$ *on* ${}^L\mathfrak{n}_i$ *decomposes as*

$$r' = \bigoplus_{j=1}^{m'_r} r'_j \text{ with } m'_r < m_r,$$

and $r_i = r'_1$*.*

Proof. To construct the pair $(\mathbf{M}_i, \mathbf{G}_i)$, we begin by taking $\mathbf{M}_i = \mathbf{M}$. For the group $\mathbf{G}_i$, we look at the unipotent subgroup

$$\mathbf{N}'_{w_0} = \prod_{l \in \Sigma'} \mathbf{N}_{0,l},$$

obtained from a subproduct of equaiton (2.8), where Σ' is the resulting indexing set. Each one unipotent group $\mathbf{N}_{0,l}$ corresponds to a block set of roots $[\beta] \in \Sigma_r^+(\theta, w_0)$. We take only groups $\mathbf{N}_{0,l}$ in the product corresponding to a $[\beta] \in \Sigma_r^+(\theta, w_0)$ such that

$$\langle \tilde{\alpha}, \beta \rangle \geq i, \ \beta \in [\beta].$$

We then set

$$\mathbf{N}_i = w_0^{-1}\mathbf{N}_{w_0}w_0.$$

For the group $\mathbf{G}_i$ we take the reductive group generated by $\mathbf{M}_i$, $\mathbf{N}_i$ and $\mathbf{N}_i^-$. The pair $(\mathbf{M}_i, \mathbf{G}_i)$ has

$$r' = \bigoplus_{j=i}^{m_r} r'_j.$$

After rearranging, we obtain the form of the proposition. □

4.3. **Proof of Theorem 4.1.** The crude functional equation of Theorem 3.3, together with Proposition 2.4, points us towards the existence part of Theorem 4.1 concerning γ-factors. Indeed, let $(F,\pi,\psi) \in \mathscr{L}_{\rm loc}(p)$ be such that π is ψ_{w_0}-generic. Then we recursively define γ-factors via the equation

$$C_\psi(s,\pi,\tilde{w}_0) = \prod_{i=1}^{m_r} \lambda_i(\psi,w_0)^{-1}\gamma(is,\pi,r_i,\psi) \tag{4.1}$$

and Lemma 4.4. For arbitrary $(F,\pi,\psi) \in \mathscr{L}_{\rm loc}(p)$, they are defined with the aid of Proposition 2.2. From Theorem 2.1 of [39], it follows that $\gamma(s,\pi,r_i,\psi) \in \mathbb{C}(q_F^{-s})$.

With the above definition of γ-factors based on equation (4.1), Properties (i) and (ii) can be readily verified. The inductive argument on the adjoint action together with Proposition 2.4 give Property (iii). Multiplicativity of the local coefficient, Proposition 2.3, leads towards Property (iv).

For Property (v), given the two characters ψ and ψ^a, $a \in F^\times$, we use Proposition 2.2 to examine the variation of π from being $\psi^a_{\tilde{w}_0}$-generic to π_x which is $\psi_{\tilde{w}_0}$-generic for a suitable x. For this we let $x \in \mathbf{T}(F_s)$ be as in equation (2.4) and such that $d \in Z_M$. In fact, d is identified with a power of a. In this case we have

$$C_\psi(s,\pi,\tilde{w}_0) = a_x(\psi^a,\mathfrak{W})C_{\psi^a}(s,\pi_x,\tilde{w}_0),$$

where $a_x(\psi^a,\mathfrak{W}) = \omega_\pi(a)^h \left\| a \right\|_F^{n(s-\frac{1}{2})}$. The recursive definition of γ-factors, allows us to obtain integers h_i and n_i for each $1 \leq i \leq m_r$. Holomorphy of tempered L-functions is proved in [17], the discussion there is valid in characteristic p.

Furthermore, we obtain individual functional equations for each of the γ-factors. Namely, this reasoning proves Property (vi) for $(k,\pi,\psi,S) \in \mathscr{L}_{\rm glob}(p)$ with $\psi_{\tilde{w}_0}$-generic π. Note we define γ-factors in a way that they are compatible with the functional equation for a χ-generic π with the help of Propositions 2.2 and 3.1.

Before continuing, we record the following important property of γ-factors:

(xi) (Local functional equation) *Let* $(F,\pi_0,\psi_0) \in \mathscr{L}_{\rm loc}(p)$, *then*

$$\gamma(s,\pi_0,r_i,\psi_0)\gamma(1-s,\tilde{\pi}_0,r_i,\overline{\psi}_0) = 1.$$

To prove this property, we start with a local triple $(F,\pi_0,\psi_0) \in \mathscr{L}_{\rm loc}(p)$. Proposition 4.1 allows us to globalize π_0 into a globally generic representation $\pi = \otimes_v \pi_v$, with $\pi_{v_0} \cong \pi_0$ and π_v unramified outside a finite set of places S of k. We take a global character $\psi : k\backslash\mathbb{A}_k \to \mathbb{C}^\times$. The character ψ_0, using Property (v) if necessary, can be assumed to be ψ_{v_0}. Applying Property (vi) twice to $(k,\pi,\psi,S) \in \mathscr{L}_{\rm glob}(p)$, we obtain

$$\prod_{v\in S} \gamma(s,\pi_v,r_{i,v},\psi_v)\gamma(1-s,\tilde{\pi}_v,r_{i,v},\overline{\psi}_v) = 1. \tag{4.2}$$

For each $v \in S - \{v_0,v_\infty\}$, the representation π_v is unramified and we have the local functional equation for the corresponding Artin γ-factors. At the place v_∞, we still obtain a generic constituent of a tamely ramified principal series

$$\pi_{v_\infty} \hookrightarrow {\rm Ind}(\chi_\infty),$$

with χ_∞ a tamely ramified character of $\mathbf{T}(k_{v_\infty})$. Property (iv) gives

$$\gamma(s,\pi_{v_\infty},r_{i,v},\psi_{v_\infty}) = \prod_{j\in\Sigma_i} \gamma(s,\pi_{j,v_\infty},r_{i,j,v_\infty},\psi_{v_\infty}). \tag{4.3}$$

Each γ-factor on the right hand side of the product is then obtained from (i) or (ii) of Proposition 1.3. The resulting abelian γ-factors are known to satisfy a functional equation as in Tate's thesis. Hence, we also have the local functional equation for v_∞. From the product of (4.2) we can thus conclude Property (xi) at the place v_0 as desired.

Property (viii), sates the relation connecting Langlands-Shahidi local factors. We show that ε-factors are well defined for tempered representations. Let $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p)$ be tempered. Let $P_{\pi,r_i}(z)$ and $P_{\tilde{\pi},r_i}(z)$ the polynomials of Property (vii) with $z = q_F^{-s}$ and write

$$\gamma(s, \pi, r_i, \psi) = e_{1,\psi}(z)\frac{P_{\pi,r_i}(z)}{Q_{\pi,r_i}(z)} \text{ and } \gamma(s, \tilde{\pi}, r_i, \psi) = e_{2,\psi}(z)\frac{P_{\tilde{\pi},r_i}(z)}{Q_{\tilde{\pi},r_i}(z)},$$

where $e_{1,\psi}(z)$ and $e_{2,\psi}(z)$ are monomials in z. From Property (xi), we have

$$Q_{\pi,r_i}(z)Q_{\tilde{\pi},r_i}(q_F^{-1}z^{-1}) = e_{1,\psi}(z)e_{2,\overline{\psi}}(q_F^{-1}z^{-1})P_{\pi,r_i}(z)P_{\tilde{\pi},r_i}(q_F^{-1}z^{-1}).$$

Property (viii) implies that the Laurent polynomials $P_{\pi,r_i}(z)$ and $P_{\tilde{\pi},r_i}(q_F z^{-1})$ have no zeros on $\Re(s) < 0$ and $\Re(s) > 1$, respectively. Then, up to a monomial in $z^{\pm 1}$, we have $L(s, \pi, r_i) = Q_{\tilde{\pi},r_i}(q_F z^{-1})^{-1}$ and $L(1 - s, \tilde{\pi}, r_i) = Q_{\pi,r_i}(z)^{-1}$. Hence, $\varepsilon(s, \pi, r_i, \psi)$ is a monomial in q_F^{-s}.

Property (ix) follows from the definitions for $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p)$ tempered, and the fact that

$$\mathrm{I}(s + s_0, \pi) = \mathrm{ind}_{P_\theta}^G(q_F^{\langle s\tilde{\alpha}+s_0\tilde{\alpha}, H_M(\cdot)\rangle} \otimes \pi).$$

To proceed to the general $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p)$, we use Langlands' classification. More precisely, π is a representation of M_θ and we let $\theta_0 \subset \theta$. Let π_0 be a tempered generic representation of $M_0 = M_{\theta_0}$ and χ a character of $A_0 = A_{\theta_0}$ which is in the Langlands' situation [4, 57]. Then π is the Langlands' quotient of the representation

$$\xi = \mathrm{Ind}(\pi_{0,\chi}),$$

with $\pi_{0,\chi} = \pi_0 \cdot \chi$.

From Proposition 2.3 we obtain $(F, \pi_j, \psi) \in \mathscr{L}_{\rm loc}(p, \mathbf{M}_j, \mathbf{G}_j)$. Each π_j is quasi-tempered. Property (ix) allows us to define $L(s, \pi_j, r_{i,j})$ and $\varepsilon(s, \pi_j, r_{i,j}, \psi)$. We then let

$$L(s, \pi, r_i) = \prod_{j \in \Sigma_i} L(s, \pi_j, r_{i,j}),$$

$$\varepsilon(s, \pi, r_i, \psi) = \prod_{j \in \Sigma_i} \varepsilon(s, \pi_j, r_{i,j}, \psi)$$

be the definition of L-functions and root numbers. This concludes the existence part of Theorem 4.1.

For uniqueness, we start with a local triple $(F, \pi_0, \psi_0) \in \mathscr{L}_{\rm loc}(p)$. We globalize π_0 into a globally generic representation $\pi = \otimes_v \pi_v$ via Proposition 4.1. We have a global character $\psi = \otimes \psi_v$, where by Property (v) if necessary, we can assume $\psi_{v_0} = \psi_0$. Notice that partial L-functions are uniquely determined. Hence, the functional equations gives a uniquely determined product

$$\prod_{v \in S} \gamma(s, \pi_v, r_{i,v}, \psi_v).$$

For each $v \in S - \{v_0, v_\infty\}$ we can use Proposition 2.4. At v_∞ equation (4.3) above reduces $\gamma(s, \pi_{v_\infty}, r_{i,v_\infty}, \psi_{v_\infty})$ to a product of uniquely determined abelian γ-factors.

Tempered L-functions and ε-factors are uniquely determined by Properties (vii) and (viii). Then in general by Properties (ix) and (x). □

4.4. **Functional equation.** Given $(k, \pi, \psi, S) \in \mathscr{L}_{\rm glob}(p)$, we define

$$L(s, \pi, r_i) = \prod_v L(s, \pi_v, r_{i,v}) \text{ and } \varepsilon(s, \pi, r_i) = \prod_v \varepsilon(s, \pi_v, r_{i,v}, \psi_v).$$

The global functional equation for Langlands-Shahidi L-functions is now a direct consequence of the existence of a system of γ-factors, L-functions and ε-factors together with Property (vi) of Theorem 4.1.

(xii) (Global functional equation) *Let* $(k, \pi, \psi, S) \in \mathscr{L}_{\rm glob}(p)$, *then*

$$L(s, \pi, r_i) = \varepsilon(s, \pi, r_i) L(1 - s, \tilde{\pi}, r_i).$$

5. Properties and applications of L-functions

5.1. **Rationality of Langlands-Shahidi L-functions.** The connection between Eisenstein series and the Langlands-Shahidi local coefficient allows us to prove the following property over function fields.

Proposition 5.1. *Let* $(k, \pi, \psi, S) \in \mathscr{L}_{\rm glob}(p)$. *Then each L-function* $L(s, \pi, r_i)$ *converges absolutely for* $\Re(s) \gg 0$ *and has a meromorphic continuation to a rational function in* q^{-s}.

Proof. Given $(k, \pi, \psi, S) \in \mathscr{L}_{\rm glob}(p)$, from Propostion 3.1, we can assume that π is globally ψ-generic. From equation (3.2), we have the connection between the global Whittaker model and the local ones. Corollary 3.4 further gives the connection with partial L-functions, where each of the local Whittaker functionals $\lambda_{\psi_v}(s, \pi_v)(\mathrm{I}(s, \pi_v)(g_v) f_{s,v})$ are polynomials in $\{q_v^s, q_v^{-s}\}$, where $q_v = q^{\deg v}$ (see Theorem 1.4 of [39]). Now, the Fourier coefficients $E_\psi(s, \Phi, g, \mathbf{P})$ are rational on q^{-s}. Also, recall that each $L^S(s, \pi, r_i)$ is absolutely convergent for $\Re(s) \gg 0$, by Theorem 13.2 of [3]. Then, we can conclude that the product

$$\prod_{i=1}^{m_r} L^S(1 + is, \pi, r_i)$$

extends to a rational function on q^{-s}. The induction step of Lemma 4.4 allows us to isolate each L-function to conclude that each

$$L^S(1 + is, \pi, r_i)$$

is rational in the variable q^{-s}. Hence, each $L^S(s, \pi, r_i)$ is rational. Furthermore, Theorem 4.1 gives that locally each L-function

$$L(s, \pi_v, r_{i,v}), \text{ for } (k_v, \pi_v, \psi_v) \in \mathscr{L}_{\rm loc}(p),$$

is a rational function on q_v^{-s}. Hence, the completed L-function

$$L(s, \pi, r_i) = \prod_{v \in S} L(s, \pi_v, r_{i,v}) L^S(s, \pi, r_i)$$

meromorphically continues to a rational function in the variable q^{-s}. □

5.2. **Intertwining operators.** We have the following connection between the intertwining operator and Langlands-Shahidi partial L-functions. Let $(k,\pi,\psi,S) \in \mathscr{L}_{\mathrm{glob}}(p)$, then

$$\mathrm{M}(s,\pi,\tilde{w}_0) = \prod_{i=1}^{m_r} \frac{L^S(is,\pi,r_i)}{L^S(1+is,\pi,r_i)} \bigotimes_{v\in S} \mathrm{A}(s,\pi_v,\tilde{w}_0). \tag{5.1}$$

The following Lemma is possible by looking into the spectral theory of Eisenstein series available over function fields.

Lemma 5.2. *Let $(k,\pi,\psi,S) \in \mathscr{L}_{\mathrm{glob}}(p)$. If $\tilde{w}_0(\pi) \not\cong \pi$, then $\mathrm{M}(s,\pi,\tilde{w}_0)$ and $E(s,\Phi,g,\mathbf{P})$ are holomorphic for $\Re(s) \geq 0$.*

Proof. Let Φ be the automorphic functions of § 1.2 that we obtain from π. We have the pseudo-Eisenstein series

$$\theta_\Phi = \int_{\Re(s)=s_0} E(s,\Phi,g,\mathbf{P})\,ds.$$

This is first defined for $s_0 > \langle \rho_{\mathbf{P}}, \alpha^\vee \rangle$ by Proposition II.1.6 (iii) and (iv) of [44]. Then II.2.1 Théorème of [*loc. cit.*] gives for self-associate $\mathbf{P}$ that

$$\langle \theta_\Phi, \theta_\Phi \rangle = \int_{\Re(s)=s_0} \sum_{\tilde{w}\in\{1,\tilde{w}_0\}} \left\langle \mathrm{M}(s,\pi,\tilde{w})\Phi_{-\tilde{w}(\bar{s}\tilde{\alpha})}, \Phi_s \right\rangle ds,$$

and is zero if $\mathbf{P}$ is not self-associate. Now, for $\mathbf{P}$ self-associate, the condition $\tilde{w}_0(\pi) \not\cong \pi$ allows us to shift the imaginary axis of integration by V.3.8 Lemme of [44] to $\Re(s) = 0$. However, by IV.1.11 Proposition we have that $M(s,\pi,\tilde{w}_0)$ is holomorphic for $\Re(s) = 0$. Thus, we have that $M(s,\pi,\tilde{w}_0)$ is holomorphic for $\Re(s) > 0$. Finally, the poles of Eisenstein series are contained by the constant terms. □

5.3. **An assumption of Kim.** Locally, let $(F,\pi,\psi) \in \mathscr{L}_{\mathrm{loc}}(p)$. We have the normalized intertwining operator

$$\mathrm{N}(s,\pi,\tilde{w}_0) : \mathrm{I}(s,\pi) \to \mathrm{I}(s',\pi')$$

defined by

$$\mathrm{N}(s,\pi,\tilde{w}_0) = \prod_{i=1}^{m_r} \varepsilon(is,\tilde{\pi},r_i,\psi) \frac{L(1+is,\tilde{\pi},r_i)}{L(is,\tilde{\pi},r_i)} \mathrm{A}(s,\pi,\tilde{w}_0).$$

Globally, for $(k,\pi,\psi,S) \in \mathscr{L}_{\mathrm{glob}}(p)$, we have

$$\mathrm{N}(s,\pi,\tilde{w}_0) = \otimes_v \mathrm{N}(s,\pi_v,\tilde{w}_0).$$

The following assumption was made by H.H. Kim in his study of local Langlands-Shahidi L-functions and normalized intertwining operators over number fields [25]. Having now the Langlands-Shahidi method available in positive characteristic, we are lead at this point to Kim's assumption.

Assumption 5.3. *Let $(k,\pi,\psi,S) \in \mathscr{L}_{\mathrm{glob}}(p)$. At every place v of k, the normalized intertwining operator $\mathrm{N}(s,\pi_v,\tilde{w}_0)$ is holomorphic and non-zero for $\Re(s) \geq 1/2$.*

It is already known to hold in many cases, including all the quasi-split classical groups. See [27] for a more detailed account. We make this assumption to prove Proposition 5.5 below.

Lemma 5.4. *Let $(k, \pi, \psi, S) \in \mathscr{L}_{\mathrm{glob}}(p)$. If $\tilde{w}_0(\pi) \not\cong \pi$, then*
$$\prod_{i=1}^{m_r} \frac{L(is, \pi, r_i)}{L(1 + is, \pi, r_i)}$$
is holomorphic for $\Re(s) \geq 1/2$.

Proof. For every place $v \notin S$, we have that
$$\mathrm{A}(s, \pi_v, \tilde{w}_0) f_{s,v}^0(e_v) = \prod_{i=1}^{m_r} \frac{L(is, \pi_v, r_i)}{L(1 + is, \pi_v, r_i)} f_{s,v}^0(e_v),$$
for $(k_v, \pi_v, \psi_v) \in \mathscr{L}_{\mathrm{loc}}(p)$. Globally, the condition $\tilde{w}_0(\pi) \not\cong \pi$ allows us to apply Lemma 5.2 to $(k, \pi, \psi, S) \in \mathscr{L}_{\mathrm{glob}}(p)$. Thus $\mathrm{M}(s, \pi, \tilde{w}_0)$ is holomorphic for $\Re(s) \geq 0$, and in this region we have
$$\mathbf{M}(s, \pi, \tilde{w}_0) f_s = \prod_{i=1}^{m_r} \varepsilon(is, \tilde{\pi}, r_i)^{-1} \frac{L(is, \tilde{\pi}, r_i)}{L(1 + is, \tilde{\pi}, r_i)} \otimes_v \mathrm{N}(s, \pi_v, \tilde{w}_0) f_{s,v}^0,$$
where $f_s = \otimes_v f_{s,v}^0 \in \mathrm{I}(s, \pi)$. Now, with an application of Assumption 5.3, we can prove the Lemma. □

5.4. **Global twists by characters.** We now extend a number fields result of Kim-Shahidi to the case of positive characteristic. In particular, Proposition 2.1 of [30] shows that, up to suitable global twists, Langlands-Shahidi L-functions become holomorphic. In the case of function fields, Harder's rationality result allows us to prove the stronger property of L-functions becoming polynomials after suitable twists.

Let $\xi \in X^*(\mathbf{M})$ be the rational character defined by
$$\xi(m) = \det \left(\mathrm{Ad}(m) | \mathfrak{n} \right),$$
where $\mathfrak{n}$ is the Lie algebra of $\mathbf{N}$. At every place v of k, we obtain a rational character $\xi_v \in X^*(\mathbf{M})_v$. Given a grössencharakter $\chi = \otimes \chi_v : k^\times \backslash \mathbb{A}_k \to \mathbb{C}^\times$, we obtain the character
$$\chi \cdot \xi = \otimes \chi_v \cdot \xi_v$$
of $\mathbf{M}(\mathbb{A}_k)$. Let $(k, \pi, \psi, S) \in \mathscr{L}_{\mathrm{glob}}(p)$. For $n \in \mathbb{Z}_{\geq 0}$, form the automorphic representation
$$\pi_{n,\chi} = \pi \otimes (\chi \cdot \xi^n).$$

Proposition 5.5. *Let $(k, \pi, \psi, S) \in \mathscr{L}_{\mathrm{glob}}(p)$ and fix a place $v_0 \in S$. Then there exist non-negative integers n and f_{v_0} such that for every grössencharakter $\chi = \otimes \chi_v$ with $\mathrm{cond}(\chi_{v_0}) \geq f_{v_0}$, we have that*
$$L\left(s, \pi_{n,\chi}, r_i\right),$$
$1 \leq i \leq m_r$, is a polynomial function on $\{q^s, q^{-s}\}$.

Proof. In order to apply Lemmas 5.2 and 5.4, we first need the existence of an integer n such that $\tilde{w}_0(\pi_{n,\chi}) \not\cong \pi_{n,\chi}$, for suitable χ. Write $\mathbf{P} = \mathbf{P}_\theta$, with $\theta = \Delta - \{\alpha\}$, so that $\mathbf{M} = \mathbf{M}_\theta$ and $\mathbf{N} = \mathbf{N}_\theta$. Let $\mathbf{A}_\theta$ be the split torus of $\mathbf{M}_\theta$ and let $A_\theta = \mathbf{A}_\theta(k_{v_0})$, the group of k_{v_0}-rational points. Let
$$A_\theta^1 = \tilde{w}_0(A_\theta) A_\theta^{-1} = \left\{ a \in A_\theta | \tilde{w}_0(a) = a^{-1} \right\}.$$
Then Lemma 2 of [54] gives
$$\xi|_{A_\theta^1} \neq 1.$$

In fact, $\xi : \mathbf{A}^1_\theta \to \mathbf{G}_m$ is onto. Given an integer n, we choose $\ell_{v_0} \in \mathbb{Z}_{\geq 0}$ such that

$$1 + \mathfrak{p}_{v_0}^{\ell_{v_0}} \subset \xi^n(A^1_\theta)$$

and ℓ_{v_0} is minimal with this property. Let ω_{v_0} be the central character of π_{v_0}. Take

$$f_{v_0} \geq \max\left\{\ell_{v_0}, \mathrm{cond}(\omega_{v_0})\right\}, \tag{5.2}$$

which depends on n. Then

$$\tilde{w}_0(\omega_{v_0} \otimes (\chi_{v_0} \cdot \xi^n)) \ncong \omega_{v_0} \otimes (\chi_{v_0} \cdot \xi^n), \tag{5.3}$$

for $\mathrm{cond}(\chi_{v_0}) \geq f_{v_0}$.

Let $(\mathbf{M}_i, \mathbf{G}_i)$ be as in Lemma 4.4, where $\mathbf{G}_i \hookrightarrow \mathbf{G}$ and we have the corresponding parabolic $\mathbf{P}_i = \mathbf{M}_i\mathbf{N}_i$ of $\mathbf{G}_i$. Let $\xi_i \in X^*(\mathbf{M}_i)$ be the rational character

$$\xi_i(m) = \det\left(\mathrm{Ad}(m)|\mathfrak{n}_i\right),$$

where $\mathfrak{n}_i$ is the Lie algebra of $\mathbf{N}_i$. There are then integers $n_1, \ldots, n_{m_r}$, such that upon restriction to $\mathbf{M}_i$ we have $\xi_i^{n_i} = \xi^{n_1}$ and $\chi \cdot \xi_i^{n_i} = \chi \cdot \xi^{n_1}$. With this choice of $n = n_1$, we choose f_{v_0} as in (5.5) at the place $v_0 \in S$. Then, for any grössencharakter $\chi : k^\times \backslash \mathbb{A}_k^\times \to \mathbb{C}^\times$ with $\mathrm{cond}(\chi_{v_0}) \geq f_{v_0}$. Equation (5.3) at v_0 ensures that

$$\tilde{w}_0(\pi_{n,\chi}) \ncong \pi_{n,\chi}.$$

Now, Lemma 5.2 together with Corollary 3.4 give that

$$\prod_{i=1}^{m_r} L^S(1 + is, \pi_{n,\chi}, r_i)$$

is holomorphic and non-zero for $\Re(s) \geq 0$. The induction step found in § 4.2 allows us to isolate each L-function and conclude that each

$$L^S(s, \pi_{n,\chi}, r_i)$$

is holomorphic and non-zero on $\Re(s) \geq 1$. Furthermore, with the aid of Lemma 5.4 we conclude that each $L^S(s, \pi_{n,\chi}, r_i)$ is holomorphic on $\Re(s) \geq 1/2$.

The functional equation of Theorem 4.1, allows us to conclude holomorphy for $\Re(s) \leq 1/2$. The automorphic L-functions $L(s, \pi_{n,\chi}, r_i)$ being now entire, in addition to being rational by Proposition 5.1, must be polynomials on $\{q^{-s}, q^s\}$. □

The following corollary is obtained from the proof of the Proposition.

Corollary 5.6. *Let $(k, \pi, \psi, S) \in \mathscr{L}_{\mathrm{glob}}(p)$ be such that $\tilde{w}_0(\pi) \ncong \pi$. Then $L(s, \pi, r_i)$, $1 \leq i \leq m_r$, is a polynomial on $\{q^{-s}, q^s\}$.*

5.5. **Local reducibility properties.** The following result is an immediate consequence of having a sound theory of local L-functions in characteristic p. It is Corollary 7.6 of [53].

Proposition 5.7. *Let $(F, \pi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p)$ be supercuspidal. If $i > 2$, then $L(s, \pi, r_i) = 1$. Also, the following are equivalent:*

(i) *The product $L(s, \pi, r_1)L(2s, \pi, r_2)$ has a pole at $s = 0$.*
(ii) *For one and only one $i = 1$ or $i = 2$, the L-function $L(s, \pi, r_i)$ has a pole at $s = 0$.*
(iii) *The representation $\mathrm{Ind}(\pi)$ is irreducible and $w_0(\pi) \cong \pi$.*

Also a consequence of the Langlands-Shahidi theory of L-functions is Shahidi's result on complimentary series. Namely, Theorem 8.1 of [53], whose proof carries through in the characteristic p case:

Theorem 5.8. *Let $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p)$ be unitary supercuspidal. With the equivalent conditions of the previous proposition, choose $i = 1$ or $i = 2$, to be such that $L(s, \pi, r_i)$ has a pole at $s = 0$, then*

(i) *For $0 < s < 1/i$, the representation $\mathrm{I}(s, \pi)$ is irreducible and in the complementary series.*
(ii) *The representation $\mathrm{I}(1/i, \pi)$ is reducible with a unique generic subrepresentation which is in the discrete series. Its Langlands quotient is never generic. It is a pre-unitary non-tempered representation.*
(iii) *For $s > 1/i$, the representations $\mathrm{I}(s, \pi)$ are always irreducible and never in the complimentary series.*

If $w_0(\pi) \cong \pi$ and $\mathrm{I}(\pi)$ is reducible, then no $\mathrm{I}(s, \pi)$, $s > 0$, is pre-unitary; they are all irreducible.

Remark 5.9. *We refer to* [14] *for a generalization of parts of this theorem to discrete series representations. Namely, we prove the basic assumption (BA) of Mœglin and Tadić* [43] *when* $\mathrm{char}(F) = p$ *in §§ 7.8 and 7.9 of* [14].

In the case of unramified principal series, we can combine the theory of local L-functions with a result of J.-S. Li [38] to obtain the following:

Lemma 5.10. *Let $(F, \pi, \psi) \in \mathscr{L}_{\rm loc}(p)$ be tempered and unramified. Then $\mathrm{I}(s, \pi)$ is irreducible for $\Re(s) > 1$.*

Proof. We have that
$$\pi \hookrightarrow \mathrm{Ind}(\chi),$$
with χ an unramified unitary character of $\mathbf{T}(F)$. From the Satake classification, the character χ corresponds to a complex semisimple conjugacy class in the dual torus, each Satake parameter having absolute value 1. Let
$$\chi_s = \chi \cdot q^{\langle s\tilde{\alpha}, H_{\mathbf{P}}(\cdot)\rangle}.$$
The function $\xi_\alpha(\chi_s)$ defined in § 2 of [38] for each non-divisible root α is either a non-zero constant or a factor appearing in
$$\prod_{i=1}^{m_r} L(1 + is, \tilde{\pi}, r_i)^{-1} \text{ or } \prod_{i=1}^{m_r} L(1 - is, \pi, r_i)^{-1}. \tag{5.4}$$
The result of Li, specifically Theorem 2.2 of [*loc. cit.*], states that $\mathrm{I}(s, \pi)$ is irreducible when each $\xi_\alpha(\chi_s) \neq 0$. The local L-functions involved are never zero, and from [17] we have that the first product $\prod_{i=1}^{m_r} L(1 + is, \tilde{\pi}, r_i)^{-1}$ is non-zero for $\Re(s) > 1$. We claim that the same is also true for the second product. For this, notice that because π is tempered we can write for each i:
$$L(s, \pi, r_i)^{-1} = \prod_j (1 - a_{i,j} q_v^{-s}),$$
where the parameters $a_{i,j}$ have absolute value 1. Then
$$\prod_{i=1}^{m_r} L(1 - is, \pi, r_i)^{-1} = \prod_{i=1}^{m_r} \prod_j (1 - a_{i,j} q_v^{-1} q_v^{is}).$$
Each factor in the latter product is non-zero for $\Re(is) > 1$. In particular, the product is non-zero for $\Re(s) > 1$. From Li's theorem, we must have $\mathrm{I}(s, \pi)$ irreducible for $\Re(s) > 1$. □

5.6. **On the holomorphy of L-functions.** We can prove a general theorem for any pair $(\mathbf{M}, \mathbf{G})$ under the assumption that π satisfies the Ramanujan conjecture, which we now recall.

Conjecture 5.11. *Let π be a globally generic cuspidal automorphic representation of a quasi-split connected reductive group $\mathbf{H}$. Then each π_v is tempered.*

We note that in this article, we will only make use of the fact that the Ramanujan conjecture is valid for GL_n from the work of L. Lafforgue [31]. In § A.2 of the Appendix, we provide examples of the following proposition involving the classical groups.

Proposition 5.12. *Let $(K/k, \pi, \psi, S) \in \mathscr{L}_{\mathrm{glob}}(p, \mathbf{M}, \mathbf{G})$, with $\mathbf{M}$ a maximal Levi subgroup of a quasi-split connected reductive group $\mathbf{G}$. Assume that π satisfies the Ramanujan conjecture. Then, for each i, $1 \leq i \leq m_r$, the automorphic L-function $L(s, \pi, r_i)$ is holomorphic for $\Re(s) > 1$.*

Proof. Let $(K/k, \pi, \psi, S) \in \mathscr{L}_{\mathrm{glob}}(p, \mathbf{M}, \mathbf{G})$. From [17], we know that at places $v \in S$ each of the L-functions $L(s, \pi_v, r_{i,v})$ is holomorphic for $\Re(s) > 0$. We begin with the known observation that local components of residual automorphic representations are unitary representations. Furthermore, if the local representation $\mathrm{I}(s, \pi_v)$ is irreducible, then it cannot be unitary. From Lemma 5.10, we conclude that the global intertwining operator $\mathrm{M}(s, \pi, \tilde{w}_0)$ must be holomorphic for $\Re(s) > 1$. Then, from equation (5.1) we conclude that the product

$$\prod_{i=1}^{m_r} \frac{L^S(is, \pi, r_i)}{L^S(1 + is, \pi, r_i)}$$

is holomorphic on $\Re(s) > 1$. Since the poles of Eisenstein series are contained in the constant terms, with an application of Corollary 3.4 we can further conclude that

$$\prod_{i=1}^{m_r} L^S(1 + is, \pi, r_i)^{-1}$$

is holomorphic for $\Re(s) > 1$. Now, the induction step found in § 4.2 allows us to isolate each L-function and prove that each

$$L^S(s, \pi, r_i)$$

is holomorphic for $\Re(s) > 1$. □

6. The quasi-split unitary groups and the Langlands-Shahidi method

We study generic L-functions $L(s, \pi \times \tau)$ in the case of representations π of a unitary group and τ of a general linear group. For this, we go through the induction step of the Langlands-Shahidi method, which gives the case of Asai L-functions [20]. The case of $L(s, \pi \times \tau)$ for representations of two unitary groups will be established in §§ 7-10.

6.1. **Definitions.** Let K be a degree-2 finite étale algebra over a field k with non-trivial involution θ. We write $\bar{x} = \theta(x)$, for $x \in K$, and extend conjugation to elements $g = (g_{i,j})$ of $\mathrm{GL}_n(K)$, i.e., $\bar{g} = (\bar{g}_{i,j})$. We fix the following hermitian forms:

$$h_{2n+1}(x,y) = \sum_{i=1}^{2n} \bar{x}_i y_{2n+2-i} - \bar{x}_{n+1} y_{n+1}, \quad x, y \in K^{2n+1},$$

$$h_{2n}(x,y) = \sum_{i=1}^{n} \bar{x}_i y_{2n+1-i} - \sum_{i=1}^{n} \bar{x}_{2n+1-i} y_i, \quad x, y \in K^{2n}.$$

Let $N = 2n+1$ or $2n$. We then have odd or even quasi-split unitary groups of rank n whose group of k-rational points is given by

$$\mathrm{U}_N(k) = \{g \in \mathrm{GL}_N(K) \,|\, h_N(gx, gy) = h_N(x,y)\}.$$

These conventions for odd and even unitary groups U_{2n+1} and U_{2n} are in accordance with those made in [11, 20, 41].

In particular, we have the two main cases to which every degree-2 finite étale algebra is isomorphic: if K is the separable algebra $k \times k$, we have $\theta(x) = \theta(x_1, x_2) = (x_2, x_1) = \bar{x}$ and $\mathrm{N}_{K/k}(x_1, x_2) = x_1 x_2$; and, if K/k is a separable quadratic extension, we have $\mathrm{Gal}(K/k) = \{1, \theta\}$ and $\mathrm{N}_{K/k}(x) = x\bar{x}$. Notice that

$$\mathrm{U}_1(k) = K^1 = \ker(\mathrm{N}_{K/k}),$$

where in the separable algebra case we embed $k \hookrightarrow K$ via $k = \{(x_1, x_2) \in K | x_1 = x_2\}$ and $k^\times \hookrightarrow K^\times$ via $k^\times = \left\{(x_1, x_2) \in K | x_1 = x_2^{-1}\right\} = \mathrm{U}_1(k)$. In these two cases we have that Hilbert's theorem 90 gives us a continuous surjection

$$\mathfrak{h} : K^\times \twoheadrightarrow K^1, \ x \mapsto x\bar{x}^{-1}. \tag{6.1}$$

Throughout this article we let $\mathbf{G}_n$ be either restriction of scalars of a general linear group or a quasi-split unitary group of rank n. We think of $\mathbf{G}_n$ as a functor taking degree-2 finite étale algebras with involution, K over k, to either $\mathrm{Res}_{K/k}\mathrm{GL}_n$ or a unitary group U_{2n+1}, U_{2n} defined over k.

Notice that in the case of the separable algebra $K = k \times k$, we have

$$\mathrm{U}_N(k) \cong \mathrm{GL}_N(k) \text{ and } \mathrm{Res}_{K/k}\mathrm{GL}_N(k) \cong \mathrm{GL}_N(k) \times \mathrm{GL}_N(k).$$

6.2. ***L*-groups.** Let K/k be a separable quadratic extension of global function fields. Let $\mathbf{G}_n$ be a unitary group of rank n. Let $N = 2n+1$ or $2n$, according to the unitary group being odd or even. Then, the L-group of $\mathbf{G}_n = \mathrm{U}_N$ has connected component ${}^L G_n^\circ = \mathrm{GL}_N(\mathbb{C})$. The L-group itself is given by the semidirect product

$${}^L G_n = \mathrm{GL}_N(\mathbb{C}) \rtimes \mathcal{W}_k.$$

To describe the action of the Weil group, let Φ_n be the $n \times n$ matrix with ij-entries $(\delta_{i,n-j+1})$. If $N = 2n+1$, we let

$$J_N = \begin{pmatrix} & & \Phi_n \\ & 1 & \\ -\Phi_n & & \end{pmatrix},$$

and, if $N = 2n$, we let

$$J_N = \begin{pmatrix} & \Phi_n \\ -\Phi_n & \end{pmatrix}.$$

Then, the Weil group $\mathcal{W}_k$ acts on ${}^L G_n$ through the quotient $\mathcal{W}_k/\mathcal{W}_K \cong \mathrm{Gal}(K/k) = \{1, \theta\}$ via the outer automorphism

$$\theta(g) = J_N^{-1 t} g^{-1} J_N.$$

The Langlands Base Change lift that we will obtain is from the unitary groups to the restriction of scalars group $\mathbf{H}_N = \mathrm{Res}_{K/k}\mathrm{GL}_N$. Its corresponding L-group is given by

$${}^L H_N = \mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C}) \rtimes \mathcal{W}_k,$$

where the Weil group $\mathcal{W}_k$ acts on $\mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C})$ through the quotient $\mathcal{W}_k/\mathcal{W}_K \cong \mathrm{Gal}(K/k) = \{1, \theta\}$ via

$$\theta(g_1 \times g_2) = g_2 \times g_1.$$

6.3. **Asai L-functions (even case).** The induction step in the Langlands-Shahidi method for the unitary groups can be seen in the when $\mathbf{M}$ is a Siegel Levi subgroup. The even case, when $(E/F, \tau, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathrm{U}_{2n})$, is thoroughly studied in [20, 41].

Assume first that E/F is a quadratic extension of non-archimedean local fields. In this case, τ is a representation of $M \cong \mathrm{GL}_n(E)$ and the adjoint representation r of ${}^L M$ on ${}^L\mathfrak{n}$ is irreducible. More precisely, let $r_{\mathcal{A}}$ be the Asai representation

$$r_{\mathcal{A}} : {}^L\mathrm{Res}_{E/F}\mathrm{GL}_n \to \mathrm{GL}_{n^2}(\mathbb{C}),$$

given by

$$r_{\mathcal{A}}(x, y, 1) = x \otimes y, \text{ and } r_{\mathcal{A}}(x, y, \theta) = y \otimes x.$$

We thus have for $(E/F, \tau, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathrm{U}_{2n})$, that Theorem 4.1 gives

$$\gamma(s, \tau, r, \psi) = \gamma(s, \tau, r_{\mathcal{A}}, \psi).$$

And, similarly we have Asai L-functions $L(s, \pi, r_{\mathcal{A}})$ and root numbers $\varepsilon(s, \pi, r_{\mathcal{A}}, \psi)$.

Now, assume E is the degree-2 finite étale algebra $F \times F$, we have for $(E/F, \pi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathrm{U}_{2n})$ that $\pi = \pi_1 \otimes \pi_2$ is a representation of $M = \mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$. Then, Proposition 4.5 of [41], together with a local-to-global argument, gives that

$$\gamma(s, \pi, r_{\mathcal{A}}, \psi) = \gamma(s, \pi_1 \times \pi_2, \psi),$$

a Rankin-Selberg γ-factor. And, similarly for the corresponding L-functions and root numbers.

Asai local factors obtained via the Langlands-Shahidi method are indeed the correct ones. Theorem 3.1 of [20] establishes their compatibility with the local Langlands correspondence [37]:

Theorem 6.1 (Henniart-Lomelí). *Let $(E/F, \pi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathrm{U}_{2n})$, with E/F a quadratic extension of non-archimedean local fields. Let σ be the Weil-Deligne representation of $\mathcal{W}_E$ corresponding to π via the local Langlands correspondence. Then*

$$\gamma(s, \pi, r_{\mathcal{A}}, \psi) = \gamma_F^{\mathrm{Gal}}(s, {}^{\otimes}\mathrm{I}(\sigma), \psi).$$

Here, ${}^{\otimes}\mathrm{I}(\sigma)$ denotes the representation of $\mathcal{W}_F$ obtained from σ via tensor induction and the Galois γ-factors on the right hand side are those of Deligne and Langlands. Local L-functions and root numbers satisfy

$$L(s, \pi, r_{\mathcal{A}}) = L(s, {}^{\otimes}\mathrm{I}(\sigma)),$$
$$\varepsilon(s, \pi, r_{\mathcal{A}}, \psi) = \varepsilon(s, {}^{\otimes}\mathrm{I}(\sigma), \psi).$$

Remark 6.2. *Since we are in the case of* GL_n*, the results of this section hold when* π *is a smooth representation, and not just generic* [20]*. Furthermore, the Rankin-Selberg products of* GL_m *and* GL_n *that appear in this article arise in the context of the Langlands-Shahidi method in positive characteristic. These are equivalent to those obtained via the integral representation of* [22] *(see* [21]*).*

6.4. **Asai L-functions (odd case).** The case $(E/F, \pi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathrm{U}_{2n+1})$, with E/F a quadratic extension of non-archimedean local fields, has $M \cong \mathrm{GL}_n(E) \times E^1$ and $r = r_1 \oplus r_2$. In this case π is of the form $\tau \otimes \nu$, where ν is a character of E^1, and we extend ν to a smooth representation of $\mathrm{GL}_1(E)$ via Hilbert's theorem 90. Then, from Theorem 4.1, we have

$$\gamma(s, \pi, r_1, \psi) = \gamma(s, \tau \times \nu, \psi_E),$$
$$\gamma(s, \pi, r_2, \psi) = \gamma(s, \tau \otimes \eta_{E/F}, r_{\mathcal{A}}, \psi).$$

Where the former γ-factor is a Rankin-Selber product of $\mathrm{GL}_n(E)$ and $\mathrm{GL}_1(E)$, while the latter is a twisted Asai γ-factor. And, similarly for the local L-functions $L(s, \pi, r_i)$ and root numbers $\varepsilon(s, \pi, r_i, \psi)$, $1 \leq i \leq 2$. This result in characteristic p is given by Theorem' 6.4 of [41] and the unramified case is proved ab initio in Proposition 4.5 there without any restriction on p.

Furthermore, the case $E = F \times F$ is also discussed in [41]. To interpret this case correctly, let ν be the character of E obtained from a character $\nu_0 : F^\times \to \mathbb{C}^\times$ and Hilbert's theorem 90 (6.1). Then π is of the form $\tau \otimes \nu$, with $\tau = \tau_1 \otimes \tau_2$ and each τ_i a representation of $\mathrm{GL}_n(F)$. We thus obtain

$$\gamma(s, \pi, r_1, \psi) = \gamma(s, \tau_1 \times \nu_0^{-1}, \psi)\gamma(s, \tau_2 \times \nu_0, \psi),$$
$$\gamma(s, \pi, r_2, \psi) = \gamma(s, \tau_1 \times \tau_2, \psi).$$

Each factor on the right hand side is a Rankin-Selberg γ-factor. In particular, the unramified case in this setting can be found in Theorem 4.5 of [*loc. cit.*]. The above equality can be obtained by combining Theorems 4.5 and Theorem' 6.4 of [*loc. cit.*] together with a local to global argument.

6.5. **Rankin-Selberg products and Asai factors.** We record a useful property of Asai factors. First for generic representations π of $\mathrm{GL}_n(E)$, and then for any smooth irreducible π.

Proposition 6.3. *Given* $(E/F, \pi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathrm{GL}_n, \mathbf{G}_n)$*, let* π^θ *be the representation of* $\mathrm{GL}_n(E)$ *given by* $\pi^\theta(x) = \pi(\bar{x})$*. Then*

$$\gamma(s, \pi, r_{\mathcal{A}}, \psi) = \gamma(s, \pi^\theta, r_{\mathcal{A}}, \psi), \tag{6.2}$$

$$\gamma(s, \pi \otimes \eta_{E/F}, r_{\mathcal{A}}, \psi) = \gamma(s, \pi^\theta \otimes \eta_{E/F}, r_{\mathcal{A}}, \psi), \tag{6.3}$$

and we have the following equation involving Rankin-Selberg and Asai γ*-factors*

$$\gamma(s, \pi \times \pi^\theta, \psi_E) = \gamma(s, \pi, r_{\mathcal{A}}, \psi)\gamma(s, \pi \otimes \eta_{E/F}, r_{\mathcal{A}}, \psi). \tag{6.4}$$

Proof. Let σ be the n-dimensional ℓ-adic Frob-semisimple Weil-Deligne representation of $\mathcal{W}_E$ corresponding to π via the local Langlands correspondence. Then, σ^θ corresponds to π^θ. And, from the definition of tensor induction (see [12]) we have that ${}^\otimes\mathrm{I}(\sigma) \cong {}^\otimes\mathrm{I}(\sigma^\theta)$. Artin L-functions and root numbers remain the same for equivalent Weil-Deligne representations, thus

$$\gamma(s, {}^\otimes\mathrm{I}(\sigma), \psi) = \gamma(s, {}^\otimes\mathrm{I}(\sigma^\theta), \psi).$$

Hence, by Theorem 6.1, the first equation of the Proposition follows.

The second equation, involving twisted Asai factors, follows from the first and the third. To prove equation (6.4), we first use multiplicativity of γ-factors to establish it for principal series representations. Then, in general, via the local-to-global technique of [20, 21]. □

Corollary 6.4. *Let π be a smooth representation of $\mathrm{GL}_n(E)$ and let π^θ be the representation of $\mathrm{GL}_n(E)$ given by $\pi^\theta(x) = \pi(\bar{x})$. Then*

$$L(s, \pi, r_{\mathcal{A}}) = L(s, \pi^\theta, r_{\mathcal{A}}),$$
$$\varepsilon(s, \pi, r_{\mathcal{A}}, \psi) = \varepsilon(s, \pi^\theta, r_{\mathcal{A}}, \psi)$$

and

$$L(s, \pi \otimes \eta_{E/F}, r_{\mathcal{A}}) = L(s, \pi^\theta \otimes \eta_{E/F}, r_{\mathcal{A}}),$$
$$\varepsilon(s, \pi \otimes \eta_{E/F}, r_{\mathcal{A}}, \psi) = \varepsilon(s, \pi^\theta \otimes \eta_{E/F}, r_{\mathcal{A}}, \psi).$$

Furthermore, we have the following equation involving Rankin-Selberg and Asai factors

$$L(s, \pi \times \pi^\theta) = L(s, \pi, r_{\mathcal{A}}, \psi)\gamma(s, \pi \otimes \eta_{E/F}, r_{\mathcal{A}})$$
$$\varepsilon(s, \pi \times \pi^\theta, \psi_E) = \varepsilon(s, \pi, r_{\mathcal{A}}, \psi)\varepsilon(s, \pi \otimes \eta_{E/F}, r_{\mathcal{A}}, \psi).$$

Proof. Since we are in the case of GL_n, the idea from § 4.2 of [20] directly applies to the cases at hand. □

6.6. **Products of GL_m and U_N.** When the maximal Levi subgroup $\mathbf{M}$ is not a Siegel Levi and we have a quadratic extension E/F of non-archimedean local fields, the adjoint representation always has two irreducible components $r = r_1 \oplus r_2$. In this case, take $(E/F, \xi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathbf{G}_l)$ in Theorem 4.1. We then have that $\mathbf{M} \cong \mathrm{Res}\,\mathrm{GL}_m \times \mathbf{G}_n$ where $\mathbf{G}_l$ and $\mathbf{G}_n$ are unitary groups of the same parity. Also, ξ is of the form $\tau \otimes \tilde{\pi}$ with τ and π representations of $\mathrm{GL}_m(E)$ and G_n, respectively.

We then have

$$\gamma(s, \xi, r_1, \psi) = \gamma(s, \tau \times \pi, \psi),$$

the Rankin-Selberg γ-factor of τ and π. For the second γ-factor we obtain Asai γ-factors

$$\gamma(s, \xi, r_2, \psi) = \begin{cases} \gamma(s, \tau, r_{\mathcal{A}}, \psi) & \text{if } N = 2n \\ \gamma(s, \tau \otimes \eta_{E/F}, r_{\mathcal{A}}, \psi) & \text{if } N = 2n+1 \end{cases}.$$

From Property (vii), given $(E/F, \xi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathbf{G}_l)$ tempered, we obtain the L-functions

$$L(s, \xi, r_1) = L(s, \pi \times \tau)$$

and

$$L(s, \xi, r_2) = \begin{cases} L(s, \tau, r_{\mathcal{A}}) & \text{if } N = 2n \\ L(s, \tau \otimes \eta_{E/F}, r_{\mathcal{A}}) & \text{if } N = 2n+1 \end{cases}.$$

Tempered root numbers

$$\varepsilon(s, \xi, r_i, \psi), \quad 1 \le i \le 2,$$

are obtained via Property (viii) of Theorem 4.1. Then, L-functions and ε-factors are defined in general as in the proof of Theorem 4.1.

Now, assume $E = F \times F$. Let $(E/F, \xi, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathbf{G}_l)$, so that $\mathbf{G}_l \cong \mathrm{U}_L \cong \mathrm{GL}_L$ and $\mathbf{M} \cong \mathrm{GL}_m \times \mathrm{U}_N \times \mathrm{GL}_m$, $l = m+n$, with L and N of the same parity. The representation ξ is of the form $\tau_1 \otimes \pi \otimes \tau_2$. Then we obtain the following equations involving Rankin-Selberg products

$$\gamma(s, \xi, r_1, \psi) = \gamma(s, \tau_1 \times \tilde{\pi}, \psi)\gamma(s, \tau_2 \times \pi, \psi)$$

and

$$\gamma(s, \xi, r_2, \psi) = \gamma(s, \tau_1 \times \tau_2, \psi).$$

And, similarly for the corresponding L-functions and root numbers.

7. Extended Langlands-Shahidi local factors for the unitary groups

Let $\mathbf{G}_1$ be either a quasi-split unitary group U_N or the group $\mathrm{Res}\,\mathrm{GL}_N$. In the case of $\mathbf{G}_1 = \mathrm{U}_N$, it is of rank n, where we write $N = 2n + 1$ or $2n$ according to wether the unitary group is odd or even. Similarly, we let $\mathbf{G}_2$ be either a unitary group of rank m or $\mathrm{Res}\,\mathrm{GL}_M$, with $M = 2m + 1$ or $2m$.

We interpret $\mathrm{Res}\,\mathrm{GL}_N$ as a functor, taking a quadratic extension E/F to the group scheme $\mathrm{Res}_{E/F}\mathrm{GL}_N$. Also, U_N takes E/F to the quasi-split reductive group scheme $\mathrm{U}(h_N)$, where h_N is the standard hermitian form of § 6.1. In order to emphasize the underlying quadratic extension, and the extended case of a system of γ-factors, L-functions and root numbers for products of two unitary groups, we modify the notation of Sections 1.1 and 3.1 accordingly. Also, given the involution θ of the quadratic extension E/F and a character $\eta : \mathrm{GL}_1(E) \to \mathbb{C}^\times$, denote by $\eta^\theta : \mathrm{GL}_1(E) \to \mathbb{C}^\times$ the character given by $\eta^\theta(x) = \eta(\bar{x})$.

In section § 7.3 below we treat the case of a separable quadratic algebra $E = F \times F$. This extends the local theory to all degree-2 finite étale algebras E over the archimedean local field F.

7.1. Local notation.

Let $\mathscr{L}_{\mathrm{loc}}(p, \mathbf{G}_1, \mathbf{G}_2)$ be the category whose objects are quadruples $(E/F, \pi, \tau, \psi)$ consisting of: a non-archimedean local field F, with $\mathrm{char}(F) = p$; a degree-2 finite étale algebra E over F; irreducible admissible representations π of G_1 and τ of G_2; and, a smooth non-trivial additive character $\psi : F \to \mathbb{C}^\times$.

We construct a character $\psi_E : E^\times \to \mathbb{C}^\times$ from the character ψ of F via the trace, i.e., $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$. When $\mathbf{G}_1$ and $\mathbf{G}_2$ are clear from context, we will simply write $\mathscr{L}_{\mathrm{loc}}(p)$ for $\mathscr{L}_{\mathrm{loc}}(p, \mathbf{G}_1, \mathbf{G}_2)$. We say $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\mathrm{loc}}(p)$ is generic (resp. supercuspidal, discrete series, tempered, principal series) if both π and τ are generic (resp. supercuspidal, discrete series, tempered, principal series) representations. We let q_F denote the cardinality of the residue field of F.

7.2. Global notation.

Let $\mathscr{L}_{\mathrm{glob}}(p, \mathbf{G}_1, \mathbf{G}_2)$ be the category whose objects are quintuples (k, π, τ, ψ, S) consisting of: a separable quadratic extension of global function fields K/k, with $\mathrm{char}(k) = p$; globally generic cuspidal automorphic representations $\pi = \otimes_v \pi_v$ of $\mathbf{G}_1(\mathbb{A}_k)$ and $\tau = \otimes_v \tau_v$ of $\mathbf{G}_2(\mathbb{A}_k)$; a non-trivial character $\psi = \otimes_v \psi_v : k\backslash\mathbb{A}_k \to \mathbb{C}^\times$; and, a finite set of places S where k, π and ψ are unramified.

We write $\mathscr{L}_{\mathrm{glob}}(p)$ when $\mathbf{G}_1$ and $\mathbf{G}_2$ are undestood. We let q be the cardinality of the field of constants of k. And, for every place v of k, we let q_v be the cardinality of the residue field of k_v.

Let $(K/k, \pi, \tau, \psi, S) \in \mathscr{L}_{\mathrm{glob}}(p)$. Then we have partial L-functions

$$L^S(s, \pi \times \tau) = \prod_{v \notin S} L(s, \pi_v \times \tau_v).$$

The case of a place v in k, which splits in K_v leads to the case of a separable algebra and we write $K_v = k_v \times k_v$.

7.3. **The case of a separable algebra.** Let $E = F \times F$, then we have the following possibilities for Langlands-Shahidi γ-factors:

(i) Let $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p, {\rm U}_M, {\rm U}_N)$. Then π is a representation of ${\rm GL}_M(F)$ and τ one of ${\rm GL}_N(F)$. The local functorial lift of π to $\mathbf{H}_M(F)$ obtained from

$$\mathbf{G}_m(F) = {\rm U}_M(F) = {\rm GL}_M(F) \rightsquigarrow \mathbf{H}_M(F) = {\rm GL}_M(F) \times {\rm GL}_M(F)$$

is given by $\pi \otimes \tilde{\pi}$. Similarly the local functorial lift of τ to $\mathbf{H}_N(F)$ obtained from

$$\mathbf{G}_n(F) = {\rm U}_N(F) = {\rm GL}_N(F) \rightsquigarrow \mathbf{H}_N(F) = {\rm GL}_N(F) \times {\rm GL}_N(F)$$

is given by $\tau \otimes \tilde{\tau}$. Then the Langlands-Shahidi local factors are

$$\begin{aligned}
\gamma_{E/F}(s, \pi \times \tau, \psi_E) &= \gamma(s, \pi \times \tau, \psi)\gamma(s, \tilde{\pi} \times \tilde{\tau}, \psi) \\
L_{E/F}(s, \pi \times \tau) &= L(s, \pi \times \tau)L(s, \tilde{\pi} \times \tilde{\tau}) \\
\varepsilon_{E/F}(s, \pi \times \tau, \psi_E) &= \varepsilon(s, \pi \times \tau, \psi)\varepsilon(s, \tilde{\pi} \times \tilde{\tau}, \psi).
\end{aligned}$$

(ii) Let $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p, {\rm U}_M, {\rm Res}\,{\rm GL}_N)$. Then π is a representation of ${\rm GL}_M(F)$ and τ one of ${\rm GL}_N(F) \times {\rm GL}_N(F)$. The local functorial lift of π to $\mathbf{H}_M(F)$ obtained from

$$\mathbf{G}_m(F) = {\rm U}_M(F) = {\rm GL}_M(F) \rightsquigarrow \mathbf{H}_M = {\rm GL}_M(F) \times {\rm GL}_M(F)$$

is given by $\pi \otimes \tilde{\pi}$. Write $\tau = \tau_1 \otimes \tau_2$ as a representation of

$${\rm Res}_{E/F}{\rm GL}_N(F) = {\rm GL}_N(F) \times {\rm GL}_N(F).$$

Then the Langlands-Shahidi local factors are

$$\begin{aligned}
\gamma_{E/F}(s, \pi \times \tau, \psi_E) &= \gamma(s, \pi \times \tau_1, \psi)\gamma(s, \tilde{\pi} \times \tau_2, \psi) \\
L_{E/F}(s, \pi \times \tau) &= L(s, \pi \times \tau_1)L(s, \tilde{\pi} \times \tau_2) \\
\varepsilon_{E/F}(s, \pi \times \tau, r, \psi_E) &= \varepsilon(s, \pi \times \tau_1, \psi)\varepsilon(s, \tilde{\pi} \times \tau_2, \psi).
\end{aligned}$$

(iii) Let $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p, {\rm Res}\,{\rm GL}_M, {\rm Res}\,{\rm GL}_N)$. Then $\pi = \pi_1 \otimes \pi_2$ is a representation of ${\rm GL}_M(F) \times {\rm GL}_M(F)$ and $\tau = \tau_1 \otimes \tau_2$ one of ${\rm GL}_N(F) \times {\rm GL}_N(F)$. Then the Langlands-Shahidi local factors are

$$\begin{aligned}
\gamma_{E/F}(s, \pi \times \tau, \psi_E) &= \gamma(s, \pi_1 \times \tau_1, \psi)\gamma(s, \pi_2 \times \tau_2, \psi) \\
L_{E/F}(s, \pi \times \tau) &= L(s, \pi_1 \times \tau_1)L(s, \pi_2 \times \tau_2) \\
\varepsilon_{E/F}(s, \pi \times \tau, r, \psi_E) &= \varepsilon(s, \pi_1 \times \tau_1, \psi)\varepsilon(s, \pi_2 \times \tau_2, \psi).
\end{aligned}$$

Remark 7.1. *We usually drop the subscripts E/F when dealing with Langlands-Shahidi local factors. Hopefully, it is clear from context what we mean by an L-function, and related local factors, at split places of a global function field.*

Remark 7.2. *Let $(K/k, \pi, \tau, \psi, S) \in \mathscr{L}_{\rm glob}(p, \mathbf{G}_1, \mathbf{G}_2)$. Then, at places v of k which are split in K we set $K_v = k_v \times k_v$. It is interesting to note that the theory of the Langlands-Shahidi local coefficient can be treated directly and uniformly for unitary groups defined over a degree-2 finite étale algebra E over a non-archimedean local field F as in* [41]. *Alternatively, one can use the isomorphism ${\rm U}_N \cong {\rm GL}_N$ in the case of a separable quadratic algebra.*

7.4. **Main theorem.** In § 6 we showed the existence of a system of γ-factors, L-functions and root numbers on $\mathscr{L}_{\rm loc}(p, \mathbf{G}, {\rm GL}_m)$. We now state our main theorem for extended factors. However, we postpone the proof until § 10. More precisely, we give a self contained proof for $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p)$ generic in § 10.1. In general, the tempered L-packet conjecture is expected to hold for the unitary groups (see Conjecture 10.3). Under this assumption, we complete the proof of existence and uniqueness of local factors on $\mathscr{L}_{\rm loc}(p)$ in § 10.2.

Theorem 7.3. *There* exist *rules γ, L and ε on $\mathscr{L}_{\rm loc}(p)$ which are* uniquely *characterized by the following properties:*

(i) (Naturality). *Let $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p)$ be generic and let $\eta : E'/F' \to E/F$ be an isomorphism on local field extensions. Let $(E'/F', \pi', \tau', \psi') \in \mathscr{L}_{\rm loc}(p)$ be the quadruple obtained via η. Then*
$$\gamma(s, \pi \times \tau, \psi_E) = \gamma(s, \pi' \times \tau', \psi'_E).$$

(ii) (Isomorphism). *Let $(E/F, \pi, \tau, \psi)$, $(E/F, \pi', \tau', \psi) \in \mathscr{L}_{\rm loc}(p)$ be generic quadruples such that $\pi \cong \pi'$ and $\tau \cong \tau'$. Then*
$$\gamma(s, \pi \times \tau, \psi_E) = \gamma(s, \pi' \times \tau', \psi_E).$$

(iii) (Compatibility with class field theory). *Let $\mathbf{G}_i$ be either ${\rm U}_1$ or ${\rm Res}\,{\rm GL}_1$, for $i = 1$ or 2, and let $(E/F, \chi_1, \chi_2, \psi) \in \mathscr{L}_{\rm loc}(p, \mathbf{G}_1, \mathbf{G}_2)$. In the case of ${\rm U}_1$, we extend a character χ_i of ${\rm U}_1(F) = E^1$ to one of ${\rm Res}_{E/F}{\rm GL}_1(F) = {\rm GL}_1(E)$ via Hilbert's theorem 90. Then*
$$\gamma(s, \chi_1 \times \chi_2, \psi_E) = \gamma(s, \chi_1\chi_2, \psi_E),$$
where the γ-factors on the right hand side are those of Tate's thesis for ${\rm GL}_1(E)$.

(iv) (Multiplicativity). *Let $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p, \mathbf{G}_1, \mathbf{G}_2)$ be generic. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be Levi subgroups of $\mathbf{G}_1$ and $\mathbf{G}_2$, respectively. Let π_0 be a generic representation of M_1 and suppose that*
$$\pi \hookrightarrow {\rm Ind}(\pi_0)$$
is the generic constituent. And let τ_0 be a generic representations of M_2 and let
$$\tau \hookrightarrow {\rm Ind}(\tau_0)$$
be the generic constituent. There exists a finite set Σ such that for each $j \in \Sigma$: there is a maximal Levi subgroup $\mathbf{M}_j$ of $\mathbf{G}_j$, where $\mathbf{G}_j$ is either ${\rm Res}\,{\rm GL}_{n_j}$ or ${\rm U}_{n_j}$; there is a generic representation ξ_j of M_j; and, the following relationship holds
$$\gamma(s, \pi \times \tau, \psi) = \prod_{j \in \Sigma} \gamma(s, \xi_j, r_1, \psi).$$

(v) (Dependence on ψ). *Let $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p)$ be generic and let $a \in E^\times$, then ψ_E^a be the character of E defined by $\psi_E^a(x) = \psi_E(ax)$. Let ω_π and ω_τ be the central characters of π and τ. Then*
$$\gamma(s, \pi \times \tau, \psi_E^a) = \omega_\pi(a)^M \omega_\tau(a)^N \, |a|_E^{MN(s-\frac{1}{2})} \, \gamma(s, \pi \times \tau, \psi_E).$$

(vi) (Functional Equation). *Let* $(K/k, \pi, \tau, \psi, S) \in \mathscr{L}_{\rm glob}(p)$*, then*

$$L^S(s, \pi \times \tau) = \prod_{v \in S} \gamma(s, \pi \times \tau, \psi_v)\, L^S(1 - s, \tilde{\pi} \times \tilde{\tau}).$$

At split places v *of* k*, where* $K_v \cong k_v \times k_v$*, the Langlands-Shahidi local factors are the ones of* § *7.3.*

(vii) (Tempered L-functions). *For* $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p)$ *tempered, let* $P_{\pi\times\tau}(t)$ *be the polynomial with* $P_{\pi\times\tau}(0) = 1$*, with* $P_{\pi\times\tau}(q_F^{-s})$ *the numerator of* $\gamma(s, \pi \times \tau, \psi_E)$*. Then*

$$L(s, \pi \times \tau) = \frac{1}{P_{\pi\times\tau}(q_F^{-s})}$$

is holomorphic and non-zero for $\mathrm{Re}(s) > 0$*.*

(viii) (Tempered ε-factors). *Let* $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p)$ *be tempered, then*

$$\varepsilon(s, \pi \times \tau, \psi_E) = \gamma(s, \pi \times \tau, \psi_E) \frac{L(s, \pi \times \tau)}{L(1 - s, \tilde{\pi} \times \tilde{\tau})}.$$

(ix) (Twists by unramified characters). *Let* $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p, \mathrm{U}_M, \mathrm{Res}\,\mathrm{GL}_N)$*. Then*

$$L(s + s_0, \pi \times \tau) = L(s, \pi \times (\tau\, |\det(\cdot)|_E^{s_0})),$$
$$\varepsilon(s + s_0, \pi \times \tau, \psi_E) = \varepsilon(s, \pi \times (\tau\, |\det(\cdot)|_E^{s_0}), \psi_E).$$

(x) (Langlands classification). *Let* $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p, \mathbf{G}_1, \mathbf{G}_2)$*. Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be Levi subgroups of* $\mathbf{G}_1$ *and* $\mathbf{G}_2$*, respectively. Let* π_0 *be a tempered representation of* M_1 *and suppose that* π *is the Langlands quotient of*

$$\mathrm{Ind}(\pi_0 \otimes \chi)$$

with $\chi \in X_{\rm nr}(\mathbf{M}_1)$ *in the Langlands situation. And let* τ_0 *be a tempered representation of* M_2 *such that* τ *is the Langlands quotient of*

$$\mathrm{Ind}(\tau_0 \otimes \mu)$$

and $\mu \in X_{\rm nr}(\mathbf{M}_2)$ *is in the Langlands situation. There exists a finite set* Σ *such that for each* $j \in \Sigma$*: there is a maximal Levi subgroup* $\mathbf{M}_j$ *of* $\mathbf{G}_j$*, where* $\mathbf{G}_j$ *is either* $\mathrm{Res}\,\mathrm{GL}_{n_j}$ *or* U_{n_j}*; there is a tempered representation* ξ_j *of* M_j*; and, the following relationship holds*

$$L(s, \pi \times \tau) = \prod_{j \in \Sigma} L(s, \xi_j, r_1)$$
$$\varepsilon(s, \pi \times \tau, \psi) = \prod_{j \in \Sigma} \varepsilon(s, \xi_j, r_1, \psi).$$

7.5. **Additional properties of L-functions and local factors.** The proof of Theorem 7.3 in the case of $\mathbf{G}_1 = \mathrm{U}_m$ and $\mathbf{G}_2 = \mathrm{Res}\,\mathrm{GL}_n$ can be obtained from that of Theorem 4.1. The proof of existence is completed in § 10.1 for generic representations, and in § 10.2 under the assumption that the tempered L-packet conjecture is valid. For uniqueness, we proceed as in Theorem 4.3 of [40].

We also obtain a local functional equation for γ-factors which is proved using only Properties (i)–(vi) of Theorem 7.3, as in § 4.2 of [40].

(xi) (Local functional equation). *Let* $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm loc}(p)$*, then*

$$\gamma(s, \pi \times \tau, \psi_E)\gamma(1 - s, \tilde{\pi} \times \tilde{\tau}, \overline{\psi}_E) = 1.$$

We can now define automorphic L-functions and root numbers for $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\rm glob}(p)$ by setting

$$L(s, \pi \times \tau) = \prod_v L(s, \pi_v \times \tau_v) \text{ and } \varepsilon(s, \pi \times \tau) = \prod_v \varepsilon(s, \pi_v \times \tau_v, \psi_v).$$

They satisfy a functional equation, whose proof is completed in § 10.3.

(xii) (Global functional equation). *Let* $(K/k, \pi, \tau, \psi, S) \in \mathscr{L}_{\rm glob}(p)$*, then*

$$L(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau) L(1 - s, \tilde{\pi} \times \tilde{\tau}).$$

The following property is Theorem 5.1 of [14] adapted to the case of unitary groups.

(xiii) (Stability). *Let* $(E/F, \pi_i, \tau_i, \psi) \in \mathscr{L}_{\rm loc}(p, {\rm U}_N, {\rm Res}\,{\rm GL}_m)$*, for* $i = 1$ *or* 2*, be generic and such that* $\omega_{\pi_1} = \omega_{\pi_2}$ *and* $\omega_{\tau_1} = \omega_{\tau_2}$*. If* $\eta : E^\times \to \mathbb{C}^\times$ *is highly ramified, then*

$$\gamma(s, \pi_1 \times (\tau_1 \cdot \eta), \psi_E) = \gamma(s, \pi_2 \times (\tau_2 \cdot \eta), \psi_E).$$

We note that we also have the corresponding stability properties for local L-functions and root numbers

$$L(s, \pi_1 \times (\tau_1 \cdot \eta)) = L(s, \pi_2 \times (\tau_2 \cdot \eta)),$$
$$\varepsilon(s, \pi_1 \times (\tau_1 \cdot \eta), \psi_E) = \varepsilon(s, \pi_2 \times (\tau_2 \cdot \eta), \psi_E).$$

The stability of local L-functions is a result of Shahidi [54]. Stability for ε-factors follows by combining stability for γ-factors and L-functions via Property (viii) above for tempered representations. Then in general, by Langlands' classification, Property (x).

7.6. **Multiplicativity and Langlands classification.** We being by making explicit the multiplicativity property of γ-factors.

(iv) *(Multiplicativity).* Let $M = m_1 + \cdots m_d + m_0$ and $N = n_1 + \cdots + n_e + n_0$. For $1 \leq i \leq d$, $1 \leq j \leq e$, let $(E/F, \pi_i, \tau_j, \psi) \in \mathscr{L}_{\rm loc}(p, {\rm Res}\,{\rm GL}_{m_i}, {\rm Res}\,{\rm GL}_{n_j})$ be generic. Take $\mathbf{G}_{1,0}$ and $\mathbf{G}_{2,0}$ be of the same kind as $\mathbf{G}_1$ and $\mathbf{G}_2$, and let $(E/F, \pi_0, \tau_0, \psi) \in \mathscr{L}_{\rm loc}(p, \mathbf{G}_{1,0}, \mathbf{G}_{2,0})$. In the case of either $\mathbf{G}_{1,0}$ or $\mathbf{G}_{2,0}$ being ${\rm U}_1$, we extend the corresponding character π_0 or τ_0 of ${\rm U}_1(F) = E^1$ to one of ${\rm Res}_{E/F}{\rm GL}_1(F) = {\rm GL}_1(E)$ via Hilbert's theorem 90. Suppose that

$$\pi \hookrightarrow {\rm ind}_{P_1}^{G_1}(\pi_1 \otimes \cdots \otimes \pi_d \otimes \pi_0)$$

is the generic constituent, where $\mathbf{P}_1$ is the parabolic subgroup of $\mathbf{G}_1$ with Levi $\mathbf{M}_1 = \prod_{i=1}^d {\rm Res}_{E/F}{\rm GL}_{m_i} \times \mathbf{G}_{1,0}$. And let

$$\tau \hookrightarrow {\rm ind}_{P_2}^{G_2}(\tau_1 \otimes \cdots \otimes \tau_e \otimes \tau_0)$$

be the generic constituent, where $\mathbf{P}_2$ is the parabolic subgroup of $\mathbf{G}_2$ with Levi $\mathbf{M}_2 = \prod_{i=1}^e {\rm Res}_{E/F}{\rm GL}_{n_i} \times \mathbf{G}_{2,0}$.

(iv.a) If both $\mathbf{G}_1$ and $\mathbf{G}_2$ are unitary groups, then

$$\begin{aligned}\gamma(s, \pi \times \tau, \psi_E) &= \gamma(s, \pi_0 \times \tau_0, \psi_E) \\ &\times \prod_{i=1}^d \gamma(s, \pi_i \times \tau_0, \psi_E)\gamma(s, \tilde{\pi}_i \times \tau_0, \psi_E) \prod_{i=1}^e \gamma(s, \pi_0 \times \tau_j, \psi_E)\gamma(s, \pi_0 \times \tilde{\tau}_j, \psi_E) \\ &\times \prod_{1 \leq h \leq d, 1 \leq l \leq e} \gamma(s, \pi_h \times \tau_l, \psi_E)\gamma(s, \pi_h \times \tilde{\tau}_l, \psi_E)\gamma(s, \tilde{\pi}_h \times \tilde{\tau}_l, \psi_E)\gamma(s, \tilde{\pi}_h \times \tau_l, \psi_E).\end{aligned}$$

(iv.b) If $\mathbf{G}_1 = \mathrm{U}_M$ and $\mathbf{G}_2 = \mathrm{Res}\,\mathrm{GL}_N$, then

$$\gamma(s, \pi \times \tau, \psi_E) = \prod_{j=1}^{e} \gamma(s, \pi_0 \times \tau_j, \psi_E) \times \prod_{1 \leq h \leq d, 1 \leq l \leq e} \gamma(s, \pi_h \times \tau_l, \psi_E)\gamma(s, \tilde{\pi}_h \times \tau_l, \psi_E).$$

(iv.c) If $\mathbf{G}_1 = \mathrm{Res}\,\mathrm{GL}_M$ and $\mathbf{G}_2 = \mathrm{Res}\,\mathrm{GL}_N$, then

$$\gamma(s, \pi \times \tau, \psi_E) = \prod_{i,j} \gamma(s, \pi_i \times \tau_j, \psi_E).$$

With the definition of L-functions and ε-factors for tempered representations of Properties (vii) and (viii), then Langlands classification and Property (ix) allows us to reduce to the tempered case. Which we now make explicit to the cases arising from the unitary groups.

(x) *(Langlands classification).* Let $M = m_1 + \cdots m_d + m_0$ and $N = n_1 + \cdots + n_e + n_0$. For $1 \leq i \leq d$, $1 \leq j \leq e$, consider quasi-tempered quadruples $(E/F, \pi_i, \tau_j, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathrm{Res}\,\mathrm{GL}_{m_i}, \mathrm{Res}\,\mathrm{GL}_{n_j})$. Take $\mathbf{G}_{1,0}$ and $\mathbf{G}_{2,0}$ be of the same kind as $\mathbf{G}_1$ and $\mathbf{G}_2$, and let $(E/F, \pi_0, \tau_0, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{G}_{1,0}, \mathbf{G}_{2,0})$ be tempered. In the case of either $\mathbf{G}_{1,0}$ or $\mathbf{G}_{2,0}$ being U_1, we extend the corresponding character π_0 or τ_0 of $\mathrm{U}_1(F) = E^1$ to one of $\mathrm{Res}_{E/F}\mathrm{GL}_1(F) = \mathrm{GL}_1(E)$ via Hilbert's theorem 90. Suppose that

$$\pi \hookrightarrow \mathrm{ind}_{P_1}^{G_1}(\pi_1 \otimes \cdots \otimes \pi_d \otimes \pi_0)$$

is the generic constituent, where $\mathbf{P}_1$ is the parabolic subgroup of $\mathbf{G}_1$ with Levi $\mathbf{M}_1 = \prod_{i=1}^{d} \mathrm{Res}\,\mathrm{GL}_{m_i} \times \mathbf{G}_{1,0}$. And let

$$\tau \hookrightarrow \mathrm{ind}_{P_2}^{G_2}(\tau_1 \otimes \cdots \otimes \tau_e \otimes \tau_0)$$

be the generic constituent, where $\mathbf{P}_2$ is the parabolic subgroup of $\mathbf{G}_2$ with Levi $\mathbf{M}_2 = \prod_{i=1}^{e} \mathrm{Res}\,\mathrm{GL}_{n_i} \times \mathbf{G}_{2,0}$.

(x.a) If both $\mathbf{G}_1$ and $\mathbf{G}_2$ are unitary groups, then

$$\begin{aligned} L(s, \pi \times \tau) &= L(s, \pi_0 \times \tau_0) \\ &\times \prod_{i=1}^{d} L(s, \pi_i \times \tau_0) L(s, \tilde{\pi}_i \times \tau_0) \prod_{i=1}^{e} L(s, \pi_0 \times \tau_j) L(s, \pi_0 \times \tilde{\tau}_j) \\ &\times \prod_{1 \leq h \leq d, 1 \leq l \leq e} L(s, \pi_h \times \tau_l) L(s, \pi_h \times \tilde{\tau}_l) L(s, \tilde{\pi}_h \times \tilde{\tau}_l) L(s, \tilde{\pi}_h \times \tau_l). \end{aligned}$$

$$\begin{aligned} \varepsilon(s, \pi \times \tau, \psi_E) &= \varepsilon(s, \pi_0 \times \tau_0, \psi_E) \\ &\times \prod_{i=1}^{d} \varepsilon(s, \pi_i \times \tau_0, \psi_E) \varepsilon(s, \tilde{\pi}_i \times \tau_0, \psi_E) \prod_{i=1}^{e} \varepsilon(s, \pi_0 \times \tau_j, \psi_E) \varepsilon(s, \pi_0 \times \tilde{\tau}_j, \psi_E) \\ &\times \prod_{1 \leq h \leq d, 1 \leq l \leq e} \varepsilon(s, \pi_h \times \tau_l, \psi_E) \varepsilon(s, \pi_h \times \tilde{\tau}_l, \psi_E) \varepsilon(s, \tilde{\pi}_h \times \tilde{\tau}_l, \psi_E) \varepsilon(s, \tilde{\pi}_h \times \tau_l, \psi_E). \end{aligned}$$

(x.b) If $\mathbf{G}_1 = \mathrm{U}_M$ and $\mathbf{G}_2 = \mathrm{Res}\,\mathrm{GL}_N$, then

$$L(s, \pi \times \tau) = \prod_{j=1}^{e} L(s, \pi_0 \times \tau_j) \times \prod_{1 \leq h \leq d, 1 \leq l \leq e} L(s, \pi_h \times \tau_l) L(s, \tilde{\pi}_h \times \tau_l).$$

$$\varepsilon(s,\pi\times\tau,\psi_E)=\prod_{j=1}^{e}\varepsilon(s,\pi_0\times\tau_j,\psi_E)\times\prod_{1\le h\le d,1\le l\le e}\varepsilon(s,\pi_h\times\tau_l,\psi_E)\varepsilon(s,\tilde{\pi}_h\times\tau_l,\psi_E).$$

(x.c) If $\mathbf{G}_1=\mathrm{Res}\,\mathrm{GL}_M$ and $\mathbf{G}_2=\mathrm{Res}\,\mathrm{GL}_N$, then

$$L(s,\pi\times\tau)=\prod_{i,j}L(s,\pi_i\times\tau_j).$$

$$\varepsilon(s,\pi\times\tau,\psi_E)=\prod_{i,j}\varepsilon(s,\pi_i\times\tau_j,\psi_E).$$

7.7. **Stable form of local factors.** The following Lemma and its Corollary, provides a stable form for the local factors after twists by highly ramified characters. It plays an important role in establishing global Base Change.

Lemma 7.4. *Let $(E/F,\pi,\tau,\psi)\in\mathscr{L}_{\mathrm{loc}}(p,\mathrm{U}_N,\mathrm{Res}\,\mathrm{GL}_m)$ be generic. Consider a quadruple $(E/F,\Pi,T,\psi)\in\mathscr{L}_{\mathrm{loc}}(p,\mathrm{Res}\,\mathrm{GL}_N,\mathrm{Res}\,\mathrm{GL}_m)$, with Π and T principal series, such that $\omega_\tau=\omega_T$ and ω_Π is the character of $E^\times$ obtained from ω_π of E^1 via Hilbert's theorem 90. Then, whenever $\eta:E^\times\to\mathbb{C}^\times$ is highly ramified, we have that*

$$\begin{aligned}L(s,\pi\times(\tau\cdot\eta))&=L(s,\Pi\times(T\cdot\eta)),\\ \varepsilon(s,\pi\times(\tau\cdot\eta),\psi_E)&=\varepsilon(s,\Pi\times(T\cdot\eta),\psi_E),\\ \gamma(s,\pi\times(\tau\cdot\eta),\psi_E)&=\gamma(s,\Pi\times(T\cdot\eta),\psi_E).\end{aligned}$$

Proof. There is always a T, which is the generic constituent of

$$\mathrm{Ind}(\mu_1\otimes\cdots\otimes\mu_m),$$

where $\mu_1,\ldots,\mu_m$ are characters of $\mathrm{GL}_1(E)$. Multiplicativity of γ-factors for the unitary groups gives

$$\gamma(s,\pi\times(\tau\cdot\eta),\psi_E)=\prod_{i=1}^{m}\gamma(s,\pi\times(\chi_i\eta),\psi_E).\tag{7.1}$$

And similarly for Rankin-Selberg products of general linear groups

$$\gamma(s,\Pi\times(\tau\cdot\eta),\psi_E)=\prod_{i=1}^{m}\gamma(s,\Pi\times(\chi_i\eta),\psi_E).\tag{7.2}$$

Now, let ξ be the representation of $G_n=\mathrm{U}_N(F)$ which is the generic constituent of either

$$\mathrm{ind}_B^{G_n}(\chi_1\otimes\cdots\otimes\chi_n)\quad\text{or}\quad\mathrm{ind}_B^{G_n}(\chi_1\otimes\cdots\otimes\chi_n\otimes\nu),$$

depending on wether $N=2n$ or $2n+1$, and such that $\omega_\xi=\omega_\pi$. Then, let

$$\Xi\hookrightarrow\mathrm{ind}_B^{\mathrm{GL}_N(E)}(\chi_1\otimes\cdots\otimes\chi_n\otimes\bar{\chi}_n^\theta\otimes\cdots\otimes\bar{\chi}_1^\theta),\tag{7.3}$$

if $N=2n$, and let

$$\Xi\hookrightarrow\mathrm{ind}_B^{\mathrm{GL}_N(E)}(\chi_1\otimes\cdots\otimes\chi_n\otimes\nu\otimes\bar{\chi}_n^\theta\otimes\cdots\otimes\bar{\chi}_1^\theta),\tag{7.4}$$

if $N=2n+1$. Then, Ξ has $\omega_\Xi=\omega_\Pi$ obtained from ω_π as in the statement of the Proposition. Using stability of γ-factors on $\mathscr{L}_{\mathrm{loc}}(p,\mathrm{U}_N,\mathrm{Res}\,\mathrm{GL}_1)$, Property (xiii) of § 7.5, we have that for each i

$$\begin{aligned}\gamma(s,\pi\times(\chi_i\cdot\eta),\psi_E) &= \gamma(s,\xi\times(\chi_i\cdot\eta),\psi_E)\\ &= \gamma(s,\Xi\times(\chi_i\cdot\eta),\psi_E)\\ &= \gamma(s,\Pi\times(\chi_i\cdot\eta),\psi_E)\end{aligned}$$

Then from equations (7.1) and (7.2), we have the desired equality of γ-factors. The corresponding relations for the L-functions and root numbers can then be proved arguing as in the proof of Lemma 9.3. □

In the course of proof, we constructed $(E/F,\Pi,T,\psi)\in\mathscr{L}_{\rm loc}(p,\mathrm{Res\,GL}_N,\mathrm{Res\,GL}_m)$ which also allows us to compute the following stable form of local factors.

Corollary 7.5. *Let $(E/F,\pi,\tau,\psi)\in\mathscr{L}_{\rm loc}(p,\mathrm{U}_N,\mathrm{Res\,GL}_m)$ be generic and let $\eta: E^\times\to\mathbb{C}^\times$ be sufficiently ramified. Let $\chi_1,\ldots,\chi_n,\mu_1,\ldots,\mu_m$ be characters of $E^\times$ and let ν be a character of E^1, which we extend to one of $E^\times$ via Hilbert's theorem 90. Assume that ξ is the generic constituent of*

$$\mathrm{ind}_B^{G_n}(\chi_1\otimes\cdots\otimes\chi_n)\quad or\quad \mathrm{ind}_B^{G_n}(\chi_1\otimes\cdots\otimes\chi_n\otimes\nu),$$

depending on wether $N=2n$ or $2n+1$, and has central character $\omega_\xi=\omega_\pi$. Suppose T is the generic constituent of

$$\mathrm{ind}_{B_m}^{\mathrm{GL}_m(E)}(\mu_1\otimes\cdots\otimes\mu_m),$$

and has central character $\omega_T=\omega_\tau$. Then, if $N=2n$, we have

$$\begin{aligned}L(s,\pi\times(\tau\cdot\eta)) &= \prod_{i=1}^m\prod_{j=1}^n L(s,\chi_j\mu_i\eta)L(s,\bar\chi^\theta\mu_i\eta),\\ \varepsilon(s,\pi\times(\tau\cdot\eta),\psi_E) &= \prod_{i=1}^m\prod_{j=1}^n \varepsilon(s,\chi_j\mu_i\eta,\psi_E)\varepsilon(s,\bar\chi_j^\theta\mu_i\eta,\psi_E),\\ \gamma(s,\pi\times(\tau\cdot\eta),\psi_E) &= \prod_{i=1}^m\prod_{j=1}^n \gamma(s,\chi_j\mu_i\eta,\psi_E)\gamma(s,\bar\chi_j^\theta\mu_i\eta,\psi_E).\end{aligned}$$

And, if $N=2n+1$, we have

$$\begin{aligned}L(s,\pi\times(\tau\cdot\eta)) &= \prod_{i=1}^m L(s,\mu_i\eta\nu)\prod_{j=1}^n L(s,\chi_j\mu_i\eta)L(s,\bar\chi_j^\theta\mu_j\eta),\\ \varepsilon(s,\pi\times(\tau\cdot\eta),\psi_E) &= \prod_{i=1}^m \varepsilon(s,\mu_i\eta\nu,\psi_E)\prod_{j=1}^n \varepsilon(s,\chi_j\mu_i\eta,\psi_E)\varepsilon(s,\bar\chi_j^\theta\mu_i\eta,\psi_E),\\ \gamma(s,\pi\times(\tau\cdot\eta),\psi_E) &= \prod_{i=1}^m \gamma(s,\mu_i\eta\nu,\psi_E)\prod_{j=1}^n \gamma(s,\chi_j\mu_i\eta,\psi_E)\gamma(s,\bar\chi_j^\theta\mu_i\eta,\psi_E).\end{aligned}$$

□

8. The converse theorem and Base Change for the unitary groups

We begin by recalling the converse theorem of Cogdell and Piatetski-Shapiro [7]. In fact, we use a variant in the function field case [49] allowing for twists by a continuous character η (see § 2 of [8]). We then combine the Langlands-Shahidi method with the Converse Theorem and establish what is known as "weak" Base Change for globally generic representations.

8.1. **The converse theorem.** Fix a finite set of places S of a global function field K, a Grössenkaracter $\eta : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ and an integer N. Let $\mathcal{T}(S;\eta)$ be the set consisting of representations $\tau = \tau_0 \otimes \eta$ of $\mathrm{GL}_n(\mathbb{A}_K)$ such that: n is an integer ranging from $1 \leq n \leq N-1$; and, τ_0 is a cuspidal automorphic representation.

Let π of $\mathbf{G}_1(\mathbb{A}_k)$ and τ of $\mathbf{G}_2(\mathbb{A}_k)$ be admissible representations whose L-function $L(s,\pi\times\tau)$ converges on some right half plane. We say $L(s,\pi\times\tau)$ is *nice* if the following properties are satisfied:

(i) $L(s,\pi\times\tau)$ and $L(s,\tilde{\pi}\times\tilde{\tau})$ are polynomials in $\{q^{-s},q^s\}$.
(ii) $L(s,\pi\times\tau) = \varepsilon(s,\pi\times\tau)L(1-s,\tilde{\pi}\times\tilde{\tau})$.

We note that Property (i) implies that $L(s,\pi\times\tau)$ and $L(s,\tilde{\pi}\times\tilde{\tau})$ have analytic continuations to entire functions to the whole complex plane and are bounded on vertical strips.

Theorem 8.1 (Converse Theorem)**.** *Let $\Pi = \otimes\,\Pi_v$ be an irreducible admissible representation of $\mathrm{GL}_N(\mathbb{A}_K)$ whose central character ω_Π is a Grössenkaracter and whose L-function $L(s,\Pi) = \prod_v L(s,\Pi_v)$ is absolutely convergent in some right half-plane. Suppose that for every $\tau \in \mathcal{T}(S;\eta)$ the L-function $L(s,\Pi\times\tau)$ is nice. Then, there exists an automorphic representation Π' of $\mathrm{GL}_N(\mathbb{A}_K)$ such that $\Pi_v \cong \Pi'_v$ for all $v \notin S$.*

8.2. **Base change for the unitary groups.** Let K/k be a separable quadratic extension of global function fields. Let $\mathbb{A}_k$ and $\mathbb{A}_K$ denote the ring of adèles of k and K, respectively. We now turn towards Base Change from $\mathbf{G}_n = \mathrm{U}_N$ to $\mathbf{H}_N = \mathrm{Res}\,\mathrm{GL}_N$. The groups $\mathbf{G}_n$ and $\mathbf{H}_N$ are related via the following homomorphism of L-groups

$$\mathrm{BC} : {}^L G_n = \mathrm{GL}_N(\mathbb{C}) \rtimes \mathcal{W}_k \hookrightarrow {}^L H_N = \mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C}) \rtimes \mathcal{W}_k. \tag{8.1}$$

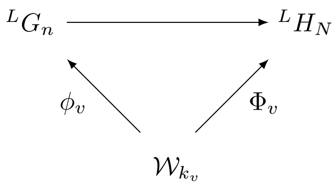
We say that a globally generic cuspidal automorphic representation $\pi = \otimes' \pi_v$ of $\mathbf{G}_n(\mathbb{A}_k)$ has a base change lift $\Pi = \otimes' \Pi_v$ to $\mathbf{H}_N(\mathbb{A}_k) = \mathrm{GL}_N(\mathbb{A}_K)$, if at every place where π_v is unramified, we have that

$$L(s,\pi_v) = L(s,\Pi_v).$$

This notion of a Base Change lift is sometimes referred to as a weak lift. The strong Base Change lift requires equality of L-functions and ε-factors at every place v of k. We will establish the strong Base Change lift in §§ 9-10.

Remark 8.2. *In order to be more precise, the base change map or Langlands functorial lift for the unitary groups obtained from* (8.1) *is known as "stable" base change. There is also "unstable" base change. See for example the "stable" and "labile" base change discussion for U_2 of* [13].

8.3. **Unramified Base Change.** Let $\pi = \otimes' \pi_v$ be a globally generic cuspidal automorphic representation of $\mathrm{U}_N(\mathbb{A}_k)$. Fix a place v of k that remains inert in K and such that π_v is unramified. Two unramified L-parameters $\phi_v : \mathcal{W}_{k_v} \to {}^L G_n$ and $\Phi_v : \mathcal{W}_{k_v} \to {}^L H_N$ are connected via the homomorphism of L-groups



given by the base change map of (8.1).

Each π_v, being uramified, is of the form

$$\pi_v \hookrightarrow \begin{cases} \mathrm{Ind}(\chi_{1,v} \otimes \cdots \chi_{n,v} \otimes \nu_v) & \text{if } N = 2n+1 \\ \mathrm{Ind}(\chi_{1,v} \otimes \cdots \chi_{n,v}) & \text{if } N = 2n \end{cases}, \tag{8.2}$$

with $\chi_{1,v}, \ldots, \chi_{n,v}$, unramified characters of $K_v^\times$. Let ϖ_v be a uniformizer and let

$$\alpha_{i,v} = \chi_{i,v}(\varpi_v), \ i = 1, \ldots, n.$$

Let Frob_v denote the Frobenius element of $\mathcal{W}_{k_v}$. We know that π_v is parametrized by the conjugacy class in ${}^L U$

$$(\phi_v(\mathrm{Frob}_v), w_\theta) = \begin{cases} \mathrm{diag}(\alpha_{1,v}^{\frac{1}{2}}, \ldots, \alpha_{n,v}^{\frac{1}{2}}, 1, \alpha_{n,v}^{-\frac{1}{2}}, \ldots, \alpha_{1,v}^{-\frac{1}{2}}) \rtimes w_\theta & \text{if } N = 2n+1 \\ \mathrm{diag}(\alpha_{1,v}^{\frac{1}{2}}, \ldots, \alpha_{n,v}^{\frac{1}{2}}, \alpha_{n,v}^{-\frac{1}{2}}, \ldots, \alpha_{1,v}^{-\frac{1}{2}}) \rtimes w_\theta & \text{if } N = 2n \end{cases}.$$

Then, from the results of [42], the L-parameter $\Phi_v = \mathrm{BC} \circ \phi_v$ corresponds a semisimple conjugacy class in $\mathrm{GL}_N(\mathbb{C})$ given by

$$\Phi_v(\mathrm{Frob}_v) = \begin{cases} \mathrm{diag}(\alpha_{1,v}, \ldots, \alpha_{n,v}, 1, \alpha_{n,v}^{-1}, \ldots, \alpha_{1,v}^{-1}) & \text{if } N = 2n+1 \\ \mathrm{diag}(\alpha_{1,v}, \ldots, \alpha_{n,v}, \alpha_{n,v}^{-1}, \ldots, \alpha_{1,v}^{-1}) & \text{if } N = 2n \end{cases}.$$

The resulting Satake parameters Φ_v, then uniquely determine an unramified representation Π_v of $\mathrm{Res}_{K_v/k_v} \mathrm{GL}_N(k_v) = \mathrm{GL}_N(K_v)$ of the form

$$\Pi_v \hookrightarrow \begin{cases} \mathrm{Ind}(\chi_{1,v} \otimes \cdots \chi_{n,v} \otimes 1 \otimes \chi_{n,v}^{-1} \otimes \cdots \otimes \chi_{1,v}^{-1}) & \text{if } N = 2n+1 \\ \mathrm{Ind}(\chi_{1,v} \otimes \cdots \chi_{n,v} \otimes \chi_{n,v}^{-1} \otimes \cdots \otimes \chi_{1,v}^{-1}) & \text{if } N = 2n \end{cases}. \tag{8.3}$$

To summarize, let $\hat{A}_v$ be the semisimple conjugacy class of $\phi_v(\mathrm{Frob}_v)$ in $\mathrm{GL}_N(\mathbb{C})$ obtained via the Satake parametrization. We have

$$\begin{array}{ccc} \pi \text{ of } \mathrm{U}_N(k_v) & \longrightarrow & \left\{(\hat{A}_v, w_{\theta,v})\right\} \text{ of } \mathrm{GL}_N(\mathbb{C}) \rtimes \mathcal{W}'_{k_v} \\ \mathrm{BC} \downarrow & & \downarrow \\ \Pi \text{ of } \mathrm{GL}_N(K_v) & \longleftarrow & \left\{(\hat{A}_v, \hat{A}_v, w_{\theta,v})\right\} \text{ of } \mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C}) \rtimes \mathcal{W}'_{k_v} \end{array}$$

Where we use the fact that there is a natural bijection between $w_{\theta,v}$-conjugacy classes of $\mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C})$ and conjugacy classes of $\mathrm{GL}_N(\mathbb{C})$.

Definition 8.3. *Let v be a place of k that remains inert in K. For every unramified π_v corresponding to ϕ_v we call the representation*

$$\mathrm{BC}(\pi_v) = \Pi_v,$$

corresponding to Φ_v as in (8.3)*, the unramified local Langlands lift or the unramified base change of π_v.*

Let τ_v be any irreducible admissible generic representation of $\mathrm{GL}_m(K_v)$. We know that, given the homomorphism of L-groups BC, we have the following equality of local factors

$$\gamma(s, \pi_v \times \tau_v, \psi_v) = \gamma(s, \Pi_v \times \tau_v, \psi_v)$$
$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v)$$
$$\varepsilon(s, \pi_v \times \tau_v, \psi_v) = \varepsilon(s, \Pi_v \times \tau_v, \psi_v).$$

8.4. **Split Base Change.** At split places v of k, we are in the case of a separable algebra, as in § 7.3. The local functorial lift of π_v to $\mathbf{H}_M(k_v)$ obtained from

$$\mathrm{U}_N(k_v) = \mathrm{GL}_N(k_v) \rightsquigarrow \mathbf{H}_N(k_v) = \mathrm{GL}_N(k_v) \times \mathrm{GL}_N(k_v).$$

Definition 8.4. *Fix a place v of k such that $K_v = k_v \times k_v$. Let π_v be an irreducible generic representation of $\mathrm{U}_N(k_v) \cong \mathrm{GL}_N(k_v)$. We call the representation*

$$\mathrm{BC}(\pi_v) = \pi_v \otimes \tilde{\pi}_v \tag{8.4}$$

the split local Langlands lift or the split base change of π_v.

Let $\tau_v = \tau_{1,v} \otimes \tau_{2,v}$ be any irreducible admissible generic representation of $\mathrm{GL}_m(K_v) = \mathrm{GL}_m(k_v) \times \mathrm{GL}_m(k_v)$. Then, from § 7.3, we have the following equality of local factors

$$\begin{aligned} \gamma(s, \pi_v \times \tau_v, \psi_v) &= \gamma(s, \pi_v \times \tau_v, \psi_v)\gamma(s, \tilde{\pi}_v \times \tilde{\tau}_v, \psi_v) \\ L(s, \pi_v \times \tau_v) &= L(s, \pi_v \times \tau_v)L(s, \tilde{\pi}_v \times \tilde{\tau}_v) \\ \varepsilon(s, \pi_v \times \tau_v, \psi_v) &= \varepsilon(s, \pi_v \times \tau_v, \psi_v)\varepsilon(s, \tilde{\pi}_v \times \tilde{\tau}_v, \psi_v). \end{aligned}$$

8.5. **Ramified Base Change.** At places v of k where the cuspidal automorphic representation π of $\mathrm{U}_N(\mathbb{A}_k)$ may have ramification, we can use the stable form hinged by Corollary 7.5. At this point, we do not have a unique ramified Base Change, even after twisting by a highly ramified character. However, we will establish a unique local Langlands lift or local Base Change completely in § 9.

Definition 8.5. *Let $\chi_{1,v}, \ldots, \chi_{n,v}$ be characters of $K_v^\times$. Let ν_v be a character of K_v^1, which we extend to one of $K_v^\times$ via Hilbert's theorem 90. Assume that the representation*

$$\Pi_v \hookrightarrow \begin{cases} \mathrm{Ind}(\chi_{1,v} \otimes \cdots \otimes \chi_{n,v} \otimes \nu_v \otimes \chi_{n,v}^{-1} \otimes \cdots \otimes \chi_{1,n}^{-1}) & \text{if } N = 2n+1 \\ \mathrm{Ind}(\chi_{1,v} \otimes \cdots \otimes \chi_{n,v} \otimes \chi_{n,v}^{-1} \otimes \cdots \otimes \chi_{1,n}^{-1}) & \text{if } N = 2n \end{cases}$$

has central character $\omega_{\Pi_v} = \omega_{\pi_v}$. Then Π_v is called a ramified local Langlands lift or a ramified Base Change of π_v.

We no longer have equality of local factors for every τ_v of $\mathrm{GL}_m(k_v)$. However, from Lemma 7.4, whenever $\eta_v : K_v^\times \to \mathbb{C}^\times$ is a highly ramified character, we have that

$$\begin{aligned} \gamma(s, \pi_v \times (\tau_v \cdot \eta_v), \psi_v) &= \gamma(s, \Pi_v \times (\tau_v \cdot \eta_v), \psi_v) \\ L(s, \pi_v \times (\tau_v \cdot \eta_v)) &= L(s, \Pi_v \times (\tau_v \cdot \eta_v)) \\ \varepsilon(s, \pi_v \times (\tau_v \cdot \eta_v), \psi_v) &= \varepsilon(s, \Pi_v \times (\tau_v \cdot \eta_v), \psi_v). \end{aligned}$$

8.6. **Weak Base Change.** We establish a preliminary version of Base Change for the unitary groups by combining the Langlands-Shahidi method with the Converse Theorem.

Theorem 8.6. *Let $\pi = \otimes' \pi_v$ be a globally generic cuspidal automorphic representation of $\mathrm{U}_N(\mathbb{A}_k)$. There exists a unique globally generic automorphic representation*

$$\mathrm{BC}(\pi) = \Pi$$

of $\mathrm{Res}_{K/k}\mathrm{GL}_N(\mathbb{A}_k) = \mathrm{GL}_N(\mathbb{A}_K)$, which is a weak Base Change lift of π.

Proof. Let $\Pi_v = \mathrm{BC}(\pi_v)$ be the local Base change of Definitions 8.3, 8.4 and 8.5, accordingly. Consider the irreducible admissible representation

$$\Pi = \otimes' \Pi_v$$

of $\mathrm{GL}_N(\mathbb{A}_K)$ whose central character ω_Π has ω_{Π_v} obtained from ω_{π_v} via Hilbert's theorem 90 at every place v of k. By construction, ω_Π is invariant under $K^\times$.

Let S be a finite set of places of k such that π_v is unramified for $v \notin S$. We abuse notation and identify S with the finite set of places of K lying above the places $v \in S$. Then, we have an equality of partial L-functions

$$L^S(s, \Pi) = L^S(s, \pi \times 1).$$

Hence, $L^S(s, \Pi)$ converges absolutely on a right hand plane; and so does $L(s, \Pi)$.

Let τ be a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_K)$. Choose a grössenkaracter $\eta = \otimes \eta_v : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ such that η_v is highly ramified for $v \in S$. Then, letting $\tau' = \tau \otimes \eta$, we have that $(K/k, \pi, \tau', \psi) \in \mathscr{L}_{\mathrm{glob}}(p, \mathrm{U}_N, \mathrm{GL}_m)$. We have seen in §§ 8.3, 8.4 and 8.5 that the following equality of local factors holds in every case:

$$L(s, \pi_v \times \tau_v') = L(s, \Pi_v \times \tau_v')$$
$$\varepsilon(s, \pi_v \times \tau_v', \psi_v) = \varepsilon(s, \Pi_v \times \tau_v', \psi_v).$$

With η as in Proposition 5.5, we know that the Langlands-Shahidi L-functions $L(s, \pi \times \tau')$ are polynomials in $\{q^s, q^{-s}\}$. They also satisfy the global functional equation, Theorem 4.1(vi). Thus, they are nice. Then, since

$$L(s, \Pi \times \tau') = L(s, \pi \times \tau') \text{ and } \varepsilon(s, \Pi \times \tau') = \varepsilon(s, \pi \times \tau'),$$

we can conclude that the L-functions $L(s, \Pi \times \tau')$ are nice, as τ' ranges through the set $\mathcal{T}(S; \eta)$. From the Converse Theorem, there now exists an automorphic representation Π' of $\mathrm{GL}_N(\mathbb{A}_K)$ such that $\Pi_v \cong \Pi_v'$ for all $v \notin S$. Then Π' gives a weak Base Change.

Now, from [33], every automorphic form Π of $\mathrm{GL}_N(\mathbb{A}_K)$ arises as a subquotient of the globally induced representation

$$\mathrm{Ind}(\Pi_1 \otimes \cdots \otimes \Pi_d), \tag{8.5}$$

with each Π_i a cuspidal automorphic representation of $\mathrm{GL}_N(\mathbb{A}_K)$. Since every Π_i is cuspidal, they are globally generic. The results on the classification of automorphic representations for general linear groups [23], shows that there exists a unique generic subquotient of (8.5), which we denote by

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d.$$

It is this automorphic representation Π which is our desired Base Change, i.e., we let $\mathrm{BC}(\pi) = \pi$. It has the property that at every place w of K where Π_w is unramified, it is generic. Hence, at places where π_v is unramified and $w = v$ remains inert, Π_w is given by the unique generic subquotient of a principal series representation and Π_w agrees with the local Base Change lift of sections § 8.3. At split places it also agrees with that of § 8.4 at almost all places. It thus agrees with Π' at almost all places and is itself a weak Base Change lift. Furthermore, by multiplicity one [50], any two globally generic automorphic representations of $\mathrm{GL}_N(\mathbb{A}_K)$ that agree at almost every place are equal. Hence Π is uniquely determined. □

9. On local Langlands functoriality and Strong Base Change

In Algebraic Number Theory there is a well known proof of existence for local class field theory from global class field theory. In an analogous fashion, we here prove the existence of the generic local Langlands functorial lift from a unitary group U_N to $\mathrm{Res}\,\mathrm{GL}_N$, i.e., local Base Change. We are guided by the discussion found in [9, 29]. The lift preserves local L-functions and root numbers. In general, we refer to § 10.2 for a discussion on reducing the study of local factors to the generic case. In § 7 of [14] we show how to establish Base Change in general, which preserves Plancherel measures for non-generic representations.

In § 9.5 we address how to strengthen the "weak" base change map of Theorem 8.6 so that it is compatibile with the local Langlands functorial lift or local base change. Throughout this section, we fix a quadratic extension E/F of non-archimedean local fields of positive characteristic. Given any general linear group $\mathrm{GL}_m(E)$, we let ν denote the unramified character obtained via the determinant, i.e., $\nu = |\det(\cdot)|_E$. Globally, we let K/k denote a separable quadratic extension of function fields.

Definition 9.1. *Let π be a generic representation of $\mathrm{U}_N(F)$. Then, we say that a generic irreducible representation Π of $\mathrm{GL}_N(E)$ is a* local base change lift *of π if for every supercuspidal representation τ of $\mathrm{GL}_m(E)$ we have that*

$$\gamma(s, \pi \times \tau, \psi_E) = \gamma(s, \Pi \times \tau, \psi_E).$$

9.1. **Uniqueness of the local base change lift.** The previous definition extends to twists by a general irreducible unitary generic representation τ of $\mathrm{GL}_m(E)$, as we show in the next lemma. For this, the clasification of [59] is very useful. It allows us to write

$$\tau = \mathrm{Ind}(\delta_1\nu^{t_1} \otimes \cdots \otimes \delta_d\nu^{t_d} \otimes \delta_{d+1} \otimes \cdots \otimes \delta_{d+k} \otimes \delta_d\nu^{-t_d} \otimes \cdots \otimes \delta_1\nu^{-t_1}), \tag{9.1}$$

where the δ_i's are unitary discrete series representations of $\mathrm{GL}_{n_i}(E)$ and $0 < t_d \leq \cdots \leq t_1 < 1/2$.

Furthermore, from the Zelevinsky classification [64], we know that every unitary discrete series representation δ of $\mathrm{GL}_m(E)$ is obtained from a segment of the form

$$\Delta = \left[\rho\nu^{-\frac{t-1}{2}}, \rho\nu^{\frac{t-1}{2}}\right],$$

where ρ is a supercuspidal representation of $\mathrm{GL}_e(E)$, $e|m$, and t is a positive integer. The representation δ is precisely the generic constituent of

$$\mathrm{Ind}(\rho\nu^{-\frac{t-1}{2}} \otimes \cdots \otimes \rho\nu^{\frac{t-1}{2}}). \tag{9.2}$$

An important result of Henniart [19] allows us to characterize the local Langlands functorial lift by the condition that it preserves local factors.

Lemma 9.2. *Let π be a generic representation of $\mathrm{U}_N(F)$ and suppose there exists Π, a local base change lift to $\mathrm{GL}_N(E)$. Then, for every irreducible unitary generic representation τ of $\mathrm{GL}_m(E)$ we have that*

$$\gamma(s, \pi \times \tau, \psi_E) = \gamma(s, \Pi \times \tau, \psi_E).$$

Furthermore, such a local base change lift Π is unique.

Proof. Given an irreducible unitary generic representation τ of $\mathrm{GL}_m(E)$, write τ in the form given by (9.1). Then, multiplicativity of γ-factors gives

$$\gamma(s, \pi \times \tau, \psi_E) = \prod_{i=1}^{k} \gamma(s, \pi \times \delta_{d+i}, \psi_E) \prod_{j=1}^{d} \gamma(s + t_j, \pi \times \delta_j, \psi_E)\gamma(s - t_j, \pi \times \delta_j, \psi_E).$$

And, similarly

$$\gamma(s, \Pi \times \tau, \psi_E) = \prod_{i=1}^{k} \gamma(s, \Pi \times \delta_{d+i}, \psi_E) \prod_{j=1}^{d} \gamma(s + t_j, \Pi \times \delta_j, \psi_E)\gamma(s - t_j, \Pi \times \delta_j, \psi_E).$$

In this way, we reduce the problem to proving the relation

$$\gamma(s, \pi \times \delta, \psi_E) = \gamma(s, \Pi \times \delta, \psi_E)$$

for discrete series representations δ of $\mathrm{GL}_m(E)$.

Now, we write the representation δ as the generic constituent of

$$\mathrm{Ind}(\rho\nu^{-\frac{t-1}{2}} \otimes \cdots \otimes \rho\nu^{\frac{t-1}{2}}),$$

as in (9.2). Then, using the multiplicativity property of γ-factors, we obtain

$$\begin{aligned} \gamma(s, \pi \times \delta, \psi_E) &= \prod_{l=0}^{t-1} \gamma(s - \frac{t-1}{2} + l, \pi \times \rho, \psi_E) \\ &= \prod_{l=0}^{t-1} \gamma(s - \frac{t-1}{2} + l, \Pi \times \rho, \psi_E) \\ &= \gamma(s, \Pi \times \delta, \psi_E). \end{aligned}$$

This shows that Π satisfies the desired relation involving γ-factors. That Π is unique then follows from Theorem 1.1 of [19]. □

Lemma 9.3. *Let π be a generic representation of $\mathrm{U}_N(F)$ and suppose it has a local Langlands functorial lift Π of $\mathrm{GL}_N(E)$. Then, for every irreducible unitary generic representation τ of $\mathrm{GL}_m(E)$ we have that*

$$\begin{aligned} L(s, \pi \times \tau) &= L(s, \Pi \times \tau) \\ \varepsilon(s, \pi \times \tau, \psi_E) &= \varepsilon(s, \Pi \times \tau, \psi_E). \end{aligned}$$

Proof. As in the previous Lemma, begin by writing the unitary generic representation τ of $\mathrm{GL}_m(E)$ as in (9.1), with discrete series as inducing data. And, write

$$\tau_0 = \mathrm{Ind}(\delta_{d+1} \otimes \cdots \otimes \delta_{d+k}).$$

Then, using Properties (ix) and (x) of Theorem 7.3, we obtain

$$L(s,\pi\times\tau)=L(s,\pi\times\tau_0)\prod_{i=1}^{d}L(s+t_i,\pi\times\delta_i)L(s-t_i,\pi\times\delta_i),$$

$$\varepsilon(s,\pi\times\tau,\psi_E)=\varepsilon(s,\pi\times\tau_0,\psi_E)\prod_{i=1}^{d}\varepsilon(s+t_i,\pi\times\delta_i,\psi_E)\varepsilon(s-t_i,\pi\times\delta_i,\psi_E).$$

Now, for the factors involving τ_0, we use Langlands classification to express π as a Langlands quotient of

$$\mathrm{Ind}(\pi_1\otimes\cdots\otimes\pi_e\otimes\pi_0).$$

Then π is the generic constituent, where $\mathbf{P}$ is the parabolic subgroup of U_N with Levi $\mathbf{M}=\prod_{i=1}^{d}\mathrm{Res}\,\mathrm{GL}_{m_i}\times\mathrm{U}_{N_0}$ and each π_i is a quasi-tempered representation of $\mathrm{GL}_{m_i}(E)$. Now, Properties (ix) and (x) of Theorem 7.3 in this situation directly give

$$L(s,\pi\times\tau_0)=L(s,\pi_0\times\tau_0)\prod_{i=1}^{e}L(s,\pi_i\times\tau_0),$$

$$\varepsilon(s,\pi\times\tau_0,\psi_E)=\varepsilon(s,\pi_0\times\tau_0,\psi_E)\prod_{i=1}^{e}\varepsilon(s,\pi_i\times\tau_0,\psi_E).$$

All representations involved in the previous two equations are quasi-tempered. Each individual local factor on the RHS of these equations can be shifted by Property (ix) of Theorem 7.3, which leads to an L-functions and ε-factor involving tempered representations. The connection to γ-factors, and the previous lemma, is now made via Property (viii) of Theorem 7.3 in addition to Property (xi). Now, multiplicativity of γ-factors leads to

$$L(s,\pi\times\tau_0)=\prod_{l=1}^{k}L(s,\pi\times\delta_{d+l}),$$

$$\varepsilon(s,\pi\times\tau_0,\psi_E)=\prod_{l=1}^{k}\varepsilon(s,\pi\times\delta_{d+l},\psi_E).$$

The properties of [21], for example, can be applied to Rankin-Selberg factors to obtain

$$L(s,\Pi\times\tau)=\prod_{l=1}^{k}L(s,\Pi\times\delta_{d+l})\prod_{i=1}^{d}L(s+t_i,\pi\times\delta_i)L(s-t_i,\pi\times\delta_i),$$

$$\varepsilon(s,\Pi\times\tau,\psi_E)=\prod_{l=1}^{k}\varepsilon(s,\Pi\times\delta_{d+l},\psi_E)\prod_{i=1}^{d}\varepsilon(s+t_i,\pi\times\delta_i,\psi_E)\varepsilon(s-t_i,\pi\times\delta_i,\psi_E).$$

Where we reduced to proving

$$L(s,\pi\times\rho)=L(s,\Pi\times\rho)$$
$$\varepsilon(s,\pi\times\rho,\psi_E)=\varepsilon(s,\Pi\times\rho,\psi_E)$$

for discrete series representations ρ of $\mathrm{GL}_m(E)$. Indeed, we now address this case in what follows.

For the irreducible unitary generic representation Π of $\mathrm{GL}_N(E)$, we use (9.1) to write

$$\Pi = \mathrm{Ind}(\xi_1\nu^{r_1} \otimes \cdots \otimes \xi_f\nu^{r_f} \otimes \xi_{f+1} \otimes \cdots \otimes \xi_{f+h} \otimes \xi_f\nu^{-r_f} \otimes \cdots \otimes \xi_1\nu^{-r_1}), \tag{9.3}$$

with each ξ_i a discrete series and $0 < r_f \le \cdots \le r_1 < 1/2$.

$$\begin{aligned}\gamma(s, \pi \times \rho, \psi_E) = &\prod_{i=1}^{h} \gamma(s, \xi_{f+i} \times \rho, \psi_E)\\ &\prod_{j=1}^{f} \gamma(s + r_j, \xi_j \times \rho, \psi_E)\gamma(s - r_j, \xi_j \times \rho, \psi_E).\end{aligned}$$

Each γ-factor on the right hand side of the previous expression involves discrete series (hence tempered) representations. Thus, each factor has a corresponding L-function and root number via Property (viii) of Theorem 4.1. The product

$$P(q_F^{-s})^{-1} = \prod_{i=1}^{h} L(s, \xi_{f+i} \times \rho) \prod_{j=1}^{f} L(s + r_j, \xi_j \times \rho)L(s - r_j, \xi_j \times \rho).$$

From the Proposition on p. 451 of [22], each $L(s, \xi_i \times \rho)$ has no poles for $\Re(s) > 0$. And, since $r_j < 1/2$, the function $P(q_F^{-s})$ is non-zero for $\Re(s) \ge 1/2$. Now, the product

$$Q(q_F^{-s})^{-1} = \prod_{i=1}^{h} L(1 - s, \tilde{\xi}_{f+i} \times \tilde{\rho}) \prod_{j=1}^{f} L(1 - s - r_j, \tilde{\xi}_j \times \tilde{\rho})L(1 - s - r_j, \tilde{\xi}_j \times \tilde{\rho})$$

is in turn non-zero for $\Re(s) \le 1/2$. Then, Property (viii) of Theorem 4.1 gives the relation

$$\gamma(s, \pi \times \rho, \psi_E) \sim \frac{P(q_F^{-s})}{Q(q_F^{-s})},$$

which is an equality up to a monomial in q_F^{-s}. More precisely, the monomial is the root number, which we can decompose as

$$\begin{aligned}\varepsilon(s, \pi \times \rho, \psi_E) = &\prod_{i=1}^{h} \varepsilon(s, \xi_{f+i} \times \rho, \psi_E)\\ &\prod_{j=1}^{f} \varepsilon(s + r_j, \xi_j \times \rho, \psi_E)\varepsilon(s - r_j, \xi_j \times \rho, \psi_E)\\ =&\varepsilon(s, \Pi \times \rho, \psi_E).\end{aligned}$$

Notice that the regions where $P(q_F^{-s})$ and $Q(q_F^{-s})$ may be zero do not intersect. Hence, there are no cancellations involving the numerator and denominator of $\gamma(s, \pi \times \rho, \psi_E)$. This shows that

$$L(s, \pi \times \rho) = \frac{1}{P(q_F^{-s})} = L(s, \Pi \times \rho),$$

where the second equality follows using the form (9.3) of Π and the multiplicativity property of Rankin-Selberg L-functions. □

Lemma 9.4. *Let π be a tempered generic representation of $\mathrm{U}_N(F)$ and suppose it has a local Langlands functorial lift Π of $\mathrm{GL}_N(E)$. Then, Π is also tempered.*

Proof. We proceed by contradiction. If Π is not tempered, then there is at least one $r_{i_0} > 0$ in the decomposition (9.3) of Π. From the previous lemma, we know that

$$L(s, \pi \times \rho) = L(s, \Pi \times \rho)$$

is valid for any discrete series representation ρ of $\mathrm{GL}_m(E)$. Take $\rho = \tilde{\xi}_{i_0}$. From the holomorphy of tempered L-functions, $L(s, \pi \times \rho)$ has no poles in the region $\Re(s) > 0$. On the other hand

$$L(s, \Pi \times \tilde{\xi}_{i_0}) = \prod_{i=1}^{h} L(s, \xi_{f+i} \times \tilde{\xi}_{i_0}) \prod_{j=1}^{f} L(s + r_j, \xi_j \times \tilde{\xi}_{i_0}) L(s - r_j, \xi_j \times \tilde{\xi}_{i_0})$$

has a pole at $s = r_{i_0}$, due to the term $L(s - r_{i_0}, \xi_{i_0} \times \tilde{\xi}_{i_0})$. And, $L(s, \Pi \times \tilde{\xi}_{i_0})$ inherits this pole, which gives the contradiction. Hence, it must be the case that $f = 0$ in equation (9.3) and we have that

$$\Pi = \mathrm{Ind}(\xi_1 \otimes \cdots \otimes \xi_h)$$

is thus tempered. □

9.2. **A global to local result.** We prove that global Base Change is compatible with local Base Change. At unramified and split places it is that of §§ 8.3 and 8.4; the central character obtained via (6.1).

Proposition 9.5. *Given a globally generic cuspidal automorphic representation $\pi = \otimes' \pi_v$ of $\mathrm{U}_N(\mathbb{A}_k)$, let $\mathrm{BC}(\pi) = \Pi = \otimes' \Pi_v$ be the base change lift of Theorem 8.6. Then, for every v we have:*

- (i) *Π_v is the uniquely determined local base change of π_v;*
- (ii) *Π_v is unitary with central character ω_{Π_v} of $K_v^\times$ obtained from the character ω_{π_v} of K_v^1 via Hilbert's theorem 90.*

Proof. The base change lift $\Pi = \mathrm{BC}(\pi)$, being globally generic, has every local Π_v generic. At every place where π_v is unramified, Π_v is a constituent of an unramified principal series representation by construction. Since Π_v is generic, it has to be the unique generic constituent of the principal series representation. Thus

$$\Pi_v = \mathrm{BC}(\pi_v) \tag{9.4}$$

is the unramified local base change of § 8.3, if v is inert, and that of § 8.4, if v is split.

Fix a place v_0 of k which remains inert in K. We wish to show that

$$\gamma(s, \pi_{v_0} \times \tau_0, \psi_{v_0}) = \gamma(s, \Pi_{v_0} \times \tau_0, \psi_{v_0})$$

for every generic $(K_{v_0}/k_{v_0}, \pi_{v_0}, \tau_0, \psi_{v_0}) \in \mathscr{L}_{\mathrm{loc}}(p, \mathrm{U}_N, \mathrm{GL}_m)$ with τ_0 supercuspidal. Let S be a finite set of places of k, not containing v_0, such that π_v is unramified for $v \notin S \cup \{v_0\}$. Via Property (v) of Theorem 7.3, we may assume that ψ_{v_0} is the component of a global additive character $\psi = \otimes \psi_v : K \backslash \mathbb{A}_K \to \mathbb{C}^\times$.

Via Lemma 4.2, there is a cuspidal automorphic representation $\tau = \otimes' \tau_v$ of $\mathrm{GL}_m(\mathbb{A}_K)$ such that $\tau_{v_0} = \tau_0$ and τ_v is unramified for all $v \notin S$. Now, using the Grunwald-Wang theorem of class field theory [2], there exists a Grössencharakter $\eta : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ such that $\eta_{v_0} = 1$ and η_v is highly ramified for all $v \in S$ such that v remains inert in K.

At places $v \in S$, which remain inert in K, we have the stable form of Lemma 7.4. Indeed, after twisting τ_v by the highly ramified character η_v we have

$$\gamma(s, \pi_v \times (\tau_v \cdot \eta_v), \psi_v) = \gamma(s, \Pi_v \times (\tau_v \cdot \eta_v), \psi_v).$$

At split places $v \in S$, we write $\tau_v = \tau_{1,v} \otimes \tau_{2,v}$ as a representation of $\mathrm{Res}_{K_v/k_v}\mathrm{GL}_m(k_v) = \mathrm{GL}_m(k_v) \times \mathrm{GL}_m(k_v)$ with each $\tau_{1,v}$ and $\tau_{2,v}$ supercuspidal. From § 7.3 (ii), the Langlands-Shahidi γ-factors are given by

$$\gamma(s, \pi_v \times \tau_v, \psi_v) = \gamma(s, \pi_v \times \tau_{1,v}, \psi_v)\gamma(s, \tilde{\pi}_v \times \tau_{2,v}, \psi_v),$$

which are compatible with the split Base Change map of § 8.4. Now, consider $\tau' = \tau \otimes \eta$. For $(K/k, \pi, \tau', \psi) \in \mathscr{L}_{\mathrm{glob}}(p, \mathrm{U}_N, \mathrm{GL}_m)$ we have the global functional equation

$$L^S(s, \pi \times \tau') = \gamma(s, \pi_{v_0} \times \tau_{v_0}, \psi_v) \prod_{v \in S - \{v_0\}} \gamma(s, \pi_v \times (\tau_v \cdot \eta_v), \psi_v) L^S(1 - s, \tilde{\pi} \times \tilde{\tau}'),$$

and for $(K/k, \Pi, \tau', \psi) \in \mathscr{L}_{\mathrm{glob}}(p, \mathrm{GL}_N, \mathrm{GL}_m)$ the functional equation for Rankin-Selberg products reads

$$L^S(s, \Pi \times \tau') = \gamma(s, \Pi_{v_0} \times \tau_{v_0}, \psi_v) \prod_{v \in S - \{v_0\}} \gamma(s, \Pi_v \times (\tau_v \cdot \eta_v), \psi_v) L^S(1 - s, \tilde{\Pi} \times \tilde{\tau}').$$

At unramified places, it follows from equation (9.4) that local L-functions agree. This gives equality of the corresponding partial L-functions appearing in the above two functional equations. We thus obtain

$$\gamma(s, \pi_{v_0} \times \tau_{v_0}, \psi_{v_0}) = \gamma(s, \Pi_{v_0} \times \tau_{v_0}, \psi_{v_0}).$$

Since our choice of supercuspidal $\tau_0 = \tau_{v_0}$ of $\mathrm{GL}_m(E)$ was arbitrary, we have that Π_0 is a local base change for π_0. Uniqueness follows from Lemma 9.2. Proving property (i) as desired.

Note that the central character $\omega_\pi = \otimes \omega_{\pi_v}$ of π is an automorphic representation of $\mathrm{U}_1(\mathbb{A}_k)$. Now, let $\chi = \otimes \chi_v$ be defined on $\mathrm{GL}_1(\mathbb{A}_K)$ from ω_π via Hilbert's theorem 90. Namely, we let $\mathfrak{h}_v : x_v \mapsto x_v \bar{x}_v^{-1}$ be the continuous map of (6.1) and let

$$\chi_v = \omega_{\pi_v} \circ \mathfrak{h}_v.$$

at every place v of k; we view K_v as a degree-2 finite étale algebra over k_v. Then $\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ is a Grössencharakter. Also, χ and ω_Π are continuous Grössencharakters such that χ_v agrees with ω_{Π_v} at every $v \notin S \cup \{v_0\}$. Hence $\chi = \omega_\Pi$. Thus, the central character of Π_{v_0} is $\omega_{\Pi_{v_0}} = \chi_{v_0}$, which is obtained via $\omega_{\pi_{v_0}}$ as in property (ii) of the Proposition. □

Remark 9.6. *Let π_0 be a generic representation of $\mathrm{U}_N(F)$. Suppose there is a globally generic cuspidal automorphic representation $\pi = \otimes' \pi_v$ of $\mathrm{U}_N(\mathbb{A}_k)$ with $\pi_0 = \pi_{v_0}$ at some place v_0 of k, where $k_{v_0} = F$. Then it follows from Proposition 9.5 that it has a uniquely determined local base change $\Pi_0 = \mathrm{BC}(\pi_0)$.*

9.3. **Supercuspidal lift.** To avoid confusion between local and global notation, from here until the end of § 9, we use π_0 to denote a local representation of a unitary group and Π_0 for its local base change, when it exists. We use $\pi = \otimes' \pi_v$ for a cuspidal automorphic representation of a unitary group and $\Pi = \otimes' \Pi_v$ for its its corresponding global base change.

Proposition 9.7. *Let π_0 be a generic supercuspidal representation of $\mathrm{U}_N(F)$. Then π_0 has a unique local base change*

$$\Pi_0 = \mathrm{BC}(\pi_0)$$

to $\mathrm{GL}_N(E)$. The central character ω_{Π_0} of Π_0 is the character of $E^\times$ obtained from ω_{π_0} of E^1 via Hilbert's theorem 90. Moreover

$$\Pi_0 = \mathrm{Ind}(\Pi_{0,1} \otimes \cdots \otimes \Pi_{0,d}),$$

where each $\Pi_{0,i}$ is a supercuspidal representation of $\mathrm{GL}_{N_i}(E)$ satisfying: $\Pi_{0,i} \cong \widetilde{\Pi}_{0,i}^{\theta}$; $\Pi_{0,i} \not\cong \Pi_{0,j}$ for $i \neq j$; and

(i) *$L(s, \Pi_{0,i}, r_{\mathcal{A}})$ has a pole at $s = 0$ if N is odd;*
(ii) *$L(s, \Pi_{0,i} \otimes \eta_{E/F}, r_{\mathcal{A}})$ has a pole at $s = 0$ if N is even.*

Proof. Let k be a global function field with $k_{v_0} = F$. From Lemma 4.2, there exists a globally generic cuspidal automorphic representation $\pi = \otimes \pi_v$ of $\mathrm{U}_N(\mathbb{A}_k)$ such that $\pi_0 = \pi_{v_0}$. Then, Remark 9.6 gives the existence of a unique base change $\Pi_0 = \mathrm{BC}(\pi_0)$.

By Lemma 9.4, Π_0 is a unitary tempered representation of $\mathrm{GL}_m(E)$. Hence, we have that

$$\Pi_0 = \mathrm{Ind}(\Pi_{0,1} \otimes \cdots \otimes \Pi_{0,d}),$$

with each $\Pi_{0,i}$ a discrete series representation. Via (9.2), each Π_i is the generic constituent of

$$\mathrm{Ind}(\rho_i \nu^{-\frac{t_i-1}{2}} \otimes \cdots \otimes \rho_i \nu^{\frac{t_i-1}{2}}),$$

where ρ_i is a supercuspidal representation of $\mathrm{GL}_{m_i}(E)$, $m_i | m$, and t_i is an integer.

We look at a fixed $\Pi_{0,j}$. Due to the fact that all of the representations involved are tempered, we have

$$L(s, \Pi_0 \times \tilde{\Pi}_{0,j}) = \prod_{i=1}^{d} L(s, \Pi_{0,i} \times \tilde{\Pi}_{0,j}) = \prod_{i=1}^{d} \prod_{l=0}^{t_i} \prod_{r=0}^{t_j} L(s + l + r - \frac{t_i - 1}{2} - \frac{t_j - 1}{2}, \rho_l \times \tilde{\rho}_r).$$

The L-function $L(s + t_j - 1, \rho_j \times \tilde{\rho}_j)$ on the right hand side gives that the product has a pole at $s = 1 - t_j$. Now, the tempered L-function

$$L(s, \Pi_0 \times \tilde{\Pi}_{0,j}) = L(s, \pi_0 \times \tilde{\Pi}_{0,j})$$

is holomorphic for $\Re(s) > 0$. This contradicts the fact that $L(s, \Pi_0 \times \tilde{\Pi}_{0,j})$ has a pole at $s = 1 - t_j$, unless $t_j = 1$. This forces $\Pi_{0,j} = \rho_j$, in addition to $L(s, \pi_0 \times \tilde{\Pi}_{0,j})$ having a pole at $s = 0$. The argument proves that our fixed $\Pi_{0,j}$ is supercuspidal.

Now, let $\mathbf{P} = \mathbf{MN}$ be the parabolic subgroup of U_{2m+N} with Levi $\mathbf{M} \cong \mathrm{Res}\,\mathrm{GL}_m \times \mathrm{U}_N$. Then Proposition 5.7 tells us that $L(s, \pi_0 \times \tilde{\Pi}_{0,j})$ has a pole at $s = 0$ if and only if

$$\mathrm{ind}_P^{\mathrm{U}_{2m+N}(F)}(\tilde{\Pi}_{0,j} \otimes \tilde{\pi}_0)$$

is irreducible and $\tilde{\Pi}_{0,j} \otimes \tilde{\pi}_0 = w_0(\tilde{\Pi}_{0,j} \otimes \tilde{\pi}_0) \cong \Pi_{0,j}^{\theta} \otimes \tilde{\pi}_0$. This gives, for each j, that $\Pi_{0,j} \cong \tilde{\Pi}_{0,j}^{\theta}$. Furthermore, we have

$$L(s, \pi_0 \times \tilde{\Pi}_{0,j}) = L(s, \Pi_0 \times \tilde{\Pi}_{0,j}) = \prod_{i=0}^{d} L(s, \Pi_{0,i} \times \tilde{\Pi}_{0,j}).$$

Each L-function in the the product of the right hand side has a pole at $s = 0$ whenever $\Pi_{0,i} \cong \Pi_{0,j}$. However, the pole of $L(s, \pi_0 \times \tilde{\Pi}_{0,j})$ at $s = 0$ being simple, forces $\Pi_{0,i} \not\cong \Pi_{0,j}$ for $i \neq j$.

From the fact that $\Pi_{0,i} \cong \tilde{\Pi}_{0,i}^{\theta}$ and Proposition 6.4, we have

$$L(s, \Pi_{0,i} \times \tilde{\Pi}_{0,i}) = L(s, \Pi_{0,i}, r_{\mathcal{A}}) L(s, \Pi_{0,i} \otimes \eta_{E/F}, r_{\mathcal{A}}).$$

Then, one and only one of $L(s, \Pi_{0,i}, r_{\mathcal{A}})$ or $L(s, \Pi_{0,i} \otimes \eta_{E/F}, r_{\mathcal{A}})$ has a simple pole at $s = 0$. With the notation of § 6.6, we have $(E/F, \tilde{\Pi}_{0,i} \otimes \tilde{\pi}_0, \psi_E) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{M}, \mathbf{G})$ for $\mathbf{M} = \mathrm{Res}\, \mathrm{GL}_{m_i} \times \mathrm{U}_N$ and $\mathbf{G} = \mathrm{U}_{2m_i+N}$. By Proposition 5.7, the product

$$L(s, \tilde{\Pi}_{0,i} \otimes \tilde{\pi}_0, r_1) L(2s, \tilde{\Pi}_{0,i}, r_2)$$

has a simple pole at $s = 0$. Using $\Pi_{0,i} \cong \tilde{\Pi}_{0,i}^{\theta}$ and Proposition 6.4, we have that

$$L(s, \Pi_{0,i}, r_2) = L(s, \tilde{\Pi}_{0,i}, r_2) = \begin{cases} L(s, \Pi_{0,i}, r_{\mathcal{A}}) & \text{if } N = 2n \\ L(s, \Pi_{0,i} \otimes \eta_{E/F}, r_{\mathcal{A}}) & \text{if } N = 2n+1 \end{cases}.$$

Since we showed above that $L(s, \pi_0 \times \tilde{\Pi}_{0,i}) = L(s, \tilde{\Pi}_{0,i} \otimes \tilde{\pi}_0, r_1)$ has a simple pole at $s = 0$, then $L(2s, \tilde{\Pi}_{0,i}, r_2)$ cannot. Thus, depending on wether N is even or odd, the other one between $L(s, \Pi_{0,i}, r_{\mathcal{A}})$ and $L(s, \Pi_{0,i} \otimes \eta_{E/F}, r_{\mathcal{A}})$ must have a pole at $s = 0$. □

9.4. **Discrete series, tempered representations and Langlands classification.** Thanks to the work of Mœglin and Tadić [43], we have the classification of generic discrete series representations for the unitary groups. Their work allows us to obtain a generic discrete series representation ξ of $\mathrm{U}_N(F)$ as a subrepresentation as follows

$$\xi \hookrightarrow \mathrm{Ind}(\delta_1 \otimes \cdots \otimes \delta_d \otimes \delta_1' \otimes \cdots \delta_e' \otimes \pi_0). \tag{9.5}$$

Here, for $1 \leq i \leq d$ and $1 \leq j \leq e$, we have essentially square integrable representations δ_i of $\mathrm{GL}_{l_i}(E)$ and δ_j' of $\mathrm{GL}_{m_j}(E)$. The representation π_0 is a generic supercuspidal of $\mathrm{U}_{N_0}(F)$, with N_0 of the same parity as N. We refer to [14] for a discussion including non-generic representations and the basic assumption (BA) that is made in [43].

We can apply the Zelevinsky classification [64], to the essentially discrete representations of general linear groups appearing in the decomposition (9.5). They are obtained via segments of the form

$$\Delta = \left[\rho\nu^{-b}, \rho\nu^{a}\right],$$

where $a, b \in \frac{1}{2}\mathbb{Z}$, $a > b > 0$, and ρ is a supercuspidal representation of $\mathrm{GL}_f(E)$, $f|l_i$ or m_j, respectively.

The Mœglin-Tadić classification involves a further refinement of the segments corresponding to each δ_i and δ_j'. More precisely, let $a_i > b_i > 0$ now be integers of the same parity. Then we have

$$\delta_i = \mathrm{Ind}\left(\rho_i \nu^{-\frac{b_i-1}{2}} \otimes \cdots \otimes \rho_i \nu^{\frac{a_i-1}{2}}\right), \tag{9.6}$$

where ρ_i is a supercuspidal representation of $\mathrm{GL}_f(E)$, $f|l_i$. Furthermore, we have $\rho_i \cong \tilde{\rho}_i^{\theta}$. Next, let a_j' be a positive integer. We set $\epsilon_j = 1/2$ if a_j' is even and $\epsilon_j = 1$ if a_j' is odd. Then we have

$$\delta_j' = \mathrm{Ind}\left(\rho_j' \nu^{\epsilon_j} \otimes \cdots \otimes \rho_j' \nu^{\frac{a_j'-1}{2}}\right), \tag{9.7}$$

with $\rho_j' \cong (\tilde{\rho}_j')^{\theta}$ a supercuspidal representation of $\mathrm{GL}_{f'}(E)$, $f'|m_j$. Furthermore, the integer a_j' will be odd if $L(s, \pi_0 \times \rho')$ has a pole at $s = 0$, and a_j' will be even otherwise. This is due to (BA) of [43] concerning the reducibility of the induced representation $\mathrm{Ind}(\rho_j' \nu^s \otimes \pi_0)$ at $s = 1/2$ or 1, which is addressed in § 5.5 for generic representations.

Proposition 9.8. *Let ξ_0 be a generic discrete series representation of $\mathrm{U}_N(F)$, which we can write as*

$$\xi_0 \hookrightarrow \mathrm{Ind}(\delta_1 \otimes \cdots \otimes \delta_d \otimes \delta_1' \otimes \cdots \otimes \delta_e' \otimes \pi_0) \tag{9.8}$$

with π_0 a generic supercuspidal of $\mathrm{U}_{N_0}(F)$ and δ_i, δ_j' as in (9.5). Then ξ_0 has a uniquely determined base change

$$\Xi_0 = \mathrm{BC}(\xi_0),$$

which is a tempered generic representation of $\mathrm{GL}_N(E)$ satisfying $\Xi_0 \cong \tilde{\Xi}_0^\theta$. The central character ω_{Ξ_0} of Ξ_0 is the character of $E^\times$ obtained from ω_{ξ_0} of E^1 via Hilbert's theorem 90. Moreover, the lift Ξ_0 is the generic constituent of an induced representation:

$$\Xi_0 \hookrightarrow \mathrm{Ind}\left(\delta_1 \otimes \cdots \otimes \delta_d \otimes \delta_1' \otimes \cdots \otimes \delta_e' \otimes \Pi_0 \otimes \tilde{\delta}_e'^{\,\theta} \otimes \cdots \otimes \tilde{\delta}_1'^{\,\theta} \otimes \tilde{\delta}_d^\theta \otimes \cdots \otimes \tilde{\delta}_1^\theta\right),$$

The representation Π_0 is the local Langlands functorial lift of π_0 of Proposition 9.7.

Proof. Let ξ_0 be as in the statement of the Proposition, and consider quadruples $(E/F, \xi_0, \rho, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathrm{U}_N, \mathrm{Res}\,\mathrm{GL}_m)$ such that ρ is an arbitrary supercuspidal representation. To π_0 there corresponds a Π_0 via Proposition 9.7. With Ξ_0 as in the Proposition, we have $(E/F, \Xi_0, \rho, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathrm{Res}\,\mathrm{GL}_N, \mathrm{Res}\,\mathrm{GL}_m)$. Then we can use multiplicativity of γ-factors to obtain

$$\begin{aligned}
\gamma(s, \xi_0 \times \rho, \psi_E) &= \gamma(s, \pi_0 \times \rho, \psi_E) \prod_{i=1}^{d} \prod_{j=1}^{e} \gamma(s, \delta_i \times \rho, \psi_E) \gamma(s, \delta_j' \times \rho, \psi_E) \\
&= \gamma(s, \Pi_0 \times \rho, \psi_E) \prod_{i=1}^{d} \prod_{j=1}^{e} \gamma(s, \delta_i \times \rho, \psi_E) \gamma(s, \delta_j' \times \rho, \psi_E) \\
&= \gamma(s, \Xi_0 \times \rho, \psi_E).
\end{aligned}$$

From Lemma 9.2, we have that Ξ_0 is the unique local Langlands lift of ξ_0. It satisfies $\Xi_0 \cong \tilde{\Xi}_0^\theta$ and has the right central character.

For each i, $1 \leq i \leq d$, let τ_i be the generic constituent of $\mathrm{Ind}(\delta_i \otimes \tilde{\delta}_i^\theta)$. After rearranging the factors coming from equation (9.6), we can see that τ_i is isomorphic to the generic constituent of

$$\mathrm{Ind}\left((\rho_i \nu^{-\frac{a_i - 1}{2}} \otimes \cdots \otimes \rho_i \nu^{\frac{a_i - 1}{2}}) \otimes (\rho_i \nu^{-\frac{b_i - 1}{2}} \otimes \cdots \otimes \rho_i \nu^{\frac{b_i - 1}{2}})\right).$$

Recall that $\rho_i \cong \tilde{\rho}_i^\theta$. Hence, each τ_i is tempered and satisfies $\tau_i \cong \tilde{\tau}_i^\theta$. We proceed similarly with τ_j', $1 \leq j \leq e$, the generic constituent of $\mathrm{Ind}(\delta_j' \otimes \tilde{\delta}_j'^{\,\theta})$, now with the aid of equation (9.7). We conclude that τ_j' is tempered and satisfies $\tau_j' \cong \tilde{\tau}_j'^{\,\theta}$. We then rearrange the inducing data for Ξ_0 to obtain the form

$$\Xi_0 \hookrightarrow \mathrm{Ind}\left(\tau_1 \otimes \cdots \otimes \tau_d \otimes \tau_1' \otimes \cdots \otimes \tau_e' \otimes \Pi_0\right).$$

Each τ_i, τ_i' and Π_0 being tempered, we conclude that Ξ_0 is also tempered. □

We now turn to the tempered case, which is crucial, since L-functions and ε-factors are defined via γ-factors in this case. The proofs of the remaining two results are now similar to the case of a discrete series, thanks to Lemma 9.2.

Proposition 9.9. *Let τ_0 be a generic tempered representation of $\mathrm{U}_N(F)$, which we can write as*

$$\tau_0 \hookrightarrow \mathrm{Ind}\left(\delta_1 \otimes \cdots \otimes \delta_d \otimes \xi_0\right),$$

with each δ_i a discrete series representation of $\mathrm{GL}_{n_i}(E)$ and ξ_0 is one of $\mathrm{U}_{N_0}(F)$. Then τ_0 has a uniquely determined base change

$$T_0 = \mathrm{BC}(\tau_0),$$

which is a tempered generic representation $\mathrm{GL}_N(E)$ satisfying $T_0 \cong \tilde{T}_0^\theta$. The central character ω_{T_0} obtained from ω_{τ_0} via Hilbert's theorem 90. Specifically, the lift T_0 is of the form

$$T_0 = \mathrm{Ind}\left(\delta_1 \otimes \cdots \otimes \delta_d \otimes \Xi_0 \otimes \tilde{\delta}_d{}^\theta \otimes \cdots \otimes \tilde{\delta}_1^\theta\right).$$

The representation Ξ_0 is the base change lift of Proposition 9.8.

Proof. The proof is now along the lines of Proposition 9.8, where we use multiplicativity of γ-factors and the fact that Ξ_0 is the local Langlands lift of the discrete series ξ_0. This way, we obtain equality of γ-factors to apply Lemma 9.2 and conclude that $T_0 \cong \tilde{T}_0^\theta$ and has the correct central character. □

In general we have the Langlands quotient [4, 57]. The work of Muić on the standard module conjecture [48] helps us to realize a general generic representation π_0 of $\mathrm{U}_N(F)$ as the unique irreducible generic quotient of an induced representation. More precisely, π_0 is the Langlands quotient of

$$\mathrm{Ind}\left(\tau_1' \otimes \cdots \otimes \tau_d' \otimes \tau_0\right), \tag{9.9}$$

with each τ_i' a quasi-tempered representation of $\mathrm{GL}_{n_i}(E)$ and τ_0 a tempered representation of $\mathrm{U}_{N_0}(F)$. We can write $\tau_i' = \tau_{i,0}\nu^{t_i}$ with $\tau_{i,0}$ tempered and the Langlands parameters have $0 \leq t_1 \leq \cdots \leq t_d$.

Theorem 9.10. *Let π_0 be a generic representation of $\mathrm{U}_N(F)$. Write π_0 as the Langlands quotient of*

$$\mathrm{Ind}\left(\tau_1' \otimes \cdots \otimes \tau_d' \otimes \tau_0\right),$$

as in (9.9). Then π_0 has a unique generic local base change

$$\Pi_0 = \mathrm{BC}(\pi_0),$$

which is a generic representation of $\mathrm{GL}_N(E)$ satisfying $\Pi_0 \cong \tilde{\Pi}_0^\theta$. The central character ω_{Π_0} obtained from ω_{π_0} via Hilbert's theorem 90. Specifically, the lift Π_0 is the Langlands quotient of

$$\mathrm{Ind}\left(\tau_1' \otimes \cdots \otimes \tau_d' \otimes T_0 \otimes \tilde{\tau}_d'^\theta \otimes \cdots \otimes \tilde{\tau}_1'^\theta\right),$$

with T_0 the Langlands functorial lift of the tempered representation τ_0. Given $(E/F, \pi_0, \tau, \psi) \in \mathscr{L}(p, \mathrm{U}_N, \mathrm{Res}\,\mathrm{GL}_m)$, we have equality of local factors

$$\begin{aligned}
\gamma(s, \pi_0 \times \tau, \psi_E) &= \gamma(s, \Pi_0 \times \tau, \psi_E) \\
L(s, \pi_0 \times \tau) &= L(s, \Pi_0 \times \tau) \\
\varepsilon(s, \pi_0 \times \tau, \psi_E) &= \varepsilon(s, \Pi_0 \times \tau, \psi_E).
\end{aligned}$$

Proof. We reason as in the case of a tempered representation, Proposition 9.9. Equality of local factors follows from the definition of base change and Lemmas 9.2 and 9.3, after incorporating twists by unramified characters. □

Theorem 9.10 summarizes our main local result. Local base change being recursively defined via the tempered, discrete series and supercuspidal cases of Propostions 9.9, 9.8 and 9.7, respectively.

9.5. **Strong base change.** Base change is now refined in such a way that it agrees with the local functorial lift of Theorem 9.10 at every place. Let

$$\mathfrak{h} : \mathrm{U}_1(\mathbb{A}_k) \to \mathrm{GL}_1(\mathbb{A}_K)$$

be the global reciprocity map such that $\mathfrak{h}_v$ is the map given by Hilbert's Theorem 90 at every place v of k, as in equation (6.1). We also have global twists by the unramified character ν of a general linear group obtained via the determinant, as in the local theory.

Theorem 9.11. *Let π be a cuspidal globally generic automorphic representation of $\mathrm{U}_N(\mathbb{A}_k)$. Then π has a unique Base Change to an automorphic representation of $\mathrm{GL}_N(\mathbb{A}_K)$, denoted by*

$$\Pi = \mathrm{BC}(\pi).$$

The central character of Π is given by $\omega_\Pi = \omega_\pi \circ \mathfrak{h}$ and is unitary. Furthermore, $\Pi \cong \tilde{\Pi}^\theta$ and there is an expression as an isobaric sum

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

where each Π_i is a unitary cuspidal automorphic representation of $\mathrm{GL}_{N_i}(\mathbb{A}_K)$ such that $\tilde{\Pi}_i \cong \Pi_i^\theta$ and $\Pi_i \not\cong \Pi_j$, for $i \neq j$. At every place v of k, we have that

$$\Pi_v = \mathrm{BC}(\pi_v)$$

is the local base change of Theorem 9.10 preserving local factors.

In the case of number fields, the method of descent is used in [58] to show how to obtain the strong lift from the weak lift. Over function fields, we can now give a self contained proof with the results of this article. Since the proof can be adapted to the classical groups in characteristic p, we prove this result in § A.3 of the Appendix, where we complete the results of [39, 40].

10. Ramanujan Conjecture and Riemann Hypothesis

Let K/k be a quadratic extension of global function fields of characteristic p. We can now complete the proof of the existence of extended γ-factors, L-functions and root number in order to include products of two unitary groups. Note that the Base Change map of Theorem 8.6 was strengthened in Theorem 9.11 so that it agrees with the local functorial lift of Theorem 9.10 at every place v of k. In this way, we can prove our main application involving L-functions for the unitary groups. The Riemann Hypothesis for L-functions associated to products of cuspidal automorphic representations of two general linear groups was proved by Laurent Lafforgue in [31].

Theorem 10.1. *Let γ, L and ε be rules on $\mathscr{L}_{\mathrm{loc}}(p)$ satisfying the ten axioms of Theorem 7.3. Given $(K/k, \pi, \tau, \psi) \in \mathscr{L}_{\mathrm{glob}}(p)$, define*

$$L(s, \pi \times \tau) = \prod_v L(s, \pi_v \times \tau_v) \quad \text{and} \quad \varepsilon(s, \pi \times \tau, \psi) = \prod_v \varepsilon(s, \pi_v \times \tau_v, \psi_v).$$

Automorphic L-functions on $\mathscr{L}_{\mathrm{glob}}(p)$ satisfy the following:

(i) (Rationality). *$L(s, \pi \times \tau)$ converges absolutely on a right half plane and has a meromorphic continuation to a rational function on q^{-s}.*
(ii) (Functional equation). *$L(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau)L(1 - s, \tilde{\pi} \times \tilde{\tau})$.*
(iii) (Riemann Hypothesis). *The zeros of $L(s, \pi \times \tau)$ are contained in the line $\Re(s) = 1/2$.*

10.1. **Extended local factors.** Let us complete the definition of extended local factors of Theorem 7.3. In this section for generic representations and in the next in general under a certain assumption. The case of a unitary group and a general linear group for generic representations already addressed in § 6. An exposition, within the Langlands-Shahidi method of the case of two general linear groups can be found in [21]. We now focus on the new case of $\mathbf{G}_1 = \mathrm{U}_M$ and $\mathbf{G}_2 = \mathrm{U}_N$.

Definition 10.2. *Given $(E/F, \pi_0, \tau_0, \psi) \in \mathscr{L}_{\mathrm{loc}}(p, \mathrm{U}_M, \mathrm{U}_N)$ generic, let*

$$\Pi_0 = \mathrm{BC}(\pi_0) \text{ and } T_0 = \mathrm{BC}(\tau_0)$$

be the corresponding base change maps of Theorem 9.10. We define

$$\gamma(s, \pi_0 \times \tau_0, \psi_E) := \gamma(s, \Pi_0 \times T_0, \psi_E)$$
$$L(s, \pi_0 \times \tau_0) := L(s, \Pi_0 \times T_0)$$
$$\varepsilon(s, \pi_0 \times \tau_0, \psi_E) := \varepsilon(s, \Pi_0 \times T_0, \psi_E).$$

The defining Properties (vii)–(x) of Theorem 7.3 allow us to construct L-functions and root numbers from γ-factors in the tempered case. This is compatible with the decomposition of π_0 and τ_0 of Theorem 9.10. The rules γ, L and ε then satisfy all of the local properties of Theorem 7.3. The remaining property, the global functional equation, is part (ii) of Theorem 10.1, addressed in § 10.2.

10.2. **Non-generic representations and local factors.** Consider an irreducible admissible representation π of $\mathrm{U}_N(F)$ that is not necessarily generic. We first look at the case when π is tempered. If $\mathrm{char}(F) = 0$, the tempered L-packet conjecture, Conjecture 10.3 below, is known (see Theorem 2.5.1 of [45]). Furthermore, it is part of the work of Ganapathy-Varma [15] on the local Langlands correspondence for the split classical groups if $\mathrm{char}(F) = p$. For the unitary groups $\mathrm{U}_N(F)$, $\mathrm{char}(F) = p$, it is thus reasonable to work under the assumption that Conjecture 10.3 holds.

Let Φ be the set of all Langlands parameters

$$\phi : \mathcal{W}'_F \to {}^L\mathrm{U}_N.$$

A parameter ϕ is tempered if its image on $\mathrm{GL}_N(\mathbb{C})$ is bounded. For any tempered L-parameter ϕ, there is an L-packet Π_ϕ which is a finite multi-set. We consider only tempered L-packets in this section, which agree with tempered Arthur packets.

Conjecture 10.3. *If an L-packet contains a tempered element, then all of its elements are tempered. Every tempered L-packet Π_ϕ of $\mathrm{U}_N(F)$ contains a representation π_0 which is generic.*

In fact, we can be more precise. Let $(\mathbf{B}, \psi)$ be Whittaker datum with $\mathbf{B} = \mathbf{TU}$ a fixed Borel subgroup of $\mathbf{G} = \mathrm{U}_N$ and $\psi : U \to \mathbb{C}^\times$ non-degenerate. Every tempered L-packet Π_ϕ of $\mathrm{U}_N(F)$ contains exactly one representation π_0 which is generic with respect to $(\mathbf{B}, \psi)$. In this article, we take the Borel subgroup of U_N consisting of upper triangular matrices.

Furthermore, L-functions are independent on how the non-degenerate character varies. Let $\mathbf{G} = \mathrm{U}_N$ and let $\widetilde{\mathbf{G}}$ be as in Proposition 2.2, sharing the same derived group as $\mathbf{G}$. Then L-functions are independent up to $\mathrm{Ad}(g)$ by elements of $\widetilde{\mathbf{G}}(F)$. And there is a formula that keeps track of how γ-factors and root numbers behave as the additive character ψ varies.

The importance of this conjecture is that it reduces the study of γ-factors, L-functions and ε-factors to the case of generic representations. Hence, under the assumption that Conjecture 10.3 is valid, we complete the existence part of Theorem 7.3 for tempered representations with the following definition.

Definition 10.4. *Let $(E/F, \pi, \tau, \psi) \in \mathscr{L}_{\mathrm{loc}}(p)$ be tempered. Let Π_{ϕ_1} and Π_{ϕ_2} be tempered L-packets with $\pi \in \Pi_{\phi_1}$ and $\tau \in \Pi_{\phi_2}$. Let $\pi_0 \in \Pi_{\phi_1}$ and $\tau_0 \in \Pi_{\phi_2}$ be generic. Then we have*

$$\begin{aligned} \gamma(s, \pi \times \tau, \psi_E) &:= \gamma(s, \pi_0 \times \tau_0, \psi_E) \\ L(s, \pi \times \tau) &:= L(s, \pi_0 \times \tau_0) \\ \varepsilon(s, \pi \times \tau, \psi_E) &:= \varepsilon(s, \pi_0 \times \tau_0, \psi_E). \end{aligned}$$

To prove Theorem 7.3 in general, we can use Langlands classification to write

$$\pi \hookrightarrow \mathrm{Ind}(\sigma_1 \otimes \chi_1)$$

and

$$\tau \hookrightarrow \mathrm{Ind}(\sigma_2 \otimes \chi_2)$$

with σ_1, σ_2 tempered and χ_1, $\chi_2 \in X_{\mathrm{nr}}(\mathbf{M})$ in the Langlands situation, as in [4, 57]. With tempered L-functions and corresponding local factors defined, then Properties (vii)–(x) of Theorem 7.3 can now be used to define L-functions and related local factors on $\mathscr{L}_{\mathrm{loc}}(p, \mathbf{G}_1, \mathbf{G}_2)$ in general.

Remark 10.5. *We refer to §§ 7 and 8 of* [14] *for a further discussion on L-parameters and the local Langlands correspondence for the classical groups, including the unitary groups. Written under certain working hypothesis, we address the local Langlands correspondence in characteristic p. First for supercuspidal representations, then for discrete series and tempered L-parameters, to end with general admissible representations.*

10.3. **Proof of Theorem 10.1.** The case of two general linear groups, i.e., for $(K/k, \pi, \tau, \psi) \in \mathscr{L}_{\mathrm{glob}}(p, \mathrm{Res}_{K/k}\mathrm{GL}_M, \mathrm{Res}_{K/k}\mathrm{GL}_N)$, is already well understood. Properties (i) and (ii) of Theorem 10.1 are attributed to Piatetski-Shapiro [49]. They can be proved in a self contained way via the Langlands-Shahidi method over function fields, see [21, 41]. The Riemann Hypothesis in this case was proved by Laurent Lafforgue in [31].

The case of $(K/k, \pi, \tau, \psi) \in \mathscr{L}_{\mathrm{glob}}(p, \mathrm{Res}_{K/k}\mathrm{GL}_m, \mathrm{U}_N)$ is included in Theorem 4.1 by taking $\mathbf{M} = \mathrm{GL}_m \times \mathrm{U}_N$ as a maximal Levi subgroup of $\mathbf{G} = \mathbf{U}_{N+2m}$ and forming the globally generic representation $\tau \otimes \tilde{\pi}$ of $\mathbf{M}(\mathbb{A}_k)$. Property (i) in this situation is Proposition 5.1. Property (ii) is the functional equation of § 4.4. To prove the Riemann Hypothesis, we let

$$\mathrm{BC}(\pi) = \Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d$$

be the base change lift of Theorem 9.11. Then

$$L(s, \pi \times \tau) = L(s, \Pi \times \tau) = \prod_{i=1}^{d} L(s, \Pi_i \times \tau),$$

with each $(K/k, \Pi_i, \tau, \psi) \in \mathscr{L}_{\rm glob}(p, {\rm Res}_{K/k}{\rm GL}_{m_i}, {\rm Res}_{K/k}{\rm GL}_{n_i})$. This reduces the problem to the Rankin-Selberg case, established by L. Lafforgue.

Given $(K/k, \pi, \tau, \psi) \in \mathscr{L}_{\rm glob}(p, {\rm U}_M, {\rm U}_N)$, let

$$\mathrm{BC}(\pi) = \Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d \quad \text{and} \quad \mathrm{BC}(\tau) = T = T_1 \boxplus \cdots \boxplus T_e$$

be the base change maps of Theorem 9.11. Then

$$L(s, \pi \times \tau) = L(s, \Pi \times T) = \prod_{i,j} L(s, \Pi_i \times T_j).$$

For each $(K/k, \Pi_i, T_j, \psi) \in \mathscr{L}_{\rm glob}(p, {\rm Res}_{K/k}{\rm GL}_{m_i}, {\rm Res}_{K/k}{\rm GL}_{n_i})$, $1 \leq i \leq d$, $1 \leq j \leq e$, we have rationality, the functional equation

$$L(s, \Pi_i \times T_j) = \varepsilon(s, \Pi_i \times T_i) L(1 - s, \tilde{\Pi}_i \times \tilde{T}_j),$$

and the Riemann Hypothesis. Hence the L-function $L(s, \pi \times \tau)$ also satisfies Properties (i)–(iii) of Theorem 10.1. □

10.4. **The Ramanujan Conjecture.** Base Change over function fields also allows us to transport the Ramanujan conjecture from the unitary groups to ${\rm GL}_N$. The Ramanujan conjecture for cuspidal representations of general linear groups, being a theorem of L. Lafforgue [31].

Theorem 10.6. *Let $\pi = \otimes' \pi_v$ be a globally generic cuspidal automorphic representation of ${\rm U}_N(\mathbb{A}_k)$. Then every π_v is tempered. Whenever π_v is unramified, its Satake parameters satisfy*

$$|\alpha_{j,v}|_{k_v} = 1, \quad 1 \leq j \leq n.$$

Proof. Fix a place v of k, which remains inert in K. We can write π_v as the generic constituent of

$$\mathrm{Ind}\left(\tau'_{1,v} \otimes \cdots \otimes \tau'_{d,v} \otimes \tau_{0,v}\right),$$

as in (9.9), with each $\tau'_{i,v}$ quasi-tempered and $\tau_{0,v}$ tempered. Furthermore, we can write $\tau'_{i,v} = \tau_{i,v} \nu^{t_{i,v}}$ with $\tau_{i,v}$ tempered and Langlands parameters $0 \leq t_{1,v} \leq \cdots \leq t_{d,v}$.

Now, let $\Pi = \mathrm{BC}(\pi)$ be the base change lift of Theorem 9.11. From Theorem 9.11, Π_v is the local Langlands functorial lift of π_v. By Theorem 9.10, Π_v is the generic constituent of

$$\mathrm{Ind}\left(\tau'_{1,v} \otimes \cdots \otimes \tau'_{d,v} \otimes T_{0,v} \otimes \tilde{\tau}'^{\theta}_{d,v} \otimes \cdots \otimes \tilde{\tau}'^{\theta}_{1,v}\right), \tag{10.1}$$

with $T_{0,v}$ the Langlands functorial lift of the tempered representation $\tau_{0,v}$. The representation $T_{0,v}$ is also tempered by Lemma 9.4.

On the other hand, the base change lift can be expressed as an isobaric sum

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_e,$$

with each Π_i a cuspidal unitary automorphic representation of ${\rm GL}_{m_i}(\mathbb{A}_K)$. Hence, Π_v is obtained from

$$\mathrm{Ind}(\Pi_{1,v} \otimes \cdots \otimes \Pi_{e,v}). \tag{10.2}$$

Then, thanks to Théorème VI.10 of [31], each $\Pi_{i,v}$ is tempered.

We then look at a fixed $\tilde{\tau}_{j,v}$ from (10.1), where we now have

$$L(s,\pi_v \times \tilde{\tau}_{j,v}) = L(s,\Pi_v \times \tilde{\tau}_{j,v})$$
$$= L(s,\tau_{0,v} \times \tilde{\tau}_{j,v}) \prod_{i=1}^{d} L(s+t_{i,v},\tau_{i,v} \times \tilde{\tau}_{j,v}) L(s-t_{i,v},\tau_{i,v} \times \tilde{\tau}_{j,v}).$$

The L-function $L(s-t_{j,v},\tau_{j,v} \times \tilde{\tau}_{j,v})$ appearing on the right hand side has a pole at $s = t_{j,v}$. However, from (10.2) we have

$$L(s,\Pi_v \times \tilde{\tau}_{j,v}) = \prod_{i=1}^{e} L(s,\Pi_{i,v} \times \tilde{\tau}_{j,v}).$$

Notice that each representation involved in the product on the right hand side is tempered. Then each $L(s,\Pi_{i,v} \times \tilde{\tau}_{j,v})$ is holomorphic for $\Re(s) > 0$. Hence, so is $L(s,\Pi_v \times \tilde{\tau}_{j,v})$. This causes a contradiction unless $t_{j,v} = 0$.

If v is split, we have that $\mathrm{BC}(\pi_v) = \pi_v \otimes \tilde{\pi}_v$ from § 8.4. For the places w_1 and w_2 of K lying above v, we have that the representation of $\mathrm{GL}_N(k_v)$

$$\pi_v = \mathrm{Ind}(\Pi_{1,w_1} \otimes \cdots \otimes \Pi_{e,w_1})$$

is tempered. Hence, so is

$$\tilde{\pi}_v = \mathrm{Ind}(\tilde{\Pi}_{1,w_2} \otimes \cdots \otimes \tilde{\Pi}_{e,w_2}).$$

Now, if v is inert and π_v is unramified, we have from § 8.3 that the unramified Base Change $\mathrm{BC}(\pi_v) = \Pi_v$ corresponds to a semisimple conjugacy class given by

$$\Phi_v(\mathrm{Frob}_v) = \begin{cases} \mathrm{diag}(\alpha_{1,v},\ldots,\alpha_{n,v},1,\alpha_{n,v}^{-1},\ldots,\alpha_{1,v}^{-1}) & \text{if } N = 2n+1 \\ \mathrm{diag}(\alpha_{1,v},\ldots,\alpha_{n,v},\alpha_{n,v}^{-1},\ldots,\alpha_{1,v}^{-1}) & \text{if } N = 2n \end{cases}.$$

Each $\alpha_{j,v}$ or $\alpha_{j,v}^{-1}$ is a Satake parameter for one of the representations $\Pi_{i,v}$, which are unramified. Since we are in the case of $\mathrm{GL}_{m_i}(K_v)$, we conclude that

$$|\alpha_{j,v}|_{k_v} = 1.$$

□

Appendix A. On functoriality for the classical groups

The main purpose of this appendix is to remove the previously present assumption made in [39, 40] that $\mathrm{char}(k) \neq 2$. In A.1 we restrict ourselves to the split case.

A.1. **Langlands functoriality.** Let us summarize the results of [39] on the globally generic functorial lift for the classical groups. Let $\mathbf{G}_n$ be a split classical group of rank n defined over a global function field k. The functorial lift of [39] takes globally generic cuspidal automorphic representations π of $\mathbf{G}_n(\mathbb{A}_k)$ to automorphic representations of $\mathbf{H}_N(\mathbb{A}_k)$, where $\mathbf{H}_N$ is chosen by the following table.

$\mathbf{G}_n$	${}^L G_n \hookrightarrow {}^L H_N$	$\mathbf{H}_N$
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C}) \times \mathcal{W}_k \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times \mathcal{W}_k$	GL_{2n}
Sp_{2n}	$\mathrm{SO}_{2n+1}(\mathbb{C}) \times \mathcal{W}_k \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C}) \times \mathcal{W}_k$	GL_{2n+1}
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C}) \times \mathcal{W}_k \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times \mathcal{W}_k$	GL_{2n}

Table 1

Theorem A.1. *Let $\mathbf{G}_n$ be a split classical group. Let π be a globally generic cuspidal representation of $\mathbf{G}_n(\mathbb{A}_k)$; $n \geq 2$ if $\mathbf{G}_n = \mathrm{SO}_{2n}$. Then, π has a functorial lift to an automorphic representation Π of $\mathbf{H}_N(\mathbb{A}_k)$. It has trivial central character and can be expressed as an isobaric sum*

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d,$$

where each Π_i is a unitary self-dual cuspidal automorphic representation of $\mathrm{GL}_{N_i}(\mathbb{A}_k)$ such that $\Pi_i \not\cong \Pi_j$ for $i \neq j$. Furthermore, Π_v is the local Langlands functorial lift of π_v at every place v of k. That is, for every $(k_v, \pi_v, \tau_v, \psi_v) \in \mathscr{L}_{\mathrm{loc}}(p, \mathbf{G}_n, \mathrm{GL}_m)$ there is equality of local factors

$$\begin{aligned} \gamma(s, \Pi_v \times \tau_v, \psi_v) &= \gamma(s, \pi_v \times \tau_v, \psi_v) \\ L(s, \Pi_v \times \tau_v) &= L(s, \pi_v \times \tau_v) \\ \epsilon(s, \Pi_v \times \tau_v, \psi_v) &= \epsilon(s, \pi_v \times \tau_v, \psi_v). \end{aligned}$$

Basically, Theorem A.1 is proved in a characteristic free way for any global function field in § A.3. For this, we build upon the results on the properties of Langlands-Shahidi L-functions of § 5. In Theorem A.2, the Ramanujan Conjecture and the Riemann Hypothesis are addressed.

Theorem A.2. *Let $\mathbf{G}_n$ be a split classical group defined over a function field k.*

(i) (Ramanujan Conjecture). *If $\pi = \otimes' \pi_v$ is a globally generic cuspidal automorphic representation of $\mathbf{G}_n(\mathbb{A}_k)$, then each local component π_v is tempered.*

Let $(k, \pi, \tau, \psi) \in \mathscr{L}_{\mathrm{glob}}(p, \mathbf{G}_n, \mathbf{G}_m)$, with $\mathbf{G}_m$ either GL_m or a split classical group of rank m. Then

(ii) (Rationality). *$L(s, \pi \times \tau)$ converges absolutely on a right half plane and has a meromorphic continuation to a rational function on q^{-s}.*

(iii) (Functional Equation). $L(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau) L(1 - s, \tilde{\pi} \times \tilde{\tau})$.

(iv) (Riemann Hypothesis). *The zeros of $L(s, \pi \times \tau)$ are contained in the line $\Re(s) = 1/2$.*

The proof of the Ramanujan conjecture is essentially that of [39]. And the proofs of Properties (ii)–(iv) are those appearing in [40]. However, they are completed with the results of this article.

A.2. **Rankin-Selberg L-functions for the classical groups.** Let us adapt the notation to include the split classical groups. Let $\mathscr{L}_{\mathrm{glob}}(p, \mathbf{G}_n, \mathrm{Res}\,\mathrm{GL}_m)$ be the class consisting of quintuples $(K/k, \pi, \tau, \psi, S)$: k a global function field of characteristic p; $K = k$ if $\mathbf{G}_n$ is split and K/k a separable quadratic extension if $\mathbf{G}_n$ is a quasi-split unitary group; $\pi = \otimes' \pi_v$ a globally generic cuspidal automorphic representation of $\mathbf{G}_n(\mathbb{A}_k)$; $\tau = \otimes' \tau_v$ a cuspidal (unitary) automorphic representation of $\mathrm{Res}\,\mathrm{GL}_m(\mathbb{A}_k) = \mathrm{GL}_m(\mathbb{A}_K)$; $\psi : k \backslash \mathbb{A}_k \to \mathbb{C}^\times$ a global additive character; and S a finite set of places of k such that π_v and ψ_v are unramified for $v \notin S$. In the case of a separable quadratic extension, we can think of the additive character $\psi_K : K \backslash \mathbb{A}_K \to \mathbb{C}^\times$ obtained via the trace.

Theorem A.3. *Let $(K/k, \pi, \tau, \psi, S) \in \mathscr{L}_{\mathrm{glob}}(p, \mathbf{G}_n, \mathrm{Res}\,\mathrm{GL}_m)$. Then $L(s, \pi \times \tau)$ is holomorphic for $\Re(s) > 1$ and has at most a simple pole at $s = 1$.*

Proof. The classification of generic unitary representations of $\mathbf{G}_n(k_v)$ [35] gives that every π_v is a constituent of

$$\text{(A.1)} \qquad \mathrm{Ind}(\delta_{1,v}\nu^{\beta_1} \otimes \cdots \otimes \delta_{d,v}\nu^{\beta_d} \otimes \delta'_{1,v}\nu^{\alpha_1} \otimes \cdots \otimes \delta'_{e,v}\nu^{\alpha_e} \otimes \pi_{0,v}).$$

Here, the representations $\delta_{i,v}$ and $\delta'_{j,v}$ are unitary discrete series, $\pi_{0,v}$ is tempered, and the Langlands parameters are of the form $1 > \beta_1 > \cdots \beta_d > 1/2 > \alpha_1 > \cdots > \alpha_e > 0$.

Now, thanks to the work of L. Lafforgue [31], each local component τ_v of the cuspidal unitary τ arises from an induced representation of the form

$$\mathrm{Ind}(\tau_{1,v} \otimes \cdots \otimes \tau_{f,v}),$$

with each $\tau_{l,v}$ tempered. The L-functions $L(s,\tau,r)$ for $r = \mathrm{Sym}^2$, $\wedge^2$, $r_{\mathcal{A}}$ or $r_{\mathcal{A}} \otimes \eta_{K/k}$ satisfy the following relationships

$$L(s,\tau \times \tau) = L(s,\tau,\mathrm{Sym}^2)L(s,\tau,\wedge^2)$$
$$L(s,\tau \times \tau^\theta) = L(s,\tau,r_{\mathcal{A}})L(s,\tau,r_{\mathcal{A}} \otimes \eta_{K/k}).$$

Also, because we are in the cases arising from GL_n, we can apply Proposition 5.12. Each L-function $L(s,\tau,r)$ appearing on the right hand sides of the previous two equations is then holomorphic for $\Re(s) > 1$. The Rankin-Selberg product L-functions on the left hand sides of the previous two equations are non vanishing for $\Re(s) > 1$ [23]. Hence, each $L(s,\tau,r)$ in turn must be non vanishing for $\Re(s) > 1$.

Then, the multiplicativity property of Langlands-Shahidi L-functions gives

$$\begin{aligned} L(s,\pi_v \times \tau_v) = L(s,\pi_{v,0} \times \tau_v) &\prod_{i=1}^{d}\prod_{j=1}^{e}\prod_{l=1}^{f} L(s+\beta_i,\delta_{i,v} \times \tau_{l,v})L(s+\alpha_j,\delta'_{j,v} \times \tau_{l,v}) \\ &\times \prod_{i=1}^{d}\prod_{j=1}^{e}\prod_{l=1}^{f} L(s-\beta_i,\delta_{i,v} \times \tau_{l,v})L(s-\alpha_j,\delta'_{j,v} \times \tau_{l,v}). \end{aligned}$$

From [17], each of the L-functions appearing in the right hand side is holomorphic for $\Re(s)$ large enough. This carries through to the left hand side and we can conclude that $L(s,\pi_v \times \tau_v)$ is holomorphic for $\Re(s) > \beta_1$. In particular, for $\Re(s) > 1$.

Now, we globally let $\sigma = \tau \otimes \tilde{\pi}$, so that $(K/k,\sigma,\psi,S) \in \mathscr{L}_{\mathrm{glob}}(p,\mathbf{M},\mathbf{G})$. Where $\mathbf{G}$ is a classical group of rank $l = m+n$ of the same type as $\mathbf{G}_n$ and $\mathbf{M} = \mathrm{Res}\,\mathrm{GL}_m \times \mathbf{G}_n$ is a maximal Levi subgroup. From [44, 46], if the global intertwining operator $\mathrm{M}(s,\Phi,g,\mathbf{P})$ has a pole at s_0, then a subquotient of $\mathrm{I}(s_0,\sigma)$ would belong to the residual spectrum and we would have that almost every $\mathrm{I}(s_0,\sigma_v)$ is unitary.

However, to obtain a contradiction, we claim for $\Re(s) > 1$ that the representation $\mathrm{I}(s,\sigma_v)$ cannot be unitary for at least one $v \notin S$ (we can actually show this for all $v \notin S$). For this, we begin by fixing a place $v \notin S$, which we assume remains inert if we are in the case of a non-split quasi-split classical group. In these cases we now apply equation (A.1) for the group $\mathbf{G}_l$. If $\mathrm{I}(s_0,\sigma_v)$ were unitary then, up to rearrangement if necessary, it would be of the form

(A.2)

$$\mathrm{Ind}(\chi_{1,v}\nu^{s_0} \otimes \cdots \otimes \chi_{m,v}\nu^{s_0} \otimes \mu_{1,v}\nu^{\beta_1} \otimes \cdots \otimes \mu_{d,v}\nu^{\beta_d} \otimes \mu'_{1,v}\nu^{\alpha_1} \otimes \cdots \otimes \mu'_{e,v}\nu^{\alpha_e} \otimes \mu_{0,v}),$$

where we now have unramified unitary characters $\mu_{0,v}$, $\mu_{i,v}$ and $\mu'_{j,v}$; the character $\mu_{0,v}$ is taken to be trivial unless we are in the odd unitary group case. The Langlands parameters are of the form $1 > \beta_1 > \cdots \beta_d > 1/2 > \alpha_1 > \cdots > \alpha_e > 0$. The classification tells us that this cannot be the case if $\Re(s_0) > 1$. Hence, the global intertwining operator $\mathrm{M}(s, \Phi, g, \mathbf{P})$ must be holomorphic for $\Re(s) > 1$.

The product

$$\prod_{i=1}^{m_r} \frac{L^S(is, \sigma, r_i)}{L^S(1 + is, \sigma, r_i)}$$

is then holomorphic on $\Re(s) > 1$. For the classical groups we precisely have that $m_r = 2$ and

$$\prod_{i=1}^{m_r} L^S(s, \sigma, r_i) = L^S(s, \sigma, r_1) L^S(2s, \tau, r_2), \tag{A.3}$$

where $r_1 = \rho_n \otimes \tilde{\rho}_m$ and $L^S(s, \sigma, r_2) = L^S(s, \tau, r_2)$ has r_2 either an exterior square, symmetric square or Asai L-function. The induction step is given by the Siegel Levi, and we showed that in these cases we have holomorphy and nonvanishing for $\Re(s) > 1$. We can then cancel the second L-functions and conclude that the quotient

$$\frac{L^S(s, \pi \times \tau)}{L^S(1 + s, \pi \times \tau)} \tag{A.4}$$

is holomorphic for $\Re(s) > 1$.

Finally, the poles of Eisenstein series are contained in the constant terms, hence Corollary 3.4 gives that $L^S(1+s, \pi \times \tau)^{-1}$ is holomorphic for $\Re(s) > 1$. In this way, we can cancel the now non-zero denominator in equation (A.4) to conclude that $L^S(s, \pi \times \tau)$ is holomorphic on $\Re(s) > 1$. We have also noted that $L(s, \pi_v \times \tau_v)$ is holomorphic for $\Re(s) > 1$ at every place $v \in S$. We then obtain the required holomorphy property for the completed L-function $L(s, \pi \times \tau)$.

Now, we have that the poles of the global intertwining operator $M(s, \sigma, \tilde{w}_0)$ are all simple [44]. Then, by equation (5.1), the quotient

$$\frac{L^S(s, \pi \times \tau) L^S(2s, \tau, r_2)}{L^S(1 + s, \tilde{\pi} \times \tilde{\tau}) L^S(1 + 2s, \tilde{\tau}, r_2)}$$

has at most a simple pole at $s = 1$. From Corollary A.4, $L^S(2, \tau, r_2)$ and $L^S(3, \tilde{\tau}, r_2)$ are different from zero. Thus, $L(s, \pi \times \tau)$ has at most a simple pole at $s = 1$. □

Corollary A.4. *Let $(K/k, \tau, \psi, S) \in \mathscr{L}_{\mathrm{glob}}(p, \mathbf{M}, \mathbf{G})$, with $\mathbf{M}$ the Siegel Levi subgroup of a quasi-split classical group. Then, for each i, $1 \le i \le m_r$, the automorphic L-function $L(s, \tau, r_i)$ is holomorphic and non vanishing for $\Re(s) > 1$.*

A.3. **Proof of Theorem A.1.** We keep the notation of § A.2, where $\mathbf{G}_n$ is either a split classical group or a quasi-split unitary group; and, similarly for K/k and $\mathrm{Res}\,\mathrm{GL}_m$. If $\mathbf{G}_n = \mathrm{U}_n$ we let $\mathbf{H}_N = \mathrm{Res}\,\mathrm{GL}_N$, with $N = n$.

Let π be a cuspidal globally generic automorphic representation of $\mathbf{G}_n(\mathbb{A}_k)$. Let Π' be the automorphic representation of $\mathrm{GL}_N(\mathbb{A}_k)$ obtained from π via the weak functorial lift of Theorem 8.5 of [39] or the weak base change lift of Theorem 8.6 for the unitary groups. This lift has the property that Π'_v is the unramified lift of π_v for all $v \notin S$ of § 8.2.1 of [*loc. cit.*] or §§ 8.3 & 8.4 for unitary groups. In the latter case, by Proposition 9.5, we know that $\Pi = \mathrm{BC}(\pi)$ has unitary central character

$$\omega_\Pi = \omega_\pi \circ \mathfrak{h}.$$

From the Jacquet-Shalika classification [23], the functorial lift Π decomposes as an isobaric sum

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_d.$$

More precisely, each cuspidal automorphic representation Π_i of $\mathrm{GL}_{N_i}(\mathbb{A}_K)$ can be written in the form

$$\Pi_i = \Xi_i \nu^{t_i},$$

with Ξ_i a unitary cuspidal automorphic representation of $\mathrm{GL}_{N_i}(\mathbb{A}_K)$. Reordering if necessary, we can assume $t_1 \leq \cdots \leq t_d$. We wish to prove that each $t_i = 0$. Notice that if $t_i < 0$ for some i, then there is a $j > i$ such that $t_j > 0$, due to the fact that Π is unitary. Also, we cannot have $t_1 > 0$. To obtain a contradiction, suppose there exists a t_{j_0} which is the smallest with the property $t_{j_0} < 0$.

Consider $(K/k, \pi, \Xi_{j_0}, \psi) \in \mathscr{L}_{\mathrm{glob}}(p, \mathbf{G}_n, \mathrm{Res}\, \mathrm{GL}_{N_{j_0}})$. Theorem A.3 above tells us that $L(s, \pi \times \tilde{\Xi}_{j_0})$ is holomorphic for $\Re(s) > 1$. However, if we consider $(K, \Pi, \Xi_{j_0}, \psi) \in \mathscr{L}_{\mathrm{glob}}(p, \mathrm{Res}\, \mathrm{GL}_N, \mathrm{Res}\, \mathrm{GL}_{N_{j_0}})$, we have that

$$L(s, \Pi \times \tilde{\Xi}_{j_0}) = \prod_{i=1}^{d} L(s + t_i, \Xi_i \times \tilde{\Xi}_{j_0}).$$

And, by Theorem 3.6 of [23] (part II) the L-function

$$L(s, \Xi_{j_0} \times \tilde{\Xi}_{j_0})$$

has a simple pole at $s = 1$. This carries over to a pole at $s = 1 - t_{j_0} > 1$ for

$$L(s, \pi \times \tilde{\Xi}_{j_0}) = L(s, \Pi \times \tilde{\Xi}_{j_0}).$$

This causes a contradiction unless there exists no such t_{j_0}. Hence, all t_i must be zero. This proves that each Π_i is a unitary cuspidal automorphic representation.

We now have that each $\Pi_i = \Xi_i$ and

$$L(s, \Pi \times \tilde{\Pi}_{l_0}) = \prod_{i=1}^{d} L(s, \Pi_i \times \tilde{\Pi}_{l_0}),$$

where l_0 is a fixed ranging from $1 \leq l_0 \leq d$. On the right hand side we have a pole at $s = 1$ every time that $\Pi_{l_0} \cong \Pi_i$, by Proposition 3.6 of [23] part II. However, on the left hand side, from Proposition A.3 of the Appendix, we have that $L(s, \Pi \times \tilde{\Pi}_{l_0}) = L(s, \pi \times \tilde{\Pi}_{l_0})$ has only a simple pole at $s = 1$. Hence, $\Pi_{l_0} \cong \Pi_i$ can only occur if $l_0 = i$.

It remains to show that, for each j, we have $\Pi_j \cong \tilde{\Pi}_j^{\theta}$. For this, let $\mathrm{T}_j = \Pi_j \otimes \pi$. From Corollary 5.6, $L(s, \pi \times \tilde{\Pi}_j)$ is a Laurent polynomial if $\tilde{w}_0(\mathrm{T}_j) \ncong \mathrm{T}_j$. In this case, would have no poles. However

$$L(s, \pi \times \tilde{\Pi}_j) = \prod_{i}^{d} L(s, \Pi_i \times \tilde{\Pi}_j)$$

has only a simple pole at $s = 1$, reasoning as before. Thus, it must be the case that $\tilde{w}_0(\mathrm{T}_j) \cong \mathrm{T}_j$, giving the desired $\Pi_j \cong \tilde{\Pi}_j^{\theta}$.

In the case of unitary groups, the compatibility of global to local base change is addressed by Proposition 9.5 (i). We have that

$$\Pi_v = \mathrm{BC}(\pi_v)$$

is unique and must be given by Theorem 9.10. Notice that $\Pi_v \cong \tilde{\Pi}_v^\theta$ for every $v \notin S$. By multiplicity one for GL_N, we globally have $\Pi \cong \tilde{\Pi}^\theta$.

Let now $\mathbf{G}_n$ be a split classical group, where $\theta = 1$. We argue as in § 9 in order to adapt the results of [39] and complete the proof of the Theorem. In particular, we proceed as in the proof of Theorem 9.5 to show that Π_v is the unique local Langlands functorial lift of π_v at every place v of k. A crucial step is the stability property of γ-factors under twists by highly ramified characters, Theorem 6.10 of [39], which is available in characteristic two. Finally, Lemmas 9.2 and 9.3 indicate how to obtain the desired equality of local factors. □

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