Do the Kontsevich tetrahedral flows preserve or destroy the space of Poisson bi-vectors?

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From the paper "Formality Conjecture" (see Ref. [1]):

I am aware of only one such a class, it corresponds to simplest good graph, the complete graph with 4 *vertices* (*and* 6 *edges*). *This class gives a remarkable vector field on the space of bi-vector fields on* $\mathbb{R}^d$. *The evolution with respect to the time* t *is described by the following non-linear partial differential equation*: [see (2a) below], *where* $\alpha = \sum_{i,j} \alpha_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ *is a bi-vector field on* $\mathbb{R}^d$. ...

It follows from general properties of cohomology that 1) ***this evolution preserves the class of*** (***real-analytic***) ***Poisson structures***, ...

In fact, I cheated a little bit. In the formula for the vector field on the space of bivector fields which one get from the tetrahedron graph, an additional term is present. This term is equal (*up to a numerical factor*) *to* [see (2b) below]. *It is possible to prove formally that* ***if*** α ***is a Poisson bracket, i.e. if*** $[\alpha, \alpha] = 0 \in T^2(\mathbb{R}^d)$, ***then the additional term shown above vanishes.***

Abstract. By using twelve Poisson structures with high-order polynomial coefficients as explicit counterexamples, we show that both the above claims are false: neither does the first flow preserve the property of bi-vectors to be Poisson nor does the second flow vanish identically at the Poisson bi-vectors. The counterexamples at hand themselves suggest a correction to the formula for the "exotic" flow on the space of Poisson bi-vectors; in fact, this flow is encoded by the balanced sum involving both the Kontsevich tetrahedral graphs (that give rise to the flows mentioned above). We reveal that it is only the balance (1 : 6) for which the flow does preserve the space of Poisson bi-vectors.

Introduction. The Kontsevich graph complex is the language of deformation quantisation on finite-dimensional Poisson manifolds [2]. Let us consider the class of oriented graphs with two sinks and $k \geqslant 1$ internal vertices (of which, each is the tail of two edges and carries a copy of the Poisson bi-vector $\mathcal{P}$). Encoding bi-differential operators, such graphs determine the flows on the space of bi-vectors on the Poisson manifold at hand. The two flows with $k = 4$ internal vertices in the graphs are provided by the two tetrahedra [1], see Fig. 1 on the reverse page. By producing 12 counterexamples, we prove that the claim [1, 2] of preservation of the Poisson property is false as stated, so that the (variational) Poisson bi-vectors are fragile with respect to the Kontsevich tetrahedral flows. Simultaneously, we reveal that the flow which is determined by the second graph is not always vanishing by virtue of the skew-symmetry and Jacobi identity for the Poisson bi-vectors $\mathcal{P}$.

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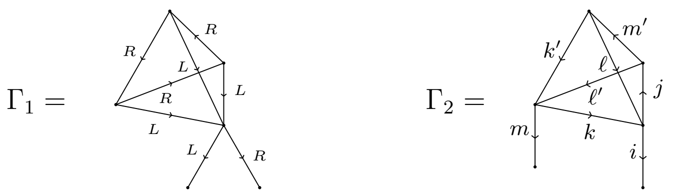


Figure 1. These tetraheral graphs encode flows (2a) and (2b), respectively. Each oriented edge carries a summation index that runs from 1 to the dimension of the Poisson manifold at hand. For each internal vertex (where a copy of the Poisson bi-vector $\mathcal{P}$ is stored), the pair of out-going edges is ordered, L $\prec$ R: the left edge (L) carries the first index and the other edge (R) carries the second index in the bi-vector coefficients, see section 1. (In retrospect, the ordering and labelling of the indexed oriented edges can be guessed from formulas (2) on p. 3.)

This paper is structured as follows. First we recall the correspondence between graphs and polydifferential operators [3, 4] and we indicate the mechanism for such an operator to vanish, cf. [5]. In section 2 we recall three constructions of Poisson brackets with polynomial coefficients of arbitrarily high degree (see [6, 7, 8]). Next, in section 3 we recite the basics of Poisson structure deformation [9]. In Tables 1–4 on pp. 10–11 we then summarise the properties of all structures from our 12 counterexamples to the claim [1] that

(i) the flow $\dot{\mathcal{P}} = \Gamma_1(\mathcal{P})$ which the first graph in Fig. 1 encodes on the space of bi-vectors $\mathcal{P}$ would preserve their property to be Poisson (in fact, it does not), and that

(ii) the flow $\dot{\mathcal{P}} = \Gamma_2(\mathcal{P})$ would always be trivial whenever the bi-vector $\mathcal{P}$ is Poisson (in fact, this is not true).

In particular, the twelfth counterexample pertains to the infinite-dimensional jet-space geometry of variational Poisson structures [10]. (Quoted from [11], the Hamiltonian differential operator for that variational Poisson bi-vector $\boldsymbol{\mathcal{P}}$ is then processed by using the techniques from [12, 13], cf. [14].)

Finally, we examine at which balance the linear combination of the Kontsevich tetrahedral flows preserves the space of Poisson structures on finite-dimensional manifolds. We argue that the ratio 1 : 6 does the job.

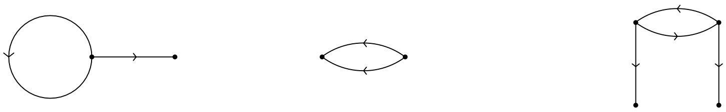
1. The graphs and operators

Let us formalise a way to encode polydifferential operators using oriented graphs. Consider the space $\mathbb{R}^n$ with Cartesian coordinates $\boldsymbol{x} = (x_1, \ldots, x_n)$, here $3 \leqslant n < \infty$; for typographical reasons only do we use the lower indices to enumerate the variables, so that $x_1^2 = (x_1)^2$, etc. By definition, the indexed edge $\bullet \xrightarrow{i} \bullet$ denotes at once the derivation $\partial/\partial x_i \equiv \partial_i$ (that acts on the content of the arrowhead vertex) and the summation $\sum_{i=1}^{n}$ (over the index i in the object which is contained within the arrowtail vertex). For example, the graph $\bullet \xleftarrow{i} \mathcal{P}^{ij}(\boldsymbol{x}) \xrightarrow{j} \bullet$ encodes the bi-differential operator $\sum_{i=1}^{n}(\cdot)\overleftarrow{\partial_i}\,\mathcal{P}^{ij}(\boldsymbol{x})\,\overrightarrow{\partial_j}(\cdot)$. If its coefficients $\mathcal{P}^{ij}$ are antisymmetric, then the graph $\bullet \xleftarrow{i} \bullet \xrightarrow{j} \bullet$ encodes the bi-vector $\mathcal{P} = \mathcal{P}^{ij}\,\partial_i \wedge \partial_j$, where $\partial_i \wedge \partial_j = \frac{1}{2}(\partial_i \otimes \partial_j - \partial_j \otimes \partial_i)$. It then specifies the Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P}}$ if the $\frac{n(n-1)}{2}$-tuple of coefficients solves the equation

$$(\mathcal{P}^{ij})\overleftarrow{\partial_\ell}\cdot\mathcal{P}^{\ell k} + (\mathcal{P}^{jk})\overleftarrow{\partial_\ell}\cdot\mathcal{P}^{\ell i} + (\mathcal{P}^{ki})\overleftarrow{\partial_\ell}\cdot\mathcal{P}^{\ell j} = 0, \tag{1}$$

that is, the bracket $\bullet \xleftarrow{i} \mathcal{P}^{ij} \xrightarrow{j} \bullet$ satisfies the Jacobi identity. Clearly, we then have $\mathcal{P}^{ij}(\boldsymbol{x}) = \{x_i, x_j\}_{\mathcal{P}}$.

From now on, let us consider only the oriented graphs whose vertices are either sinks, with no issued edges, or tails for an ordered pair of arrows, each carrying its own index (see Fig. 1 on p. 2). Allowing the only exception in footnote 1 on p. 4 below, we shall always assume that there are neither tadpoles, nor double oriented edges, nor two-edge loops so that none of the three graphs which are shown here (or similar graphs) will be considered in what follows:



(excluded).

We also postulate that every vertex which is not a sink carries a copy of a given Poisson bi-vector $\mathcal{P} = \mathcal{P}^{ij}(\boldsymbol{x})\,\partial_i \wedge \partial_j$; the ordering of indexed out-going edges coincides with the ordering "first $\prec$ second" of the indexes in the coefficients of $\mathcal{P}$.

Example 1. Under all these assumptions, the two tetrahedra which are portrayed in Fig. 1 are, up to a symmetry, the only admissible graphs with $k = 4$ internal vertices, $2k = 6 + 2$ edges, and two sinks.

The first graph in Fig. 1 encodes the bi-vector

$$\Gamma_1(\mathcal{P}) = \sum_{i,j=1}^{n}\left(\sum_{k,\ell,m,k',\ell',m'=1}^{n} \frac{\partial^3 \mathcal{P}^{ij}}{\partial x_k \partial x_\ell \partial x_m}\frac{\partial \mathcal{P}^{kk'}}{\partial x_{\ell'}}\frac{\partial \mathcal{P}^{\ell\ell'}}{\partial x_{m'}}\frac{\partial \mathcal{P}^{mm'}}{\partial x_{k'}}\right)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}. \tag{2a}$$

Likewise, the second graph in Fig. 1 yields the bi-vector

$$\Gamma_2(\mathcal{P}) = \sum_{i,m=1}^{n}\left(\sum_{j,k,\ell,k',\ell',m'=1}^{n} \frac{\partial^2 \mathcal{P}^{ij}}{\partial x_k \partial x_\ell}\frac{\partial^2 \mathcal{P}^{km}}{\partial x_{k'} \partial x_{\ell'}}\frac{\partial \mathcal{P}^{k'\ell}}{\partial x_{m'}}\frac{\partial \mathcal{P}^{m'\ell'}}{\partial x_j}\right)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_m}. \tag{2b}$$

In this paper we examine

(i) whether the respective flows $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\mathcal{P}) = \Gamma_\alpha(\mathcal{P})$ at $\alpha = 1, 2$ preserve or, in fact, destroy the property of bi-vectors $\mathcal{P}(\varepsilon)$ to be Poisson, provided that the Cauchy datum $\mathcal{P}\big|_{\varepsilon=0}$ is such; we also inspect

(ii) whether the second flow is (actually, it is not) vanishing identically at all ε, provided that the Cauchy datum is a Poisson bi-vector.

Remark 1. Whenever the bi-vector $\mathcal{P}$ in every internal vertex of a non-empty graph Γ is Poisson, the bi-differential operator which is encoded by Γ can vanish identically.

First, this occurs due to the skew-symmetry of coefficients of the bi-vector.[1] Second, the operators encoded using graphs (with a copy of the Poisson bi-vector $\mathcal{P}$ at every internal vertex) can vanish by virtue of the Jacobi identity, see (1), or its differential consequences. This mechanism has been illustrated in [5]; making a part of our present argument (see section 3), it will be a key to the (re-)proof of the fact that the balanced flow $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\mathcal{P}) = \Gamma_1(\mathcal{P}) + 6\Gamma_2(\mathcal{P})$ does preserve the property of bi-vectors $\mathcal{P}(\varepsilon)$ to be Poisson whenever the Cauchy datum $\mathcal{P}\big|_{\varepsilon=0}$ is such.

So, each of the two claims (*i*–*ii*) is false if it does not hold for at least one Poisson structure (itself already known to have skew-symmetric coefficients and turn the left-hand side of the Jacobi identity into zero for any triple of arguments of the Jacobiator). To examine both claims, we clearly need a store of Poisson structures such that the coefficients $\mathcal{P}^{ij}(\boldsymbol{x})$ are not mapped to zero by the third or second order derivatives in (2a) and (2b), respectively. For that, a regular generator of Poisson structures with polynomial coefficients of arbitrarily high degree would suffice.

2. The generators

Let us recall three regular ways to generate the Poisson brackets or modify a given one, thus obtaining a new such structure. These generators will be used in section 4 to produce the counterexamples to both claims from [1].

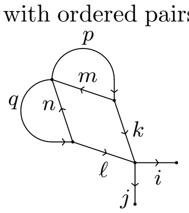
2.1. **The determinant construction.** This generator of Poisson bi-vectors is described in [6], cf. [15] and references therein. The construction goes as follows. Let $x_1, \ldots, x_n$ be the Cartesian coordinates on $\mathbb{R}^{n\geqslant 3}$. Let $\vec{g} = (g_1, \ldots, g_{n-2})$ be a fixed tuple of smooth functions in these variables. For any a, $b \in \mathcal{C}^\infty(\mathbb{R}^n)$, put

$$\{a, b\}_{\vec{g}} = \det\bigl(\mathbf{J}(g_1, \ldots, g_{n-2}, a, b)\bigr)$$

where $\mathbf{J}(\cdot, \ldots, \cdot)$ is the Jacobian matrix. Clearly, the bracket $\{\cdot,\cdot\}_{\vec{g}}$ is bi-linear and skew-symmetric. Moreover, it is readily seen to be a derivation in each of its arguments: $\{a, b\cdot c\}_{\vec{g}} = \{a,b\}_{\vec{g}}\cdot c + b\cdot\{a,c\}_{\vec{g}}$. For the validity mechanism of the Jacobi identity for this particular instance of the Nambu bracket we refer to [15] again (see also [16]).

To obtain the coefficients $\mathcal{P}^{ij}(\boldsymbol{x})$ of the respective Poisson bi-vector $\mathcal{P}$, one evaluates the bracket at the coordinate functions: $\mathcal{P}^{ij}(\boldsymbol{x}) = \{x^i, x^j\}_{\vec{g}}\big|_{\boldsymbol{x}}$.

[1] For example, consider the oriented graph with ordered pairs of indexed edges ($i \prec j$, $k \prec \ell$, $m \prec n$, $p \prec q$):



We claim that due to the antisymmetry of $\mathcal{P}$ which is contained in each of the four internal vertices, the operator (which this graph encodes) vanishes identically. Indeed, it equals minus itself:

$$\begin{aligned}\partial_m\partial_n(\mathcal{P}^{pq})\partial_p(\mathcal{P}^{km})\partial_q(\mathcal{P}^{\ell n})\partial_k\partial_\ell(\mathcal{P}^{ij})\,\partial_i\wedge\partial_j &= -\partial_m\partial_n(\mathcal{P}^{qp})\partial_p(\mathcal{P}^{km})\partial_q(\mathcal{P}^{\ell n})\partial_k\partial_\ell(\mathcal{P}^{ij})\,\partial_i\wedge\partial_j\\ &= -\partial_n\partial_m(\mathcal{P}^{pq})\partial_q(\mathcal{P}^{\ell n})\partial_p(\mathcal{P}^{km})\partial_\ell\partial_k(\mathcal{P}^{ij})\,\partial_i\wedge\partial_j = 0.\end{aligned}$$

To establish the second equality, we interchanged the labelling of indices ($p \rightleftarrows q$, $k \rightleftarrows \ell$, and $m \rightleftarrows n$) and we recalled that the partial derivatives commute.

Example 2 (see entry 3 in Table 2 on p. 10)**.** Fix the functions $g_1 = x_2^3 x_3^2 x_4$ and $g_2 = x_3^4 x_4 x_1$, and insert them in the determinant generator of Poisson bi-vectors. We thus obtain the bi-vector $\mathcal{P}$, the coefficients of which are given in the matrix

$$\begin{pmatrix} 0 & -2\,x_2^3x_3^5x_4x_1 & -3\,x_2^2x_3^6x_4x_1 & 12\,x_2^2x_3^5x_4^2x_1 \\ 2\,x_2^3x_3^5x_4x_1 & 0 & -x_3^6x_4x_2^3 & 2\,x_3^5x_4^2x_2^3 \\ 3\,x_2^2x_3^6x_4x_1 & x_3^6x_4x_2^3 & 0 & -3\,x_3^6x_4^2x_2^2 \\ -12\,x_2^2x_3^5x_4^2x_1 & -2\,x_3^5x_4^2x_2^3 & 3\,x_3^6x_4^2x_2^2 & 0 \end{pmatrix}.$$

By construction, the above matrix is skew-symmetric. The validity of Jacobi identity (1) is straightforward: indexed by i, j, k, all the components $[\![\mathcal{P},\mathcal{P}]\!]^{ijk}$ of the tri-vector vanish.[2] This Poisson bi-vector $\mathcal{P}$ will be used in section 4 in the list of our counterexamples to the claims under study.

2.2. **Pre-multiplication in the 3-dimensional case.** Let x, y, z be the Cartesian coordinates on the vector space $\mathbb{R}^3$. For every bi-vector $\mathcal{P} = \mathcal{P}^{ij}\,\partial_i \wedge \partial_j$, introduce the differential one-form $\mathrm{P} = \mathrm{P}_1\,\mathrm{d}x + \mathrm{P}_2\,\mathrm{d}y + \mathrm{P}_3\,\mathrm{d}z$ by setting $\mathrm{P} := -\mathcal{P} \lrcorner\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, so that $\mathrm{P}_1 = -\mathcal{P}^{23}$, $\mathrm{P}_2 = \mathcal{P}^{13}$, and $\mathrm{P}_3 = -\mathcal{P}^{12}$. It is readily seen [7] that the original Jacobi identity for the bi-vector $\mathcal{P}$ now reads[3] $\mathrm{dP} \wedge \mathrm{P} = 0$ for the respective one-form P. But let us note that the pre-multiplication $\mathrm{P} \mapsto f \cdot \mathrm{P}$ of the form P by a smooth function f preserves this reading of the Jacobi identity:

$$\mathrm{d}(f\mathrm{P}) \wedge (f\mathrm{P}) = f \cdot \big[\mathrm{d}f \wedge \mathrm{P} \wedge \mathrm{P} + f \cdot \mathrm{dP} \wedge \mathrm{P}\big] = f^2 \cdot \mathrm{dP} \wedge \mathrm{P} = 0.$$

This shows that the bi-vector $f\mathcal{P}$ which the form $f\mathrm{P}$ yields on $\mathbb{R}^3$ is also Poisson.

This pre-multiplication trick provides the examples of Poisson structures of arbitrarily high polynomial degree coefficients (in a manifestly non-symplectic three-dimensional set-up).[4]

2.3. **The Vanhaecke construction.** In [8], Vanhaecke created another construction of high polynomial degree Poisson bi-vectors. Let u be a monic degree d polynomial

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[2]Indeed, there are four tuples of distinct values of the indices i, j, and k up to permutations; we let $1 \leqslant i < j < k \leqslant n = 4$ so that the check runs over the set of triples $\{(1,2,3), (1,2,4), (1,3,4), (2,3,4)\}$. For example,

$$\begin{aligned}[\![\mathcal{P},\mathcal{P}]\!]^{123} &= 6x_2^5x_3^{11}x_4^2x_1 - 6x_2^5x_3^{11}x_4^2x_1 - 6x_2^5x_3^{11}x_4^2x_1 + 6x_2^5x_3^{11}x_4^2x_1 \\ &\quad - 18x_2^5x_3^{11}x_4^2x_1 + 18x_2^5x_3^{11}x_4^2x_1 + 12x_2^5x_3^{11}x_4^2x_1 - 6x_2^5x_3^{11}x_4^2x_1 - 6x_2^5x_3^{11}x_4^2x_1 = 0.\end{aligned}$$

Therefore, $[\![\mathcal{P},\mathcal{P}]\!] = \sum_{1\leqslant i<j<k\leqslant 4} [\![\mathcal{P},\mathcal{P}]\!]^{ijk}(\boldsymbol{x})\,\partial_i \wedge \partial_j \wedge \partial_k = 0$.

[3]The exterior differential dP is equal to

$$\mathrm{dP} = (\partial_x \mathrm{P}^{13} + \partial_y \mathrm{P}^{23})\,\mathrm{d}x \wedge \mathrm{d}y + (-\partial_x \mathrm{P}^{12} + \partial_z \mathrm{P}^{23})\,\mathrm{d}x \wedge \mathrm{d}z + (-\partial_y \mathrm{P}^{11} - \partial_z \mathrm{P}^{13})\,\mathrm{d}y \wedge \mathrm{d}z.$$

The wedge product is

$$\begin{aligned}\mathrm{dP} \wedge \mathrm{P} &= \big(\partial_x \mathrm{P}^{31}\,\mathrm{P}^{12} + \partial_y \mathrm{P}^{23}\,\mathrm{P}^{21} + \partial_x \mathrm{P}^{12}\,\mathrm{P}^{13} + \partial_z \mathrm{P}^{23}\,\mathrm{P}^{31} + \partial_y \mathrm{P}^{12}\,\mathrm{P}^{23} + \partial_z \mathrm{P}^{31}\,\mathrm{P}^{32}\big)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ &= (-[\![\mathcal{P},\mathcal{P}]\!] \lrcorner\, \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.\end{aligned}$$

[4]In dimension three, this pre-multiplication procedure also provides the examples of Poisson bi-vectors at which the second flow (2b) does not vanish identically.

in λ and v be a polynomial of degree $d-1$ in λ:

$$\begin{aligned} u(\lambda) &= \lambda^d + u_1\lambda^{d-1} + \ldots + u_{d-1}\lambda + u_d, \\ v(\lambda) &= \qquad v_1\lambda^{d-1} + \ldots + v_{d-1}\lambda + v_d. \end{aligned}$$

Consider the space $\Bbbk^{2d}$ (e.g., set $\Bbbk := \mathbb{R}$) with Cartesian coordinates u_1, $\ldots$, u_n, v_1, $\ldots$, v_d. To define the Poisson bracket, fix a bivariate polynomial $\phi(\cdot,\cdot)$ and for all $1 \leqslant i, j \leqslant d$ set

$$\{u_i, u_j\} = \{v_i, v_j\} = 0, \tag{3a}$$

$$\{u_i, v_j\} = \text{coeff. of } \lambda^j \text{ in } \left(\phi\bigl(\lambda, v(\lambda)\bigr) \cdot \left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+ \mod u(\lambda)\right), \tag{3b}$$

where we denote by $[\ldots]_+$ the argument's polynomial part and where the remainder modulo the degree d polynomial $u(\lambda)$ is obtained using the Euclidean division algorithm.

Let us emphasise that these Poisson bi-vector are defined on the even-dimensional spaces. Indeed, the coefficients of Poisson bracket (3) are arranged in the block matrix $\left(\begin{smallmatrix} 0 & U \\ -U & 0 \end{smallmatrix}\right)$, where the components of the matrix U are $U^{ij} = \{u_i, v_j\}$.

2.4. **The Hamiltonian differential operators on jet spaces.** The variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ for functionals of sections of fibre bundles generalise the notion of Poisson brackets $\{\cdot,\cdot\}_{\mathcal{P}}$ for functions on finite-dimensional Poisson manifolds $(N^n, \{\cdot,\cdot\}_{\mathcal{P}})$. Namely, let us consider the space $J^\infty(\pi)$ of infinite jets of sections for a given bundle π over a manifold M^n of positive dimension m. The variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ on $J^\infty(\pi)$ are then specified by using the Hamiltonian differential operators (which we shall denote by A and the order of which is typically positive).[5] The formalism of variational Poisson bi-vectors $\boldsymbol{\mathcal{P}} = \frac{1}{2}\langle \boldsymbol{\xi} \cdot \vec{A}(\boldsymbol{\xi})\rangle$ and the variational Schouten bracket $[\![\cdot,\cdot]\!]$ is standard (see [10, 18] and section 3 below). The geometry of iterated variations is revealed in [12]; the correspondence between the Kontsevich graphs and local variational polydifferential operators is explained in [13].

Example 3. For an inspection whether any of the two claims (which we quoted from [1] on the title page) would hold in the variational set-up, it is enough to consider a Hamiltionian differential operator with (differential-)polynomial coefficients of degree $\geqslant$ 3. Let us conveniently take the Hamiltonian operator[6]

$$A = u^2 \circ \frac{\mathrm{d}}{\mathrm{d}x} \circ u^2$$

for the Harry–Dym equation (see [11]); here u is the fibre coordinate in the trivial bundle $\pi\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and x is the base variable. This operator is obviously skew-adjoint, whence the variational Poisson bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ is skew-symmetric. The Jacobi identity for $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ is also easy to check: the variational master equation $[\![\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}]\!] \cong 0$ does hold for the variational bi-vector $\boldsymbol{\mathcal{P}} = \frac{1}{2}\langle \xi \cdot \vec{A}(\xi)\rangle$.

[5]In fact, the Poisson geometry of finite-dimensional manifolds $(N^n, \{\cdot,\cdot\}_{\mathcal{P}})$ is a zero differential order sub-theory in the variational Poisson geometry of infinite jet spaces $J^\infty(\pi)$. Indeed, let the fibres in the bundle π be N^n and proclaim that only *constant* sections are allowed.

[6]More examples of variational Poisson structures, which are relevant for our present purpose, can be found in [19] or, e.g., in [20] (see also the references contained therein).

3. The deformation theory

Now that the generators of high polynomial degree Poisson structures are at our disposal, let us recall several necessary facts from the deformation theory; this material is standard [9, 17]. Denote by ξ_i the parity-odd canonical conjugate of the variable x_i for every $i = 1, \ldots, n$ (see [18] for discussion about the reverse parity symplectic duals). Every bi-vector is then realised in terms of the local coordinates x_i and ξ_i on ΠT^*N^n by using $\mathcal{P} = \frac{1}{2}\langle \xi_i \, \mathcal{P}^{ij}(\boldsymbol{x}) \, \xi_j \rangle$. We denote by $[\![\cdot,\cdot]\!]$ the Schouten bracket, i.e., the parity-odd Poisson bracket which is determined on $\Pi T^*\mathbb{R}^n$ by the canonical symplectic structure $\mathrm{d}\boldsymbol{x} \wedge \mathrm{d}\boldsymbol{\xi}$ (see [12] for details and [14] for illustration). Currently, our working formula is[7]

$$[\![\mathcal{P}, \mathcal{Q}]\!] = (\mathcal{P})\frac{\overleftarrow{\partial}}{\partial x_i} \cdot \frac{\overrightarrow{\partial}}{\partial \xi_i}(\mathcal{Q}) - (\mathcal{P})\frac{\overleftarrow{\partial}}{\partial \xi_i} \cdot \frac{\overrightarrow{\partial}}{\partial x_i}(\mathcal{Q}).$$

To be Poisson, a bi-vector $\mathcal{P}$ must satisfy the master-equation $[\![\mathcal{P},\mathcal{P}]\!] = 0$, of which formula (1) is the component expansion with respect to the indices (i, j, k) in the tri-vector $[\![\mathcal{P},\mathcal{P}]\!](\boldsymbol{x},\boldsymbol{\xi})$.

Under an infinitesimal deformation $\mathcal{P}(\varepsilon) = \mathcal{P}+\varepsilon\mathcal{Q}+\bar{o}(\varepsilon)$ of the bi-vector $\mathcal{P}$ satisfying $[\![\mathcal{P},\mathcal{P}]\!] = 0$, the bi-vector $\mathcal{P}(\varepsilon)$ remains Poisson only if $[\![\mathcal{P}(\varepsilon),\mathcal{P}(\varepsilon)]\!] = \bar{o}(\varepsilon)$, whence $[\![\mathcal{P},\mathcal{Q}]\!] = 0$. The violation of this requirement will be exemplified in what follows by the deformation leading terms $\Gamma_1(\mathcal{P})$ and $\Gamma_2(\mathcal{P})$ given by (2a) and (2b), respectively, for the Poisson bi-vectors $\mathcal{P}$ which we generate using the techniques from section 2.

Remark 2. For a Poisson bi-vector $\mathcal{P}$, the operator $\boldsymbol{\partial}_{\mathcal{P}} = [\![\mathcal{P},\cdot]\!]$ is readily seen to be a differential: by virtue of the Jacobi identity for the Schouten bracket $[\![\cdot,\cdot]\!]$ we have that $\boldsymbol{\partial}_{\mathcal{P}}^2 = 0$. Therefore, the leading order terms $\mathcal{Q}$ in the deformations $\mathcal{P}(\varepsilon) = \mathcal{P}+\varepsilon\mathcal{Q}+\bar{o}(\varepsilon)$ can be trivial in the second $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology, meaning that $\mathcal{Q} = [\![\mathcal{P},\mathcal{X}]\!]$ for some one-vector $\mathcal{X}$ (whence $[\![\mathcal{P},[\![\mathcal{P},\mathcal{X}]\!]]\!] \equiv 0$). Alternatively, for the $\boldsymbol{\partial}_{\mathcal{P}}$-cocycles $\mathcal{Q}$ which are not $\boldsymbol{\partial}_{\mathcal{P}}$-coboundaries, the flows $\mathcal{P}(\varepsilon)$ stay infinitesimally Poisson but leave the $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology class of the Poisson bi-vector $\mathcal{P}$ at $\varepsilon = 0$.

For consistency, let us recall that generally speaking, not every infinitesimal deformation $\mathcal{P} \mapsto \mathcal{P} + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ of a Poisson bi-vector $\mathcal{P}$ can be completed to a Poisson deformation $\mathcal{P} \mapsto \mathcal{P} + \mathcal{Q}(\varepsilon)$ at all orders in ε. All the obstructions are contained in the third $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology group $\mathrm{H}^3_{\mathcal{P}} = \{\mathrm{T} \in \Gamma(\bigwedge^3 TN) \mid \boldsymbol{\partial}_{\mathcal{P}}(\mathrm{T}) = 0\} \;/\; \{\mathrm{T} = \boldsymbol{\partial}_{\mathcal{P}}(\mathrm{R}), \mathrm{R} \in \Gamma(\bigwedge^2 TN)\}$. Indeed, cast the master-equation $[\![\mathcal{P} + \mathcal{Q}(\varepsilon),\mathcal{P} + \mathcal{Q}(\varepsilon)]\!] = 0$ for the Poisson deformation to the coboundary statement $[\![\mathcal{Q}(\varepsilon),\mathcal{Q}(\varepsilon)]\!] = \boldsymbol{\partial}_{\mathcal{P}}(-\mathcal{P} - 2\mathcal{Q}(\varepsilon))$ within $O^*(\varepsilon^2)$, whence $\boldsymbol{\partial}_{\mathcal{P}}([\![\mathcal{Q}(\varepsilon),\mathcal{Q}(\varepsilon)]\!] \equiv 0$ by $\boldsymbol{\partial}_{\mathcal{P}}^2 = 0$. Therefore, the vanishing of the third $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology group guarantees the existence of a power series solution $\mathcal{Q}(\varepsilon)$ to the cocycle-coboundary equation $[\![\mathcal{Q}(\varepsilon),\mathcal{Q}(\varepsilon)]\!] = -2\boldsymbol{\partial}_{\mathcal{P}}(\mathcal{Q}(\varepsilon))$: known to be a cocycle, the left-hand side has been proven to be a coboundary as well.

Remark 3. Nowhere above should one expect that the leading deformation term $\mathcal{Q}$ in $\mathcal{P}(\varepsilon) = \mathcal{P} + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ itself would be a Poisson bi-vector. This may happen for $\mathcal{Q}$ only incidentally.

[7] In the set-up of infinite jet spaces $J^\infty(\pi)$ (see [10] and [12, 13, 18]) the four partial derivatives in the formula for $[\![\cdot,\cdot]\!]$ become the variational derivatives with respect to the same variables, which now parametrise the fibres in the Whitney sum $\pi \times_{M^m} \Pi\widehat{\pi}$ of (super-)bundles over the m-dimensional base M^m.

4. The counterexamples

We now examine the properties of both tetrahedral flows (2) whenever each of them is evaluated at a given Poisson bi-vector (the examples of such bi-vectors are produced by using the techniques from section 2). To motivate the composition of Tables 1–4 and clarify the meaning of their content, let us consider an example: namely, we first take the Poisson bi-vector which was obtained in section 2.1 (see p. 5).

Example 4 (continued)**.** Rewriting the Poisson bi-vector $\mathcal{P}_0 \in \Gamma\big(\bigwedge^2 TN^4\big)$ in terms of the parity-odd variables $\boldsymbol{\xi}$, we obtain that under the isomorphism $\Gamma\big(\bigwedge^\bullet TN^n\big) \simeq C^\infty(\Pi T^* N^n)$ the bi-vector $\mathcal{P}_0^{ij}(\boldsymbol{x})\,\partial_i \wedge \partial_j$ becomes $\frac{1}{2}\mathcal{P}_0^{ij}(\boldsymbol{x})\,\xi_i\xi_j$:

$$\begin{aligned}\mathcal{P}_0 = -2\,x_2^3x_3^5x_4x_1\xi_1\xi_2 - 3\,x_2^2x_3^6x_4x_1\xi_1\xi_3 + 12\,x_2^2x_3^5x_4^2x_1\xi_1\xi_4 \\ - x_3^6x_4x_2^3\xi_2\xi_3 + 2\,x_3^5x_4^2x_2^3\xi_2\xi_4 - 3\,x_3^6x_4^2x_2^2\xi_3\xi_4.\end{aligned}$$

Now, we calculate the right-hand sides $\mathcal{P}_1 := \Gamma_1(\mathcal{P}_0)$ and $\mathcal{P}_2 := \Gamma_2(\mathcal{P}_0)$ of tetrahedral flows (2). The coefficient matrix of the bi-vector $\mathcal{P}_1$ is

$$\mathcal{P}_1^{ij} = \begin{pmatrix} 0 & -24480\,x_2^9x_3^{20}x_4^4x_1 & -51840\,x_3^{21}x_4^4x_2^8x_1 & 12960\,x_3^{20}x_4^5x_2^8x_1 \\ 24480\,x_2^9x_3^{20}x_4^4x_1 & 0 & -15480\,x_3^{21}x_4^4x_2^9 & 2448\,x_3^{20}x_4^5x_2^9 \\ 51840\,x_3^{21}x_4^4x_2^8x_1 & 15480\,x_3^{21}x_4^4x_2^9 & 0 & -18144\,x_3^{21}x_4^5x_2^8 \\ -12960\,x_3^{20}x_4^5x_2^8x_1 & -2448\,x_3^{20}x_4^5x_2^9 & 18144\,x_3^{21}x_4^5x_2^8 & 0 \end{pmatrix}.$$

In a similar way, the polydifferential operator Γ_2 (encoded by the second tetrahedral graph in Fig. 1) yields the matrix

$$\mathcal{P}_2^{ij} = \begin{pmatrix} 16920x_2^8x_3^{20}x_4^4x_1^2 & -12060\,x_2^9x_3^{20}x_4^4x_1 & -16380\,x_3^{21}x_4^4x_2^8x_1 & 42840\,x_3^{20}x_4^5x_2^8x_1 \\ 2700\,x_2^9x_3^{20}x_4^4x_1 & -7200\,x_2^{10}x_3^{20}x_4^4 & 4680\,x_3^{21}x_4^4x_2^9 & -252\,x_3^{20}x_4^5x_2^9 \\ -13140\,x_3^{21}x_4^4x_2^8x_1 & 5040\,x_3^{21}x_4^4x_2^9 & -12060\,x_2^8x_3^{22}x_4^4 & 13716\,x_3^{21}x_4^5x_2^8 \\ -80280\,x_3^{20}x_4^5x_2^8x_1 & -18036\,x_3^{20}x_4^5x_2^9 & 21708\,x_3^{21}x_4^5x_2^8 & -58104\,x_3^{20}x_4^6x_2^8 \end{pmatrix}.$$

Notice that this coefficient matrix is not yet antisymmetric, but its *symmetric* counterpart is skipped out in the construction of the bi-vector $\mathcal{P}_2$ and its transcription by using the anticommuting variables $\boldsymbol{\xi}$. Therefore, we antisymmetrise the above matrix at once, the output to be used in what follows. We obtain that the bi-vector $\mathcal{P}_2$ is

$$\begin{aligned}\mathcal{P}_2 = -7380x_2^9x_3^{20}x_4^4x_1\xi_1\xi_2 - 1620x_3^{21}x_4^4x_2^8x_1\xi_1\xi_3 + 61560x_3^{20}x_4^5x_2^8x_1\xi_1\xi_4 \\ - 180x_3^{21}x_4^4x_2^9\xi_2\xi_3 + 8892x_3^{20}x_4^5x_2^9\xi_2\xi_4 - 3996x_3^{21}x_4^5x_2^8\xi_3\xi_4.\end{aligned}$$

We now see that for the Poisson bi-vector $\mathcal{P}_0$ from Example 2 on p. 5, ***the bi-vector*** $\mathcal{P}_2$ ***does not vanish***, thereby disavowing the second claim from [1].

To check the compatibility of the original Poisson bi-vector $\mathcal{P}_0$ with the newly obtained bi-vector $\mathcal{P}_1$, we calculate their Schouten bracket:

$$\begin{aligned}[\![\mathcal{P}_0, \mathcal{P}_1]\!] = 46008\,x_2^{11}x_3^{26}x_4^5x_1\xi_1\xi_2\xi_3 + 852768\,x_1x_2^{11}x_3^{25}x_4^6\xi_1\xi_2\xi_4 \\ + 1246752\,x_1x_2^{10}x_3^{26}x_4^6\xi_1\xi_3\xi_4 + 340200\,x_2^{11}x_3^{26}x_4^6\xi_2\xi_3\xi_4 \neq 0.\end{aligned}$$

The above expression is not identically zero. Therefore, ***the leading term*** $\mathcal{P}_1$ ***in the deformation*** $\mathcal{P}_0 \mapsto \mathcal{P}(\varepsilon) = \mathcal{P}_0 + \varepsilon\mathcal{P}_1 + \bar{o}(\varepsilon)$ ***destroys the property of bi-vector*** $\mathcal{P}(\varepsilon)$ ***to be Poisson*** at $\varepsilon \neq 0$ for all $\boldsymbol{x} \in \mathbb{R}^4$.

The same compatibility test for $\mathcal{P}_0$ and its second flow (2b) yields that

$$[\![\mathcal{P}_0, \mathcal{P}_2]\!] = -7668\, x_2^{11} x_3^{26} x_4^5 x_1 \xi_1 \xi_2 \xi_3 - 142128\, x_1 x_2^{11} x_3^{25} x_4^6 \xi_1 \xi_2 \xi_4 - 207792\, x_1 x_2^{10} x_3^{26} x_4^6 \xi_1 \xi_3 \xi_4 - 56700\, x_2^{11} x_3^{26} x_4^6 \xi_2 \xi_3 \xi_4.$$

Again, this expression does not vanish identically at points $\boldsymbol{x}$ of the Poisson manifold $\big(\mathbb{R}^4, \{\cdot,\cdot\}_{\mathcal{P}_0}\big)$. We conclude that neither of two flows (2) preserve the property of bi-vector $\mathcal{P}(\varepsilon)$ to stay (infinitesimally) Poisson at $\varepsilon \neq 0$ for this example of Poisson bi-vector.[8]

Remark 4. In the above example, the Schouten brackets $[\![\mathcal{P}_0, \mathcal{P}_1]\!]$ and $[\![\mathcal{P}_0, \mathcal{P}_2]\!]$ are determined by the same polynomials in the variables $\boldsymbol{x}$ and $\boldsymbol{\xi}$: we see that $[\![\mathcal{P}_0, \mathcal{P}_1]\!] = -6 \cdot [\![\mathcal{P}_0, \mathcal{P}_2]\!]$. This implies that for this example of Poisson bi-vector $\mathcal{P}_0$, the leading term $\mathcal{Q} := \mathcal{P}_1 + 6\mathcal{P}_2$ does (infinitesimally) preserve the property of $\mathcal{P}(\varepsilon)$ to be Poisson in the course of deformation $\mathcal{P}_0 \mapsto \mathcal{P}_0 + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$. Indeed, we have that

$$[\![\mathcal{P}_0, \mathcal{Q}]\!] = [\![\mathcal{P}_0, \mathcal{P}_1 + 6\mathcal{P}_2]\!] = [\![\mathcal{P}_0, \mathcal{P}_1]\!] + 6[\![\mathcal{P}_0, \mathcal{P}_2]\!] = 0$$

due to the linearity of the Schouten bracket.

Moreover, it is readily seen that the ratio 1 : 6 is the *only* way to balance the two flows, (2a) vs (2b), such that their nontrivial linear combination $\mathcal{Q}$ is compatible with the given Poisson bi-vector $\mathcal{P}_0$ from Example 2.[9]

Remark 5. The linear combination $\mathcal{Q}$ of two flows (2) is compatible with the initial Poisson bi-vector $\mathcal{P}_0$ in a nontrivial manner, that is, the bi-vector $\mathcal{Q} = \mathcal{P}_1 + 6\mathcal{P}_2 \neq 0$ is not identically equal to zero. (For other examples this may happen incidentally.) We expect that the leading term $\mathcal{Q}$ in the infinitesimal deformation $\mathcal{P}_0 \mapsto \mathcal{P}_0 + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ is nontrivial in the Poisson cohomology with respect to $\boldsymbol{\partial}_{\mathcal{P}_0}$, that is, $\mathcal{Q} \neq [\![\mathcal{P}_0, \mathcal{X}]\!]$ for any vector $\mathcal{X}$ on the four-dimensional space.[10]

In the three tables below we summarise the results about the flows $\mathcal{P}_1$ and $\mathcal{P}_2$, which we evaluate at the examples of Poisson bi-vectors $\mathcal{P}_0$. Our special attention is paid to the leading deformation term $\mathcal{Q} = \mathcal{P}_1 + 6\mathcal{P}_2$ in each case: we inspect whether this bi-vector incidentally vanishes and whether it is (indeed, always) compatible with the original Poisson structure $\mathcal{P}_0$.

[8]Let us also inspect whether the Jacobi identity holds for any of the bi-vectors $\mathcal{P}_1$ and $\mathcal{P}_2$. For $\mathcal{P}_1$ we have that the left-hand side of the Jacobi identity is equal to

$$[\![\mathcal{P}_1, \mathcal{P}_1]\!] = -2963589120 \cdot \big(x_3^{41} x_4^8 x_2^{17} x_1 \xi_1 \xi_2 \xi_3 + 5\, x_3^{40} x_4^9 x_2^{17} x_1 \xi_1 \xi_2 \xi_4 - 2\, x_3^{41} x_4^9 x_2^{16} x_1 \xi_1 \xi_3 \xi_4\big),$$

which does not vanish. (Therefore the Jacobi identity is not satisfied for $\mathcal{P}_1$.) For $\mathcal{P}_2$ the left-hand side of the Jacobi identity equals

$$[\![\mathcal{P}_2, \mathcal{P}_2]\!] = -262517760 \cdot \big(x_3^{41} x_4^8 x_2^{17} x_1 \xi_1 \xi_2 \xi_3 + 5\, x_3^{40} x_4^9 x_2^{17} x_1 \xi_1 \xi_2 \xi_4 - 2\, x_3^{41} x_4^9 x_2^{16} x_1 \xi_1 \xi_3 \xi_4\big).$$

This expression also does not vanish, so that neither $\mathcal{P}_1$ nor $\mathcal{P}_2$ are Poisson bi-vectors.

[9]The balance $1 : \frac{4}{3}$ was advocated in [21, §5.2] for the linear combination of flows (2a) and (2b), respectively; our present argument and the counterexamples which follow withdraw that claim.

[10]In all the two-dimensional Poisson geometries, the first flow $\mathcal{P}_1$ is always cohomologically trivial, i.e., it is of the form $\mathcal{P}_1 = [\![\mathcal{P}_0, \mathcal{X}]\!]$ for some one-vector $\mathcal{X}$, see [1].

TABLE 1. The Poisson bi-vectors $\mathcal{P}_0$ are generated using the determinant method from section 2.1 (the dimension is equal to 3, so we specify the fixed argument g_1); that generator is combined with the pre-multiplication $(f\cdot)$ as explained in section 2.2.

№	dim	Argument & pre-factor	$[\![\mathcal{P}_0, \mathcal{P}_1]\!] = 0\,?$	$\mathcal{P}_2 \stackrel{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{P}_2]\!] = 0\,?$	$\mathcal{Q} \stackrel{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{Q}]\!] = 0\,?$
1.	3	$[x_1^5x_2^3x_3^4 + x_1^2x_3^5 + x_1x_2^5x_3]$ $x_1^3 + x_2^2$	✗	✗	✗	✗	✓
2.	3	$[x_1x_2 + x_3x_1 + x_2x_3]$ $x_1^2 + x_2$	✗	✗	✗	✗	✓

For both examples in Table 1 we have that neither does $\mathcal{P}_1$ preserve the property of $\mathcal{P}_0 + \varepsilon\mathcal{P}_1 + \bar{o}(\varepsilon)$ to be (infinitesimally) Poisson nor does $\mathcal{P}_2$ vanish identically — which is in contrast with both the claims from [1].

TABLE 2. In dimensions higher than 3, we generate the Poisson bi-vectors $\mathcal{P}_0$ by using the determinant method from section 2.1: the auxiliary arguments $g_1, \ldots, g_{n-2}$ are specified.

№	dim	Arguments	$[\![\mathcal{P}_0, \mathcal{P}_1]\!] = 0\,?$	$\mathcal{P}_2 \stackrel{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{P}_2]\!] = 0\,?$	$\mathcal{Q} \stackrel{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{Q}]\!] = 0\,?$
3.	4	$[x_2^3x_3^2x_4, x_3^4x_4x_1]$	✗	✗	✗	✗	✓
4.	4	$[x_1^2x_2^3x_3^4x_4^5, x_1x_2x_3x_4]$	✗	✗	✗	✓	✓
5.	4	$[x_2^2x_3^2x_4^2, x_1^2x_3^2x_4^2]$	✗	✗	✗	✓	✓
6.	5	$[x_2^3x_3^2x_4, x_3^4x_4x_1, x_5^4x_4^2x_3^3]$	✗	✗	✗	✗	✓

In Table 2 we again have that neither is the property preserved for $\mathcal{P}_0 + \varepsilon\mathcal{P}_1 + \bar{o}(\varepsilon)$ to be (infinitesimally) Poisson nor is the bi-vector $\mathcal{P}_2$ vanishing identically.

TABLE 3. The results for the Vanhaecke method from section 2.3: we here specify the bivariate polynomials ϕ.

№	dim	$\phi(x, y)$	$[\![\mathcal{P}_0, \mathcal{P}_1]\!] \stackrel{?}{=} 0$	$\mathcal{P}_2 \stackrel{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{P}_2]\!] \stackrel{?}{=} 0$	$\mathcal{Q} \stackrel{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{Q}]\!] \stackrel{?}{=} 0$
7.	4	$[x^2y^2]$	✗	✗	✗	✗	✓
8.	4	$[x^2y]$	✗	✗	✗	✗	✓
9.	4	$[x^3y^2]$	✗	✗	✗	✗	✓
10.	4	$[x^3y^3]$	✗	✗	✗	✗	✓
11.	6	$[x^2y^2]$	✗	✗	✗	✗	✓

The entries in Table 3 report on the use of the generator from section 2.3: experimentally established, the properties of these Poisson bi-vectors do not match both the claims from [1].

TABLE 4. The results for the infinite-dimensional case.

№	dim	Operator	$[\![\boldsymbol{\mathcal{P}}_0, \boldsymbol{\mathcal{P}}_1]\!] \overset{?}{\cong} 0$	$\boldsymbol{\mathcal{P}}_2 \overset{?}{\cong} 0$
12.	∞	$u^2 \circ \frac{\mathrm{d}}{\mathrm{d}x} \circ u^2$	✗	✓

The variational bi-vector $\boldsymbol{\mathcal{P}}_1 = \frac{1}{2}\langle \xi \cdot \vec{A}_1(\xi)\rangle$, which we construct from the variational Poisson bi-vector $\boldsymbol{\mathcal{P}}_0 = \frac{1}{2}\langle \xi \cdot u^2 \frac{\mathrm{d}}{\mathrm{d}x}(u^2\,\xi)\rangle$ by using the geometric technique from [12] (see also [13]), is determined by the (skew-adjoint part of the) first order differential operator $A_1 = 192\,\bigl(9u^8 u_x u_{xx} - u^9 u_{xxx}\bigr)\,\frac{\mathrm{d}}{\mathrm{d}x}$ in total derivatives. Again, the two variational bi-vectors are *not* compatible: we check that $[\![\boldsymbol{\mathcal{P}}_0, \boldsymbol{\mathcal{P}}_1]\!] \not\cong 0$ under the variational Schouten bracket.

Remarkably, the variational bi-vector $\boldsymbol{\mathcal{P}}_2$ is specified by the second-order total differential operator whose skew-adjoint component vanishes, whence the respective variational bi-vector *is* equal to zero (modulo exact terms within its horizontal cohomology class [10]).

Conclusion. The linear combination $\mathcal{Q} = \mathcal{P}_1 + 6\mathcal{P}_2$ of the Kontsevich tetrahedral flows preserves the space of Poisson bi-vectors $\mathcal{P}_0$ under the infinitesimal deformations $\mathcal{P}_0 \mapsto \mathcal{P}_0 + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$. This is manifestly true for all the examples of Poisson bi-vectors on finite-dimensional (vector or affine) spaces $\mathbb{R}^n$ which we have considered so far. We now conjecture that the leading deformation term $\mathcal{Q} = \mathcal{Q}(\mathcal{P}_0)$ always has this property, that is, the bi-vector $\mathcal{Q}$ marks a $\boldsymbol{\partial}_{\mathcal{P}_0}$-cohomology class for every Poisson bi-vector $\mathcal{P}_0$ on a finite-dimensional affine manifold. (Recall that such class can be $\boldsymbol{\partial}_{\mathcal{P}_0}$-trivial; moreover, it can vanish identically — yet the above examples confirm the existence of Poisson geometries where neither of the two options is realised.)

Let us conclude that every claim of an object's vanishing by virtue of the skew-symmetry and Jacobi identity for a given Poisson bi-vector, which that object depends on by construction, must be accompanied with an explicit description of that factorisation mechanism (e.g., see [5]) or at least, with a proof of that mechanism's existence. Apart from the trivial case (here, $\mathcal{Q} = 0$ so that $[\![\mathcal{P}_0, \mathcal{Q}]\!] \equiv 0$), such factorisation through the master-equation $[\![\mathcal{P}_0, \mathcal{P}_0]\!] = 0$ can be immediate: here, we have that[11]

$$[\![\mathcal{P}_0, \mathcal{Q}]\!] = [\![\mathcal{P}_0, [\![\mathcal{P}_0, \mathcal{X}]\!]]\!] = \tfrac{1}{2}[\![[\![\mathcal{P}_0, \mathcal{P}_0]\!], \mathcal{X}]\!] = \bigl(\tfrac{1}{2}[\![\cdot, \mathcal{X}]\!]\bigr)\,\bigl([\![\mathcal{P}_0, \mathcal{P}_0]\!]\bigr)$$

for all $\boldsymbol{\partial}_{\mathcal{P}_0}$-exact infinitesimal deformations $\mathcal{Q} = \boldsymbol{\partial}_{\mathcal{P}_0}(\mathcal{X})$ of the Poisson bi-vectors $\mathcal{P}_0$. Elaborated in [5], the Poisson cohomology estimate mechanism of the vanishing $[\![\mathcal{P}_0, \mathcal{Q}]\!] \doteq 0$ via $[\![\mathcal{P}_0, \mathcal{P}_0]\!] = 0$ works – for the nontrivial cocycles $\mathcal{Q} \notin \operatorname{im} \boldsymbol{\partial}_{\mathcal{P}_0}$ in the $\boldsymbol{\partial}_{\mathcal{P}_0}$-cohomology – due to much more refined principles. We shall address this mechanism in a subsequent paper.

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[11]Otherwise speaking, the flow $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\mathcal{P}) = [\![\mathcal{P}, \mathcal{X}(\mathcal{P})]\!]$ is tautologically Poisson with respect to its native structure $\mathcal{P}$.

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