Introduction to Higher Cubical Operads. First Part: The Cubical Operad of Cubical Weak ∞-Categories

Camel KACHOUR

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Décembre 2017

IHES/M/17/19

Introduction to Higher Cubical Operads. First Part : The Cubical Operad of Cubical Weak ∞-Categories

Camell Kachour

December 20, 2017

Abstract

In this article, divided in two parts, we show how to build main aspects of the article [1] but with the cubical geometry. This first part is devoted to build the contractible $\mathbb{S}$-operad $\mathbb{B}^0_C$ equipped with a cubical C^0-system, where $\mathbb{S}$ is the monad of free strict cubical ∞-categories on cubical sets. Actions of this monad are on cubical sets with no notions of reflexivities (the classical and the connections) in order to be sure that it is cartesian (see [14]). In our approach, classical reflexivities plus connections, appear in the level of algebras. This operad is free on this C^0-system (which itself is a specific cubical pointed $\mathbb{S}$-collection). We exhibit some simple coherences cells of $\mathbb{B}^0_C$ and show how they provide more richness compare to its globular analogue (see [1]).

Keywords. cubical (∞, n)-categories, weak cubical ∞-groupoids, homotopy types.
Mathematics Subject Classification (2010). 18B40,18C15, 18C20, 18G55, 20L99, 55U35, 55P15.

Contents

Introduction

In this article, divided in two parts, we show how to build main aspects of the article [1] but with the cubical geometry. This first part is devoted to build the contractible $\mathbb{S}$-operad $\mathbb{B}^0_C$ equipped with a cubical C^0-system, where $\mathbb{S}$ is the monad of free strict cubical ∞-categories on cubical sets. Actions of this monad are on cubical sets with no notions of reflexivities (the classical and the connections) in order to be sure that it is cartesian (see [14]). In our approach, classical reflexivities plus connections, appear in the level of algebras. This operad is free on this C^0-system (which itself is a specific cubical pointed $\mathbb{S}$-collection). We exhibit some simple coherences cells of $\mathbb{B}^0_C$ and show how they provide more richness compare to its globular analogue (see [1]). In the second part of this article (see [13]), we use it as a fundamental step to associate to any topological space X its fundamental cubical weak ∞-groupoids $\Pi_\infty(X)$, and this endows a functor $Top \xrightarrow{\Pi_\infty(-)} \infty\text{-}\mathbb{C}\mathbb{G}\text{rp}$ which has a left adjoint functor CN_∞. This pair of adjunction $(CN_\infty, \Pi_\infty(-))$ should put an equivalence between the homotopy category of homotopy types and the homotopy category of ∞-$\mathbb{C}\mathbb{G}$rp of cubical weak ∞-groupoids equipped with an adapted Quillen model structure. This was shown to be true but in the context of the Cisinski model structure on the category of cubical sets with connections (see [19]).

Acknowledgement. This work was doing thanks to the beautiful ambiance I got during my two months visiting in the IHES. I especially want to mention Maxim Kontsevich whose questions on my work were deep and push me to clarify a lot of aspects of my work. I want also to mention Pierre Cartier, who patiently, has taken his time to understand my work and who shared with me many of his very deep point of views of Higher Category Theory. During my stay, Vasily Pestun has shared with me some deep questions which relate mathematics and theoretical physics. I was especially impressed by the huge imagination that a physicist can have, starting with simple questions in mathematics, but which answers is highly non-trivial and should open new mathematics, that mathematicians cannot see when "jailed" in mathematics. I cannot also forget my friendly discussions with Thibault Damour ! I want also to point out how Laurent Lafforgue, Fanny Kassel, Emmanuel Ullmo and Elisabeth Jasserand, where all very kind with me. Many other mathematicians or physicists open my eyes on other point of views of science. I want to mention Thiago Araujo (Strings theory), Yang Liu (Non-commutative geometry), and Benjamin Ward (Operads and homology). Last but not least, this work wouldn't exist without my natural interactions with Ross Street and Mark Weber.

1 Cubical sets

See also [11] for more references on cubical sets.

1.1 The cubical category

Consider the small category $\mathbb{C}$ with integers $\underline{n} \in \mathbb{N}$ as objects. Generators for $\mathbb{C}$ are, for all $\underline{n} \in \mathbb{N}$ given by *sources* $\underline{n} \xrightarrow{s^n_{n-1,j}} \underline{n-1}$ for each $j \in \{1,..,n\}$ and *targets* $\underline{n} \xrightarrow{t^n_{n-1,j}} \underline{n-1}$ for each $j \in \{1,..,n\}$ such that for $1 \leq i < j \leq n$ we have the following cubical relations

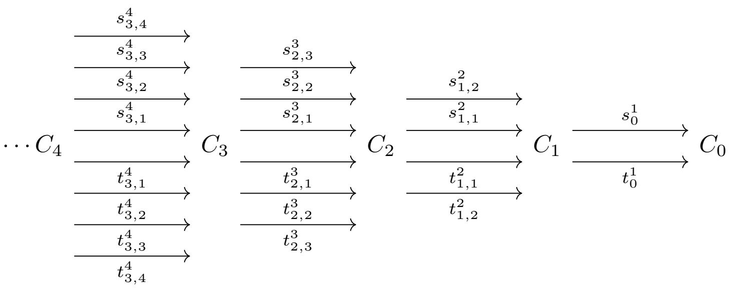
(i) $s^{n-1}_{n-2,i} \circ s^n_{n-1,j} = s^{n-1}_{n-2,j-1} \circ s^n_{n-1,i}$,

(ii) $s^{n-1}_{n-2,i} \circ t^n_{n-1,j} = t^{n-1}_{n-2,j-1} \circ s^n_{n-1,i}$,

(iii) $t^{n-1}_{n-2,i} \circ s^n_{n-1,j} = s^{n-1}_{n-2,j-1} \circ t^n_{n-1,i}$,

(iv) $t^{n-1}_{n-2,i} \circ t^n_{n-1,j} = t^{n-1}_{n-2,j-1} \circ t^n_{n-1,i}$

These generators plus these relations give the small category $\mathbb{C}$ called the *cubical category* that we may represent schematically with the low dimensional diagram :



and this category $\mathbb{C}$ gives also the sketch $\mathcal{E}_{\mathbb{S}}$ of cubical sets used especially in 2.2, **??** and **??** to produce the monads $\mathbb{S}=(S,\lambda,\mu)$, $\mathbb{W}=(W,\eta,\nu)$ and $\mathbb{W}^m=(W^m,\eta^m,\nu^m)$ on $\mathbb{C}\mathbb{S}$ets, which algebras are respectively cubical strict ∞-categories, cubical weak ∞-categories and cubical weak (∞,m)-categories.

Definition 1 The category of cubical sets $\mathbb{C}\mathbb{S}$ets is the category of presheaves $[\mathbb{C};\mathbb{S}\text{ets}]$. The terminal cubical set is denoted 1. □

Occasionally a cubical set shall be denoted with the notation

$$\mathcal{C}=(C_n,s^n_{n-1,j},t^n_{n-1,j})_{1\leq j\leq n,\ n\in\mathbb{N}}$$

in case we want to point out its underlying structures.

1.2 Reflexive cubical sets

Reflexivity for cubical sets are of two sorts : one is "classical" in the sense that they are very similar to their globular analogue; thus we shall use the notation $(1^n_{n+1,j})_{n\in\mathbb{N},j\in\{1,..,n\}}$ to denote these maps $C(n)\xrightarrow{1^n_{n+1,j}}C(n+1)$ which formally behave like globular reflexivity ([15]); the others are called *connections* and are given by maps $C(n)\xrightarrow{\Gamma}C(n+1)$ where the notation using the greek letter *"Gamma"* seems to be the usual notation. However we do prefer to use instead the notation $C(n)\xrightarrow{1^{n,\gamma}_{n+1,j}}C(n+1)$ $(\gamma\in\{+,-\})$ in order to point out the reflexive nature of connections.

Consider the cubical category $\mathbb{C}$. For all $n\in\mathbb{N}$ we add in it generators $\underline{n-1}\xrightarrow{1^{n-1}_{n,j}}\underline{n}$ for each $j\in\{1,..,n\}$ subject to the relations :

(i) $1^n_{n+1,i}\circ 1^{n-1}_{n,j}=1^n_{n+1,j+1}\circ 1^{n-1}_{n,i}$ if $1\leq i\leq j\leq n$;

(ii) $s^n_{n-1,i}\circ 1^{n-1}_{n,j}=1^{n-2}_{n-1,j-1}\circ s^{n-1}_{n-2,i}$ if $1\leq i<j\leq n$;

(iii) $s^n_{n-1,i}\circ 1^{n-1}_{n,j}=1^{n-2}_{n-1,j}\circ s^{n-1}_{n-2,i-1}$ if $1\leq j<i\leq n$;

(iv) $s^n_{n-1,i}\circ 1^{n-1}_{n,j}=id(\underline{n-1})$ if $i=j$.

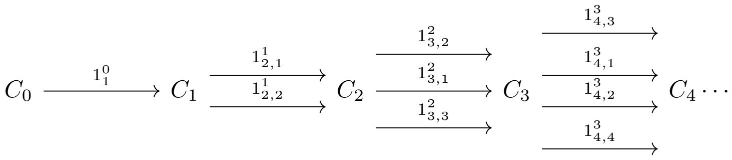
(i) $1^n_{n+1,i}\circ 1^{n-1}_{n,j}=1^n_{n+1,j+1}\circ 1^{n-1}_{n,i}$ if $1\leq i\leq j\leq n$;

(ii) $t^n_{n-1,i}\circ 1^{n-1}_{n,j}=1^{n-2}_{n-1,j-1}\circ t^{n-1}_{n-2,i}$ if $1\leq i<j\leq n$;

(iii) $t^n_{n-1,i}\circ 1^{n-1}_{n,j}=1^{n-2}_{n-1,j}\circ t^{n-1}_{n-2,i-1}$ if $1\leq j<i\leq n$;

(iv) $t^n_{n-1,i}\circ 1^{n-1}_{n,j}=id(\underline{n-1})$ if $i=j$.

These generators and relations give the small category $\mathbb{C}_{\text{sr}}$ called the *semireflexive cubical category* where a quick look at its underlying semireflexive structure is given by the following diagram :



Definition 2 The category of semireflexive cubical sets $\mathbb{C}_{\mathrm{sr}}\mathbb{S}\mathrm{ets}$ is the category of presheaves $[\mathbb{C}_{\mathrm{sr}}; \mathbb{S}\mathrm{ets}]$. The terminal semireflexive cubical set is denoted 1_{sr} ◻

Consider the semireflexive cubical category $\mathbb{C}_{\mathrm{sr}}$. For all integers $n \geq 1$ we add in it generators $\underline{n-1} \xrightarrow{1_{n,j}^{n-1,\gamma}} \underline{n}$ for each $j \in \{1,..,n-1\}$ subject to the relations :

(i) for $1 \leq j < i \leq n$, $1_{n+1,i}^{n,\gamma} \circ 1_{n,j}^{n-1,\gamma} = 1_{n+1,j+1}^{n,\gamma} \circ 1_{n,i}^{n-1,\gamma}$;

(ii) for $1 \leq i \leq n-1$, $1_{n+1,i}^{n,\gamma} \circ 1_{n,i}^{n-1,\gamma} = 1_{n+1,i+1}^{n,\gamma} \circ 1_{n,i}^{n-1,\gamma}$;

(iii) for $1 \leq i,j \leq n$, $\begin{cases} 1_{n+1,i}^{n,\gamma} \circ 1_{n,j}^{n-1} & = 1_{n+1,j+1}^{n} \circ 1_{n,i}^{n-1,\gamma} \quad \text{if} \quad 1 \leq i < j \leq n \\ & = 1_{n+1,j}^{n} \circ 1_{n,i-1}^{n-1,\gamma} \quad \text{if} \quad 1 \leq j < i \leq n \end{cases}$;

(iv) for $1 \leq j \leq n$, $1_{n+1,j}^{n,\gamma} \circ 1_{n,j}^{n-1} = 1_{n+1,j}^{n} \circ 1_{n,j}^{n-1}$;

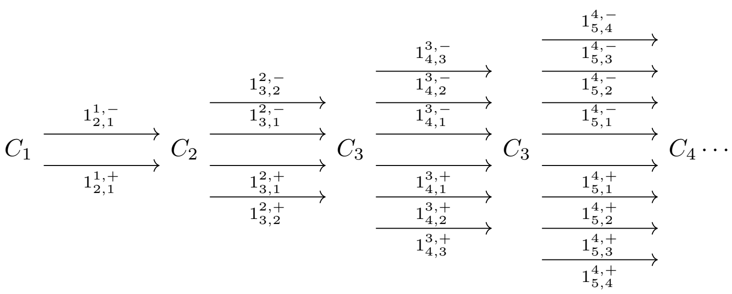
(v) for $1 \leq i,j \leq n$,
$\begin{cases} s_{n-1,i}^{n} \circ 1_{n,j}^{n-1,\gamma} & = 1_{n-1,j-1}^{n-2,\gamma} \circ s_{n-2,i}^{n-1} \quad \text{if} \quad 1 \leq i < j \leq n-1 \\ & = 1_{n-1,j}^{n-2,\gamma} \circ s_{n-2,i-1}^{n-1} \quad \text{if} \quad 2 \leq j+1 < i \leq n \end{cases}$;
and
$\begin{cases} t_{n-1,i}^{n} \circ 1_{n,j}^{n-1,\gamma} & = 1_{n-1,j-1}^{n-2,\gamma} \circ t_{n-2,i}^{n-1} \quad \text{if} \quad 1 \leq i < j \leq n-1 \\ & = 1_{n-1,j}^{n-2,\gamma} \circ t_{n-2,i-1}^{n-1} \quad \text{if} \quad 2 \leq j+1 < i \leq n \end{cases}$;

(vi) for $1 \leq j \leq n-1$, $s_{n-1,j}^{n} \circ 1_{n,j}^{n-1,-} = s_{n-1,j+1}^{n} \circ 1_{n,j}^{n-1,-} = 1_{\underline{n-1}}$ and $t_{n-1,j}^{n} \circ 1_{n,j}^{n-1,+} = t_{n-1,j+1}^{n} \circ 1_{n,j}^{n-1,+} = 1_{\underline{n-1}}$;

(vii) for $1 \leq j \leq n-1$, $s_{n-1,j}^{n} \circ 1_{n,j}^{n-1,+} = s_{n-1,j+1}^{n} \circ 1_{n,j}^{n-1,+} = 1_{n-1,j}^{n-2} \circ s_{n-2,j}^{n-1}$;

(viii) for $1 \leq j \leq n-1$, $t_{n-1,j}^{n} \circ 1_{n,j}^{n-1,-} = t_{n-1,j+1}^{n} \circ 1_{n,j}^{n-1,-} = 1_{n-1,j}^{n-2} \circ t_{n-2,j}^{n-1}$.

These generators and relations give the small category $\mathbb{C}_r$ called the *reflexive cubical category* and in it, connections have the following shape :



Definition 3 The category of reflexive cubical sets $\mathbb{C}_{\mathrm{r}}\mathbb{S}\mathrm{ets}$ is the category of presheaves $[\mathbb{C}_{\mathrm{r}}; \mathbb{S}\mathrm{ets}]$. The terminal reflexive cubical set is denoted 1_{r} ◻

2 The category of strict cubical ∞-categories

Cubical strict ∞-categories have been studied in [2, 21].

In [2] the authors proved that the category of cubical strict ∞-categories with cubical strict ∞-functors as morphisms is equivalent to the category of globular strict ∞-categories with globular strict ∞-functors as morphisms. Consider a cubical reflexive set

$$(C, (1^n_{n+1,j})_{n\in\mathbb{N}, j\in[\![1,n+1]\!]}, (1^{n,\gamma}_{n+1,j})_{n\geq 1, j\in[\![1,n]\!]})$$

equipped with partial operations $(\circ^n_j)_{n\geq 1, j\in[\![1,n]\!]}$ where if $a, b \in C(n)$ then $a \circ^n_j b$ is defined for $j \in \{1, ..., n\}$ if $s^n_j(b) = t^n_j(a)$. We also require these operations to follow the following axioms of positions :

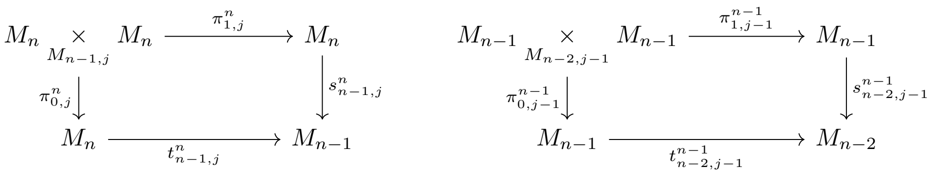
(i) For $1 \leq j \leq n$ we have : $s^n_{n-1,j}(a \circ^n_j b) = s^n_{n-1,j}(a)$ and $t^n_{n-1,j}(a \circ^n_j b) = t^n_{n-1,j}(a)$,

(ii) $s^n_{n-1,i}(a \circ^n_j b) = \begin{cases} s^n_{n-1,i}(a) \circ^{n-1}_{j-1} s^n_{n-1,i}(b) \text{ if } 1 \leq i < j \leq n \\ s^n_{n-1,i}(a) \circ^{n-1}_{j} s^n_{n-1,i}(b) \text{ if } 1 \leq j < i \leq n \end{cases}$

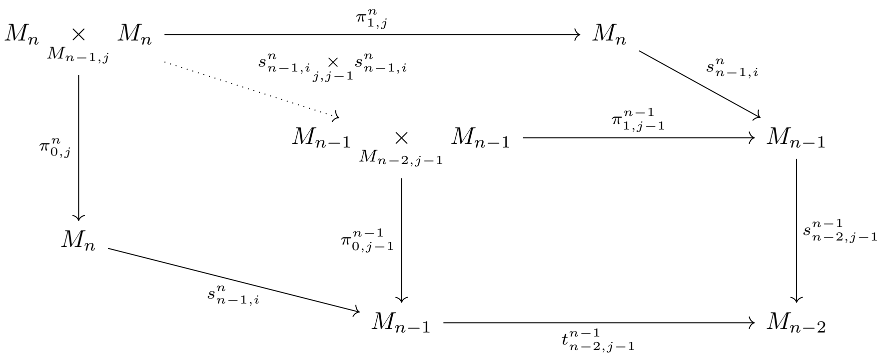
(iii) $t^n_{n-1,i}(a \circ^n_j b) = \begin{cases} t^n_{n-1,i}(a) \circ^{n-1}_{j-1} t^n_{n-1,i}(b) \text{ if } 1 \leq i < j \leq n \\ t^n_{n-1,i}(a) \circ^{n-1}_{j} t^n_{n-1,i}(b) \text{ if } 1 \leq j < i \leq n \end{cases}$

The following sketch $\mathcal{E}_M$ of *axioms of positions* as above shall be used in 2.2 to justify the existence of the monad on $\mathbb{C}\mathbb{S}$ets of cubical strict ∞-categories. It is important to notice that the sketch just below has only *one generation* which means that diagrams and cones involved in it are not build with previous data of other diagrams and cones.

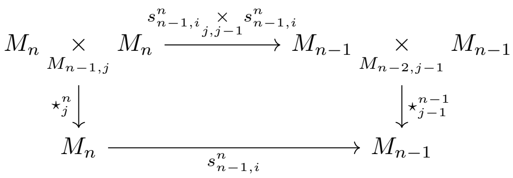
- For $1 \leq i < j \leq n$ we consider the following two cones :



and the following commutative diagram (definition of $s^n_{n-1,i} \underset{j,j-1}{\times} s^n_{n-1,i}$)



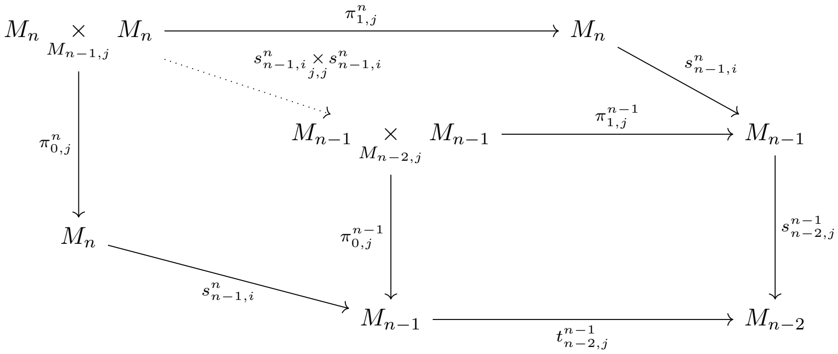
which gives the following commutative diagram



- For $1 \leq j < i \leq n$ we consider the following two cones :

$$\begin{array}{ccc} M_n \underset{M_{n-1,j}}{\times} M_n & \xrightarrow{\pi^n_{1,j}} & M_n \\ {\scriptstyle \pi^n_{0,j}}\downarrow & & \downarrow{\scriptstyle s^n_{n-1,j}} \\ M_n & \xrightarrow[t^n_{n-1,j}]{} & M_{n-1} \end{array} \qquad \begin{array}{ccc} M_{n-1} \underset{M_{n-2,j}}{\times} M_{n-1} & \xrightarrow{\pi^{n-1}_{1,j}} & M_{n-1} \\ {\scriptstyle \pi^{n-1}_{0,j}}\downarrow & & \downarrow{\scriptstyle s^{n-1}_{n-2,j}} \\ M_{n-1} & \xrightarrow[t^{n-1}_{n-2,j}]{} & M_{n-2} \end{array}$$

and the following commutative diagram (definition of $s^n_{n-1,i} \underset{j,j}{\times} s^n_{n-1,i}$)



The previous datas gives the following commutative diagram of axioms

$$\begin{array}{ccc} M_n \underset{M_{n-1}}{\times} M_n & \xrightarrow{s^n_{n-1,i} \underset{j,j}{\times} s^n_{n-1,i}} & M_{n-1} \underset{M_{n-2}}{\times} M_{n-1} \\ {\scriptstyle \star^n_j}\downarrow & & \downarrow{\scriptstyle \star^{n-1}_j} \\ M_n & \xrightarrow[s^n_{n-1,i}]{} & M_{n-1} \end{array}$$

and for $1 \leq j \leq n$ we have the following commutative diagram of axioms

$$\begin{array}{ccc} M_n \underset{M_{n-1}}{\times} M_n & \xrightarrow{\pi_1} & M_n \\ {\scriptstyle \star^n_j}\downarrow & & \downarrow{\scriptstyle s^n_{n-1,j}} \\ M_n & \xrightarrow[s^n_{n-1,j}]{} & M_{n-1} \end{array}$$

which actually complete the description of $\mathcal{E}_M$

Definition 4 Cubical reflexive ∞-magmas are cubical reflexive set equipped with partial operations like just above which follow axioms of positions. A morphism between two cubical reflexive ∞-magmas is a morphism of their underlying cubical reflexive sets. The category of cubical reflexive ∞-magmas is noted ∞-$\mathbb{C}\mathbb{M}\mathsf{ag}_{\mathrm{r}}$

Remark 1 Cubical ∞-magmas are poorer structure : they are cubical sets equipped with partial operations like above with these axioms of positions. A morphism between two cubical ∞-magmas is a morphism of their underlying cubical sets.The category of cubical ∞-magmas is noted ∞-$\mathbb{C}\mathbb{M}\mathsf{ag}$ □

2.1 Definition

Strict cubical ∞-categories are cubical reflexive ∞-magmas such that partials operations are associative and also we require the following axioms :

(i) The interchange laws : $(a \circ_i^n b) \circ_j^n (c \circ_i^n d) = (a \circ_j^n c) \circ_i^n (b \circ_j^n d)$ whenever both sides are defined

(ii) $1^n_{n+1,i}(a \circ^n_j b) = 1^n_{n+1,i}(a) \circ^{n+1}_{j+1} 1^n_{n+1,i}(b)$ if $1 \leq i \leq j \leq n$
$1^n_{n+1,i}(a \circ^n_j b) = 1^n_{n+1,i}(a) \circ^{n+1}_{j} 1^n_{n+1,i}(b)$ if $1 \leq j < i \leq n+1$

(iii) $1^{n,\gamma}_{n+1,i}(a \circ^n_j b) = 1^{n,\gamma}_{n+1,i}(a) \circ^{n+1}_{j+1} 1^{n,\gamma}_{n+1,i}(b)$ if $1 \leq i < j \leq n$
$1^{n,\gamma}_{n+1,i}(a \circ^n_j b) = 1^{n,\gamma}_{n+1,i}(a) \circ^{n+1}_{j} 1^{n,\gamma}_{n+1,i}(b)$ if $1 \leq j < i \leq n$

(iv) First transport laws : for $1 \leq j \leq n$

$$1^{n,+}_{n+1,j}(a \circ^n_j b) = \begin{bmatrix} 1^{n,+}_{n+1,j}(a) & 1^n_{n+1,j}(a) \\ 1^n_{n+1,j+1}(a) & 1^{n,+}_{n+1,j}(b) \end{bmatrix}$$

(v) Second transport laws : for $1 \leq j \leq n$

$$1^{n,-}_{n+1,j}(a \circ^n_j b) = \begin{bmatrix} 1^{n,-}_{n+1,j}(a) & 1^n_{n+1,j+1}(b) \\ 1^n_{n+1,j}(b) & 1^{n,-}_{n+1,j}(b) \end{bmatrix}$$

(vi) for $1 \leq j \leq n$, $1^{n,+}_{n+1,i}(x) \circ^{n+1}_i 1^{n,-}_{n+1,i}(x) = 1^n_{n+1,i+1}(x)$ and $1^{n,+}_{n+1,i}(x) \circ^{n+1}_{i+1} 1^{n,-}_{n+1,i}(x) = 1^n_{n+1,i}(x)$

The category ∞-$\mathbb{C}$CAT of strict cubical ∞-categories is the full subcategory of ∞-$\mathbb{CM}\text{ag}_\text{r}$ spanned by strict cubical ∞-categories. A morphism in ∞-$\mathbb{C}$CAT is called a *strict cubical ∞-functor*. We study it more specifically in **??** with the perspective to weakened it and to obtain cubical model of weak ∞-functors.

2.2 The monad of cubical strict ∞-categories

In this section we describe cubical strict ∞-categories as algebras for a monad on $\mathbb{C}$Sets. We hope it to be a specific ingredient to compare globular strict ∞-categories with cubical strict ∞-categories

Consider the forgetful functor : $\infty\text{-}\mathbb{C}\text{CAT} \xrightarrow{U} \mathbb{C}\text{Sets}$ which associate to any strict cubical ∞-category its underlying cubical set and which associate to any strict cubical ∞-functor its underlying morphism of cubical sets.

Proposition 1 *The functor U is right adjoint* □

Its left adjoint is denoted F

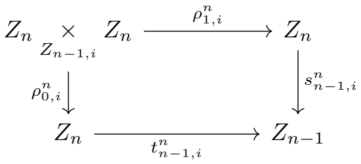
PROOF We are going to use Sketch Theory as explain in [8] : Actually it is not difficult to see that the category ∞-$\mathbb{C}$CAT and the category $\mathbb{C}$Sets are both projectively sketchable. Let us denote by $\mathcal{E}_\text{C}$ the sketch of ∞-$\mathbb{C}$CAT and $\mathcal{E}_\text{S}$ the sketch of $\mathbb{C}$Sets. Main parts of $\mathcal{E}_\text{C}$ are described just below and we see that $\mathcal{E}_\text{C}$ contains $\mathcal{E}_\text{S}$, and that this inclusion induces a forgetful functor $\infty\text{-}\mathbb{C}\text{CAT} \xrightarrow{U} \mathbb{C}\text{Sets}$ which has a left adjunction thanks to the sheafification theorem of Foltz [10]. Now we have the commutative diagram

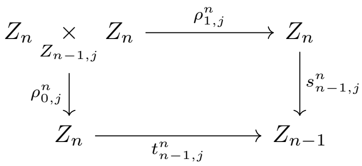
$$\begin{array}{ccc} Mod(\mathcal{E}_\text{C}) & \longrightarrow & Mod(\mathcal{E}_\text{S}) \\ {\scriptstyle iso}\downarrow & & \downarrow{\scriptstyle iso} \\ \infty\text{-}\mathbb{C}\text{CAT} & \xrightarrow{U} & \mathbb{C}\text{Sets} \end{array}$$

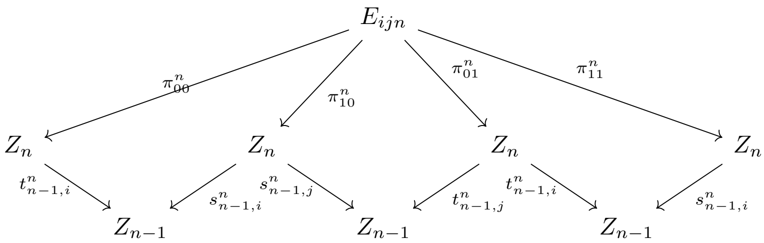
which shows that U is right adjoint.

The description of $\mathcal{E}_\text{C}$ started with the description of $\mathcal{E}_\text{M}$ in 2. We carry on to it in describing the sketch behind the interchange laws, which shall complete main parts of $\mathcal{E}_\text{C}$:

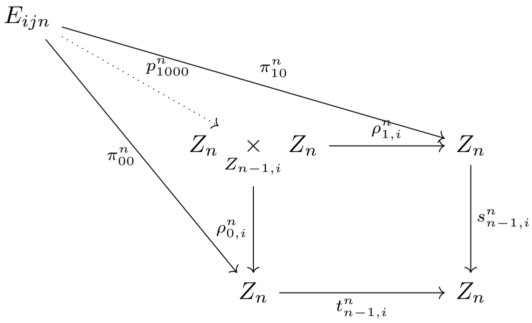
- In the first generation of $\mathcal{E}_\text{C}$ we start with three cones :

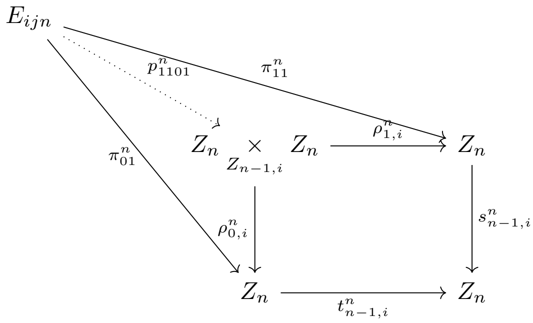


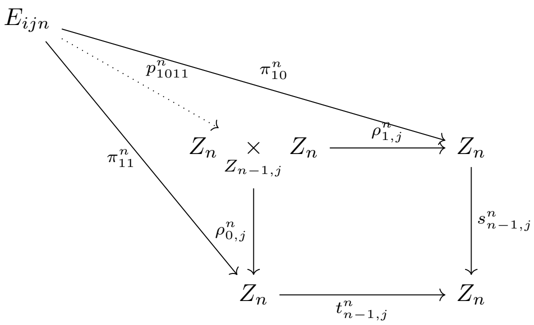


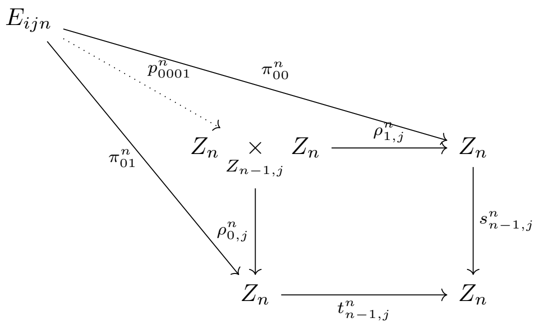


- Then we consider the following commutative diagrams :

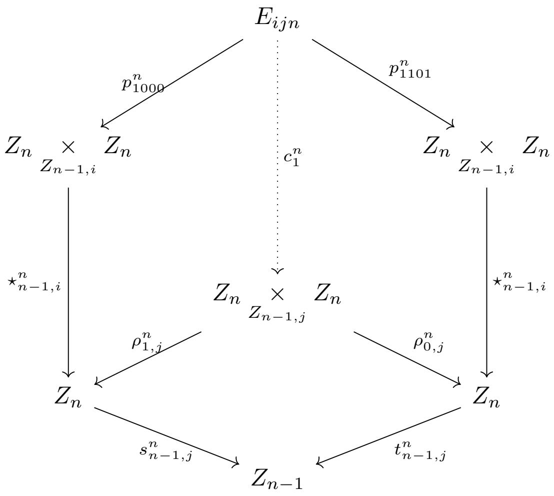


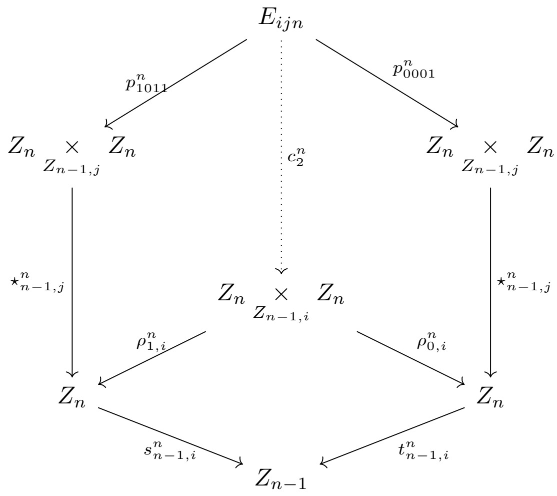




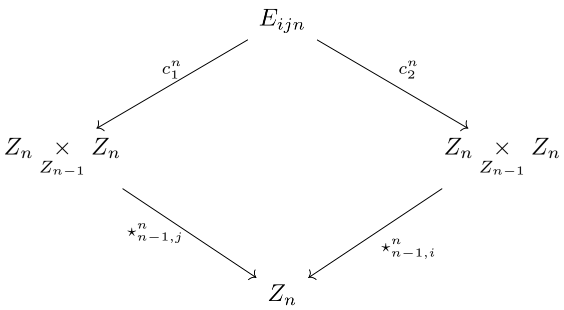


- We consider then (still in the first generation) the following two commutative diagrams :





- Finally we consider the following commutative diagram of interchange laws



■

The monad of strict cubical ∞-categories on cubical sets is denoted $\mathbb{S} = (S, \lambda, \mu)$. Here λ is the unit map of $\mathbb{S}$: $1_{\mathbb{C}\mathbb{S}\text{ets}} \xrightarrow{\lambda} S$ and μ is the multiplication of $\mathbb{S}$: $S^2 \xrightarrow{\mu} S$

3 The cubical higher operad of cubical weak ∞-categories

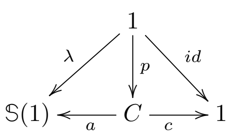
3.1 The monoidal category of cubical pointed $\mathbb{S}$-collections

In [14] we will prove that the free cubical strict ∞-categories monad $\mathbb{S} = (S, \lambda, \mu)$ on $\mathbb{C}\mathbb{S}$ets built in 2.2 is cartesian. Thanks to this cartesianess we can build the monoidal category $\mathbb{S}$-$\mathbb{C}oll_p$ of pointed $\mathbb{S}$-collections

If S is a cartesian monad on a category $\mathcal{G}$ then *S-collections* are kind of S-graphs defined in [18], where their domains of arities is an object $S(1)$ such that 1 is a terminal object of the category $\mathcal{G}$. The category of S-collections is denoted S-$\mathbb{C}oll$. The category of pointed S-collections is also defined in [18] and is denoted S-$\mathbb{C}oll_p$. In this section we accept the following result

Conjecture *The monad $\mathbb{S} = (S, \lambda, \mu)$ (see 2.2) of strict cubical ∞-categories on cubical sets is cartesian*

Thus we can work with the locally finitely presentable category $\mathbb{S}$-$\mathbb{C}oll_p$ of pointed $\mathbb{S}$-collections ($n \in \mathbb{N}$). An object of $\mathbb{S}$-$\mathbb{C}oll_p$ is denoted $(C, a, c; p)$, and described by a commutative diagram in $\mathbb{C}\mathbb{S}ets$



The category $\mathbb{S}$-$\mathbb{C}oll_p$ is monoidal and described in [18]. Monoids in it are $\mathbb{S}$-operads.

Definition 5 The category of $\mathbb{S}$-operads is given by the category of monoids of the monoidal category $\mathbb{S}\text{-}\mathbb{C}oll_p$. We denote is by $\mathbb{COper}$. □

Sometimes we shall call it *cubical operads* in order to make clear the geometry involved for this kind of higher operads.

3.2 Cubical contractions

Consider a pointed $\mathbb{S}$-collection $(C, a, c; p)$, and for each $n \geq 1$ and for all integer $k \geq 1$, we define the following subsets of $C_n \times C_n$

- $\underline{C}_n = \{(\alpha, \beta) \in C_n \times C_n : a_n(\alpha) = a_n(\beta)\}$
- $\underline{C}^s_{n,j} = \{(\alpha, \beta) \in C_n \times C_n : s^n_{n-1,j}(\alpha) = s^n_{n-1,j}(\beta) \text{ and } a_n(\alpha) = a_n(\beta)\}$
- $\underline{C}^t_{n,j} = \{(\alpha, \beta) \in C_n \times C_n : t^n_{n-1,j}(\alpha) = t^n_{n-1,j}(\beta) \text{ and } a_n(\alpha) = a_n(\beta)\}$

and also we consider $\underline{C}_0 = \{(\alpha, \beta) \in C_0 \times C_0 : \alpha = \beta\}$

Then $(C, a, c; p)$ is equipped with a *cubical contractibility structure* if they are extra structures given by maps :

$$([-;-]^n_{n+1,j} : \underline{C}_n \longrightarrow C_{n+1})_{n\in\mathbb{N};j\in\{1,\dots,n+1\}}$$

$$([-;-]^{n,-}_{n+1,j} : \underline{C}^s_{n,j} \longrightarrow C_{n+1})_{n\geq1;j\in\{1,\dots,n\}},\ ([-;-]^{n,+}_{n+1,j} : \underline{C}^t_{n,j} \longrightarrow C_{n+1})_{n\geq1;j\in\{1,\dots,n\}}$$

such that

- If $1 \leq i < j \leq n+1$, then
  $$s^{n+1}_{n,i}([\alpha,\beta]^n_{n+1,j}) = [s^n_{n-1,i}(\alpha), s^n_{n-1,i}(\beta)]^{n-1}_{n,j-1}, \text{ and } t^{n+1}_{n,i}([\alpha,\beta]^n_{n+1,j}) = [t^n_{n-1,i}(\alpha), t^n_{n-1,i}(\beta)]^{n-1}_{n,j-1}$$
- If $1 \leq j < i \leq n+1$ then
  $$s^{n+1}_{n,i}([\alpha,\beta]^n_{n+1,j}) = [s^n_{n-1,i-1}(\alpha), s^n_{n-1,i-1}(\beta)]^{n-1}_{n,j}, \text{ and } t^{n+1}_{n,i}([\alpha,\beta]^n_{n+1,j}) = [t^n_{n-1,i-1}(\alpha), t^n_{n-1,i-1}(\beta)]^{n-1}_{n,j}$$
- If $i = j$ then
  $$s^{n+1}_{n,i}([\alpha,\beta]^n_{n+1,j}) = \alpha \text{ and } t^{n+1}_{n,i}([\alpha,\beta]^n_{n+1,j}) = \beta$$
- $a_{n+1}([\alpha,\beta]^n_{n+1,j}) = 1^n_{n+1,j}(a_n(\alpha)) = 1^n_{n+1,j}(a_n(\beta))$,
- $\forall\alpha \in C_n, [\alpha,\alpha]^n_{n+1,j} = 1^n_{n+1,j}(\alpha)$.

and such that

- for $1 \leq j \leq n$ we have :
  - $s^{n+1}_{n,j}([\alpha;\beta]^{n,-}_{n+1,j}) = \alpha$ and $s^{n+1}_{n,j+1}([\alpha;\beta]^{n,-}_{n+1,j}) = \beta$
  - $t^{n+1}_{n,j}([\alpha;\beta]^{n,+}_{n+1,j}) = \alpha$ and $t^{n+1}_{n,j+1}([\alpha;\beta]^{n,-}_{n+1,j}) = \beta$
  - $s^{n+1}_{n,j}([\alpha;\beta]^{n,+}_{n+1,j}) = s^{n+1}_{n,j+1}([\alpha;\beta]^{n,+}_{n+1,j}) = [s^{n}_{n-1,j}(\alpha);s^{n}_{n-1,j}(\beta)]^{n-1}_{n,j}$
  - $t^{n+1}_{n,j}([\alpha;\beta]^{n,-}_{n+1,j}) = t^{n+1}_{n,j+1}([\alpha;\beta]^{n,-}_{n+1,j}) = [t^{n}_{n-1,j}(\alpha);t^{n}_{n-1,j}(\beta)]^{n-1}_{n,j}$
- for $1 \leq i,j \leq n+1$
  - $s^{n+1}_{n,i}([\alpha;\beta]^{n,\gamma}_{n+1,j}) = \begin{cases} [s^{n}_{n-1,i}(\alpha);s^{n}_{n-1,i}(\beta)]^{n-1,\gamma}_{n,j-1} \text{ if } 1 \leq i < j \leq n \\ [s^{n}_{n-1,i-1}(\alpha);s^{n}_{n-1,i-1}(\beta)]^{n-1,\gamma}_{n,j} \text{ if } 2 \leq j+1 < i \leq n+1 \end{cases}$
  - $t^{n+1}_{n,i}([\alpha;\beta]^{n,\gamma}_{n+1,j}) = \begin{cases} [t^{n}_{n-1,i}(\alpha);t^{n}_{n-1,i}(\beta)]^{n-1,\gamma}_{n,j-1} \text{ if } 1 \leq i < j \leq n \\ [t^{n}_{n-1,i-1}(\alpha);t^{n}_{n-1,i-1}(\beta)]^{n-1,\gamma}_{n,j} \text{ if } 2 \leq j+1 < i \leq n+1 \end{cases}$
- $a_{n+1}([\alpha;\beta]^{n,\gamma}_{n+1,j}) = 1^{n,\gamma}_{n+1,j}(a_n(\alpha)) = 1^{n,\gamma}_{n+1,j}(a_n(\beta))$
- $\forall \alpha \in C_n, [\alpha,\alpha]^{n,\gamma}_{n+1,j} = 1^{n,\gamma}_{n+1,j}(\alpha)$.

Such $\mathbb{S}$-collection $(C,a,c;p)$ is called *contractible* where its contractibilty structure is usually denoted by :

$$([-;-]^{n}_{n+1,j})_{n\in\mathbb{N};j\in\{1,\dots,n+1\}}, ([-;-]^{n,\gamma}_{n+1,j})_{n\geq 1;j\in\{1,\dots,n\};\gamma\in\{-,+\}}$$

A morphism of pointed contractible $\mathbb{S}$-collections is given by a morphism of $\mathbb{S}$-$\mathbb{C}oll_p$:
$(C,a,c;p) \xrightarrow{\quad f \quad} (C',a',c';p')$
which preserves their structures of contractibility, i.e it is given by a map :
$C \xrightarrow{\quad f \quad} C'$
such that :

$$f([\alpha;\beta]^{n}_{n+1,j}) = [f(\alpha);f(\beta)]^{n}_{n+1,j} \text{ and } f([\alpha;\beta]^{n,\gamma}_{n+1,j}) = [f(\alpha);f(\beta)]^{n,\gamma}_{n+1,j}$$

The category of pointed contractible $\mathbb{S}$-collections is denoted $\mathcal{C}\mathbb{S}$-$\mathbb{C}oll_p$.

Proposition 2 (G.M. Kelly) *Let K be a locally finitely presentable category, and $Mnd_f(K)$ the category of finitary monads on K and strict morphisms of monads. Then $Mnd_f(K)$ is itself locally finitely presentable. If T and S are object of $Mnd_f(K)$, then the coproduct $T \coprod S$ is algebraic, which means that $K^T \underset{K}{\times} K^S$ is equal to $K^{T \amalg S}$ and the diagonal of the pullback square*

$$\begin{array}{ccc} K^T \underset{K}{\times} K^S & \xrightarrow{\quad p_1 \quad} & K^S \\ \big\downarrow{\scriptstyle p_2} & & \big\downarrow{\scriptstyle U} \\ K^T & \xrightarrow[\quad V \quad]{} & K \end{array}$$

is the forgetful functor $K^{T \amalg S} \longrightarrow K$. Furthermore the projections $K^T \underset{K}{\times} K^S \longrightarrow K^T$ and $K^T \underset{K}{\times} K^S \longrightarrow K^S$ are monadic. □

But the following forgetful functors are monadic :
$\mathbb{C}\mathbb{O}\text{per} \xrightarrow{\quad U \quad} \mathbb{S}\text{-}\mathbb{C}oll_p$
$\mathcal{C}\mathbb{S}\text{-}\mathbb{C}oll_p \xrightarrow{\quad V \quad} \mathbb{S}\text{-}\mathbb{C}oll_p$

Lets denote by $\mathbb{B}$ the monad on $\mathbb{S}$-$\mathbb{C}oll_p$ which algebras are cubical higher operads, and denote by $\mathbb{T}_V$ the monad on $\mathbb{S}$-$\mathbb{C}oll_p$ which algebras are pointed contractible $\mathbb{S}$-collections. We are in the situation of the above proposition, which shows that algebras of the sum $\mathbb{B}\coprod\mathbb{T}_V$ is the following pullback in $\mathbb{C}AT$:

$$\begin{array}{ccc} \mathbb{COper} \underset{\mathbb{S}\text{-}\mathbb{C}oll_p}{\times} \mathcal{C}\mathbb{S}\text{-}\mathbb{C}oll_p & \longrightarrow & \mathcal{C}\mathbb{S}\text{-}\mathbb{C}oll_p \\ \big\downarrow & & \big\downarrow V \\ \mathbb{COper} & \underset{U}{\longrightarrow} & \mathbb{S}\text{-}\mathbb{C}oll_p \end{array}$$

This pullback is denoted $\mathcal{C}\mathbb{COper}$ for short. It is the category of contractible cubical higher operads. The left adjoint functor F of the monadic forgetful functor :

$$\mathcal{C}\mathbb{COper} \xrightarrow{W} \mathbb{S}\text{-}\mathbb{C}oll_p$$

gives free contractible cubical higher operads. In particular it gives the free contractible cubical higher operad B^0_C on the specific pointed $\mathbb{S}$-collection $(C^0,a^0,c^0;p^0)$ that we shall describe in the next section. This operad B^0_C is the cubical analogue of the operad of Michael Batanin, the one which algebras are the globular weak ∞-categories. Similarly algebras for this cubical operad B^0_C are cubical weak ∞-categories.

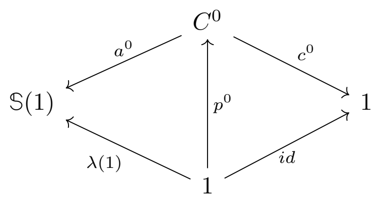
3.3 The cubical operad of cubical weak ∞-categories

Cubical pasting diagrams or *Cubical trees* are cells of the free cubical strict ∞-category $\mathbb{S}(1)$ on the terminal object 1 of $\mathbb{CS}ets$. For example $1(n)\star^n_j 1(n)$ for $j\in\{1,...,n\}$ are basic examples of cubical trees, and they are such that

$$s^n_{n-1,i}(1(n)\star^n_j 1(n)) = \begin{cases} 1(n-1)\star^{n-1}_{j-1} 1(n-1) \text{ if } 1\leq i<j\leq n \\ 1(n-1)\star^{n-1}_{j} 1(n-1) \text{ if } 1\leq j<i\leq n \end{cases}$$

Now we are going to build a specific pointed $\mathbb{S}$-collection $(C^0,a^0,c^0;p^0)$ which is the underlying pointed $\mathbb{S}$-collection of the contractible cubical higher operad B^0_C which algebras are weak cubical ∞-categories. This collection is build as follow :

- $C^0(1)$ contains a cell u_1 such that $s^1_0(u_1)=t^1_0(u_1)=u_0$, and for each integer $n\geq 2$ we have an n-cell $u_n\in C^0(n)$ which is such that : $\forall j\in\{1,...,n\}$, $s^n_{n-1,j}(u_n)=t^n_{n-1,j}(u_n)=u_{n-1}$. Arities and coarities of such cells are easy : $\forall n\in\mathbb{N}, a^0_n(u_n)=c^0_n(u_n)=1(n)$
- C^0_n contains, for all $n\geq 1$ and all $j\in\{1,...,n\}$ an n-cell μ^n_j which is such that :
  - $\begin{cases} s^n_{n-1,j}(\mu^n_j)=u_{n-1}, t^n_{n-1,j}(\mu^n_j)=u_{n-1} \\ s^n_{n-1,i}(\mu^n_j)=t^n_{n-1,i}(\mu^n_j)=\mu^{n-1}_{j-1} \text{ if } 1\leq i<j\leq n \\ s^n_{n-1,i}(\mu^n_j)=t^n_{n-1,i}(\mu^n_j)=\mu^{n-1}_{j} \text{ if } 1\leq j<i\leq n \end{cases}$
  - $a_n(\mu^n_j)=1(n)\star^n_j 1(n)$
- The pointing



  is given by $p^0_n(1(n))=u_n$.

Definition 6 The free contractible cubical higher operad B^0_C on the pointed $\mathbb{S}$-collection $(C^0,a^0,c^0;p^0)$ described just above, is the operad for cubical weak ∞-categories. Its underlying monad is denoted $(\mathbb{B}^0_C,\eta^0,\nu^0)$[1]. The category of cubical weak ∞-categories is denoted B^0_C-$\mathbb{A}$lg □

[1] We use this short notation, but the reader has to have in mind that it means in particular that the underlying cubical set of the operad B^0_C is the value of this monad on the terminal object 1 of $\mathbb{CS}$ets

Let us give simple examples of cells in B^0_C : consider the 1-cells $x = \gamma(\mu^1_0; \mu^1_0 \star^1_0 u_1)$ and $y = \gamma(\mu^1_0; u_1 \star^1_0 \mu^1_0)$. Because the couple (x, y) belongs to $\underline{C^-_{1,0}} \cap \underline{C^+_{1,0}}$ we get the following 2-cells in B^0_C :

$$\begin{array}{ccc} u_0 & \xrightarrow{\gamma(\mu^1_0;\mu^1_0\star^1_0 u_1)} & u_0 \\ {\scriptstyle [u_0,u_0]^0_1}\Big\downarrow & [x,y]^1_{2,1} & \Big\downarrow{\scriptstyle [u_0,u_0]^0_1} \\ u_0 & \xrightarrow[\gamma(\mu^1_0;u_1\star^1_0\mu^1_0)]{} & u_0 \end{array} \qquad \begin{array}{ccc} u_0 & \xrightarrow{[u_0,u_0]^0_1} & u_0 \\ {\scriptstyle \gamma(\mu^1_0;\mu^1_0\star^1_0 u_1)}\Big\downarrow & [x,y]^1_{2,2} & \Big\downarrow{\scriptstyle \gamma(\mu^1_0;u_1\star^1_0\mu^1_0)} \\ u_0 & \xrightarrow[{[u_0,u_0]^0_1}]{} & u_0 \end{array}$$

$$\begin{array}{ccc} u_0 & \xrightarrow{\gamma(\mu^1_0;\mu^1_0\star^1_0 u_1)} & u_0 \\ {\scriptstyle \gamma(\mu^1_0;u_1\star^1_0\mu^1_0)}\Big\downarrow & [x,y]^{1,-}_{2,1} & \Big\downarrow{\scriptstyle [u_0,u_0]^0_1} \\ u_0 & \xrightarrow[{[u_0,u_0]^0_1}]{} & u_0 \end{array} \qquad \begin{array}{ccc} u_0 & \xrightarrow{[u_0,u_0]^0_1} & u_0 \\ {\scriptstyle [u_0,u_0]^0_1}\Big\downarrow & [x,y]^{1,+}_{2,1} & \Big\downarrow{\scriptstyle \gamma(\mu^1_0;u_1\star^1_0\mu^1_0)} \\ u_0 & \xrightarrow[\gamma(\mu^1_0;\mu^1_0\star^1_0 u_1)]{} & u_0 \end{array}$$

These coherences show that if G is an object of $\mathbb{C}\mathbb{S}$ets, and if :

$B^0_C(G) \xrightarrow{v} G$

is a B^0_C-algebra, then for any string in $G(1)$:

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

where :

$$X = (h \circ^1_0 g) \circ^1_0 f := v(\gamma(\mu^1_0; \mu^1_0 \star^1_0 u_1); \eta^0(h) \star^1_0 \eta^0(g) \star^1_0 \eta^0(f))$$
$$Y = h \circ^1_0 (g \circ^1_0 f) := v(\gamma(\mu^1_0; u_1 \star^1_0 \mu^1_0); \eta^0(h) \star^1_0 \eta^0(g) \star^1_0 \eta^0(f))$$

the contractions of the operad B^0_C derive the following 2-cubes in G :

$$\begin{array}{ccc} a & \xrightarrow{(h\circ^1_0 g)\circ^1_0 f} & d \\ {\scriptstyle 1_a}\Big\downarrow & [X,Y]^1_{2,1} & \Big\downarrow{\scriptstyle 1_d} \\ a & \xrightarrow[h\circ^1_0(g\circ^1_0 f)]{} & d \end{array} \qquad \begin{array}{ccc} a & \xrightarrow{1_a} & a \\ {\scriptstyle (h\circ^1_0 g)\circ^1_0 f}\Big\downarrow & [X,Y]^1_{2,2} & \Big\downarrow{\scriptstyle h\circ^1_0(g\circ^1_0 f)} \\ d & \xrightarrow[1_d]{} & d \end{array}$$

$$\begin{array}{ccc} a & \xrightarrow{(h\circ^1_0 g)\circ^1_0 f} & d \\ {\scriptstyle h\circ^1_0(g\circ^1_0 f)}\Big\downarrow & [X,Y]^{1,-}_{2,1} & \Big\downarrow{\scriptstyle 1_d} \\ d & \xrightarrow[1_d]{} & d \end{array} \qquad \begin{array}{ccc} a & \xrightarrow{1_a} & d \\ {\scriptstyle 1_a}\Big\downarrow & [X,Y]^{1,+}_{2,1} & \Big\downarrow{\scriptstyle h\circ^1_0(g\circ^1_0 f)} \\ d & \xrightarrow[(h\circ^1_0 g)\circ^1_0 f]{} & d \end{array}$$

These simple examples show us how cubical weak ∞-categories provide more richness of coherences than the globular weak ∞-categories.

References

[1] Michael Batanin, *Monoidal globular categories as a natural environment for the theory of weak-n-categories*, Advances in Mathematics (1998), volume 136, pages 39–103 . 1, 2

[2] Fahd Ali Al-Agl and Ronald Brown and Richard Steiner, *Multiple Categories: The Equivalence of a Globular and a Cubical Approach*, Advances in Mathematics 170, 71–118 (2002) . 5

[3] Brown, R.,*A new higher homotopy groupoid: the fundamental globular ω-groupoid of a filtered space*, Homology, Homotopy Appl. 10 (1) (2008) 327–343. 219 .

[4] Brown, R.,*double modules?, double categories and groupoids, and a new homotopy double groupoid*, arXiv Math (0903.2627) (2009) 8 pp. .

[5] Brown, R. and Higgins, P. J.*The equivalence of ∞-groupoids and crossed complexes*, Cahiers Topologie Géom. Différentielle 22 (4) (1981) 371–386. .

[6] Brown, R. and Loday, J.-L*Homotopical excision, and Hurewicz theorems for n-cubes of spaces*, Proc. London Math. Soc. (3) 54 (1) (1987) 176–192. .

[7] Denis-Charles Cisinski, *Batanin higher groupoids and homotopy types*, Contemporary Mathematics (2007) volume 431, pages 171–186.

[8] Laurent Coppey and Christian Lair, *Leçons de théorie des esquisses*, Université Paris VII, (1985). 7

[9] Eduardo Dubuc, *Adjoint triangles*, Lecture Notes in Mathematics (Springer-Verlag 1968) volume 61, pages 69–91.

[10] F. Foltz, *Sur la catégorie des foncteurs dominés*, Cahiers de Topologie et de Géométrie Différentielle Catégorique (1969), volume 11(2), pages 101–130). 7

[11] Marco Grandis and Mauri ,*Cubical sets and their site*, Theory Applic. Categories 11 (2003) 185–201. . 2

[12] Kamel Kachour, *Définition algébrique des cellules non-strictes*, Cahiers de Topologie et de Géométrie Différentielle Catégorique, volume 1 (2008), pages 1–68.

[13] Camell Kachour, *Introduction to Higher Cubical Operads. Second Part : The Functor of Fundamental Cubical Weak ∞-Groupoids for Spaces* 2

[14] Camell Kachour and Mark Weber, *2-topos view for higher operads*, Work in progress 1, 2, 9

[15] Camell Kachour, *Algebraic definition of weak (∞, n)-categories*, Published in Theory and Applications of Categories (2015), Volume 30, No. 22, pages 775-807 3

[16] Camell Kachour, *An algebraic approach to weak ω-groupoids*, Australian Category Seminar, 14 September 2011. http://web.science.mq.edu.au/groups/coact/seminar/cgi-bin/speaker-info.cgi?name=Camell+Kachour

[17] G.M.Kelly, *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bulletin of the Australian Mathematical Society (1980), volume 22, pages 1–83.

[18] Tom Leinster, *Higher Operads, Higher Categories*, London Mathematical Society Lecture Note Series, Cambridge University Press (2004), volume 298. 9

[19] Georges Maltsiniotis,*La catégorie cubique avec connections est une catégorie test stricte*, (preprint) (2009) 1–16 . 2

[20] Timothy Porter ,*n-types of simplicial groups and crossed n-cubes*, Topology 32 (1) (1993) 5–24.

[21] Richard Steiner,*Thin fillers in the cubical nerves of omega-categories*, Theory Appl. Categ. 16 (2006) No. 8, 144–173 . 5

[22] Ross Street,*The petit topos of globular sets*, Journal of Pure and Applied Algebra. Volume 154, pages 299–315 (2000) .

[23] Mark Weber,*Yoneda structures from 2-toposes*, Applied Categorical Structures, Volume 15, pages 259–323 (2007).

[24] Mark Weber,*Operads Within Monoidal Pseudo-algebras*, Applied Categorical Structures, Volume 13(5), page 389–420 (2005) .

Camell KACHOUR
Department of Mathematics, Macquarie University
North Ryde, NSW 2109 Australia.
Phone: 00 612 9850 8942
Email:camell.kachour@gmail.com