On Emergent Geometry from Entanglement Entropy in Matrix Theory

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Abstract

Using Matrix theory, we compute the entanglement entropy between a supergravity probe and modes on a spherical membrane. We demonstrate that a membrane stretched between the probe and the sphere entangles these modes and leads to an expression for the entanglement entropy that encodes information about local gravitational geometry seen by the probe. We propose in particular that this entanglement entropy measures the rate of convergence of geodesics at the location of the probe.

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1 Introduction and Highlights

Classical gravity is a phenomenon of geometrical origin, encoded in the curvature of spacetime. Quantum considerations however, whether in the setting of string theory or otherwise, suggest that the geometrical picture of gravity may be an effective long distance approximation scheme. At Planckian distances, a fundamental rethinking of the nature of gravity sets in. There have also been recent suggestions that the perception of gravity is entropic, arising from quantum entanglement [1]-[8]. And subsequently, one talks about the concept of 'emergent geometry': the idea that gravitational geometry is a collective phenomenon associated with underlying microscopic degrees of freedom.

In attempting to understand these ideas in a concrete computational setting, the Banks-Fishler-Shenker-Susskind (BFFS) Matrix model [9] – and its related cousin, the Berenstein-Maldacena-Nastase (BMN) system [10] – provide for a rich playground. They purport to describe quantum gravity in a non-perturbative and complete framework, that of light-cone M-theory. The degrees of freedom are packaged into matrices that, in principle, encode geometrical gravity data at long enough distances. Spacetime curvature is then expected to arise from the collective dynamics of these matrix degrees of freedom. Unfortunately, the map between emergent geometry and matrix dynamics has proven to be a difficult one to unravel (but see recent progress in this direction [11]-[18]).

A crude cartoon of Matrix theory dynamics goes as follows. The degrees of freedom, arranged in matrices, represent an interlinked complex web of membranes and fivebranes. At low energies, one can find settings where a hierarchy separates the different matrix degrees of freedom. Sub-blocs of the matrices, modes that remain light and slow, describe localized and widely separated lumps of energy; while other 'off-diagonal' modes become heavy and frozen in their ground states – heuristically corresponding to membranes and fivebranes stretched between the lumps. The effective dynamics of the lumps leads to the expected low energy supergravity dynamics, and hence a notion of emergent geometry. From this perspective, it is not surprising that a mechanism of entanglement across the degrees of freedom in sub-blocs of the matrices is key to the notion of emergent spacetime geometry. However, to our knowledge the role of quantum entanglement has not yet been explored in this context. In this work, our goal is to take the first steps in understanding how geometry is encoded in Matrix theory degrees of freedom through the phenomenon of quantum entanglement.

We consider a particularly simple setup in an attempt to make the otherwise challenging computation feasible. We will arrange a spherical membrane in light-cone M-theory, stabilized artificially so as to source a smooth spherically symmetric curved spacetime; and then we will add a probe supergravity particle a large distance away from the spherical source. Realizing the setup in matrices, we are immediately led to explore fluctuations of matrix modes that describes membranes stretched between sphere and probe. Using the configuration as a background scaffolding, we then focus on the dynamics of the fermionic modes. Zero modes of the fermionic degrees of freedom describe a system of qubits with a dense network of interactions. The qubit states map onto the eleven dimensional supergravity multiplet; hence, one is describing the interactions of supergravity modes in the given background. The setup for example has been used recently to demonstrate fast scrambling of supergravity modes in Matrix theory [19, 20].

We then can pose a concrete physical question. Given a bosonic matrix configuration, what is the vacuum of the qubit system? Some of the qubits will be associated with the spherical membrane, others with the probe. We can then compute the entanglement entropy between the sphere and probe qubits in the vacuum. Effectively, through the BFSS conjecture, we would be computing the entanglement between supergravity modes on the sphere and on the probe. The computation can be carried out using an expansion in a small parameter, the ratio of the sphere-probe distance to the radius of the sphere. That is, we compute the entanglement entropy in the regime the probe is far away from the sphere.

We then analyze the result in relation to spin chain systems arising in the literature. We see suggestive qualitative similarities – a logarithmic dependence on the sphere-probe distance – that allow us to present an interpretation of the results. Our conclusion is that space geometry is indeed encoded in the computed entanglement entropy. Unlike a similar map between entanglement and geometry through an area law [2, 1], our suggestion has a local character – possibly relating the rate of convergence of geodesics to a coefficient in the entanglement entropy.

The presentation is organized as follows. Section 1 gives an overview of the Matrix theory of interest and sets up the sphere-probe background configuration. Section 2 presents the dynamics of the fermionic modes in the given background, shows the derivation of the effective Hamiltonian and the corresponding vacuum. Section 3 shows the computation of the Von Neumann entropy in the vacuum, summarizes an overview of what is known about similar observables in the condensed matter literature, and collects a series of observations and speculations about the interpretation of the result. Finally, section 4 collects some concluding thoughts and directions for the future. Two appendices summarize several technical details that arise in the main text.

2 The setup

2.1 BFSS matrix theory

The BFSS theory is a $0+1$ dimensional supersymmetric matrix theory (the dimensional reduction of the 10d super Yang-Mills) describing the dynamics of D0 branes. For N D0 branes, the Lagrangian is given by [9, 21, 22, 23]

$$L = \mathrm{Tr}\left[\frac{1}{2\,R} D_t X_i D_t X_i + \frac{R}{4}\left[X_i, X_j\right]^2 + i\Psi^{\dagger\,I}\cdot D_t\Psi^I - R\Psi^{\dagger\,I}\cdot\sigma_i\cdot\left[X_i, \Psi^I\right]\right] . \tag{1}$$

The X_i's are bosonic matrices, while the Ψ^I_α's are fermionic – both in the adjoint of $U(N)$. The full theory has $SO(9)$ symmetry, but for the purposes of the current work, we focus on a scenario where six of the nine target space directions are compactified and the corresponding bosonic excitations are frozen. Hence, we are left with $SO(3)$ – the index i on X_i runs from 1 to 3. Correspondingly, we also write the spinors using $SU(4)\times SU(2)$ notation: on the Ψ^I_α's, I denotes the $SU(4)$ index and α is the $SU(2)$ label (not shown in the equation above). The σ_i's are then the 2×2 Pauli matrices. Fixing the static gauge, the covariant derivative D_t becomes simply the time derivative – at the cost of the constraint

$$i[X_i, \Pi^i] + 2\Psi^I\cdot\Psi^{\dagger\,I} = 0\ , \tag{2}$$

where the Π^i's are the X_i's canonical momenta. The system is parameterized by

$$R = \frac{g_s}{l_s}\ , \tag{3}$$

where g_s is the string coupling and l_s is the string length. In our conventions, X and Ψ are dimensionless, and time has unit of length[2].

The BFSS conjecture purports that this Lagrangian *fully* describes M-theory in the light-cone frame with N units of light-cone momentum, $p_{\mathrm{LC}} = N/R$. The matrix Hamiltonian is then identified with M-theory's light-cone energy

$$H = \frac{M^2 + p_i^2}{2\,p_{\mathrm{LC}}}\ . \tag{4}$$

While the BFSS conjecture was originally formulated in the large N regime, the BFSS Matrix theory at finite N is also believed to described discrete light-cone quantized (DLCQ) M-theory [24, 22]. However, to make our computation more concise, we will assume that we are dealing with large matrices

$$N \gg 1\ . \tag{5}$$

This is necessary because we want to arrange for a configuration of matrices which enough energy content to source a smooth curved spacetime for a probe.

[2]To change to the conventions used in [9], write $X = g^{-1/3}\overline{X}$, $t = g^{-2/3}\overline{t}$, and hence $H = g^{2/3}\overline{H}$ where the variables with bars correspond to the ones in [9] in $l_s = 1$ units. X is then length scale in eleven dimensional Planck units.

A massless supergravity particle with one unit of light-cone momentum is realized through a $U(1)$ configuration: the bosonic part of the Hamiltonian reproduces the expected light-cone dispersion relation for a massless particle, while the zero modes of Ψ^I_α's give rise to the eleven dimensional supergravity multiplet – the 256 polarizations of the gravitons, gravitinos, and the 3-form gauge field. For more interesting setups, one starts with block diagonal configurations that break $U(N) \rightarrow U(N_1) \times U(N_2) \times \cdots$, and each matrix block can realize supergravitons or membranes or fivebranes or black holes carrying different amounts of light-cone momentum [9, 25, 26, 27, 28]. By developing the quantum effective Hamiltonian for these blocks, one then reproduces eleven dimensional light-cone M-theory interactions.

Consider the matrix configuration given by

$$X_i = r L_i \quad , \quad \Psi = 0 \tag{6}$$

The L_is are the angular momenta matrices, satisfying $[L_i, L_j] = i\,\varepsilon_{ijk} L_k$, in an N dimensional representation. This configuration represents an M-theory spherical membrane with N units of light-cone momentum – sometimes called a non-commutative or fuzzy sphere – of radius

$$\text{Radius} \equiv \mathcal{R} = l_P \sqrt{\frac{\text{Tr} X_i^2}{N}} \simeq l_P \frac{rN}{2} \quad \text{for large } N \gg 1 \tag{7}$$

where l_P is the eleven dimensional Planck scale. The matrix Hamiltonian then leads to the light-cone energy

$$H = \frac{R}{2\,N} M^2 \Rightarrow M = \frac{1}{l_P} \frac{N^2 r^2}{2} = T_2 \times 4\pi\,\mathcal{R}^2 \tag{8}$$

where $T_2 = 1/2\,\pi\, l_P^3$ is the tension of the membrane. This configuration however is not a solution to the equations of motion. In particular, the potential $[X_i, X_j]^2$ appearing in the Hamiltonian provides for flat directions corresponding to mutually commuting matrices. Physically, this implies that it is energetically and entropically advantageous for this spherical membrane to explode into widely separated supergravitons.

The BFSS conjecture has survived numerous checks (see for example [9, 25, 21]) and may be considered to be a background-dependent non-perturbative definition of light-cone M-theory. More recently, the BMN matrix model extended the setup to light-cone M theory in a plane wave background – with the additional flux and curvature of the background geometry lifting the flat directions we alluded to above. The fuzzy sphere configuration then becomes a BPS stable configuration in the BMN system. These M-theory inspired matrix models can also be related to the AdS/CFT or gravitational holography conjecture [29, 30]. In practice however, the latter provides for a more precise dictionary between a gauge theory and string theory, while computations in the BFSS and BMN matrix theories quickly become technically very challenging and conceptually more difficult to interpret from the M-theory side.

Paramount amongst the difficulties plaguing the BFSS/BMN settings is the challenge of understanding how the perception of spacetime is to emerge from matrix degrees of freedom. One natural approach is to identify the eigenvalues of the bosonic matrices X_i as position labels (after all, they are related to the position of the underlying D0 branes). Implicit in this is that the usual notion of space geometry arises in the limit of commuting matrices. When matrix eigenvalues (or D0 brane positions) are widely separated, the off-diagonal matrix modes become heavy and frozen, leading to an effective dynamics of the eigenvalues that reproduces the supergravity interactions. Hence, in Matrix theory language, it seems the key to emergent gravitational geometry – that is, the encoding of spacetime curvature information into matrix degrees of freedom – lies in the interplay between the matrix eigenvalues *and* heavy off-diagonal modes. In the dual M-theory language, off-diagonal matrix degrees of freedom correspond to membranes stretched between the gravitating parts of the system. Presumably, it is then such a network of stretched M-theory membranes that underlies – in the right low energy limit – the perception of an emergent curved space. This may appear like an unusual perspective on emergent geometry, yet it syncs well with a seemingly independent line of thought that has recently risen in various other contexts: the concept of geometry or gravity emergent from quantum entanglement [1]-[7]. A network of stretched membranes as represented by off-diagonal modes of matrices provides for a natural mechanism for entangling supergravity modes. Hence, in the right setting, we may be expect to read off spacetime geometry data by looking at entanglement entropy in Matrix theory.

It is worthwhile noting that there have been several other different yet related approaches to the problem of emergent geometry in Matrix theory. In [13, 31], the focus has been on the bosonic dynamics of the matrix degrees of freedom. The system is highly non-linear and known to be chaotic [32, 33] and the idea here is that geometry emerges once one averages over the complex chaotic evolution of the matrix degrees of freedom. In the commuting matrix regime, methods from random matrix theory (see for example [34] for a review) can be employed to extract statistical information about the eigenvalue distribution and corresponding geometry. In [14], the role of the fermionic degrees were also considered in decoding geometry from matrices in the context of matrix black holes. Beyond the details, our approach also differs conceptually from previous attempts in that it focuses on a key new quantity – the entanglement entropy of the fermionic matrix degrees of freedom.

2.2 Bosonic scaffolding

We want to set up a computational framework that allows us to extract spacetime geometry from matrix degrees of freedom and entanglement entropy. For this purpose, we want to arrange for a *fixed* configuration in light-cone M-theory that seeds a simple yet non-trivial background geometry. We consider a spherical membrane of radius $\mathcal{R}$ with $N \gg 1$ units of light-cone momentum, plus a massless particle with one unit of light-cone momentum a distance x from the center of the membrane. We consider the regime $x \gg \mathcal{R}$ so that $\mathcal{R}/x \ll 1$ will serve as a small expansion parameter[3]. The configuration is obviously unstable but we are not concerned with dynamics: we pin down the membrane and probe using the necessary external forces and we ask within Matrix theory: what is the geometry experienced by the probe as a function of $\mathcal{R}/x$?

In Matrix theory language, we start with a static arrangement of $(N+1) \times (N+1)$ matrices of the form

$$X_i, \Psi^I \to \begin{pmatrix} \boxed{\quad N \times N \quad} & \boxed{} \\ \boxed{\qquad\qquad} & \boxed{} \end{pmatrix} . \tag{9}$$

For the bosonic degrees of freedom, we write

$$X_i = \left(\begin{array}{c|c} r\, L_i & 0 \\ \hline 0 & x_i \end{array} \right) + \left(\begin{array}{c|c} 0 & \delta x_i \\ \hline \delta x_i^\dagger & 0 \end{array} \right) \tag{10}$$

where the δx_i's are N dimensional vectors, and the x_i's are numbers. The first part of equation (10) represents a fuzzy sphere at the origin, plus a probe at x_i. Using external means, we pin down the sphere and the probe: that is, we assume the necessary terms are added to the Lagrangian so that perturbations within the $N \times N$ and the 1×1 blocks can classically be set to zero. The second part of equation (10) can be thought of as fluctuations corresponding to stretched membranes between the sphere and the probe. Note however that we will *not* need to assume that the δx_i's are small, hence allowing for even a classical profile for these off-diagonal modes. Qualitatively, the dynamics is very similar to the usual story in Matrix theory. We can guess that the δx_i's will settle in their quantum oscillator ground state where contributions from their zero point energy cancels with the zero point energy from the fermionic degrees of freedom thanks to supersymmetry [35, 22]. At one and two loop levels in these fluctuations, one would then reproduce the momentum dependent effective gravitational potential between probe and sphere. Hence, the $\langle \delta x^2 \rangle$'s end up with non-zero quantum expectation values as is typical for Gaussian ground states. From the perspective of the fermionic degrees of freedom, we will see that this non-zero quantum

[3] We expect that, when the probe is close to the sphere, it will become sensitive to the non-commutative matrix nature of the spherical configuration. We are trying to focus on extracting smooth geometry data, where the Planckian substructure of D0 branes is smoothed over.

expectation value of the off-diagonal modes results in quantum entanglement between the supergravity modes associated with the spherical membrane and the probe.

Without loss of generality, we may arrange the probe so that $x_1 = x_2 = 0$, and $x_3 \equiv x$. Using (7), the condition that the probe is far away from the sphere then reads

$$x \gg r\,N \ . \tag{11}$$

Furthermore, we also have

$$x \gg 1 \quad , \quad r\,N \gg 1 \tag{12}$$

which translate in terms of dimensionfull coordinates to distance and radius being much greater than the eleven dimensional Planck length. The configuration should still have rotational symmetry about the 3 axis. In matrix language, we require

$$\left[\left(\begin{array}{c|c} L_3 & 0 \\ \hline 0 & 0 \end{array}\right), X_i\right] = i\,\epsilon_{ij3} X_j \ , \tag{13}$$

which is the algebra obeyed for the sphere configuration alone – without the probe and the δx_i's. This immediately leads to the conditions on the δx_i's

$$L_3 \delta x_3 = 0 \quad , \quad L_3 \delta x_+ = \delta x_+ \quad , \quad L_3 \delta x_- = -\delta x_- \tag{14}$$

where $\delta x_\pm \equiv \delta x_1 \pm i \delta x_2$. We then write

$$\delta x_3 = C_3 \left|J 0\right\rangle \quad , \quad \delta x_+ = C_+ \left|J 1\right\rangle \quad , \quad \delta x_- = C_- \left|J -1\right\rangle \tag{15}$$

where $N = 2\,J + 1$, the C's are complex constants to be determined, and the $\left|JM\right\rangle$ are the usual angular momentum eigenstates of spin J. An alternative approach is to require this azimuthal symmetry on expectation values in the vacuum, allowing for the spin J to be integer or half-integer – and hence N to be odd or even. At large N, we expect there would be no difference in conclusions between the case where N is odd and where N is even; and it is more computationally convenient to impose the symmetry conditions directly as in (15): this however means that we will be dealing with integer J, and henceforth N is an odd integer.

The bosonic potential is given by

$$V_{\rm pot} = -\frac{R}{4} \mathrm{Tr} \left[X_i, X_j\right]^2 \ , \tag{16}$$

where the commutator becomes

$$\left[X_i, X_j\right] = \left(\begin{array}{cc} i\epsilon_{ijk} r^2 L_k + \delta x_i \otimes \delta x_j^\dagger - \delta x_j \otimes \delta x_i^\dagger & x_j \delta x_i - x_i \delta x_j + r\, L_i \cdot \delta x_j - r\, L_j \cdot \delta x_i \\ -x_j \delta x_i^\dagger + x_i \delta x_j^\dagger - \delta x_j^\dagger \cdot r\, L_i + \delta x_i^\dagger \cdot r\, L_j & \delta x_i^\dagger \cdot \delta x_j - \delta x_j^\dagger \cdot \delta x_i \end{array} \right) . \tag{17}$$

We can then write the potential in terms of the C's

$$\begin{aligned}
V_{\text{pot}} = \quad & \frac{R}{8}\left[r^2\left(J^2+4\frac{x}{r}+2\frac{x^2}{r^2}\right)|C_-|^2+r^2\left(J^2-4\frac{x}{r}+2\frac{x^2}{r^2}\right)|C_+|^2\right. \\
& +4J^2r^2|C_3|^2-2J^2r^2C_+C_-^*+4Jr^2C_3C_-^*\left(1+\frac{x}{r}\right)-4Jr^2C_3C_+^*\left(1-\frac{x}{r}\right) \\
& \left.+|C_-|^4+|C_+|^4-2|C_-|^2|C_+|^2\right]+\text{c.c.}\ , \qquad (18)
\end{aligned}$$

where we have made use of $J = N/2 \gg 1$, and we have dropped the constant term arising from the energy of the spherical membrane (the probe does not contribute energy, it is a massless supergravity particle with a single unit of light-cone momentum). We note that the potential has a symmetry under the exchange of $C_+ \leftrightarrow C_-$ in the regime $x \gg r\,N$.

The minimum of (18) is at $C_+ = C_- = C_3 = 0$ as one can easily verify. The normal modes, labeled a, b, and c, are straightforward to identify and one gets, in the regime $x \gg r\,N$, to second order in $r\,N/x$

$$\begin{aligned}
C_3 &= \frac{1}{4}\sqrt{R}\left[\sqrt{2}\left(2-J^2\frac{r^2}{x^2}\right)C_a+2J\frac{r}{x}\left(1-\frac{r}{x}\right)C_b+2J\frac{r}{x}\left(1+\frac{r}{x}\right)C_c\right] \\
C_+ &= -\frac{1}{4}\sqrt{R}\left[2\sqrt{2}J\frac{r}{x}\left(1+\frac{r}{x}\right)C_a+J^2\frac{r^2}{x^2}C_b-\left(4-J^2\frac{r^2}{x^2}\right)C_c\right] \\
C_- &= \frac{1}{4}\sqrt{R}\left[-2\sqrt{2}J\frac{r}{x}\left(1-\frac{r}{x}\right)C_a+\left(4-J^2\frac{r^2}{x^2}\right)C_b-J^2\frac{r^2}{x^2}C_c\right] \qquad (19)
\end{aligned}$$

with associated frequencies

$$\omega_a = R\,r\,\frac{J^2r^2}{x^2} \quad , \quad \omega_b=\omega_c=R\,x\ . \qquad (20)$$

written to leading order in $r\,N/x$. The kinetic terms for the C_a, C_b, and C_c modes are normalized to unity, which then quickly allows us to write the vacuum expectation values of the normal mode oscillators in the Gaussian ground states

$$\langle C_a^*C_a\rangle = \frac{1}{2\,R\,r}\frac{1}{J^2}\frac{x^2}{r^2} \quad , \quad \langle C_c^*C_c\rangle = \langle C_b^*C_b\rangle = \frac{1}{2\,R\,r}\frac{r}{x}\ . \qquad (21)$$

This then allows us to estimate the vacuum expectation values of the original oscillator degrees of freedom

$$\begin{aligned}
&\langle C_3^*C_3\rangle = \frac{1}{4\,r}\frac{1}{J^2}\frac{x^2}{r^2} \quad , \quad \left\langle C_+^*C_+\right\rangle = \left\langle C_-^*C_-\right\rangle = \left\langle C_+^*C_-\right\rangle = \frac{1}{4\,r} \\
&\langle C_3^*C_+\rangle = \langle C_3^*C_-\rangle = -\frac{1}{4\,r}\frac{1}{J}\frac{x}{r}\ . \qquad (22)
\end{aligned}$$

This scaling of the correlators of the C's with x may at first appear problematic given the form of (18) and the inherent small fluctuation assumption in arriving at (22); but a quick comparison of the terms in the potential confirms that the quartic terms are sub-leading since $x \gg r$, $N \gg 1$, and $r \gg 1$. Note also that these relations imply, at the quadratic order in expectation values, the following

$$C_+ = C_- \quad , \quad C_3 = -\frac{x}{J\,r} C_- \ . \tag{23}$$

It will also be helpful, as we shall see, if we can understand the global structure of the potential at the classical level when the fluctuations are not small. Extremizing equation (18) with respect to C_3, for $x \gg r\,N$, we get

$$C_3 = -\frac{x}{2\,J\,r}\left(C_- + C_+\right) \ . \tag{24}$$

Extremizing with respect to C_+ and C_-, once again for $x \gg r\,N$, gives

$$4\,C_+ C_+^* = -2\,C_- C_-^* + \frac{x^2}{r^2} \pm \sqrt{\left(2\,C_- C_-^* - \frac{x^2}{r^2}\right)^2 + 8\,C_+^* C_- \frac{x^2}{r^2}} \tag{25}$$

and a similar expression with $C_+ \leftrightarrow C_-$. These equations imply that the phases of C_+, C_-, and C_3 must be in sync; and we confirm that there is only one minimum of the potential, located at $C_3 = C_- = C_+ = 0$. However, there is a valley in the potential with a shallow curvature in the regime where $x \gg r\,N$ (in the $x \to \infty$ limit, it is a flat direction for the δx_i's). Focus on the case where all the C's have the same phase[4], and minimizing the resulting potential in the C_3, C_+, and C_- *in the large x regime*, we easily obtain the conditions

$$C_+ = C_- = -\frac{J\,r}{x} C \quad , \quad C_3 = C \tag{26}$$

where C is a yet to be determined real parameter which runs along the shallow-slopped valley of the potential. This is indeed what we found near the origin by looking at small quantum fluctuations in the vacuum from (23). We now see that the shape of these fluctuations sits along a valley that extends from the origin on a *global scale*. In particular, evaluating the full non-linear potential along this valley, we get

$$V_{\text{pot}} = r^2 R \frac{J^2 r^2}{x^2} C^2 \tag{27}$$

[4]Note that there is no non-trivial topological profile that we can construct through the overall phase. Hence, we may safely take all the C's to be real.

for all C. The curvature of the valley goes as $J^2 r^2/x^2 \ll 1$ indicating a parametrically small cost in energy for $C \sim 1$. For ground state quantum fluctuations, we have from (22)

$$C = -\frac{1}{2\,J}\frac{x}{r^{3/2}} \ . \tag{28}$$

Due to the form of the fermionic terms in (1), we will see that only terms quadratic in the C's will appear in the final effective action. Hence, we will be able to treat quantum and classical profiles for the δx_i's in a unified notation by writing the off-diagonal profiles as

$$\delta x_+ = C_+ \left|J1\right\rangle = \frac{J\,r}{x} C \left|J1\right\rangle \ , \ \delta x_- = C_- \left|J-1\right\rangle = \frac{J\,r}{x} C \left|J-1\right\rangle \ , \ \delta x_3 = C_3 \left|J0\right\rangle = -C \left|J0\right\rangle \ . \tag{29}$$

For C of quantum mechanical origin, we can use (28): all quadratic expressions in the C's should then be interpreted as quantum expectation values in the harmonic oscillator ground states. On the other hand, in the same final expression, we can also take C as a arbitrary fixed classical profile along the shallow valley of the potential that extends from the origin.

In summary, our setup qualitatively corresponds to a spherical membrane, a probe, and a membrane stretched between them. Equation (10), along with (29) and (28), constitute our bosonic configuration that will serve as scaffolding for the fermionic matrix degrees of freedom. The latter represent the supergravity modes on the sphere, the probe, and the stretched membrane.

3 Fermion dynamics

Based on the scaffolding from the previous section, we arrange the fermionic degrees of freedom as

$$\Psi^I = \left(\begin{array}{c|c} \Psi^I & \delta\psi^I \\ \hline \overline{\delta\psi}^I & \psi^I \end{array} \right) \tag{30}$$

where the Ψ^I's are $N \times N$ matrices, the $\delta\psi^I$'s are N-vectors, and the $\delta\psi^I$ is a 1×1 matrix entry. Note that $\overline{\delta\psi}^I$ is *not* the complex conjugate of $\delta\psi^I$. The zero modes of these fermions determine the supergravity modes on the spherical membrane (from the Ψ^I's), the probe (from the ψ^I), and the stretched membrane (from the $\delta\psi^I$'s). The Hamiltonian for the static setup then takes the form

$$\mathcal{H}_F = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 \ , \tag{31}$$

where the subscript labels the number of off-diagonal fermion variables in the corresponding term. Substituting the bosonic scaffolding into (1), we get

$$\frac{\mathcal{H}_0}{R} = r\,\mathrm{Tr}\left(\Psi^\dagger \cdot \sigma_i \cdot L_i \Psi - \Psi^\dagger \cdot \sigma_i \cdot \Psi L_i\right) \tag{32}$$

$$\frac{\mathcal{H}_1}{R} = -\delta x_i^\dagger(\Psi-\psi)\cdot\sigma_i^T\cdot\overline{\delta\psi}^\dagger - \delta x_i^\dagger(\Psi^\dagger-\psi^\dagger)\cdot\sigma_i\cdot\delta\psi - \delta\psi^\dagger\cdot\sigma_i\cdot(\Psi-\psi)\delta x_i - \overline{\delta\psi}\cdot\sigma_i^T\cdot(\Psi^\dagger-\psi^\dagger)\delta x_i \quad (33)$$

$$\frac{\mathcal{H}_2}{R} = \overline{\delta\psi}\cdot\sigma_i^T\cdot(r\,L_i - x_i)\overline{\delta\psi}^\dagger + \delta\psi^\dagger\cdot\sigma_i\cdot(r\,L_i - x_i)\delta\psi \quad (34)$$

where we henceforth suppress the $SU(4)$ indices. As we shall see, this describes a system of qubits with a dense network interactions. Our task is to find the vacuum of this system. We are eventually aiming to compute the entanglement entropy in this vacuum – between the qubits associated with the probe and the qubits associated with the sphere.

3.1 Diagonalization I

We start by diagonalizing the Hamiltonian in the off-diagonal fermions, the $\delta\psi$'s. We need to solve the eigenvalue problem

$$\sigma_i\cdot(r\,L_i - x_i)\delta\psi = \lambda\delta\psi \quad (35)$$

We denote the corresponding eigenvalues as $\lambda_M^{(i)}$ for $i=1,2$ with the eigenvectors written as $|M\rangle_{(1)}$ and $|M\rangle_{(2)}$. Similarly, we need to solve the eigenvalue problem

$$\sigma_i^T\cdot(r\,L_i - x_i)\overline{\delta\psi}^\dagger = \lambda\overline{\delta\psi}^\dagger \quad (36)$$

We denote these eigenvalues as $\overline{\lambda}_M^{(i)}$ for $i=1,2$ with the corresponding eigenvectors written as $\overline{|M\rangle}_{(1)}$ and $\overline{|M\rangle}_{(2)}$. We first unravel the $SU(2)$ spinorial structure, and write

$$\delta\psi = \begin{pmatrix}\delta\psi^+\\ \delta\psi^-\end{pmatrix} = \sum_M \delta\eta_M\,|M\rangle_{(1)} + \sum_M \delta\chi_M\,|M\rangle_{(2)} \quad (37)$$

where $\delta\psi^+$, $\delta\psi^-$, $\delta\eta_M$, and $\delta\chi_M$ are vectors in the fundamental of $U(N)$. Explicitly, we find the solution

$$\delta\psi = \sum_M \begin{pmatrix}\delta\eta_M\,k_1^+(M)\,|JM\rangle + \delta\chi_M\,k_2^+(M)\,|JM\rangle\\ \delta\eta_M\,k_1^-(M)\,|JM+1\rangle + \delta\chi_M\,k_2^-(M)\,|JM+1\rangle\end{pmatrix} \quad (38)$$

where $2J+1=N$, $M=-J\ldots J$, and we will henceforth write $|JM\rangle\to|M\rangle$. Similarly, we easily find

$$\begin{aligned}\overline{\delta\psi}^\dagger &= \sum_M \overline{\delta\eta}_M^\dagger\overline{|M\rangle}_{(1)} + \sum_M \overline{\delta\chi}_M^\dagger\overline{|M\rangle}_{(2)}\\ &= \sum_M \begin{pmatrix}\overline{\delta\eta}_M^\dagger\,\overline{k}_1^+(M)\,|M\rangle + \overline{\delta\chi}_M^\dagger\,\overline{k}_2^+(M)\,|M\rangle\\ \overline{\delta\eta}_M^\dagger\,\overline{k}_1^-(M)\,|M-1\rangle + \overline{\delta\chi}_M^\dagger\,\overline{k}_2^-(M)\,|M-1\rangle\end{pmatrix}\end{aligned} \quad (39)$$

The full form of the eigenvalues and the constants $k^{\pm}_{1,2}(M)$ are given in Appendix A. The diagonalized Hamiltonian then takes the form

$$\begin{aligned}\mathcal{H}_1+\mathcal{H}_2 &= r\,R\sum_M \lambda^{(1)}_M\delta\eta^\dagger_M\delta\eta_M+\lambda^{(2)}_M\delta\chi^\dagger_M\delta\chi_M+\overline{\lambda}^{(1)}_M\overline{\delta\eta}^\dagger_M\overline{\delta\eta}_M+\overline{\lambda}^{(2)}_M\overline{\delta\chi}^\dagger_M\overline{\delta\chi}_M\\ &+ r\,R\sum_M J^\dagger_{\chi\,M}\delta\chi_M+\delta\eta^\dagger_M J_{\eta\,M}+\delta\chi^\dagger_M J_{\chi\,M}+J^\dagger_{\eta\,M}\delta\eta_M\\ &+ r\,R\sum_M \overline{\delta\chi}_M J^\dagger_{\overline{\chi}\,M}+J_{\overline{\eta}\,M}\overline{\delta\eta}^\dagger_M+J_{\overline{\chi}\,M}\overline{\delta\chi}^\dagger_M+\overline{\delta\eta}_M J^\dagger_{\overline{\eta}\,M}\end{aligned}\tag{40}$$

with the 'currents' defined as

$$\begin{aligned}J_{\eta\,M} &= k^-_1(M)\Big(\langle M+1|\,\Psi'^-\,|0\rangle\,C_3-\langle M+1|\,\Psi'^+\,|1\rangle\,C_+\Big)\\ &- k^+_1(M)\Big(\langle M|\,\Psi'^-\,|-1\rangle\,C_-+\langle M|\,\Psi'^+\,|0\rangle\,C_3\Big)\end{aligned}\tag{41}$$

$$\begin{aligned}J_{\chi\,M} &= k^-_2(M)\Big(\langle M+1|\,\Psi'^-\,|0\rangle\,C_3-\langle M+1|\,\Psi'^+\,|1\rangle\,C_+\Big)\\ &- k^+_2(M)\Big(\langle M|\,\Psi'^-\,|-1\rangle\,C_-+\langle M|\,\Psi'^+\,|0\rangle\,C_3\Big)\end{aligned}\tag{42}$$

$$\begin{aligned}J_{\overline{\eta}\,M} &= \overline{k}^-_1(M)\Big(\langle 0|\,\Psi'^-\,|M-1\rangle\,C_3-\langle -1|\,\Psi'^+\,|M-1\rangle\,C_-\Big)\\ &- \overline{k}^+_1(M)\Big(\langle -1|\,\Psi'^-\,|M\rangle\,C_-+\langle 0|\,\Psi'^+\,|M\rangle\,C_3\Big)\end{aligned}\tag{43}$$

$$\begin{aligned}J_{\overline{\chi}\,M} &= \overline{k}^-_2(M)\Big(\langle 0|\,\Psi'^-\,|M-1\rangle\,C_3-\langle -1|\,\Psi'^+\,|M-1\rangle\,C_-\Big)\\ &- \overline{k}^+_2(M)\Big(\langle -1|\,\Psi'^-\,|M\rangle\,C_-+\langle 0|\,\Psi'^+\,|M\rangle\,C_3\Big)\end{aligned}\tag{44}$$

In these expressions, we use

$$\Psi'\equiv\Psi-1\,\psi\ .\tag{45}$$

We can now integrate out the off-diagonal fermionic modes and write an effective Hamiltonian for the remaining fermionic degrees of freedom arising from the sphere and probe. Generally, for a Hamiltonian of the form

$$\mathcal{H}=\Lambda\,f^\dagger f+J^\dagger\,f+f^\dagger\,J\tag{46}$$

where the f is a fermionic mode, the partition function becomes

$$\log Z = -\int dt \int ds J^\dagger(t) D_F(t,s) J(s) \tag{47}$$

where the Feynman propagator is given by [36]

$$D_F(t,s) = \theta(t-s) e^{i\Lambda(t-s)} \ . \tag{48}$$

This generally leads to an effective Hamiltonian non-local in time. But for the regime where $x \gg r\, N$, our eigenvalues are given by (see Appendix A)

$$\lambda_M^{(i)} \to (-1)^i \frac{x}{r} \quad , \quad \overline{\lambda}_M^{(i)} \to (-1)^i \frac{x}{r} \ . \tag{49}$$

The exponent in the Feynman propagator (48) takes the form $\sim e^{iRx\,(t-s)}$. In units of light-cone energy R, $x \gg 1$ then implies that the off-diagonal modes are heavy (remember that in M-theory dimensionfull variables, this condition is the statement that the probe is much further away from the sphere than a Planck distance.). Hence, integrating out these heavy modes generates a contact Fermi-like interaction

$$D(t,s) \to \frac{i}{\Lambda} \delta(t-s) \ . \tag{50}$$

The partition function then takes the form

$$\log Z_{eff} \to \frac{i}{\Lambda} \int dt\, J^\dagger(t) J(t) \tag{51}$$

with the effective Hamiltonian given by

$$\mathcal{H}_{eff} = i \, \log Z_{eff} \ . \tag{52}$$

We then want to read equation (51) in two possible ways. For the bosonic degrees of freedom in their ground state, we imply

$$J^\dagger J \to \left\langle J^\dagger J \right\rangle \tag{53}$$

where we take the expectation value of the C's appearing in (41)-(44) in the harmonic oscillator ground state using (22). Alternatively, treating the C's as describing a fixed classical profile along the valley of the potential, we can use (26) instead. Given the quadratic form of the expression, and the similar local and global structure of the bosonic potential along the valley of interest, we can see from (23) and (26) that we can treat both cases simultaneously by using (26) and writing the effective Hamiltonian in terms of the external parameter C. But then, for tackling the case of ground state quantum fluctuations, we simply replace C using (28).

Putting things together for our scenario, we have

$$\mathcal{H}_{eff} = r\,R \sum_M \frac{1}{\lambda^{(1)}_M} J^\dagger_{\eta\,M} J_{\eta\,M} + \frac{1}{\lambda^{(2)}_M} J^\dagger_{\chi\,M} J_{\chi\,M} + \frac{1}{\overline{\lambda}^{(1)}_M} J^\dagger_{\overline{\eta}\,M} J_{\overline{\eta}\,M} + \frac{1}{\overline{\lambda}^{(2)}_M} J^\dagger_{\overline{\eta}\,M} J_{\overline{\eta}\,M}\ . \tag{54}$$

To quadratic order in $r\,N/x$, we then get

$$\begin{aligned}
\mathcal{H}_{eff} \quad = \quad & r\,R\,\frac{C^2}{x^2} J \left(\psi^{-\dagger} \langle -1|\,\Psi^+\,|0\rangle - \psi^{-\dagger} \langle 0|\,\Psi^+\,|1\rangle - \psi^{+\dagger} \langle 0|\,\Psi^-\,|-1\rangle + \psi^{+\dagger} \langle 1|\,\Psi^-\,|0\rangle\right) \\
& + r\,R\,\frac{C^2}{r^2} \sum_{M=-J}^{J} \frac{r^2}{x^2} \Big(-\sqrt{(J-M)(J+M+1)}\,\langle 0|\,\Psi^{-\dagger}\,|M+1\rangle \langle M|\,\Psi^+\,|0\rangle \\
& + \sqrt{(J-M+1)(J+M)}\,\langle M|\,\Psi^{+\dagger}\,|0\rangle \langle 0|\,\Psi^-\,|M-1\rangle \\
& J\left(-\langle M|\,\Psi^{+\dagger}\,|-1\rangle \langle 0|\,\Psi^-\,|M\rangle - \langle M|\,\Psi^{+\dagger}\,|0\rangle \langle 1|\,\Psi^-\,|M\rangle\right. \\
& \left. + \langle -1|\,\Psi^{-\dagger}\,|M\rangle \langle M|\,\Psi^+\,|0\rangle - \langle 0|\,\Psi^{-\dagger}\,|M\rangle \langle M|\,\Psi^+\,|1\rangle\right) \\
& + \frac{1}{2}\left(\frac{r}{x} + \frac{r^2}{x^2} M\right) \Big(\langle 0|\,\Psi^{-\dagger}\,|M\rangle \langle M|\,\Psi^-\,|0\rangle - \langle M|\,\Psi^{-\dagger}\,|0\rangle \langle 0|\,\Psi^-\,|M\rangle \\
& - \langle 0|\,\Psi^{+\dagger}\,|M\rangle \langle M|\,\Psi^+\,|0\rangle + \langle M|\,\Psi^{+\dagger}\,|0\rangle \langle 0|\,\Psi^+\,|M\rangle\Big) \\
& + \text{c.c.}
\end{aligned} \tag{55}$$

The $\psi^\pm$ are the fermionic modes associated with the probe, while the $\langle M|\,\Psi^\pm\,|M'\rangle$'s are associated with the sphere. We then see that, integrating out the off-diagonal fermions, the $\delta\psi$'s and the $\overline{\delta\psi}$'s, we generate qubit couplings between probe and sphere. In addition, we also get corrections to the masses of the $\langle M|\,\Psi^\pm\,|M'\rangle$ modes. The task is now to add this effective Hamiltonian to $\mathcal{H}_0$ from (32) and compute the entanglement between sphere and probe qubits.

3.2 Diagonalization II

The form of equation (32) suggests an alternate decomposition of the qubits on the sphere: instead of looking at matrix elements $\langle M|\,\Psi^\pm\,|M'\rangle$, it is advantageous to decompose the $\Psi^\pm$'s in terms of matrix spherical harmonics [37, 38, 39]

$$\Psi = \sum_{j,m} \Psi_{jm} Y^j_m \tag{56}$$

The matrix spherical harmonics Y^j_m are $N\times N$ matrices with $j=1,\dots,N-1$ and $m=-j,\dots,j$ [37, 38] and they form a complete basis for $U(N)$ [5]

$$\sum_{j=0}^{N-1}(2j+1)=\dim[U(N)] \tag{57}$$

They satisfying the algebra [39]

$$\left[Y^j_m,Y^{j'}_{m'}\right]=\frac{2}{N}\sqrt{(2j+1)(2j'+1)(2j''+1)}\,f^{jm\,j'm'}_{j''m''}(-1)^{m''}Y^{j''}_{-m''} \tag{58}$$

where

$$f^{jm\,j'm'}_{j''m''}=(-1)^N N^{3/2}\times\begin{pmatrix} j & j' & j''\\ m & m' & m''\end{pmatrix}\times\begin{Bmatrix} j & j' & j''\\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2}\end{Bmatrix} \tag{59}$$

written in terms of $3j$ and $6j$ symbols. We also have the normalization condition

$$\mathrm{Tr}\left(Y^j_mY^{j'}_{m'}\right)=(-1)^mN\delta_{jj'}\delta_{-mm'} \tag{60}$$

We then write each of the two components of the fundamental of the $SU(2)$, $\Psi^\pm$, in terms of $SU(4)$ spinors $\eta_{jm}(t)$ (fundamental) and $\chi_{jm}(t)$ (anti-fundamental) [37]

$$\Psi=\begin{pmatrix}\sum_{j=0}^{N-1}\sum_{m=-j-1}^{j}Y^j_m\frac{\sqrt{j+m+1}}{\sqrt{(2j+1)N}}\eta_{j\,m+1}-\sum_{j=1}^{N-1}\sum_{m=-j}^{j-1}Y^j_m\frac{\sqrt{j-m}}{\sqrt{(2j+1)N}}\chi_{j\,m+1}\\ \sum_{j=0}^{N-1}\sum_{m=-j}^{j+1}Y^j_m\frac{\sqrt{j-m+1}}{\sqrt{(2j+1)N}}\eta_{j\,m}+\sum_{j=1}^{N-1}\sum_{m=-j+1}^{j}Y^j_m\frac{\sqrt{j+m}}{\sqrt{(2j+1)N}}\chi_{j\,m}\end{pmatrix}, \tag{61}$$

once again suppressing the $SU(4)$ index to avoid clutter. This diagonalizes $\mathcal{H}_0$ and we get

$$\mathcal{H}_0=r\,R\sum_{j=0}^{N-1}\sum_{m=-j}^{j+1}j\,\eta^\dagger_{j\,m}\eta_{j\,m}-r\,R\sum_{j=1}^{N-1}\sum_{m=-j+1}^{j}(1+j)\,\chi^\dagger_{j\,m}\chi_{j\,m} \tag{62}$$

To combine this expression with (55), we then need to express the matrix elements $\langle M|\,\Psi^\pm\,|M'\rangle$ in terms of the η_{jm}'s and the χ_{jm}'s. This is easily achieved using [39]

$$\begin{aligned}\langle M|\,\Psi\,|M'\rangle &= \sum_{j,m}\Psi_{jm}\,\langle M|\,Y^j_m\,|M'\rangle\\ &= \sum_{j,m}(-1)^{J-M}\sqrt{2j+1}\sqrt{2J+1}\begin{pmatrix} J & j & J\\ -M & m & M'\end{pmatrix}\Psi_{jm}\end{aligned} \tag{63}$$

[5]The $j=0$ case corresponds to the center of mass degree of freedom – the $U(1)$ in $U(N)$.

Putting things together, our final Hamiltonian for the sphere and probe qubits takes the form

$$\frac{1}{r\,R}\mathcal{H}^{tot}_{eff} = \frac{1}{2}\sum_j j\,\eta^\dagger_{j0}\eta_{j0} + j\,\eta^\dagger_{j1}\eta_{j1} - \frac{1}{2}\sum_j (j+1)\chi^\dagger_{j0}\chi_{j0} + (j+1)\chi^\dagger_{j1}\chi_{j1}$$
$$+C^2\frac{(-1)^J J}{x^2}\sum_j \left((-1)^{j+2J}+1\right)T(j)\left[\sqrt{j+1}\left(\chi^\dagger_{j0}\psi^- - \chi^\dagger_{j,1}\psi^+\right) - \sqrt{j}\left(\eta^\dagger_{j0}\psi^- + \eta^\dagger_{j1}\psi^+\right)\right]$$
$$+\frac{C^2(-1)^{2J}}{r\,x}\sum_j\sum_{j'}\left((-1)^{j+j'+1}+1\right)T(j)T(j')\left[\sqrt{j}\sqrt{j'+1}\left(\chi^\dagger_{j',0}\cdot\eta_{j,0} - \chi^\dagger_{j',1}\cdot\eta_{j,1}\right)\right.$$
$$\left.-\frac{1}{2}\sqrt{j}\sqrt{j'}\left(\eta^\dagger_{j',0}\cdot\eta_{j,0} + \eta^\dagger_{j',1}\cdot\eta_{j,1}\right) - \frac{1}{2}\sqrt{j+1}\sqrt{j'+1}\left(\chi^\dagger_{j',0}\cdot\chi_{j,0} + \chi^\dagger_{j',1}\cdot\chi_{j,1}\right)\right]$$
$$+\frac{C^2}{r\,x}\sum_j (2\,j+1)\,T_j^2 + \text{c.c.} + \text{decoupled higher harmonic terms} \tag{64}$$

We have defined the following $3j$ symbols

$$T_j \equiv \begin{pmatrix} J & J & j \\ -1 & 0 & 1 \end{pmatrix} \tag{65}$$

And the sums over j's run from 0 to $N-1$ when the index appears on a η_{jm}, and it runs from 1 to $N-1$ when it appears on a χ_{jm}. In arriving at this expression, we have also expanded in $r\,N/x$ to the order we will need later on in computing the leading contribution to the entanglement entropy. However, consistent to the order of expansion we have, there are also additional mass terms for the η's and χ's that we have not shown coming it at order $1/x^2$: as we shall see, they will be inconsequential to our analysis and we have not shown them to avoid unnecessary clutter. Finally, we have used several $3j$ symbol identities listed in Appendix B to simplify the result.

Quantizing the fermionic modes in the usual manner, we get

$$\left\{\eta^\dagger_{jm}, \eta_{j'm'}\right\} = \delta_{jj'}\delta_{mm'} \quad , \quad \left\{\chi^\dagger_{jm}, \chi_{j'm'}\right\} = \delta_{jj'}\delta_{mm'} \quad , \quad \left\{\psi^{+\dagger}, \psi^+\right\} = \left\{\psi^{-\dagger}, \psi^-\right\} = 1\;. \tag{66}$$

We then see that each mode represents a qubit that can be turned on or off: the $\psi^\pm$ on the probe, the η_{j0}, η_{j1}, χ_{j0}, and χ_{j1} on the sphere. We have a total of $4\times(2\,N-1)$ qubits on the sphere and 4×2 qubits on the probe – entangled together through a dense network of interactions (the 4 arises from the suppressed $SU(4)$ structure on each qubit): basically each $\psi^\pm$ is interacting with all the other qubits. Modes for higher spherical harmonics $|m|>1$ decouple from this entanglement dynamics. The entanglement coupling is tuned by C^2, the expectation value of the off-diagonal bosonic modes – *i.e.* a contribution that we can interpret as a stretched membrane between sphere and probe. For this mode frozen in its harmonic oscillator ground state, we know from (28) that $C \sim x$. At first, it may seem that our large x expansion is then in trouble. But note that the fermionic Hamiltonian in Matrix theory involves only a coupling bilinear in the fermions with linear dependence on the bosonic modes. This immediately implies that the effective Hamiltonian would scale as C^2 and no higher power of C would come in. Hence, we are guaranteed a well defined large x expansion of the form $C^2 \times \sum_n a_n x^n$ for negative powers n.

3.3 Spectrum and vacuum

3.3.1 Diagonalization III

The Hamiltonian (64) is our central result. We want to find its vacuum; then, we want to compute the Von Neumann entanglement entropy between the probe qubits $\psi^\pm$ and the sphere qubits η_{j0}, η_{j1}, χ_{j0}, and χ_{j1} in this vacuum. To do this, we need to diagonalize the Hamiltonian (64) one more time and determine the sign of the eigenvalues. This will allow us to identify the Fermi vacuum, and then we can compute the correlators of the $\psi^\pm$ in this vacuum. We will demonstrate in the next section that this would then tell us the Von Neumann entropy of interest.

Before we proceed, note that (64) involves a constant term (the last line of the equation). For $C \sim 1$, this term is sub-leading to energies appearing in the first line of the equation; hence, it is inconsequential to the analysis in the case where we treat C as a classical fixed paramater. On the other hand, for the bosonic modes frozen in the ground state, we have $C \sim x$ and it appears that this constant term affects the total balance of vacuum energy – competing with the first terms in the expression. However, this constant term is the zero point energy from the fermions, scaling as $R\,x$: if we were to include the ground state energy from the perturbations of the bosonic modes, this ground state energy (scaling also as $R\,x$) would cancel this fermionic contribution. This is the mechanism by which Matrix theory overcomes a confining potential and instead generates light-cone supergravity interactions that die down with distance as needed [22, 23]. Hence, the constant term in (64) is a red herring and can safely be dropped.

To diagonalize (64), we proceed by defining the $4 \times (4N-1)$ dimensional vector (the additional factor of 4 comes from $SU(4)$.

$$(\underbrace{\cdots \eta_{j0}^\dagger \cdots}_{j=0\cdots N-1} \quad \underbrace{\cdots \eta_{j1}^\dagger \cdots}_{j=1\cdots N-1} \quad \underbrace{\cdots \chi_{j0}^\dagger \cdots}_{j=1\cdots N-1} \quad \underbrace{\cdots \chi_{j1}^\dagger \cdots}_{j=1\cdots N-1} \quad \psi^{+\dagger} \quad \psi^{-\dagger}) \tag{67}$$

We then write (64) as

$$\frac{1}{r\,R}\mathcal{H}_{eff}^{tot} = H_0 + V \tag{68}$$

where H_0 is the block-diagonal matrix form

$$H_0 = r\,R \begin{pmatrix} j\delta_{jj'} & 0 & 0 & 0 & 0 & 0 \\ 0 & j\delta_{jj'} & 0 & 0 & 0 & 0 \\ 0 & 0 & (-j-1)\delta_{jj'} & 0 & 0 & 0 \\ 0 & 0 & 0 & (-j-1)\delta_{jj'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{69}$$

and V is defined as

$$V = r\,R \left(\begin{matrix} -\sqrt{jj'}A_{jj'} & 0 & \sqrt{j'(j+1)}A_{jj'} \\ 0 & \sqrt{jj'}A_{jj'} & 0 \\ \sqrt{j(j'+1)}A_{j'j} & 0 & -\sqrt{(j'+1)(j+1)}A_{jj'} \\ 0 & \sqrt{j(j'+1)}A_{j'j} & 0 \\ 0 & -\sqrt{j}D_j^* & 0 \\ -\sqrt{j}D_j^* & 0 & \sqrt{j+1}D_j^* \end{matrix} \right. \cdots$$

$$\cdots \left. \begin{matrix} 0 & 0 & -\sqrt{j'}D_{j'} \\ \sqrt{j'(j+1)}A_{jj'} & -\sqrt{j'}D_{j'} & 0 \\ 0 & 0 & \sqrt{j'+1}D_{j'} \\ \sqrt{(j+1)(j'+1)}A_{jj'} & -\sqrt{j'+1}D_{j'} & 0 \\ -\sqrt{j+1}D_j^* & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right), \tag{70}$$

where we have defined

$$\begin{aligned} A_{jj'} &= \frac{C^2}{r^2}(-1)^{2J}\frac{r}{x}\left(1-(-1)^{j+j'}\right)T_jT_{j'} + \frac{C^2}{r^2}\frac{r^2}{x^2}f_{jj'} \\ D_j &= \frac{C^2}{r^2}\frac{r^2}{x^2}(-1)^J((-1)^{j+2J}+1)J\,T_j \end{aligned} \tag{71}$$

The $f_{jj'}$'s are not shown since they play no role in the analysis, but must be included for consistency in the expansion. The essential structure of this potential tells us the following: the A terms perturb the masses of modes on the sphere; the D terms are the key terms that introduce couplings between qubits on the sphere and the probe. Note that they come at order $1/x^2$: the $1/x$ contribution exactly cancels. To determine the vacuum, we need to diagonalize this system. We will do this treating V as a perturbation to H_0, which implies we need to verify that all terms in V are smaller than terms in H_0. Looking at the matrix entries, we see two types of terms, and correspondingly two needed conditions for a well-defined perturbation problem

$$|J\,A_{jj'}| \sim \frac{J\,C^2}{r^2}\frac{r}{x}|T_j|^2 < \frac{C^2}{r^2}\frac{r}{x} \ll 1 \quad , \quad ||\sqrt{J}D_j|| \sim \sqrt{J}\frac{C^2}{r^2}\frac{r^2}{x^2}J\,|T_j| < \frac{C^2}{r^2}\frac{r^2}{x^2}J \ll 1\ . \tag{72}$$

where we used the asymptotic behavior of the T_j for large J from Appendix B. For $C \sim 1$, each entry in V is much less than those in H_0 in the regime we have been considering. When the bosonic modes are in their quantum ground state however, we know from (28) that $C \sim x/Jr^{3/2}$. This means a perturbation analysis requires in addition the condition

$$1 \ll \frac{x}{r\,J} \ll J\ . \tag{73}$$

This is a parametrically controllable regime as long as J is large. Hence, we can safely proceed with a diagonalization procedure using V as a perturbation to H_0 as long as we add (73) to our regime of interest.

A glance ahead reveals that, to capture the leading order qubit mixing effect between probe and sphere, we will need to consider *second* order in perturbation. The second order perturbation relations are shown at the end of Appendix A, equations (123) and (124). But these expressions in their full glory are actually unnecessary for our purposes. First, looking at the eigenvalues, we only care about their *sign* since this is what determine where the Fermi sea level is. By construction, the perturbations are small and cannot change the signs. Denoting the new modes that diagonalize the potential with under-bars, *i.e.* $\underline{\eta}_{j0}$, $\underline{\eta}_{j1}$, $\underline{\chi}_{j0}$, $\underline{\chi}_{j1}$, $\underline{\psi}^+$, and $\underline{\psi}^-$, we can immediately write the eigenvalues as[6]

$$\begin{aligned}
&\underline{\eta}_{j0} : j + \text{small} \quad , \quad \underline{\eta}_{j1} : j + +\text{small} \\
&\underline{\chi}_{j0} : -j - 1 + \text{small} \quad , \quad \underline{\chi}_{j1} : -j - 1 + \text{small} \\
&\underline{\psi}^+ : 0 \quad , \quad \underline{\psi}^- : 0
\end{aligned} \tag{74}$$

In the regime we are considering, the modes $\underline{\chi}_{j0}$, $\underline{\chi}_{j1}$ continue to have negative eigenvalues, implying that the Fermi vacuum contains non-zero condensates

$$\langle\Omega|\,\underline{\chi}_{j0}^\dagger\underline{\chi}_{j0}\,|\Omega\rangle = \langle\Omega|\,\underline{\chi}_{j1}^\dagger\underline{\chi}_{j1}\,|\Omega\rangle = 1 \quad \text{for } j = 1, \ldots, N \tag{75}$$

[6]Note that the perturbation for the $j = 0$ eigenvalue vanishes identically.

Next, looking at the new eigenvectors, for the purposes of computing entanglement entropy we will only need to look at the correlators of $\psi^{\pm}$ in the vacuum. For this purpose, only the boxed term in (124) gives a non-zero contribution. We get

$$\psi^{+} = \underline{\psi}^{+} - \sum_{j} \frac{D_j}{\sqrt{j}} \underline{\eta}_{j1} + \sum_{j} \frac{D_j}{\sqrt{j+1}} \underline{\chi}_{j1} \tag{76}$$

$$\psi^{-} = \underline{\psi}^{-} - \sum_{j} \frac{D_j}{\sqrt{j}} \underline{\eta}_{j0} - \sum_{j} \frac{D_j}{\sqrt{j+1}} \underline{\chi}_{j0} \ . \tag{77}$$

In short, while consistency in expanding for large x requires a second order perturbation treatment, for the quantity we end up computing, only first order terms contribute due to the form of the effective Hamiltonian. The key point here is that, coming in at order x^{-2}, there is mixing between the probe qubits $\psi^{\pm}$ and the mode excited in the Fermi vacuum $\underline{\chi}_{j0}$ and $\underline{\chi}_{j1}$. This is the origin of entanglement between sphere and probe. As we shall see next, this allows us to compute the entanglement entropy between the supergravity modes on the probe and on the sphere.

4 Spectral analysis and entropy

Let us illustrate the key to the entanglement mechanism between probe and sphere in more general terms. Consider a qubit system with Hamiltonian

$$H = \sum a_{mn} f_m^{\dagger} f_n \tag{78}$$

for arbitrary a_{mn}. Diagonalizing the system through

$$F_k = \sum_{m} c_{km} f_m \tag{79}$$

we end up with a Hamiltonian of the form

$$H = \sum_{k} \lambda_k F_k^{\dagger} F_k \ . \tag{80}$$

The Fermi vacuum $|\Omega\rangle$ of the system then has a condensate of fermionic modes for all $\lambda_n < 0$

$$\langle\Omega| F_k^{\dagger} F_l |\Omega\rangle = \delta_{kl} \quad \text{for all } \lambda_k < 0 \ . \tag{81}$$

This in turn in general implies a non-zero vacuum expectation value for the original fermionic modes

$$\langle\Omega| f_m^\dagger f_n |\Omega\rangle \neq 0\ . \tag{82}$$

for f_m modes that overlap with the excited F_k's as determined from (79). Now imagine that we pick a subset of the f_m modes and ask for the entanglement entropy for these modes with the rest of the system in the vacuum. Because the original system is that of free fermions – with a Hamiltonian that is quadratic in the f_m's, we can proceed as follows. The reduced density matrix must take the form [40]

$$\rho' = \frac{1}{Z} e^{-\mathcal{H}} = \frac{1}{Z} e^{-h_{mn} f_m^\dagger f_n} \tag{83}$$

where Z is the normalization constant so that $\mathrm{Tr}\rho' = 1$; and $\mathcal{H}$ is known as the entanglement Hamiltonian. The sum in the exponent includes only the qubits in the subsystem of interest. And the coefficient h_{mn} can be found by computing the relevant correlators from the original Hamiltonian (78), that is the $\langle\Omega| f_m^\dagger f_n |\Omega\rangle$'s for m and n in the subsystem of interest. Wick's theorem guarantees that all correlator data is indeed packed in these two-point correlators. It is easier to derive the entanglement Hamiltonian if we diagonalize it so that

$$\rho' = \frac{1}{Z} e^{-\sum_k \epsilon_k F_k^\dagger F_k} \tag{84}$$

where the sum is over the subsystem degrees of freedom. Writing

$$\langle\Omega| F_k^\dagger F_l |\Omega\rangle = c_k \delta_{kl} \tag{85}$$

we easily find

$$\epsilon_k = \ln \frac{1 - c_k}{c_k}\ . \tag{86}$$

Hence, by computing two point correlators in the original Hamiltonian, we find the c_k's and we can construct the reduced density matrix from (84) and (86). And the Von Neumann entropy then takes the standard form

$$S = -\mathrm{Tr}\rho' \ln \rho' = \sum_k \ln\left(1 + e^{-\epsilon_k}\right) + \frac{\epsilon_k}{e^{\epsilon_k} + 1}\ . \tag{87}$$

Now, let us come back to the system at hand. From (84), and choosing the probe qubits $\psi^\pm$ as our subsystem, we can immediately write the reduced density matrix as

$$\rho' = \frac{1}{Z} e^{-\epsilon\, \psi^{+\dagger\, I} \psi^{+\, I} - \epsilon\, \psi^{+\dagger\, I} \psi^{+\, I}} \tag{88}$$

Noting that the Fermi vacuum has a condensate of χ modes as determined in (75), we can now compute the two point correlators of the original probe modes using (76) and (77)

$$\begin{aligned}\left\langle\psi^{+\dagger}\psi^{+}\right\rangle &= \left\langle\psi^{-\dagger}\psi^{-}\right\rangle = \frac{1}{r^2R^2}\frac{1}{x^4}\sum_{j=1}^{N-1}\frac{\left|D_j\right|^2}{j+1} \\ &= \frac{4\,J^2C^4}{x^4}\sum_{j=1}^{N-1}\frac{\left|1+(-1)^{j+2J}\right|^2}{j+1}\left|\left(\begin{array}{ccc} J & J & j \\ -1 & 0 & 1\end{array}\right)\right|^2 \equiv c \end{aligned} \tag{89}$$

Since J is a large integer, and noting that we chose earlier $N = 2J+1$ to be odd for convenience, we have

$$c = \frac{C^4}{x^4}4\,J^2\sum_{k=1}^{J}\frac{1}{2k+1}\left|\left(\begin{array}{ccc} N/2 & N/2 & 2k \\ -1 & 0 & 1\end{array}\right)\right|^2 . \tag{90}$$

For large N or J, we get from Appendix B

$$c = \xi^2\frac{C^4}{x^4} \tag{91}$$

where ξ^2 is a number of order one, independent of the parameters of the problem. Summarizing, we have found that the non-zero correlators of interest are

$$\left\langle\psi^{+\dagger\,I}\psi^{+\,I'}\right\rangle = \left\langle\psi^{-\dagger\,I}\psi^{-\,I'}\right\rangle = \xi^2\frac{C^4}{x^4}\;\delta_{II'} \tag{92}$$

where we have added the $SU(4)$ indices to the qubit modes. The Von Neumann entanglement entropy (87) then takes the form

$$S = 8\left(c\ln\left(\frac{1}{c}-1\right)+\ln\left(\frac{1}{1-c}\right)\right) . \tag{93}$$

with the factor of 8 coming from counting all modes with equal contribution (2 from $\psi^{\pm}$, and 4 from the $SU(4)$ multiplicity). It is worthwhile noting that the maximum of this entanglement entropy occurs at

$$c = \frac{1}{2} . \tag{94}$$

For the regime we have been focusing on, whether due to quantum fluctuations or classical profile for the off-diagonal bosonic modes, we can verify that we always have $c \ll 1$. We can then write this expression to leading order in small c

$$S = 8\,c\,\ln c^{-1} . \tag{95}$$

For quantum fluctuations, we have from (28) and (91), to leading order

$$c = \frac{\xi^2}{N^4 r^6} \Rightarrow S = \xi^2 \frac{32}{N^4 r^6} \ln N r^{3/2} \ . \tag{96}$$

For a classical profile with fixed C, we get instead

$$c = \xi^2 \frac{C^4}{x^4} \Rightarrow S = \xi^2 \frac{32\, C^4}{x^4} \ln x \ , \tag{97}$$

where C is then to be viewed as an externally fixed classical parameter. These expressions represent the entanglement entropy between supergravity modes from the probe and from the sphere. We next will try to interpret these results and see how geometrical information may be encoded in the entropy expressions.

4.1 Some background

In attempting to understand the results given by (96) and (97), it is instructive to look at other related systems that have been extensively studied in the literature – in the hopes to get inspired. The investigation of entanglement entropy in the vacuum for qubit systems is a vast area of research in condensed matter physics (see for example [41]-[45]) where entanglement entropy is used to probe criticality and spin chain dynamics. In general, one can interpret the entanglement entropy as (see [46] for a nice review)

$$S \sim \ln M_{eff} \tag{98}$$

where M_{eff} is the number of states in the Schmidt decomposition of the reduced density matrix – or alternatively an estimate of the effective number of entangled states between the subsystems in question. In most cases, one studies one dimensional setups with sparse qubit-qubit interactions. Ours is unusual in that the network of qubit interactions is dense. It is believed [47, 48] that this attribute is key in making the BFSS Matrix model a candidate for a theory of quantum gravity. Effectively, the dense network of qubit interactions can be thought of as describing an infinite dimensional spin chain – leading for example to the fast scrambling phenomena required of black holes [19, 20]. We note that, in *fermionic* critical systems, the entanglement entropy of a sub-block of L qubits is expected to scale as

$$S \sim L^{d-1} \ln L \tag{99}$$

where $L \gg 1$ and d is dimensionality of space. For $d = 1$, the coefficient of the logarithm is related to the central charge of the system. For $d > 1$, the coefficient is also believe to be related to some measure of the number of degrees of freedom. Beside the size of the sub-system, the setup can involve coupling parameters on which the entanglement entropy can depend. In our case for example, the separation between probe and sphere x appears in the effective qubit Hamiltonian as a coupling constant. To compare and contrast, consider a specific one dimensional spin chain, the Ising model with transverse field, described by the Hamiltonian

$$H = -\sum_n \sigma_n^z - \lambda \sum_n \sigma_n^x \sigma_{n+1}^x \tag{100}$$

with a sparse nearest neighbor interaction tuned by the parameter λ. For $\lambda < 1$, the entanglement entropy for a half-chain is given by [40]

$$S = -\frac{c}{6} \ln(1-\lambda) \sim \ln \xi \tag{101}$$

where c is the central charge, and we see a typical logarithmic dependence on the coupling parameter λ. ξ is the correlation length and we also see the entanglement entropy typically scales as the logarithm of the correlation length.

To make better contact with our BFSS system, let us look at another spin chain model that has a key shared feature: a dense network of qubit interactions. The Lipkin-Meshkov-Glick system [49, 50] is described by the Hamiltonian

$$H = -\frac{\lambda}{N} \sum_{i<j} \sigma_x^i \sigma_x^j + \gamma \sigma_y^i \sigma_y^j - h \sum_i \sigma_x^i \tag{102}$$

where λ, γ, and h are coupling parameters. For the isotropic case $\gamma = 1$, the entanglement entropy can be computed exactly [49]. To mimic our BFSS scenario, take the subsystem to be one qubit, then one finds

$$S = 1 - \frac{1}{2}(1-h)\ln(1-h) - \frac{1}{2}(1+h)\ln(1+h) \ , \tag{103}$$

where for simplicity we have chosen h to be an integer multiple of $2/N$. Once again, we see a logarithmic dependence on the coupling h, with a coefficient that also depends on h. For $\gamma \neq 1$, the system can be studied numerically. For a large number of qubits and a large size of the subsystem, one finds once again a logarithmic behavior

$$S \sim \ln(1-\gamma) \ . \tag{104}$$

These are very interesting results since the system is not one dimensional; yet, the entanglement entropy continues to exhibit logarithmic behaviour in the parameters, characteristic of a one dimensional system.

The general theme from a survey of qubit systems leads to the following observations. The entanglement entropy often scales logarithmically with the size of the subsystem and the coupling parameters of the theory. The pre-factor of the logarithm can often be related to a measure of the effective number of degrees of freedom that are entangled. We will now proceed to use these insights for proposing an interpretation of (96) and (97).

4.2 Speculations

Let us start by focusing on (97), the case where the off-diagonal bosonic modes are fixed classically along a shallow valley of the potential. The x dependence of the entropy takes the form

$$S \sim \frac{1}{x^4} \ln x \ . \tag{105}$$

Here, x plays the role of a coupling parameter in the effective Hamiltonian, and we see a characteristic logarithmic dependence on x – in addition to the x dependent pre-factor. x is also the correlation length between the supergravity modes for $x \gg 1$, which also syncs well with expectations from our previous discussion. The x^{-4} pre-factor is however potentially much more interesting for the purposes of unravelling geometry. If we are to interpret it as some sort of measure of effective number of entangled degrees of freedom, its x dependence may be similar to connections between entropy and geometry from proposals such as [4]. One's natural inclination would then to be to suggest a connection to area. Yet, unlike the proposal of [4], our setup has a local character: the probe sits at a point in space, as opposed to being a surface area extending globally. There is a natural local geometrical quantity that is known to be related to area and that has risen previously in the context of gravitational holography in describing effective number of degrees of freedom: the rate of convergence of a congruence of null geodesics.

Consider the M-theory metric seen by the probe for the setup at hand[7]

$$ds_{11}^2 = \left(1+\frac{q}{r}\right)^{-1/2}\left[-dt^2+dx_{11}^2\right] + \left(1+\frac{q}{r}\right)^{1/2}\left[dr^2+r^2 d\Omega^2\right] \tag{106}$$

where x_{11} is the light-cone direction. This is easily obtained by uplifting the D0 brane metric from IIA supergravity [51]: at asymptotically large distances from the D0 branes, we expect the metric of the spherical shell to look like that of N D0 branes at the origin. In the BFSS limit, we also have $1+q/r \to q/r$. If we were to project a congruence of null geodesics radially toward $r=0$, the rate of convergence of the geodesics is given by (see for example [52])

$$\theta = \nabla_a n^a = -\frac{1}{r} \tag{107}$$

[7]Note that seven of the ten space dimensions have been compactified, one being the light-cone direction

where n^a is the tangent to the geodesics, normalized to absorb numerical factors and the energy on the right hand side. Geometrically, θ is the rate of change of a transverse area element A along the geodesic flow

$$\theta \propto \frac{1}{A}\frac{dA}{d\lambda} \tag{108}$$

where λ is a geodesic affine parameter.

In the context of gravitational holography, the condition that $\theta < 0$ has been proposed to determine whether a region of space can be holographically encoded in a dual boundary theory [53]. In another work, θ was related to a c-function of the boundary theory [52]. Along with the null energy condition, it can be shown that this c-function is monotonically decreasing and has the right scaling with the parameters of the dual theory [52, 54, 55].

These observations suggest that the pre-factor of the logarithm in (105) should perhaps be related to the rate of convergence of geodesics. However, there is a conceptual obstacle that we need to understand first: when the off-diagonal modes are allowed to relax in the harmonic oscillator ground state, as they would want to, the entropy expression is given by (96) instead, with no x dependence. Furthermore, if we were to allow C to settle instead in a configuration that maximizes entanglement entropy, this leads to $C \sim x$ – corresponding to an excited harmonic oscillator state for the off-diagonal modes – and once again a maximal entanglement entropy agnostic of the coordinate distance x between probe and sphere (and even of N). Why do we then 'see geometry' in the entanglement entropy expression in some of these scenarios but not in others?

Our setup corresponds to artificially fixed source and probe. If the probe was to be allowed to in-fall, it would not feel gravity by the equivalence principle. On the other, if held up by necessarily x-dependent external forces that exactly cancel the gravitational pull, it would also experience no net force. Finally, it should feel an x dependence force if the gravitational pull is not balanced, say by virtue of some x-independent external forces. Entanglement entropy – which focuses on a particular subsystem – is naturally tied to a fixed perspective, a fixed observer of the underlying space geometry. Hence, this may suggest that when looking at entanglement entropy of the supergravity modes one is implicitly focusing on the local geometry or gravity seen by the probe in question. When C is allowed to freeze in the ground state of the off-diagonal mode oscillators, this corresponds to the physical situation of an exact balance of gravitational and external forces, the stable setup: no net force is experienced by the probe and we have an entanglement entropy that is independent of x given by (96). If C is allowed to have an arbitrary x-independent classical value, this corresponds to an unstable configuration as C sits in a shallow potential that is not strictly flat: the gravitational and external forces on the probe are not balanced, there is a net force depending on C and on x, and hence we see the entanglement entropy depend on both parameters. Furthermore, C being independent of x, the dependence on x in the entropy should reflect local gravitational geometry. We can take this line of thought one step further. When the classical profile for C is allowed to run so as to *maximize* entanglement entropy, we use (94) and find $C \sim x$. We propose that this scenario – where maximizing the entanglement entropy 'back reacts' on the off-diagonal modes – corresponds to switching to the perspective of the *in-falling* probe: hence there is no gravity or forces to be detected and the entropy expression takes an x and N independent form.

Our suggestion is then that the entanglement entropy quantity we computed between supergravity modes on the probe and the sphere *does* encode net force or gravitational geometry information. But to demonstrate this in the setup at hand where source and probe are fixed externally, we needed to allow for arbitrary C, an unbalanced dynamical situation, and hence consider the expression given by (97). Once this is arranged, we can try to extract local geometry information from the entropy expression.

Hence, coming back to the x dependence of the pre-factor in (97), we can now hint at a possible map:

$$S \sim |\theta|^D \ln x \ , \tag{109}$$

where D is the number of spacetime dimensions. The D depedence is at best an educated guess. This implies that this entanglement entropy of supergravity modes can perhaps be used to extract *local* geometry information. If true, it would be a local version of an area-entropy proposal akin to [1].

Finally, we note that the entanglement entropy (96) for the case where the off-diagonal matrix modes are dynamically frozen in the ground state scales logarithmically with the number of entangled qubits N. This is as expected from our previous discussion of spin chains. The N dependence of the prefactor is curious, coming in as N^{-4}; the fourth power may be reflecting the dimensionality of the target spacetime. We have however no further substantive interpretation of this observation.

5 Conclusions and Outlook

In this work, we have taken the first steps into developing a new approach for probing emergent geometry in Matrix theory: looking at the entanglement of the fermionic modes. We have identified a controllable computational regime – corresponding to widely separated source and probe – where one can decode geometric information from the entanglement entropy. This map is local in character, allowing one in principle to read off the rate of geodesic convergence at any point in space.

This however is far from a complete story, and we are lead to a myriad of new open questions in the grand problem of relating geometry to matrices and entanglement entropy. First, to establish and confirm the connection between geodesic convergence rate and entanglement entropy, other scaffolding configurations need to be set up and the corresponding entropies need to be computed. For example, one can arrange for a string or cylindrical source, or a non-commutative plane – each having a different scaling of convergence rates with distance. The matrix configurations for these shapes are known [25, 56, 57, 58, 59] and the technology to compute the corresponding entanglement entropies would be a straightforward extension of the work presented here. One can also consider higher dimensional target spaces as well, and even consider configurations involving five brane source. However, we expect that the latter case would present a significant technical challenge [60, 61, 23].

Another parallel issue that needs attention has to do with the fact that our setups are not dynamical, and they involve external pinning forces. We saw that this created subtelties in reading off geometry from entanglement entropy as the latter seems to be sensitive to the net forces acting on the probe. This required us to consider an unstable, unbalanced snapshot of the configuration to be able to decode geometry from the entropy expression. However, one can also consider a full dynamical situation and see how the entanglement entropy evolves in time. This would be an alternate approach to extracting geometrical data as a probe sifts through patches of curved spacetime.

Furthermore, in developing an interpretation of the entanglement entropy, we suggested that switching perspectives to an in-falling probe perhaps involves maximizing the entanglement entropy. This was inspired by the realization that: (1) the entropy is by its nature a probe-specific observable; and (2) the maximal entropy is independent of all parameters in the problem, distance as well as mass of the source. These may indicate a potential link with the idea of the equivalence principle, the abscence of gravitational forces for point-like in-falling probes. Physically, the entropy maximization procedure corresponds to exciting the off-diagonal matrix modes that correspond to a membrane stretched between probe and source. In the dual picture, the in-falling perspective sees flat space – known to be characterized by strong quantum entanglement (see for example [62]). This syncs well with notions of black hole complementary [63] and may be a hint at a resolution of the firewall issue [64]. Indeed, a wormwhole connecting modes inside and far away from a black hole has been proposed already as a means to evade the firewall issue [6]. Computations involving Matrix black holes and qubits have appeared recently in [65, 14, 20]. Hence, it would be very interesting to explore the Matrix-black-hole-plus-probe scenario along the line we have developed in this work, looking at qubit entanglement.

6 Appendices

6.1 Appendix A: Diagonalization

We collect in this appendix the coefficients arising in the procedure of diagonalizing the off-diagonal fermionic modes in the BFSS Hamiltonian. The eigenvalues are found (without any expansion for large x)

$$\lambda_M^{(1)} = -\frac{1}{2} - \frac{1}{2}\sqrt{N^2 - 4\frac{x}{r}(2M-1) + 4\frac{x^2}{r^2}} \ , \tag{110}$$

$$\lambda_M^{(2)} = -\frac{1}{2} + \frac{1}{2}\sqrt{N^2 - 4\frac{x}{r}(2M-1) + 4\frac{x^2}{r^2}} \ ; \tag{111}$$

and also

$$\overline{\lambda}_M^{(1)} = \frac{1}{2} - \frac{1}{2}\sqrt{N^2 - 4\frac{x}{r}(2M-1) + 4\frac{x^2}{r^2}} \ , \tag{112}$$

$$\overline{\lambda}_M^{(2)} = \frac{1}{2} + \frac{1}{2}\sqrt{N^2 - 4\frac{x}{r}(2M-1) + 4\frac{x^2}{r^2}} \ . \tag{113}$$

The coefficients in the expressions for the eigenvectors take the form

$$k_i^+(M) = \frac{\lambda_M^{(i)} + M + 1 - (x/r)}{\sqrt{(\lambda_M^{(i)} - (x/r) + M + 1)^2 - (J-M)(J+M+1)}} \ ; \tag{114}$$

$$k_i^-(M) = \frac{\sqrt{(J-M)(J+M+1)}}{\sqrt{(\lambda_M^{(i)} - (x/r) + M + 1)^2 + (J-M)(J+M+1)}} \ ; \tag{115}$$

$$\overline{k}_i^+(M) = \frac{\overline{\lambda}_M^{(i)} + M - 1 - (x/r)}{\sqrt{(\overline{\lambda}_M^{(i)} - (x/r) + M - 1)^2 - (J+M)(J-M+1)}} \ ; \tag{116}$$

$$\overline{k}_i^-(M) = \frac{\sqrt{(J+M)(J-M+1)}}{\sqrt{(\overline{\lambda}_M^{(i)} - (x/r) + M - 1)^2 + (J+M)(J-M+1)}} \ ; \tag{117}$$

Once again, before any large x expansion. For $x \gg r\,N$, we find to order x^{-2}, the expressions for the eigenvalues

$$\lambda_M^{(i)} \to (-1)^i \frac{x}{r} \quad , \quad \overline{\lambda}_M^{(i)} \to (-1)^i \frac{x}{r} \ . \tag{118}$$

Similarly, in this regime, the eigenvector coefficients become

$$k_1^+(M) \to -1 \quad , \quad r\, k_2^+(M) \to \frac{N^2 - 4M(M+1) - 1}{8\sqrt{(J-M)(J+M+1)}} \frac{r}{x} \ , \tag{119}$$

$$k_1^-(M) \to \frac{(J-M)(J+M+1)}{2} \frac{r}{x} \quad , \quad r\, k_2^-(M) \to 1 \ , \tag{120}$$

$$\overline{k}_1^+(M) \to -1 \quad , \quad r\, \overline{k}_2^+(M) \to \frac{N^2 - 4M(M-1) - 1}{8\sqrt{(J+M)(J-M+1)}} \frac{r}{x} \ , \tag{121}$$

$$\overline{k}_1^-(M) \to \frac{(J+M)(J-M+1)}{2} \frac{r}{x} \quad , \quad r\, \overline{k}_2^-(M) \ . \to 1 \tag{122}$$

We also will need to implement diagonalization through a second order perturbation scheme. We collect here the formal expressions for the eigenvalues

$$\lambda_n = \lambda_n^{(0)} + V_{nn} - \sum_{m\neq n} \frac{|V_{mn}|^2}{\lambda_m^{(0)} - \lambda_n^{(0)}} \ . \tag{123}$$

And for the eigenvectors, we get[8]

$$\begin{aligned} |n\rangle^{(0)} &= \left(1 - \tfrac{1}{2}\textstyle\sum_{m\neq n} \frac{|V_{mn}|^2}{\left(\lambda_m^{(0)} - \lambda_n^{(0)}\right)^2}\right) |n\rangle \\ &+ \sum_{m\neq n} \left(\boxed{\frac{V_{nm}}{\lambda_m^{(0)} - \lambda_n^{(0)}}} - \frac{V_{nm}V_{mm}}{\left(\lambda_m^{(0)} - \lambda_n^{(0)}\right)^2} + \textstyle\sum_{m'\neq m} \frac{V_{nm'}V_{m'm}}{\left(\lambda_m^{(0)} - \lambda_n^{(0)}\right)\left(\lambda_m^{(0)} - \lambda_{m'}^{(0)}\right)} \right) |m\rangle \ . \end{aligned} \tag{124}$$

Only the boxed term however will play a role in computing the entanglement entropy.

6.2 Appendix B: $3j$ symbol identities

In dealing with matrix spherical harmonics, we invariably encounter Wigner's $3j$ symbols. To simplify expression, we make use of the following identities. Under even permutations of columns, we have

$$P_{even} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \ . \tag{125}$$

[8]Note also that our Hamiltonian is already diagonal in the subspace with degenerate eigenvalues.

Under odd permutations of columns, we instead have

$$P_{odd}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} . \tag{126}$$

Furthermore, we can also flip lower row signs

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} . \tag{127}$$

Note also that the $3j$ symbols vanish unless $m_1 + m_2 + m_3 = 0$.

We also need asymptotic forms of some of these $3j$ symbols. In particular, the following expression arises throughout

$$T_j \equiv \begin{pmatrix} J & J & j \\ -1 & 0 & 1 \end{pmatrix} \tag{128}$$

in the regime where $J \gg 1$. For j odd or even, this expression has different asymptotic form in J. However, we can look for a bound on the maximum of $|T_j|$ for all j. We find

$$\max |T_j| < J^{-1/2} . \tag{129}$$

We can establish this by numerical evaluation: the result is very robust for $J > 10$, with numerical fit errors being less than a fraction of a percent. We note that the standard asymptotic expression for $3j$ symbols – sometimes called Flude or Edmonds asymptotics – is not applicable for the expression of interest to us, hence the need for a numerical determination.

We also will need the large integer J asymptotic form of

$$K = 4\,J^2 \sum_{k=1}^{J} \frac{1}{2k+1}\left|\begin{pmatrix} N/2 & N/2 & 2k \\ -1 & 0 & 1 \end{pmatrix}\right|^2 . \tag{130}$$

Once again, through straightforward numerical fits, it is very easy to determine

$$K = \xi^2 \tag{131}$$

where ξ^2 is a J independent numerical factor of order one. That is, the sum goes as J^{-2} and cancels the pre-factor.

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