Three Hopf algebras from number theory, physics and topology, and their common operadic, simplicial and categorical background

Imma IMMA GALVEZ-CARRILLO, Ralph M. KAUFMANN and Andy TONKS

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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THREE HOPF ALGEBRAS FROM NUMBER THEORY, PHYSICS & TOPOLOGY, AND THEIR COMMON OPERADIC, SIMPLICIAL & CATEGORICAL BACKGROUND

IMMA GÁLVEZ–CARRILLO, RALPH M. KAUFMANN, AND ANDREW TONKS

Abstract. We consider three *a priori* totally different setups for Hopf algebras from number theory, mathematical physics and algebraic topology. The primary examples are the Hopf algebras of Goncharov for multiple zeta values, that of Connes–Kreimer for renormalization, and a Hopf algebra constructed by Baues to study double loop spaces. We show that these examples can be successively unified by considering simplicial objects, cooperads with multiplication and Feynman categories at the ultimate level. These considerations open the door to new constructions and reinterpretation of known constructions in a large common framework.

Introduction

Hopf algebras have long been known to be a highly effective tool in classifying and methodologically understanding complicated structures. In this vein, we start by recalling three Hopf algebra constructions, two of which are rather famous and lie at the center of their respective fields. These are Goncharov's Hopf algebra of multiple zeta values [Gon05] whose variants lie at the heart of the recent work [Bro17], for example, and the ubiquitous Connes–Kreimer Hopf algebra of rooted forests [CK98]. The third Hopf algebra predates them but is not as well publicized: it is a Hopf algebra discovered and exploited by Baues [Bau81] to model double loop spaces. We will trace the existence of the first and third of these algebras back to a fact known to experts[1], namely that simplices form an operad. It is via this simplicial bridge that we can push the understanding of the Hopf algebra of Goncharov to a deeper level and relate it to Baues' construction which comes from an *a priori* totally different setup. Here, we prove a general theorem, that any simplicial object gives rise to bialgebra.

The Hopf algebra of Connes and Kreimer fits into this picture through a map given by contracting all the internal edges of the trees. This map also furnishes an example *par excellence* of the complications that arise in this story. A simpler example is given by restricting to the sub-Hopf algebra of three-regular trees. In this case the contraction map exhibits the corresponding Hopf algebra as a pull-back of a simplicial object. This relationship is implicit in [Gon05] and is now put into a more general framework.

We show that the essential ingredient to obtain a Hopf structure in all three examples is our notion of *cooperad with multiplication*. For the experts, we wish to point out that due

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[1] As one expert put it: "Yes this is well–known, but not to many people".

to different gradings (in the operad degree) this is neither what is known as a Hopf operad nor its dual. We prove a general theorem which states that a cooperad with multiplication always yields a bialgebra. In the general setting, these bialgebras are neither unital nor counital. While there is no problem adjoining a unit, the counit is a subtle issue in general and we discuss the conditions for their existence in detail. We show that the conditions are met in the special cases at hand. This is due to the fact that the three examples are free constructions of a cooperad with multiplication from a cooperad with a cooperadic unit. Examples of the latter are abundant and are furnished for instance by the dual of unital operads. An upshot of the more general case is that there is a natural 'depth' filtration. We furthermore elucidate the relation of the general case to the free case by proving that there is always a surjection from a free construction to the associated graded. Going further, we prove the following structural theorem: if the bialgbra has a left coalgebra counit, then it is a deformation of its associated graded and moreover this associated graded is a quotient of the free construction of its first graded piece.

Another nice result comes about by noticing that just as there are operads and pseudo-operads, there are cooperads and pseudo-cooperads. We show that these dual structures lead to bialgebras and a version of infinitesimal bialgebras. The operations corresponding to the dual of the partial compositions of pseudo-operads are then dual to the infinitesimal action of Brown. In other words they give the Lie-coalgebra structure dual to the pre-Lie structure.

Moving from the constructed bialgebras to Hopf algebras is possible under the extra condition of almost connectedness. If the cooperad satisfies this condition, which technically encompasses the existence of a split bialgebraic counit, then there is a natural quotient of the bialgebra which is connected and hence Hopf. Indeed in the three examples taking this quotient is implemented in the original constructions by assigning values to degenerate expressions.

A further level of complexity is reflected in the fact that there are several variations of the construction of the Connes–Kreimer Hopf algebra based for example on planar labelled trees, labelled trees, unlabelled trees and trees whose external legs have been "amputated" — a term common in physics. We show, in general, these correspond to non-Sigma cooperads, coinvariants of symmetric cooperads and certain colimits, which are possible in semisimplicial cooperads. This is also a natural realm in which to study coactions. Coactions have become an important tool in computing Feynman amplitudes [Bro17] and are discussed as arising naturally from several points of view.

An additional degree of understanding is provided by the insight that the underlying cooperads for the Hopf algebras of Goncharov and of Baues are given by a cosimplicial structure. This also allows us to understand the origin of the shuffle product and other relations commonly imposed in theory of multiple zeta values and motives from this angle. For the shuffle product, in the end it is as Broadhurst remarked, the product comes from the fact that we want to multiply the integrals, which are the amplitudes of connected components of disconnected graphs. In simplicial terms this translates to the compatibility of different naturally occurring free monoid constructions, in the form of the Alexander–Whitney map and a multiplication based on the relative cup product. There are more surprising direct correspondences between the extra relations, like the contractibility of a 2-skeleton used by Baues and a relation on multiple zeta values essential for the motivic coaction.

These digressions into mathematical physics bring us to the ultimate level of abstraction and source of Hopf algebras of this type: the Feynman categories of [KW17]. We show that under reasonable assumptions a Feynman category gives rise to a Hopf algebra formed by the free Abelian group of its morphisms. Here the coproduct, motivated by a discussion with D. Kreimer, is deconcatenation. With hindsight, this type of coproduct goes back at least as far as [JR79] or [Ler75], who considered a deconcatenation coproduct from a combinatorial point of view. Feynman categories are monoidal, and this monoidal structure yields a product. Although it is not true in general for any monoidal category that the multiplication and comultiplication are compatible and form a bialgebra, it is for Feynman categories, and hence also for their opposites. This also gives a new understanding for the axioms of a Feynman category. The case relevant for cooperads with multiplication is the Feynman category of finite sets and surjections and its enrichments by operads. The constructions of the bialgebra then correspond to the pointed free case considered above if the cooperad is the dual of an operad. Invoking opposite categories, one can treat cooperads directly. For this one notices that the opposite Feynman category, that for coalgebras, can be enriched by cooperads. It is here that we can also say that the two constructions of Baues and Goncharov are related by Joyal duality to the operad of surjections.

There are quotients that are obtained by "dividing out isomorphisms", which amounts to dividing out by certain coideals. This again allows us to distinguish the levels between planar, symmetric, labelled and unlabelled versions. To actually get the Hopf algebras, rather than just bialgebras, one again has to take quotients and require certain connectedness assumptions. Here the conditions become very transparent. Namely, the unit, hidden in the three examples by normalizations, will be given by the unit endomorphism of the monoidal unit $\mathbb{1}$ of the Feynman category, viz. $id_{\mathbb{1}}$. Isomorphisms keep the coalgebra from being conilpotent. Even if there are no isomorphisms, still all identities are group–like and hence the coalgebra is not connected. This explains the necessity of taking quotients of the bialgebra to obtain a Hopf algebra. We give the technical details of the two quotients, first removing isomorphisms and then identifying all identity maps.

There is also a distinction here between the non–symmetric and the symmetric case. While in the non–symmetric case, there is a Hopf structure before taking the quotient, the passing to the quotient, viz. coinvariants is necessary in the symmetric case.

These construction are more general than those of the first chapter in the sense that there are other Feynman categories besides those which yield cooperads with multiplication. One of the most interesting examples going deeper into mathematical physics is the Feynman category whose morphisms are graphs. This allows us to obtain the graph Hopf algebras of Connes and Kreimer. Going further, there are also the Hopf algebras corresponding to cyclic operads, modular operads, and new examples based on 1-PI graphs and motic graphs, which yield the new Hopf algebras of Brown [Bro17]. Here several general constructions on Feynman categories, such as enrichment, decoration, universal operations, and free construction come into play and give interrelations between the examples.

The paper is organized as follows. We begin by recalling the three Hopf algebras and their variations in §1. We give all the necessary details and add a discussion after each example indicating its position within the whole theory. In §2, we give the main definition of a cooperad with multiplication and the constructions of bialgebras and Hopf algebras. To be self-contained, we write out the relevant definitions at work in the background at each step. This paragraph also contains a discussion of the filtered and graded cases. This setup is strictly more general than the three examples, which are all of a free type that we define. We also discuss and analyze the coactions in this chapter. Given that the origin of the cooperad structure for Goncharov's and Baues' Hopf algebras is simplicial, we develop the general theory for the simplicial setting in §3. It is §4 that contains the generalization to Feynman categories. Here we realize the examples in the more general setting and give several pertinent constructions. Having the whole theory at hand, we give a detailed discussion in §5. To be self-contained the paper also has three appendices: one on graphs, one on coalgebras and Hopf algebras and one on Joyal duality. The latter is of independent interest, since this duality explains the ubiquitous occurrence of two types of formulas, those with repetition and those without repetition, in the contexts of number theory, mathematical physics and algebraic topology. This also explains the two graphical versions used in this type of calculations, polygons vs. trees, which are now just Joyal duals of each other.

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Before the final product, there have been many earlier results, which have been recorded in the arXiv versions and final publication of [KW17] and several talks starting in 2013. It was after a talk in July 2013 at the Humboldt University Berlin that discussion with D. Kreimer led us to generalize our constructions from cooperads to Feynman categories. There were numerous talks at Berlin and other places around the world including at CUNY Einstein Chair seminar in September 2013, Rutgers in October 2013, the MPI in Bonn in Feb 2014 and the "Homotopy Summer Days" in July 2014 where many details, such as the quotienting procedures to go from Bi- to Hopf algebras, were already presented. We thank the Max–Planck–Institut for Mathematics for the activity on "Higher structures in Geometry and Physics" in Spring 2016 which allowed our collaboration to finally put the first arXiv version of the preprint after having presented the results for years prior. The results have grown since then and RK thanks the HIM trimester program "Periods in Number Theory, Algebraic Geometry and Physics" in Spring 2018, in Bonn where the coaction parts and several details about the grading were written.

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1. Preface: Three Hopf Algebras

In this section, we will review the construction of the main Hopf algebras which we wish to put under one roof and generalize. After each example we will give a discussion paying special attention to their unique features.

1.1. **Multiple zeta values.** We briefly recall the setup of Goncharov's Hopf algebra of multiple zeta values. Given r natural numbers $n_1, \dots, n_{r-1} \geq 1$ and $n_r \geq 2$, one considers the real numbers

$$\zeta(n_1, \dots, n_r) := \sum_{1 \leq k_1 \leq \cdots \leq k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}} \tag{1.1}$$

The value $\zeta(2) = \pi^2/6$, for example, was calculated by Euler.

Kontsevich remarked that there is an integral representation for these, given as follows. If $\omega_0 := \frac{dz}{z}$ and $\omega_1 := \frac{dz}{1-z}$ then

$$\zeta(n_1, \dots, n_r) = \int_0^1 \omega_1 \underbrace{\omega_0 \dots \omega_0}_{n_1 - 1} \omega_1 \underbrace{\omega_0 \dots \omega_0}_{n_2 - 1} \dots \omega_1 \underbrace{\omega_0 \dots \omega_0}_{n_r - 1} \tag{1.2}$$

Here the integral is an iterated integral and the result is a real number. The *weight* of (1.2) is $N = \sum_1^r n_i$ and its *depth* is r.

Example 1.1. As was already known by Leibniz,

$$\zeta(2) = \int_0^1 \omega_1 \omega_0 = \int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1}{1-t_1} \frac{dt_2}{t_2}$$

One of the main interests is the independence over $\mathbb{Q}$ of these numbers: some relations between the values come directly from their representation as iterated integrals, see e.g. [Bro12b] for a nice summary. As we will show in Chapter 3 many of these formulas can be understood from the fact that simplices form an operad and hence simplicial objects form a cooperad.

1.1.1. **Formal symbols.** Following Goncharov, one turns the iterated integrals into formal symbols $\hat{I}(a_0; a_1, \dots, a_{n-1}; a_n)$ where the $a_i \in \{0,1\}$. That is, if w is an arbitrary word in $\{0,1\}$ then $\hat{I}(0; w; 1)$ represents the iterated integral from 0 to 1 over the product of forms according to w, so that

$$\hat{I}(0; 1, \underbrace{0, \dots, 0}_{n_1 - 1}, 1, \underbrace{0, \dots, 0}_{n_2 - 1}, \dots, 1, \underbrace{0 \dots 0}_{n_r - 1}; 1)$$

is the formal counterpart of (1.2). The weight is now the length of the word and the depth is the number of 1s. Note that the integrals (1.2) converge only for $n_r \ge 2$, but may be extended to arbitrary words using a regularization described e.g. in [Bro12b, Lemma 2.2].

1.1.2. **Goncharov's first Hopf algebra.** Taking a more abstract viewpoint, let $\mathscr{H}_G$ be the polynomial algebra on the formal symbols $\hat{I}(a; w; b)$ for elements a, b and any nonempty word w in the set $\{0,1\}$, and let

$$\hat{I}(a; \varnothing; b) = \hat{I}(a; b) = 1 \tag{1.3}$$

On $\mathscr{H}_G$ define a comultiplication Δ whose value on a polynomial generator is

$$\Delta(\hat{I}(a_0; a_1, \dots, a_{n-1}; a_n)) = \sum_{\substack{k \ge 0 \\ 0 = i_0 < i_1 < \dots < i_k = n}} \hat{I}(a_{i_0}; a_{i_1}, \dots; a_{i_k}) \otimes \hat{I}(a_{i_0}; a_{i_0+1}, \dots; a_{i_1}) \hat{I}(a_{i_1}; a_{i_1+1}, \dots; a_{i_2}) \cdots \hat{I}(a_{i_{k-1}}; a_{i_{k-1}+1}, \dots; a_{i_k}) \tag{1.4}$$

Theorem 1.2. [Gon05] *If we assign $\hat{I}(a_0; a_1, \dots, a_m; a_{m+1})$ degree m then $\mathscr{H}_G$ with the coproduct (1.4) (and the unique antipode) is a connected graded Hopf algebra.*

Remark 1.3. The fact that it is unital and connected follows from (1.3).

Remark 1.4. The letters $\{0,1\}$ are actually only pertinent insofar as to get multiple zeta values at the end; the algebraic constructions work with any finite set of letters S. For instance, if S are complex numbers, one obtains polylogarithms.

1.1.3. **Goncharov's second Hopf algebra and the version of Brown.** There are several other conditions one can impose, which are natural from the point of view of iterated integrals or multiple zeta values, by taking quotients. They are

(1) The shuffle formula

$$\hat{I}(a;a_1,\dots,a_m;b)\hat{I}(a;a_{m+1},\dots,a_{m+n};b) = \sum_{\sigma\in Ш_{m,n}} \hat{I}(a;a_{\sigma(1)},\dots,a_{\sigma(m+n)};b) \tag{1.5}$$

where $Ш_{m,n}$ is the set of (m,n)-shuffles.

(2) The path composition formula

$$\forall x\in\{0,1\}: \hat{I}(a_0;a_1,\dots,a_m;a_{m+1}) = \sum_{k=1}^{m} \hat{I}(a_0;a_1,\dots,a_k;x)\hat{I}(x;a_{k+1},\dots,a_m;a_{m+1}) \tag{1.6}$$

(3) The triviality of loops

$$\hat{I}(a;w;a)=0 \tag{1.7}$$

(4) The inversion formula

$$\hat{I}(a_0;a_1,\dots,a_n;a_{n+1}) = (-1)^n\hat{I}(a_{n+1},a_n,\dots,a_1;a_0) \tag{1.8}$$

(5) The exchange formula

$$\hat{I}(a_0;a_1,\dots,a_n;a_{n+1}) = \hat{I}(1-a_{n+1};1-a_n,\dots,1-a_1;1-a_0) \tag{1.9}$$

Here the map $a_i\mapsto 1-a_i$ interchanges 0 and 1.

(6) 2–skeleton equation

$$\hat{I}(a_0;a_1;a_2)=0 \tag{1.10}$$

Definition 1.5. $\tilde{\mathscr{H}}_G$ be the quotient of $\mathscr{H}_G$ with respect to the following homogeneous relations (1),(2),(3) and (4), let $\mathscr{H}_B$ be the quotient of $\mathscr{H}_G$ with respect to (1), (3), (4) and let $\tilde{\mathscr{H}}_B$ be the quotient by (1),(2),(4),(5) and (6).

Again one can generalize to a finite set S.

Theorem 1.6. [Gon05, Bro12a, Bro12b] *Δ and the grading descend to $\tilde{\mathscr{H}}_G$ and using the unique antipode is a graded connected Hopf algebra. Furthermore (1), (2), (3) imply (4). $\mathscr{H}_B$ and $\tilde{\mathscr{H}}_B$ are graded connected Hopf algebras as well.*

1.1.4. **Discussion.** In the theory of multiple zeta values it is essential that there are two parts to the story. The first is the motivic level. This is represented by the Hopf algebras and comodules over them. The second are the actual real numbers that are obtained through the iterated integrals. The theory is then an interplay between these two worlds, where one tries to get as much information as possible from the motivic level. This also explains the appearance of the different Hopf algebras since the evaluation in terms of iterated integrals factors through these quotients. In our setting, we will be able to explain many of the conditions naturally. The first condition (1.3) turns a naturally occurring non-connected bialgebra into a connected bialgebra and hence a Hopf algebra. The existence of the bialgebra itself follows from a more general construction stemming from cooperad structure with multiplication. One example of this is given by simplicial objects and the particular coproduct (1.4) is of this simplicial type. This way, we obtain the generalization of $\mathscr{H}_G$. Condition (1.3) is understood in the simplicial setup in Chapter 3 as the contraction of a 1-skeleton of a simplicial object. The relation (2) is actually related to a second algebra structure, the so-called path algebra structure [Gon05], which we will discuss in the future. The relation (3) is a normalization, which is natural from iterated integrals. The condition (1) is natural within the simplicial setup, coming from the Eilenberg–Zilber and Alexander–Whitney maps and interplay between two naturally occurring monoids. That is we obtain a generalization of $\mathscr{H}_B$ used in the work of Brown [Bro15, Bro12a].

The Hopf algebra $\tilde{\mathscr{H}}_B$ is used in [Bro12b]. The relation (5), in the simplicial case, can be understood in terms of orientations. Finally, the equation (6) corresponds to contracting the 2-skeleton of a simplicial object. It is intriguing that on one hand (6) is essential for the coaction [Bro16] while it is essential in a totally different context to get a model for chains on a double loopspace [Bau98], see below.

Moreover, in his proofs, Brown essentially uses operators D_r which we show to be equal to the dual of the $\circ_i$ map used in the definition of a pseudo-cooperad, see §2.8.1. There is a particular normalization issue with respect to $\zeta(2)$ which is handled in [Bro15] by regarding the Hopf comodule $\mathscr{H}_B \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta^{\mathfrak{m}}(2))$ of $\mathscr{H}_B$. The quotient by the second factor then yields the Hopf algebra above, in which the element representing $\zeta(2)$ vanishes. Natural coactions are discussed in §2.10.

1.2. Connes–Kreimer.

1.2.1. Rooted forests without tails.

We will consider graphs to be given by vertices, flags or half-edges and their incidence conditions; see Appendix A for details. There are two ways to treat graphs: either with or without tails, that is, free half-edges. In this section, we will recapitulate the original construction of Connes and Kreimer and hence use graphs without tails.

A tree is a contractible graph and a forest is a graph all of whose components are trees. A rooted tree is a tree with a marked vertex. A rooted forest is a forest with one root per tree. A rooted subtree of a rooted tree is a subtree which shares the same root.

1.2.2. Connes–Kreimer's Hopf algebra of rooted forests.

We now fix that we are talking about isomorphism classes of trees and forests. In particular, the trees in a forest will have no particular order.

Let $\mathscr{H}_{CK}$ be the free *commutative* algebra, that is, the polynomial algebra, on rooted trees, over a fixed ground commutative ground ring k. A forest is thus a monomial in trees and the empty forest $\varnothing$, which is equal to "the empty rooted tree", is the unit 1_k in k. We denote the commutative multiplication by juxtaposition and the algebra is graded by the number of vertices.

Given a rooted subtree τ_0 of a rooted tree τ, we define $\tau \setminus \tau_0$ to be the forest obtained by deleting all of the vertices of τ_0 and all of the edges incident to vertices of τ_0 from τ:

it is a rooted forest given by a collection of trees whose root is declared to be the unique vertex that has an edge in τ connecting it to τ_0.

One also says that $\tau \setminus \tau_0$ is given by an admissible cut [CK98].

Define the coproduct on rooted trees as:

$$\Delta(\tau) := \tau \otimes 1_k + 1_k \otimes \tau + \sum_{\substack{\tau_0 \text{ rooted subtree of } \tau \\ \tau_0 \neq \tau}} \tau_0 \otimes \tau \setminus \tau_0 \tag{1.11}$$

and extend it multiplicatively to forests, $\Delta(\tau_1\tau_2) = \tau_1^{(1)}\tau_2^{(1)} \otimes \tau_1^{(2)}\tau_2^{(2)}$ in Sweedler notation. One may include the first two terms in the sum by considering also $\tau_0 = \tau$ and $\tau_0 = \varnothing = 1_k$ (the empty subforest of τ), respectively, by declaring the empty forest to be a valid rooted sub–tree. In case τ_0 is empty $\tau \setminus \tau_0 = \tau$ and in case $\tau_0 = \tau$: $\tau \setminus \tau_0 = \varnothing = 1_k$.

Theorem 1.7. [CK98] *The comultiplication above together with the grading define a structure of connected graded Hopf algebra.*

Note that, since the Hopf algebra is graded and connected, the antipode is unique.

1.2.3. **Other variants.** There is a planar variant, using planar planted trees. Another variant which is important for us is the one using trees with tails. This is discussed in §2 and §5 and Appendix A. There is also a variant where one uses leaf labelled trees. For this it is easier not to pass to isomorphism classes of trees and just keep the names of all the half edges during the cutting. These will be introduced in the text, see also [Foi02b, Foi02a].

Finally there are algebras based on graphs rather than trees, which are possibly super-graded commutative by the number of edges. In this generality, we will need Feynman categories to explain the naturality of the constructions. Different variants of interest to physics and number theory are discussed in §5.

1.2.4. **Discussion.** This Hopf algebra, although similar, is more complicated than the example of Goncharov. This is basically due to three features which we would like to discuss. First, we are dealing with isomorphism classes, secondly, in the original version, there are no tails and lastly there is a sub-Hopf algebra of linear trees. Indeed the most natural bialgebra that will occur will be on planar forests with tails. To make this bialgebra into a connected Hopf algebra, one again has to take a quotient analogous to the normalization (1.3), implemented by the identification of the forests with no vertices (just tails) with the unit in k. To obtain the commutative, unlabelled case, one has to pass to coinvariants. Finally, if one wants to get rid of tails, one has to be able to 'amputate' them. This is an extra structure, which in the case of labelled trees is simply given by forgetting a tail together with its label. Taking a second colimit with respect to this forgetting construction yields the original Hopf algebra of Connes and Kreimer. The final complication is given by the Hopf subalgebra of forests of linear, i.e. trees with only binary vertices. This Hopf subalgebra is again graded and connected. In the more general setting, the connectedness will be an extra check that has to be performed. It is related to the fact that for an operad $\mathcal{O}$, $\mathcal{O}(1)$ is an algebra and dually for a cooperad $\check{\mathcal{O}}$, $\check{\mathcal{O}}(1)$ is a coalgebra, as we will explain.

If $\mathcal{O}$ or $\check{\mathcal{O}}$ is not reduced (i.e. one dimensional generated by a unit, if we are over k), then this extra complication may arise and in general leads to an extra connectedness condition.

1.3. **Baues' Hopf algebra for double loop spaces.** The basic starting point for Baues [Bau81] is a simplicial set X, from which one passes to the chain complex $C_*(X)$. It is well known that $C_*(X)$ is a coalgebra under the diagonal approximation chain map $\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$, and to this coalgebra one can apply the cobar construction: $\Omega C_*(X)$ is the free algebra on $\Sigma^{-1}C_*(X)$, with a natural differential which is immaterial to the discussion at this moment.

The theorem by Adams and Eilenberg–Moore is that if ΩX is connected then $\Omega C_*(X)$ is a model for chains on the based loop space ΩX of X. This raises the question of iterating the construction, but, unlike ΩX, which can be looped again, $\Omega C_*(X)$ is now an algebra and thus does not have an obvious cobar construction. To remedy this situation Baues introduced the following comultiplication map:

$$\Delta(x) = \sum_{\substack{k\geq 0 \\ 0=i_0<i_1<\cdots<i_k=n}} x_{(i_0,i_1,\dots,i_k)} \otimes x_{(i_0,i_0+1,\dots,i_1)}x_{(i_1,i_1+1,\dots,i_2)}\cdots x_{(i_{k-1},i_{k-1}+1,\dots,i_k)},$$

where $x \in X_n$ is an $(n-1)$-dimensional generator of $\Omega C_*(X)$, and $x_{(\alpha)}$ denotes its image under the simplicial operator specified by a monotonic sequence α.

Theorem 1.8. [Bau81] *If X has a reduced one skeleton $|X|^1 = *$, then the comultiplication, together with the free multiplication and the given grading, make $\Omega C_*(X)$ into a Hopf algebra. Furthermore if $\Omega\Omega|X|$ is connected, i.e. $|X|$ has trivial 2-skeleton, then $\Omega\Omega C_*(X)$ is a chain model for $\Omega\Omega|X|$.*

1.3.1. **Discussion.** Historically, this is actually the first of the type of Hopf algebra we are considering. With hindsight, this is in a sense the graded and noncommutative version of Goncharov and gives the Hopf algebra of Goncharov a simplicial backdrop. There are several features, which we will point out. In our approach, the existence of the diagonal (coproduct), written by hand in [Bau81], is derived from the fact that simplices form an operad. This can then be transferred to a cooperad structure on any simplicial set. Adding in the multiplication as a free product (as is done in the cobar construction), we obtain a bialgebra with our methods. The structure can actually be pushed back into the simplicial setting, rather than just living on the chains, which then explains the appearance of the shuffle products.

To obtain a Hopf algebra, we again need to identify 1 with the generators of the one skeleton. This quotient passes through the contraction of the one skeleton, where one now only has one generator. This is the equivalent to the normalization (1.3). We speculate that the choice of the *chemin droit* of Deligne can be seen as a remnant of this in further analysis. We expect that this gives an interpretation of (1.9). The condition (1.8) can be viewed as an orientation condition, which suggests to work with dihedral instead of non-Σ operads, see e.g. [KL16]. Again this will be left for the future.

Lastly, the condition (1.10) corresponds to the triviality of the 2-skeleton needed by Baues for the application to double loop spaces. At the moment, this is just an observation, but we are sure this bears deeper meaning.

2. Hopf Algebras from Cooperads with Multiplication

In this section, we give a general construction, which encompasses all the examples discussed in §1. We start by collecting together the results needed about operads, which we will later dualize to cooperads, as these are the main actors. There are many sources for further information about operads. A standard reference is [MSS02] and [Kau04] contains the essentials with figures for the relevant examples.

The construction is more general than we would need for the examples, which all correspond to a free non-connected construction on the dual of an operad, where the free construction furnishes the compatible multiplication. As such they carry additional structure, such as a double grading. These gradings reduce to filtrations in the general case. Another complication is the existence of units and counits. We can prove a structure theorem saying that if the units and counits exist, then we are dealing with a deformation of a quotient of the free connected construction on a cooperad.

2.1. **Recollections on operads.**

2.1.1. **Non-Σ pseudo-operads.** Loosely an operad is a collection of "somethings" with n inputs and one output, like functions of several variables. And just like for functions there are permutations of variables and substitution operations. To make things concrete: consider the category $g\mathcal{A}b$ of graded Abelian groups with the tensor product $\otimes_{\mathbb{Z}}$. This is a symmetric monoidal category, if one adds the so-called associativity constraints $(G\otimes H)\otimes K \xrightarrow{\cong} G\otimes(H\otimes K) : (g\otimes h)\otimes k \mapsto g\otimes(h\otimes k)$ and the commutativity $g\otimes h \mapsto (-1)^{|g||h|}h\otimes g$, where $|g|$ is the degree of g. *A non-Σ pseudo-operad* in this category is given by a collection $\mathcal{O}(n)$ of graded Abelian groups, together with structure maps

$$\circ_i : \mathcal{O}(k) \otimes \mathcal{O}(m) \to \mathcal{O}(k+m-1) \text{ for } 1 \leq i \leq k \tag{2.1}$$

which are associative in the appropriate sense,

$$(- \circ_i -) \circ_j - = \begin{cases} - \circ_i (- \circ_{j-i+1} -) & \text{if } i \leq j < m+i \\ ((- \circ_j -) \circ_{i+n-1} -)\pi & \text{if } 1 \leq j < i. \end{cases}$$

Here $\pi = (23) : \mathcal{O}(k) \otimes \mathcal{O}(m) \otimes \mathcal{O}(n) \cong \mathcal{O}(k) \otimes \mathcal{O}(n) \otimes \mathcal{O}(m)$.

We call $\mathcal{O}$ connected if $\mathcal{O}(1)$ is $\mathbb{Z}$ or in general the unit of the monoidal category.

2.1.2. **Pseudo-operads.** If we add the condition that each $\mathcal{O}(n)$ has an action of the symmetric group $\mathbb{S}_n$ and that the $\circ_i$ are equivariant with respect to the symmetric group actions in the appropriate sense, we arrive at the definition of a pseudo-operad.

Example 2.1. As previously mentioned, the most instructive example is that of multivariate functions, given by the collection $\{End(X)(n) = Hom(X^{\otimes n}, X)\}$. The $\circ_i$ act as substitutions, that is, $f_1 \circ_i f_2$ substitutes the function f_2 into the ith variable of f_1. The symmetric group action permutes the variables. The equivariance then states that it does

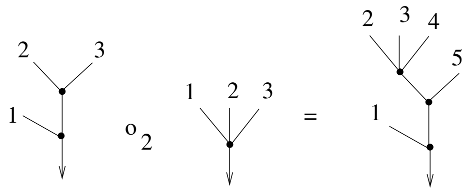


FIGURE 1. Grafting trees with labelled leaves. The tree is grafted onto the leaf number 2.

not matter if one permutes first and then substitutes or the other way around, provided that one uses the correct permutation. If one takes X to be a set or a compactly generated Hausdorff space $\otimes$ stands for the Cartesian product. If X is a vector space over k, then $\otimes$ is the tensor product over k and the functions are multilinear.

Remark 2.2. The only thing we needed in the definitions is that the underlying category is symmetric monoidal, in particular there is a monoidal, aka. tensor, product. We obviously need monoidality to write down the structure morphisms. In the axioms, we need to consider the switching and re-bracketing of factors, i.e. the symmetric monoidal structure. The other categories we will consider are $\mathcal{S}et$ with $\sqcup$, $\mathcal{V}ect_k$ with $\otimes_k$. If one works with Feynman categories, one does not need the symmetric monoidal structure in the non-symmetric case. The associativity is then associativity of morphisms.

2.1.3. **The three main examples.** Here we give the main examples which underlie the three Hopf algebras above. Notice that not all of them directly live in $\mathcal{A}b$ or $\mathcal{V}ect_k$, but for instance live in $\mathcal{S}et$. There are then free functors, which allow one to carry these over to $\mathcal{A}b$ or $\mathcal{V}ect_k$ as needed.

Example 2.3. The operad of leaf-labelled rooted trees. We consider the set of rooted trees with n-labelled leaves, which means a bijection is specified between the set of leaves and $\{1, \dots, n\}$. Given a n-labelled tree τ and an m-labelled tree τ', we define an $(m+n-1)$-labelled tree $\tau \circ_i \tau'$ by grafting the root of τ' onto the ith leaf of τ to form a new edge. The root of the tree is taken to be the root of τ and the labelling first enumerates the first $i-1$ leaves of τ, then the leaves of τ' and finally the remaining leaves of τ, see Figure 1.

The action of $\mathbb{S}_n$ is given by permuting the labels.

There are several interesting suboperads, such as that of trees whose vertices all have valence k. Especially interesting are the cases $k = 2$ and 3: also known as the linear and the binary trees respectively. Also of interest are the trees whose vertices have valence at least 3.

Example 2.4. The non-Σ operad of (unlabelled) planar planted trees. A planar planted tree is a planar rooted tree with a linear order at the root. Planar means that there is a cyclic order for the flags at each vertex. Adding a root promotes the cyclic order at all of the non-root vertices to a linear order, the flag in the direction of the root being the first element. For the root vertex itself, there is no canonical choice for a first vertex, and planting makes a choice for first flag, which sometimes called the root flag. With these

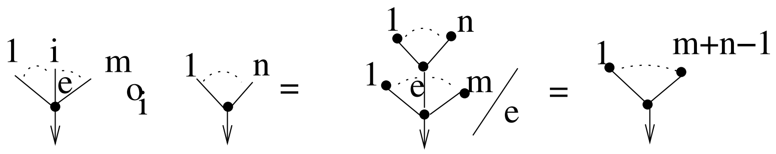


FIGURE 2. Grafting of corollas as first grafting trees and then contracting the new edge.

choices, there is a linear order on all the flags and in particular there is a linear order to all the leaves. Thus, we do not have to give them extra labels for the gluing: there is an unambiguous i-th leaf for each planar planted tree with $\geq i$ leaves, and $\tau \circ_i \tau'$ is the tree obtained by grafting the root flag of τ' onto that i-th leaf.

The suboperads above given by restricting the valency exist as well.

Example 2.5. The operad of surjections, also known as planar labelled corollas, or just the associative operad. Consider n-labelled planar corollas, that is, rooted trees with one vertex. For an n-labelled corolla τ and an m-labelled corolla τ' define $\tau \circ_i \tau'$ to the the $(n+m-1)$-labelled planar corolla with the same relabelling scheme as in example 2.3 above. This can be thought of as the gluing on labelled trees followed by the edge contraction of the new edge, see Figure 2.

Alternatively we can think of such a corolla as the unique map of ordered sets from the set $\underline{n} = \{1, \dots, n\}$, with the order given by the planar structure, to the one element set $\underline{1} = \{1\}$. The composition of the maps is now just given by using the composition of the orders according to the labelling scheme above. That is splicing in the orders.

The $\mathbb{S}_n$ action permutes the labels and acts effectively on the possible orders. There is the non-Σ version, in which case we are dealing with unlabelled planar corollas. This is then the non–Σ operad of order preserving surjections of the sets $\underline{n}$ with the natural order.

Example 2.6. Simplices form a non-Σ operad (see also Proposition 3.3 for another dual operad structure). We consider $[n]$ to be the category with $n+1$ objects $\{0, \dots, n\}$ and morphisms generated by the chain $0 \to 1 \to \cdots \to n$. The i–th composition of $[m]$ and $[n]$ is given by the following functor $\circ_i : [m] \sqcup [n] \to [m+n-1]$. On objects of $[m] : \circ_i(l) = l$ for $l < i$ and $\circ_i(l) = l+n-1$ for $l \geq i$. On objects of $[n] : \circ_i(l) = i-1+l$. Finally on morphisms: the morphism $l-1 \to l$ of $[m]$ is sent to the morphism $l-1 \to l$ of $[m+n-1]$ for all $l < i$, the morphism $i-1 \to i$ of $[m]$ is sent to the composition of $i-1 \to i \cdots \to i+n-1$ in $[m+n-1]$, the morphism $l-1 \to l$ of $[m]$ to $l+n-1 \to l+n$ of $[m+n-1]$ for $l > i$ and finally sends the morphism $k \to k+1$ of $[n]$ to $k+i \to k+1+i$.

In words, one splices the chain $[n]$ into $[m]$ by replacing the i-th link, see Figure 3. This is of course intimately related to the previous discussion of order preserving surjections. In fact the two are related by Joyal duality as we will explain in §3 and Appendix C.

2.1.4. **The $\circ$-product aka. pre-Lie structure.** One important structure going back to Gerstenhaber [Ger64] is the following bilinear map:

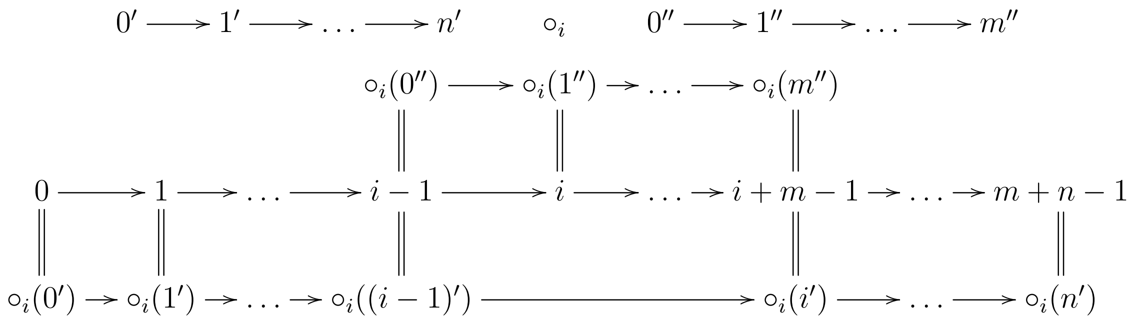


FIGURE 3. Splicing together simplices. Primes and double primes are mnemonics only

$$a \circ b := \sum_{i=1}^{n} a \circ_i b \text{ if } a \text{ has operad degree } n \tag{2.2}$$

This product is neither commutative nor associative but preLie, which means that it satisfies the equation $(a \circ b) \circ c - a \circ (b \circ c) = (a \circ c) \circ b - a \circ (c \circ b)$.

An important consequence is that $[a, b] = a \circ b - b \circ a$ is a Lie bracket.

Remark 2.7. One often shifts degrees as in the cobar construction, such that $\mathcal{O}(n)$ obtains degree $n-1$ and the operation obtains degree 1, see [KWZ12] for a full discussion

$$a \circ b := \sum_{i=1}^{n} (-1)^{(i-1)(n-1)} a \circ_i b \text{ if } a \text{ has operad degree } n \tag{2.3}$$

The algebra is graded pre–Lie [Ger64] and the commutator is odd Lie.

2.1.5. **(Non-Σ) Operads: γ.** Another almost equivalent way to encode the above data is as follows. A non–Σ operad is a collection $\mathcal{O}(n)$ together with structure maps

$$\gamma = \gamma_{n_1,\dots,n_k} : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \dots \otimes \mathcal{O}(n_k) \to \mathcal{O}(\sum_{i=1}^{k} n_i) \tag{2.4}$$

Such that map γ is associative in the sense that if $(n_1, \dots, n_k)$ is a partition of n, and $(n^i_1, \dots, n^i_{l_i})$ are partitions of the n_i, $l = \sum_{i=1}^{k} l_i$ then

$$\gamma_{n_1,\dots,n_k} \circ id \otimes \gamma_{n^1_1,\dots,n^1_{l_1}} \otimes \gamma_{n^2_1,\dots,n^2_{l_2}} \otimes \dots \otimes \gamma_{n^k_1,\dots,n^k_{l_k}} = \gamma_{n^1_1,\dots,n^1_{l_1},n^2_1,\dots,n^2_{l_2},\dots,n^k_1,\dots,n^k_{l_k}} \circ \gamma_{l_1,\dots,l_k} \otimes id^{\otimes l} \circ \pi \tag{2.5}$$

as maps $\mathcal{O}(k) \otimes \bigotimes_{i=1}^{k} (\mathcal{O}(l_i) \otimes \bigotimes_{j=1}^{l_i} \mathcal{O}(n^j_i)) \to \mathcal{O}(n)$, where π permutes the factors of the $\mathcal{O}(l_i)$ to the right of $\mathcal{O}(k)$. Notice that we chose to index the operad maps, since this will make the operations easier to dualize. The source and target of the map are then determined by the length k of the index, the indices n_i and their sum.

For an operad one adds the data of an $\mathbb{S}_n$ action on each $\mathcal{O}(n)$ and demands that the map γ is equivariant, again in the appropriate sense, see Example 2.1 or [MSS02, Kau04].

2.1.6. **Morphisms.** Morphisms of (pseudo)–operads $\mathcal{O}$ and $\mathcal{P}$ are given by a family of morphisms $f_n : \mathcal{O}(n) \to \mathcal{P}(n)$ that commute with the structure maps. E.g. $f_n(a) \circ_i^{\mathcal{P}} f_m(b) = f_{n+m-1}(a \circ_i b)$. If there are symmetric group actions, then the maps f_n should be $\mathbb{S}_n$ equivariant.

Example 2.8. If we consider the operad of rooted leaf labelled trees $\mathcal{O}$ there is a natural map to the operad of corollas $\mathcal{P}$ given by $\tau \mapsto \tau/E(\tau)$, where $\tau/E(\tau)$ is the corolla that results from contracting all edges of τ. This works in the planar and non–planar version as well as in the pseudo-operad setting, the operad setting and the symmetric setting. This map contracts all linear trees and identifies them with the unit corolla. Furthermore, it restricts to operad maps for the suboperads of k-regular or at least k-valent trees.

An example of interest considered in [Gon05] is the map restricted to planar planted 3-regular tress (sometimes called binary). The kernel of this map is the operadic ideal generated by the associativity equation between the two possible planar planted binary trees with three leaves.

2.1.7. **Units.** The two notions of pseudo-operads and operads become equivalent if one adds a unit.

Definition 2.9. A *unit* for a pseudo-operad is an element $u \in \mathcal{O}(1)$ such that $u \circ_1 b = b$ and $b \circ_i u = b$ for all m, for all $1 \le i \le m$ and all $b \in \mathcal{O}(m)$.

A unit for an operad is an element $u \in \mathcal{O}(1)$ such that

$$\gamma(u, a) = a \text{ and } \gamma(a; u, \dots, u) = a \tag{2.6}$$

There is an equivalence of categories between unital pseudo–operads and unital operads. It is given by the following formulas;

$$a \circ_i b = \gamma(a; u, \dots, u, b, u, \dots, u) \ b \text{ in the } i\text{–th place} \tag{2.7}$$

and vice–versa:

$$\gamma(a; b_1, \dots, b_k) = (\dots((a \circ_k b_k) \circ_{k-1} b_{k-1}) \dots) \circ_1 b_1) \tag{2.8}$$

Morphisms for (pseudo)–operads with units should preserve the unit.

Remark 2.10. The component $\mathcal{O}(1)$ always forms an algebra via $\gamma : \mathcal{O}(1) \otimes \mathcal{O}(1) \to \mathcal{O}(1)$. If there is an operadic unit, then this algebra is unital.

2.2. From non-Σ cooperads to bialgebras.

2.2.1. **Non-Σ cooperads $\check{\gamma}$.** Dualizing the equation for γ, we obtain the notion of a cooperad. That is, there are structure maps for all m, k and partitions $(n_1, \dots, n_k)$ of m,

$$\check{\gamma}_{n_1,\dots,n_k} : \check{\mathcal{O}}(m) \to \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n_1) \otimes \dots \otimes \check{\mathcal{O}}(n_k) \tag{2.9}$$

which satisfy the dual relations. That is,

$$id \otimes \check{\gamma}_{n^1_1,\dots,n^1_{l_1}} \otimes \check{\gamma}_{n^2_1,\dots,n^2_{l_2}} \otimes \cdots \otimes \check{\gamma}_{n^k_1,\dots,n^k_{l_k}} \circ \check{\gamma}_{n_1,\dots,n_k} = \pi \circ \check{\gamma}_{l_1,\dots,l_k} \otimes id^{\otimes l} \circ \check{\gamma}_{n^1_1,\dots,n^1_{l_1},n^2_1,\dots,n^2_{l_2},\dots,n^k_1,\dots,n^k_{l_k}} \tag{2.10}$$

as maps $\check{\mathcal{O}}(n) \to \check{\mathcal{O}}(k) \otimes \bigotimes_{i=1}^k (\check{\mathcal{O}}(l_i) \otimes \bigotimes_{j=1}^{l_i} \check{\mathcal{O}}(n_i^j))$, for any k-partition $(n_1,\dots,n_k)$ of n and l_i-partitions $(n^i_1,\dots,n^i_{l_i})$ of n_i. Either side of the relation determines these partitions and hence determines the other side. Here $l = \sum l_i$ and π is the permutation permuting the factors $\check{\mathcal{O}}(l_i)$ to the left of the factors $\check{\mathcal{O}}(n_i^j)$.

2.2.2. **Morphisms.** Morphisms of cooperads $\check{\mathcal{O}}$ and $\check{\mathcal{P}}$ are given by a family of morphisms $f_n : \check{\mathcal{O}}(n) \to \check{\mathcal{P}}(n)$ that commute with the structure maps

$$\check{\gamma}^{\check{\mathcal{P}}}_{n_1,\dots,n_k} \circ f_n = (f_k \otimes f_{n_1} \otimes \cdots \otimes f_{n_k}) \circ \check{\gamma}^{\check{\mathcal{O}}}_{n_1,\dots,n_k}$$

Remark 2.11. If the monoidal category in which the cooperad lives is complete and certain limits (in particular, products) commute with taking tensors, then we can define

$$\check{\gamma} : \check{\mathcal{O}}(m) \to \lim_k \lim_{(n_1,\dots,n_k):\sum_{i=1}^k n_i = m} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k). \tag{2.11}$$

Definition 2.12. A *non–Σ cooperad with multiplication* μ is a non–Σ cooperad $(\check{\mathcal{O}}, \check{\gamma})$ together with a family of maps, $n, m \geq 0$,

$$\mu_{n,m} : \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(m) \to \check{\mathcal{O}}(n+m),$$

which satisfy the following compatibility equations:

(1) For any $n, n' \geq 1$ and partitions $m_1 + \cdots + m_k = n$ and $m'_1 + \cdots + m'_{k'} = n'$, write $\check{\gamma}$ and $\check{\gamma}'$ for $\check{\gamma}_{m_1,\dots,m_k}$ and $\check{\gamma}_{m'_1,\dots,m'_{k'}}$ respectively, and write $\check{\gamma}''$ for $\check{\gamma}_{m_1,\dots,m_k,m'_1,\dots,m'_{k'}}$. Then the following diagram commutes

$$\begin{array}{ccc}
\check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(n') & \xrightarrow{\pi\,(\check{\gamma}\otimes\check{\gamma}')} & \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(k') \otimes \bigotimes_{r=1}^{k} \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'}) \\
\downarrow{\scriptstyle \mu_{n,n'}} & & \downarrow{\scriptstyle \mu_{k,k'}\otimes \mathrm{id}} \\
\check{\mathcal{O}}(n+n') & \xrightarrow{\check{\gamma}''} & \check{\mathcal{O}}(k+k') \otimes \bigotimes_{r=1}^{k} \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'})
\end{array} \tag{2.12}$$

Here π is the isomorphism which permutes the $k + k' + 2$ tensor factors according to the $(k+1)$-cycle $(2\,3\,\dots\,k+2)$.

(2) If $m''_1 + \cdots + m''_{k''} = n + n'$ is a partition of $n + n'$ which does not arise as the concatenation of a partition of n and a partition of n' (that is, there is no k such that $m''_1 + \cdots + m''_k = n$ and $m''_{k+1} + \cdots + m''_{k''} = n'$) then the composite

$$\check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(n') \xrightarrow{\mu_{n,n'}} \check{\mathcal{O}}(n+n') \xrightarrow{\check{\gamma}_{m''_1,\dots,m''_{k''}}} \check{\mathcal{O}}(k'') \otimes \bigotimes_{r''=1}^{k''} \check{\mathcal{O}}(m''_{r''})$$

is zero.

Under the completeness assumption, the $\mu_{n,m}$ assemble into a map μ satisfying the compatibility relation

$$\check{\gamma}(\mu(a \otimes b)) = \mu(\pi(\check{\gamma}(a) \otimes \check{\gamma}(b))) \tag{2.13}$$

where π is the permutation that permutes the first factor of $\check{\gamma}(b)$ next to the first factor of $\check{\gamma}(a)$.

A *morphism of cooperads with multiplication* $f : \check{\mathcal{O}} \to \check{\mathcal{P}}$ is a morphism of cooperads which commutes with the multiplication, $f_{m+n}\mu_{n,m} = \mu_{n,m}(f_n \otimes f_m)$.

Assumption 2.13. In order to simplify the situation, we will make the following assumptions. There is no $\check{\mathcal{O}}(0)$. This means that there are only finitely many maps and the limits reduce to finite limits.

In order to write down the multiplication and the comultiplication, we will need to take products over all $\mathcal{O}(n)$ and identify them with coproducts. Since the main applications of the Hopf algebras lie in the abelian monoidal categories of (graded) vector spaces k-$Vect$, differential graded vector spaces dg-Vect, abelian groups Ab, or gAb graded Abelian groups, we will thus assume:

Assumption 2.14. We will further assume that we are in abelian monoidal categories whose biproduct distributes over tensors. and use $\bigoplus$ for the biproduct.

Theorem 2.15. *Let $\check{\mathcal{O}}$ be a cooperad with compatible multiplication μ in an abelian symmetric monoidal category with unit $\mathbb{1}$. Then*

$$\mathscr{B} := \bigoplus_n \check{\mathcal{O}}(n)$$

is a (non-unital, non-counital) bialgebra, with multiplication μ, and comultiplication Δ given by $(\mathrm{id} \otimes \mu)\check{\gamma}$*:*

$$\check{\mathcal{O}}(n) \xrightarrow{\check{\gamma}} \bigoplus_{\substack{k \geq 1, \\ n = m_1 + \cdots + m_k}} \left(\check{\mathcal{O}}(k) \otimes \bigotimes_{r=1}^{k} \check{\mathcal{O}}(m_r) \right) \xrightarrow{\mathrm{id} \otimes \mu} \bigoplus_{k \geq 1} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n). \qquad \Delta := (\mathrm{id} \otimes \mu)\check{\gamma} \tag{2.14}$$

Morphisms of cooperads with comultiplication induce homomorphisms of bialgebras.

Proof. The multiplication μ is associative by definition. The compatibility of μ with $\check{\gamma}$, together with the associativity of μ, shows that μ is a morphism of coalgebras, $\Delta\mu =$

$(\mu \otimes \mu)\pi(\Delta \otimes \Delta)$:

$$
\begin{array}{ccccc}
\check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(n') & \xrightarrow{\pi\,\check{\gamma}^{\otimes 2}} & \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(k') \otimes \bigotimes_{r=1}^{k} \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'}) & \xrightarrow{\mathrm{id}\otimes\mu\otimes\mu} & \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(k') \otimes \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(n') \\
\downarrow{\scriptstyle \mu_{n,n'}} & \text{compatibility} & \downarrow{\scriptstyle \mu_{k,k'}\otimes \mathrm{id}} & \text{associativity} & \downarrow{\scriptstyle \mu_{k,k'}\otimes\mu_{n,n'}} \\
\check{\mathcal{O}}(n+n') & \xrightarrow{\check{\gamma}} & \check{\mathcal{O}}(k+k') \otimes \bigotimes_{r=1}^{k} \check{\mathcal{O}}(m_r) \otimes \bigotimes_{r'=1}^{k'} \check{\mathcal{O}}(m'_{r'}) & \xrightarrow{\mathrm{id}\otimes\mu} & \check{\mathcal{O}}(k+k') \otimes \check{\mathcal{O}}(n+n').
\end{array}
$$

For the coassociativity, we notice that Δ just like $\check{\gamma}$ can be written in components $\Delta = \sum_n \Delta_n = \sum_n \sum_k \Delta_{k,n}$ with $\Delta_{k,n} : \check{\mathcal{O}}(n) \to \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n)$ and these can be decomposed further as $\Delta_{k,n} = \sum_{(n_1,\dots,n_k):\sum n_i = n} \Delta_{n_1,\dots,n_k}$ with $\Delta_{n_1,\dots,n_k} = (id \otimes \mu^{\otimes k-1}) \circ \check{\gamma}_{n_1,\dots,n_k}$.

One now has to prove that $(id \otimes \Delta_{l,n})\Delta_{k,n} = (\Delta_{k,l} \otimes id)\Delta_{l,n} : \check{\mathcal{O}}(n) \to \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(l) \otimes \check{\mathcal{O}}(n)$, which can be done term by term using (2.10) and (2.12).

Explicitly fix a k–partition $n_1, \dots n_k$ of n an l partition $(m_1, \dots, m_l)$ of n. by compatibility the left hand side vanishes unless $(m_1, \dots, m_l)$ naturally decomposes into the list $(n^1_1, \dots, n^1_{l_1}, n^2_1, \dots, n^2_{l_2}, \dots, n^k_1, \dots, n^k_{l_k})$ where n^i_j is a partition of n_i. This yields the k partition $(l_1, \dots l_k)$ of l. Starting on the rhs that is with $(m_1, \dots, m_l)$ and $(l_1, \dots l_k)$, we decompose the list $(m_1, \dots, m_l)$ as above, which determines the $n_i = \sum_j n^i_j$. The proof is then:

$$
\begin{aligned}
&(id \otimes \Delta_{m_1,\dots,m_l})\Delta_{n_1,\dots n_k} = (id \otimes id \otimes \mu^{l-1})(id \otimes [\check{\gamma}_{m_1,\dots,m_l} \circ \mu^{k-1}]) \circ \check{\gamma}_{n_1,\dots,n_k} \\
&= (id \otimes id \otimes \mu^{l-1})(id \otimes \mu^{k-1} \otimes id^{\otimes l}) \circ \pi \circ (id \otimes \check{\gamma}_{n^1_1,\dots,n^1_{l_1}} \otimes \check{\gamma}_{n^2_1,\dots,n^2_{l_2}} \otimes \dots \otimes \check{\gamma}_{n^k_1,\dots,n^k_{l_k}}) \circ \check{\gamma}_{n_1,\dots,n_k} \\
&= (id \otimes \mu^{k-1} \otimes id)(id \otimes id^{\otimes k} \otimes \mu^{l-1})(\check{\gamma}_{l_1,\dots,l_k} \otimes id^{\otimes l})\check{\gamma}_{n^1_1,\dots,n^1_{l_1},n^2_1,\dots,n^2_{l_2},\dots,n^k_1,\dots,n^k_{l_k}} \\
&= [([id \otimes \mu^{k-1}]\check{\gamma}_{l_1,\dots,l_k}) \otimes id](id \otimes \mu^{l-1})\check{\gamma}_{m_1,\dots,m_l} = (\Delta_{l_1,\dots,l_k} \otimes id)\Delta_{m_1,\dots,m_l} \qquad (2.15)
\end{aligned}
$$

where π is the permutation that shuffles all the right factors next to each other as before. □

2.2.3. **Examples from a free construction.** In this section, we provide a large class of examples of the structure above. We show that for any cooperad, there exists a non-connected version, which is a cooperad with multiplication and hence furnishes a bialgebra as above. For finiteness, we assume that there is no cooperadic degree 0 part, as above.

Cooperads themselves can be obtained by dualizing operads. Namely, starting with a non–Σ operad $\mathcal{O}$ and let $\check{\mathcal{O}}$ be its linear dual, that is assuming the existence of internal homs, set $\check{\mathcal{O}}(n) = (\mathcal{O}(n))^\vee = Hom(\mathcal{O}(n), \mathbb{1})$. In particular, we can use the examples from 2.1.3. In order to transport *Set* cooperads with multiplication to Abelian categories, we can take the free construction, adjoint to the forgetful functor [Kel82]. Similarly, we can induce cooperads in different categories, by extending coefficients, say from $\mathbb{Z}$ to $\mathbb{Q}$, and other free constructions.

Construction 2.16. Let $\check{\mathcal{O}}$ be a non-Σ cooperad. Consider

$$\check{\mathcal{O}}^{nc}(n) := \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} \check{\mathcal{O}}(n_1)\otimes\cdots\otimes\check{\mathcal{O}}(n_k) \tag{2.16}$$

and define μ to be the concatenation of tensors: $\mu(a,b) = a\otimes b$. This means that $\mathscr{B} = \bigoplus_n \check{\mathcal{O}}^{nc}(n)$ is the tensor algebra on $\check{\mathcal{O}} := \bigoplus_n \check{\mathcal{O}}(n)$. The collection $\check{\mathcal{O}}^{nc}(n)$ is a non-Σ cooperad, by using (2.12) to extend $\check{\gamma}$ from $\check{\mathcal{O}}$ to its free tensor algebra $\mathscr{B}$.

$$\mathscr{B} = \bigoplus_n \check{\mathcal{O}}^{nc}(n) = \bigoplus_{n,k} \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} \check{\mathcal{O}}(n_1)\otimes\cdots\otimes\check{\mathcal{O}}(n_k) \tag{2.17}$$

Since $\check{\mathcal{O}}^{nc}$ as a cooperad with multiplication satisfies the conditions of Theorem 2.15, we obtain:

Proposition 2.17. *$\mathscr{B}$, as defined in (2.17), with tensor multiplication and the associated Δ is a (non-unital, non-counital) bialgebra and this association is functorial. If p is the word length grading and n is the operad degree grading, then the bialgebra is a graded bialgebra with respect to $n-p$.*

Proof. It is clear from the construction that $\check{\mathcal{O}}^{nc}$ is a cooperad with multiplication. It is also straightforward that any map $\check{\mathcal{O}} \to \check{\mathcal{P}}$ of cooperads induces a map $\check{\mathcal{O}}^{nc} \to \check{\mathcal{P}}^{nc}$ of cooperads with multiplication and hence bialgebras. For the grading, we compute as follows. For the multiplication we obtain $n_1+n_2-p_1-p_2$ as the degree for both sides. For the coproduct, we can restrict to the case of word legth one. Then the lhs. has degree $n-1$ while the kth term on the rhs. has degree $k-1+\sum_{i=1}^k (n_i-1) = n-1$, since $\sum_i n_i = n$. □

Remark 2.18.

(1) This type of non-connected version of (co)–operads is one of the variations for non-connected operads studied in detail in [KWZ12].
(2) This type of example is also the type of example that comes from the enriched Feynman categories $\mathfrak{F}_{\mathcal{O}}$, see [KW17] and §4.
(3) This example has the several extra properties not present in the general situation. There is an induced double grading by length of the tensor word and cooperadic degree. In general, as we show below, there will just be a depth filtration replacing the tensor length. Furthermore the bialgebra is generated by $\check{\mathcal{O}}$ as an algebra, that is words of length one. Some of these additional properties will be reappear as necessary conditions to construct units, counits and an antipode on a suitable quotient.

Remark 2.19. Our main examples of operads of §2.1.3 all define bialgebras by first taking their duals and then performing the free construction. Notice, they are all unital pseudo–operads and hence equivalently are unital operads. Notice that there is always a grading by operad degree, so that we are considering graded duals.

Operad maps between them induce maps of bialgebras going in the other direction, since we are taking duals.

Example 2.20. In the example of the operad of leaf–labelled trees. Taking the duals, we view each tree as the characteristic function of itself, $\tau \leftrightarrow \delta_\tau$ where $\delta_\tau(\tau) = 1$ and $\delta_\tau(\tau') = 0$ for all $\tau' \neq \tau$. Taking the tensor algebra corresponds to regarding ordered forests. Looking at forests, we see that the word length p is the number of roots, while the operad degree n is the number of non root tails aka. input flags. These all interact to give different gradings, see §2.88.

Via the morphisms of operads from leaf–labelled trees to corrolas, defined by collapsing the internal edges, we obtain a morphisms from the bialgebra of forests of corollas to the bialgebra of forests of (binary) trees.

2.2.4. **Shifted version.** One obtains the above grading naturally if one introduces the shifted graded version of the construction 2.16. For this one uses the suspensions $\check{\mathcal{O}}(n)[1]$ of the $\check{\mathcal{O}}(n)$ in (2.17). This is analogous to the use of signs in the pre–Lie structure [KWZ12]. Incorporating an additional internal grading is also possible.

2.2.5. **Cobar versions.** Another way to obtain a cooperad from an operad it given by the operadic bar transform, see e.g. [MSS02]. One can then plug this cooperad into the non–connected construction. This is much bigger than just doing the tensor algebra on the dual, see §4.

Remark 2.21. The shifted version above is similar to a algebra cobar transform without the differential, but is only a part of this as the operadic cobar construction which would have components for any tree and the ones in the shifted construction are only those that are of height 2. The two constructions are related by enrichment of Feynman categories and B_+ operators. We will not go in to full details here.

A similar situation is what happens in Baues' construction. Here one can think of a cobar transform of an algebra of simplicial objects, where the simplicial structure gives the (co)operad structure, see the next section.

2.3. **A natural depth filtration and the associated graded.** In the free construction $\check{\mathcal{O}}^{nc}$ of §2.2.3 there is a natural grading by tensor length. In the general case, there is only a filtration, the depth filtration. The grading appears, as expected, on the associated graded object. This adds complications that the reader interested in only the main examples may skip.

Definition 2.22. We define the *decreasing depth filtration* on a cooperad $\check{\mathcal{O}}$ as follows: $a \in F^{\geq p}$ if $\check{\gamma}(a) \in \bigoplus_{k\geq p} \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = m} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k)$. So $\mathscr{B} = F^{\geq 1} \supset F^{\geq 2} \supset \dots$ and $\bigcap_p F^{\geq p} = 0$, since we assumed that there is no $\check{\mathcal{O}}(0)$.

We define the depth of an element a to be the maximal p such that $a \in F^{\geq p}$.

This filtration induces a depth filtration $F^{\geq p}T\mathscr{B}$ on the tensor algebra $T\mathscr{B}$ by giving $F^{\geq p_1} \otimes \cdots \otimes F^{\geq p_k}$ depth $p_1 + \cdots + p_k$. Note that any element in $T^p\mathscr{B}$ will have depth at least p.

Proposition 2.23. *The following statements hold for a cooperad with multiplication with empty* $\check{\mathcal{O}}(0)$*:*

a) *The algebra structure is filtered:* $F^{\geq p} \cdot F^{\geq q} \subset F^{\geq p+q}$.
b) *The cooperad structure satisfies* $\check{\gamma}(F^{\geq p}) \subset F^{\geq p} \otimes T^{\geq p}\mathscr{B}$ *where* $T^{\geq p}\mathscr{B} = \bigoplus_{i=p}^{\infty}(\mathscr{B})^{\otimes i} \subset F^{\geq p}T\mathscr{B}$ *and more precisely* $\check{\gamma}_{n_1,\ldots,n_k} : \check{\mathcal{O}}(n) \cap F^{\geq p} \to [\check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k)] \cap F^{\geq p} \otimes F^{\geq k}T\mathscr{B}$.
c) *The coalgebra structure satisfies:* $\Delta(F^{\geq p}) \subset F^{\geq p} \otimes F^{\geq p}$ *and more precisely* $\Delta_k(F^{\geq p}) \subset F^{\geq p} \otimes F^{\geq k}$.
d) $\check{\mathcal{O}}(n) \cap F^{\geq n+1} = \varnothing$.

Proof. The first statement follows from the compatibility (2.12). The second statement follows from the Lemma 2.24 below. The more precise statement on the right part of the filtration stems from the fact that $T^k\mathscr{B} \subset F^{\geq k}T\mathscr{B}$. The third statement then follows from a) and b), since there are at least p factors on the right before applying the multiplication and the filtration starts at 1. This shows that the right factor is in F^p. Finally, for $\check{\mathcal{O}}(n)$ the greatest depth that can be achieved happens when all the $n_i = 1 : i = 1, \ldots, k$ and since they sum up to n this is precisely at $k = n$. □

Lemma 2.24. *If* $a^p \in \mathscr{B}$ *of depth* p *let* $\check{\gamma}_{n_1,\ldots,n_k}(a^p) = \sum a^{(0)}_{(p_0)} \otimes a^{(1)}_{(p_1)} \otimes \cdots \otimes a^{(k)}_{(p_k)}$, *where we used Sweedler notation for both the cooperad structure and the depth. Then the terms of lowest depth will satisfy* $p_0 = \sum_{i=1}^{k} p_i \geq p$.

Proof. To show the equation, we use coassociativity of the cooperad structure. If we apply $\mathrm{id} \otimes \check{\gamma}^{\otimes k}$ we get least $1 + k + \sum_{i=1}^{k} p_i$ tensor factors from the lowest depth term, since we assumed that $\check{\mathcal{O}}(0)$ is empty. On the other hand applying $\check{\gamma} \otimes id^{\otimes k}$ to the terms of lowest depth, we obtain elements with at least $1 + p_0 + k$ tensor factors. Since elements of higher depth due to equation (2.10) produce more tensor factors these numbers have to agree. Since all the $p_i \geq 1$ their sum is $\geq p$.

□

2.3.1. **The associated graded bialgebra.** We now consider the associated graded objects $Gr^p := F^{\geq p}/F^{\geq p+1}$ and denote the image of $\check{\mathcal{O}}(n) \cap F^p$ in Gr^p by $\check{\mathcal{O}}(n,p)$. An element of depth p will have non–trivial image in Gr^p under this map. We denote the image of an element a^p of depth p under this map by $[a^p]$ and call it the principal part.

We set $Gr = \bigoplus Gr^p$, by part d) of 2.23: $Gr = \bigoplus_p \bigoplus_{n=1}^{p} \check{\mathcal{O}}(n,p)$ and define a grading by giving the component $\check{\mathcal{O}}(n,p)$ the total degree $n - p$.

Corollary 2.25. *By the Proposition 2.23 above we obtain maps*

- $\mu : Gr^p \otimes Gr^q \to Gr^{p+q}$ *by taking the quotient by* $F^{\geq p+1} \otimes F^{q+1}$ *on the left and* $F^{\geq p+q+1}$ *on the right*
- $\check{\gamma}^{p,k} : Gr^p \to Gr^p \otimes (Gr^1)^{\otimes k}$ *by taking the quotient by* $F^{\geq p+1}$ *on the left and* $F^{\geq k+1}T\mathscr{B} \cap T^{\otimes k}\mathscr{B}$ *on the right. In particular* $\check{\gamma}(Gr^1) \subset Gr^1 \otimes TGr^1$
- $\Delta^{p,k} : Gr^p \to Gr^p \otimes Gr^k$ *by taking the quotient by* $F^{\geq p+1}$ *on the left and* $F^{\geq k+1}$ *on the right.*
- $\Delta^p : Gr^p \to Gr^p \otimes Gr$ *via* $\Delta^p = \sum_k \Delta^{p,k}$
- $\Delta : Gr \to Gr \otimes Gr$ *via* $\Delta = \sum_p \Delta^p$

Proposition 2.26. *Gr inherits the structure of a non-unital, non-counital graded bialgebra. Each Gr^p is a non-counital comodule over Gr, and Gr^1 is a cooperad.*

Proof. Most claims are straightforward from the definitions in the corollary. For the grading we notice the multiplication preserves grading: $\check{\mathcal{O}}(n,p) \otimes \check{\mathcal{O}}(m,q) \dot{\to} \check{\mathcal{O}}(n+m, p-q)$. For the comultiplication we have that $\Delta_k(\check{\mathcal{O}}(n,p)) \subset \check{\mathcal{O}}(k,p) \otimes \check{\mathcal{O}}(n,k)$. The degree on the left is $n-p$ and on the right is $k-p+n-k = n-p$ and hence the comultiplication also preserves degree. □

Example 2.27. For the free construction $\check{\mathcal{O}}^{nc}$ of §2.2.3 we obtain

$$F^{\geq p} = \bigoplus_{k \geq p} \bigoplus_{(n_1,\dots,n_k)} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k) \tag{2.18}$$

$$Gr^k = \bigoplus_{(n_1,\dots,n_k)} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k) \tag{2.19}$$

$$\check{\mathcal{O}}^{nc}(n,k) = \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k) \tag{2.20}$$

This means that the depth of an element of $\mathscr{B}$ given by an elementary tensor is its length. The associated graded is isomorphic to the $\mathscr{B}$ which has a double grading by depth and operadic degree. Furthermore $Gr^1 = \check{\mathcal{O}}$ and $\mathscr{B} = (Gr^1)^{nc} = \check{\mathcal{O}}^{nc}$.

Corollary 2.28. *Since Gr^1 is a cooperad $(Gr^1)^{nc}$ yields a cooperad with multiplication. Multiplication gives a morphism $(Gr^1)^{nc} \to Gr$ of cooperads with multiplication preserving the filtrations and hence gives a morphism of (non-unital, non–counital) bialgebras.*

Proof. Indeed the multiplication map gives such a map of algebras, since Gr^{nc}. The compatibility map (2.13) ensures that this is also a map of cooperads with multiplication. The compatibility with the filtration is clear. □

2.4. **Unital and counital bialgebra structure.** The general construction gives a multiplication and a comultiplication which are compatible. What is missing for a bialgebra are the unit and counit. In the case of the free construction of $\check{\mathcal{O}}^{nc}$ of §2.2.3, these are fairly easily described, see §2.4.6. There is no problem in adding a unit and that the existence of a bialgebraic counit in the free case $\check{\mathcal{O}}^{nc}$ is equivalent to the existence of a cooperadic counit for $\check{\mathcal{O}}$.

For the general case, things are more complicated and worked out in detail in this section, which can again be skipped by the reader only interested in the free examples. The existence of a right cooperadic counit is a necessary condition and such a cooperadic counit determines a bialgebra counit uniquely if it exists. But, the unique candidate does not automatically work. We give several conditions that are necessary for this, treating the cases of left and right counits separately with care.

The existence of a right bialgebra counit, is equivalent to the cooperad having a right counit, which extends to a multiplicative family. Having a left coalgebra counit for $\mathscr{B}$ fixes the structure of the associated graded as a quotient of the free construction on Gr^1 via the map of Corollary 2.28 and $\mathscr{B}$ is a deformation of this quotient, see Theorem 2.39.

2.4.1. **Unit.** If there is no element of operad degree 0 then, as the multiplication preserves operad degree, $(\mathscr{B},\mu)$ cannot have a unit. In this case we may formally adjoin a unit 1 to $\mathscr{B}$: $\mathscr{B}' = \mathbb{1}\oplus\mathscr{B}$, with η be the inclusion of $\mathbb{1}$ and pr the projection to $\mathscr{B}$. We extend μ in the obvious way, and set $\Delta(1) = 1\otimes 1$, making $\mathscr{B}'$ into a unital bialgebra. In the full detail: $1 = id_{\mathbb{1}} \in Hom(\mathbb{1},\mathbb{1})$ which is the ground ring/field. In the free construction, we think of $\mathbb{1}$ as the tensors of length 0 and in the Feynman category interpretation indeed $1 = id_{\mathbb{1}}$ where $\mathbb{1}$ is the emtpy word.

2.4.2. **Counit and multiplicativity.** We will denote putative counits on $\mathscr{B}$ by ϵ_{tot} : $\mathscr{B}\to\mathbb{1}$ and decompose $\epsilon_{tot} = \sum_{k\geq 1}\epsilon_k$ according to the direct sum decomposition on $\mathscr{B}$: $\epsilon_k : \check{\mathcal{O}}(k)\to\mathbb{1}$ extended to zero on all other components. We will also use the truncated sum $\epsilon_{\geq p} = \sum_{k\geq p}\epsilon_k$ which is set to 0 on all $\check{\mathcal{O}}(k)$ for $k<p$.

Remark 2.29. There is a 1–1 correspondence between (left/right) counits on $\mathscr{B}$ and on $\mathscr{B}'$. This is given by adding ϵ_0 on the identity component via the definition $\epsilon_0\circ\eta = id$ and vice–versa truncating the extended sum $\epsilon_{tot} = \sum_{k\geq 0}\epsilon_k$ at $k=1$.

A family of morphisms $\epsilon_k : \check{\mathcal{O}}(k)\to\mathbb{1}$ is called multiplicative if $\kappa\circ(\epsilon_k\otimes\epsilon_l) = \epsilon_{k+l}\circ\mu$, where $\kappa : \mathbb{1}\otimes\mathbb{1}\to\mathbb{1}$ is the unit constraint — e.g. multiplication in the ground field in case we are in k-$Vect$ — which we will omit from now on.

Lemma 2.30. *If ϵ_{tot} is a counit (left or right) then the ϵ_k are a multiplicative family. More generally $\epsilon_{n_1}\otimes\cdots\otimes\epsilon_{n_k} = \epsilon_{\sum n_i}\circ\mu^{k-1}$ and in particular $\epsilon_1^{\otimes k} = \epsilon_k\otimes\mu^{k-1}$. If ϵ_k is a any multiplicative family and η_1 is a section of ϵ_1 then $\mu^{k-1}\circ\eta^{\otimes k}$ is a section of ϵ_k.*

Furthermore ϵ_{tot} descends to the associated graded.

Proof. The first statement is equivalent to ϵ being an algebra morphism. The other equations follow readily. Now $\epsilon_p(F^{\geq p+1}) = 0$, since $\check{\mathcal{O}}(p,p+1) = 0$ and hence each ϵ_p descends to Gr^p. The sum ϵ_{tot} then descends as the sum of the ϵ_p with each ϵ_p defined on the summand Gr^p. □

2.4.3. **Recollection on cooperadic counits.** A morphism $\epsilon : \mathscr{B}\to\mathbb{1}$ with support in $\check{\mathcal{O}}(1)$ is a left and right cooperadic counit if it satisfies[2]:

$$\sum_k(\epsilon\otimes id^{\otimes k})\circ\check{\gamma} \;=\; id \tag{2.21}$$

$$\sum_k(id\otimes\epsilon^{\otimes k})\circ\check{\gamma} \;=\; id \tag{2.22}$$

[2]Here and in the following, we suppress the unit constraints in the monoidal category and tacitly identify $V\otimes\mathbb{1}\simeq V\simeq\mathbb{1}\otimes V$.

Remark 2.31. The notion of cooperadic counits is the dual to a unit $u \in \mathcal{O}(1)$, thought of as a map of $u : \mathbb{1} \to \mathcal{O}(1)$, where $\mathbb{1}$ is $\mathbb{Z}$ for Abelian groups or in general the unit object, e.g. k for $\mathcal{V}ect_k$. Its dual is then a morphism $\check{u} := \check{\mathcal{O}}(1) \to \mathbb{1}$. We will use $\epsilon : \mathscr{B} \to \mathbb{1}$ for its extension by 0 on all $\check{\mathcal{O}}(n) : n \neq 1$. ϵ is a left/right cooperadic counit if it satisfies the diagrams dual to the equations (2.6), that is the equations (2.21) and (2.22).

Remark 2.32. Note, if there is only one tensor factor on the right, then the left factor has to be $\check{\mathcal{O}}(1)$ by definition. If ϵ would have support outside $\check{\mathcal{O}}(1)$, the $\check{\gamma}$ would have to vanish on the right side for all elements having that left hand side, which is rather non–generic. This is why we assume ϵ vanishes outside $\check{\mathcal{O}}(1)$.

2.4.4. Right counits.

Lemma 2.33. *If $\mathscr{B}$ has a right bialgebra counit ϵ_{tot}, then ϵ_1 is a right cooperadic counit. If there are elements of depth greater than one, there can be no left cooperadic counit.*

Proof. For the first statement, we verify (2.22) using Lemma 2.30:

$$\sum_k (id \otimes \epsilon_1^{\otimes k}) \circ \check{\gamma} = \sum_k (id \otimes \epsilon_k) \circ \mu^{k-1} \circ \check{\gamma} = (id \otimes \epsilon_{tot}) \circ \Delta = id \tag{2.23}$$

The second statement just says that using ϵ on the left, we would need exactly one tensor factor on the right after applying $\check{\gamma}$ in order to get an identity. Indeed, if we apply $\check{\gamma}$ to $a \in F^{\geq p}$ then there are at least $p+1$ tensor factors, and ϵ will only take the leftmost tensor factor to the ground field. Thus there can be no left counit for elements in $F^{\geq 2}$. □

A necessary condition for the existence of a right counit for $\mathscr{B}$ is hence

Proposition 2.34. *ϵ_{tot} is a right bialgebraic counit if and only if ϵ_1 is a right cooperadic counit which extends to a multiplicative family ϵ_k.*

Proof. This follows by reading equation (2.23) right to left. □

2.4.5. Left counits.

Proposition 2.35. *If $\mathscr{B}$ as a coalgebra has a left counit ϵ_{tot}, then $F^{\geq p} = (F^{\geq 1})^{\geq p}$, where the latter denotes the sum of the k–th powers of $F^{\geq 1}$ with $k \geq p$. Moreover, the morphism of cooperads with multiplication and of bialgebras $(Gr^1)^{nc} \to Gr$ given by Corollary 2.28 is surjective.*

Proof. The inclusion $F^{\geq p} \supset (F^{\geq 1})^{\geq p}$ is in Proposition 2.23. For the reverse inclusion, let $a \in F^{\geq p}$, then after applying $(\epsilon_{tot} \otimes id) \circ \Delta$ we are left with a sum of products of at least p factors and hence the reverse inclusion follows.

In the same way, we see that $Gr^p = (Gr^1)^p$ and that the map in question is surjective. □

We recall from [Ger64] that a filtered algebra/ring $(\mathscr{B}, F^{\geq p})$ is predevelopable if there exists for each p an additive mapping $q_p : Gr^p \to F^{\geq p}$ which is a section of $p_p : F^{\geq p} \to Gr^p = F^{\geq p}/F^{\geq p+1}$ i.e. $p_p \circ q_p(a) = a$ for all $a \in Gr^p$. It is developable if also $\bigcap_p F^{\geq p} = 0$ and the ring is complete in the topology induced by the filtration. In our case, due to the assumption the there is no $\check{\mathcal{O}}(0)$, the first condition is true and also since we only took finite sums, the algebra is complete.

Proposition 2.36. *If $\mathscr{B}$ has a left coalgebra counit then $P_p = (\epsilon_{\geq p} \otimes id) \circ \Delta$ is a projector to $F^{\geq p}$. Hence the short exact sequence $0 \to F^{\geq p+1} \to F^{\geq p} \to Gr^p \to 0$ splits and $\mathscr{B}$ is predevelopable.*

Proof. If ϵ_{tot} is a left coalgebra counit then using multi–Sweedler notation for $a \in \check{\mathcal{O}}(n)$: $a = (\epsilon_{tot} \otimes id) \circ \Delta(a) = \sum_k \epsilon_k(a_k^{(0)}) \otimes a_{n_1}^{(1)} \cdots a_{n_k}^{(1)} =: \sum_k a_k$ with a_k a product of k factors and hence in $F^{\geq k}$. Since $\epsilon_{\geq p} = 0$ on $\check{\mathcal{O}}(k) : k < p$, we see that $P_p(a) = \sum_{k \geq p}^n a_k$ and hence the image of P_p lies in $F^{\geq p}$. If on the other hand $a \in F^{\geq p}$ then $a = \sum_k a_k = \sum_{k \geq p} a_k = P_p(a)$, since all lower terms do not exist as the summation for Δ stands at p.

□

Note that $T_i(a) = [P_{i-1} \cdots P_1(a)]$ gives the development of a in Gr in the notation of [Ger64].

Corollary 2.37. *If ϵ_{tot} is a left bialgebra unit, then for $a \in \check{\mathcal{O}}(n) \cap F^{\geq p}$ there is a decomposition $a = \sum_{k \geq p}^n a_k$ with each $a_k \in F^{\geq k}$ and (after possibly collecting terms) this gives the development of a.* □

Corollary 2.38. *If ϵ_{tot} is a left coalgebra counit for $\mathscr{B}$, then ϵ_p descends to a well defined map $Gr^p \to \mathbb{1}$. and on Gr^p : $(\epsilon_p \otimes id) \circ \Delta_p = id$. Thus ϵ_{tot} understood as acting on Gr^p with ϵ_p is a left counit for Gr. Furthermore $(\epsilon_k \otimes id) \circ \Delta|_{Gr^p} = \delta_{k,p} id$.*

Proof. First $\epsilon_p(F^{\geq p+1}) = 0$, since $\check{\mathcal{O}}(p, p+1) = 0$. The statements then follows from the development. □

It is known [Ger64] that if $\mathscr{B}$ is developable then Gr is a deformation of $\mathscr{B}$. Coupled with the results above one has:

Theorem 2.39. *if $\mathscr{B}$ has a left coalgebra counit, then $\mathscr{B}$ is a deformation Gr, which is a quotient of the free construction on Gr^1.*

□

2.4.6. **Units and counits for the free case $\check{\mathcal{O}}^{nc}$.** In this section, we let $\check{\mathcal{O}}$ be a cooperad and consider $\check{\mathcal{O}}^{nc}(n) = \bigoplus_k \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k)$ and its bialgebra $\mathscr{B} = \bigoplus \check{\mathcal{O}}^{nc}(n)$.

Proposition 2.40. *For the bialgebra $\mathscr{B} = \bigoplus_n \check{\mathcal{O}}^{nc}(n)$ to have a bialgebraic counit it is sufficient and necessary that $\check{\mathcal{O}}$ has a cooperadic counit.*

Proof. We already know that a right cooperadic counit for $\check{\mathcal{O}}^{nc}$ is necessary. This yields a right cooperadic coounit for $\check{\mathcal{O}}$ by restriction to $Gr^1 = \check{\mathcal{O}}$. Then for $a \in \check{\mathcal{O}} = Gr^1$ $a = \epsilon_1 \otimes id \circ \Delta(a) = \sum_k \epsilon_1 \otimes id^{\otimes k} \circ \check{\gamma}$, since all terms with $k \neq 1$ vanish and for the term with $k = 1$ $\Delta = \check{\gamma}$. Thus ϵ_1 is also a left cooperadic counit for $\check{\mathcal{O}}$. We stress for $\check{\mathcal{O}}$ not for $\check{\mathcal{O}}^{nc}$.

Now assume that ϵ_1 is a cooperadic counit for $\check{\mathcal{O}}$. It follows that ϵ_1 is a right cooperadic counit for $\check{\mathcal{O}}^{nc}$ by compatibility. Now since $\mu = \otimes$: the extension $\epsilon_k = \epsilon_1^{\otimes k}$ is multiplicative and hence a right bialgebra counit. It remains to check whether it is bialgebraic, which reduces to checking that it is a left coalgebraic unit. The multiplicativity is clear, so, we only need to check on Gr^1, that is for all $a \in \check{\mathcal{O}}^{nc}(n,1) = \check{\mathcal{O}}(n)$. On the $\check{\mathcal{O}}(n)$ the equation says exactly that ϵ_1 is a left cooperadic unit for $\check{\mathcal{O}}$. □

Corollary 2.41. *If $\mathcal{O}$ has an operadic unit, then $\check{\mathcal{O}}$ has a cooperadic counit and hence $\mathscr{B}' = \mathbb{1} \oplus \bigoplus \check{\mathcal{O}}^{nc}(n)$ is a unital, counital bialgebra.*

This encompasses all the examples of §2.1.3.

2.4.7. **Counits summary.** If $\mathscr{B}$ comes from $\check{\mathcal{O}}^{nc}$ then having a bialgebra unit ϵ_{tot} is equivalent to ϵ_1 being a cooperad counit on $\check{\mathcal{O}}$.

In general, for $\mathscr{B}$ to have a bialgebra counit, it is necessary, that

(1) ϵ_1 is a right cooperadic counit.
(2) $F^{\geq p} = (F^{\geq 1})^{\geq p}$.
(3) $P_k = (\epsilon_{\geq k} \otimes id) \circ \Delta$ are projectors onto $F^{\geq k}$.
(4) $\mathscr{B}$ is developable and a deformation of the associated graded Gr

On the associated graded Gr. If ϵ_{tot} is a putative bialgebra counit

(1) ϵ_p is uniquely determined from ϵ_1.
(2) Lifted to $(Gr^1)^{nc}$, ϵ_1 is a cooperadic unit, which ensures that the lift of ϵ_{tot} is a bialgebra unit.
(3) For ϵ_{tot} to descend to Gr, it needs to vanish on the kernel of the, by (2) surjective, map $\mu^{\otimes p-1} : (Gr^1)^{\otimes p} \to Gr^p$.

The first statement holds by Proposition 2.35 and Corollary 2.38 which says that $Gr^p = (Gr^1)^p$ and hence Lemma 2.30 determines ϵ_p. Since counits are multiplicative, they lift via Proposition 2.40.

Definition 2.42. In general, we say that a cooperadic right counit ϵ_1 is *bialgebraic*, if it extends to a bialgebraic counit ϵ_{tot} for $\mathscr{B}$. If such an ϵ_{tot} exists, we will call $\check{\mathcal{O}}$ bialgebraic.

2.5. **The pointed case.** In the end, we would like to produce Hopf algebras, by showing that appropriate quotients of the bialgebras above are connected. For this one actually needs distinguished elements, which will be called $|$ or sections, see Appendix B. Even if these exist, the bialgebra is usually not connected, since the powers $|^p$ keep it from being so. However, taking a quotient remedies the situation up to a possible problem in the coalgebra $\check{\mathcal{O}}(1)$. We now set the stage and do the construction in the next section.

We will also give further necessary conditions for the existence of bialgebraic counits in the pointed case.

Definition 2.43. A cooperad $\check{\mathcal{O}}$ with a right cooperadic counit ϵ_1 is called *pointed* if the counit ϵ_1 is split, i.e. there is a section $\eta_1 : \mathbb{1} \to \check{\mathcal{O}}(1)$ of ϵ_1.

We call $\check{\mathcal{O}}$ *reduced* if it is pointed and η_1 is an isomorphism $\mathbb{1} \simeq \check{\mathcal{O}}(1)$; it is then automatically pointed.

A bialgebra unit will be called pointed if the associated right cooperadic unit ϵ_1 is pointed.

We will denote $| := \eta_1(1)$.[3] For pointed cooperads Lemma 2.30 applies and we split each $\check{\mathcal{O}}(n) = \mathbb{1} \oplus \bar{\check{\mathcal{O}}}(n)$ where $\bar{\check{\mathcal{O}}}(n) = ker(\epsilon_n) = ker(\epsilon_{tot}|_{\check{\mathcal{O}}(n)})$ and $\mathbb{1}$ is the component of $|^n$. We set $\bar{B} = \bigoplus \bar{\check{\mathcal{O}}}(n)$.

Notice that this is smaller than the augmentation ideal $\mathscr{B}^{red} = ker(\epsilon_{tot})$.

Example 2.44. Any cooperad with multiplication $\check{\mathcal{O}}^{nc}$ that is the free construction of dual $\check{\mathcal{O}}$ of a unital operad $\mathcal{O}$ is pointed if the unit morphism $u : \mathbb{1} \to \mathcal{O}(1)$ split via a morphism c. We call such an operad *split unital*. In the notation above $\check{u} = \epsilon_1$ and $\check{c} = \eta_1$. The element $|$ is then the dual element to the unit $u(1) \in \mathcal{O}(1)$. Here $| = \check{c}(1) = \eta_1(1)$ and being the dual element means that $\check{u}(|) = \epsilon_1 \circ \eta_1(1) = (c \circ u)^\vee(1) = 1$.

Again all of the examples of §2.1.3 have this property.

Lemma 2.45. *If $\mathscr{B}$ has a split bialgebraic counit, then have $\Delta(|) = | \otimes | + \bar{\Delta}(|)$ with $\bar{\Delta}(|) \in \bar{\check{\mathcal{O}}}(1) \otimes \bar{\check{\mathcal{O}}}(1)$ and hence $\Delta(|^p) = |^p \otimes |^p +$ terms of lower order in $|$. Thus the image of $|^p$ is not 0 in Gr^p and we can split $Gr^p = \mathbb{1} \oplus \overline{Gr}^p$ where $\mathbb{1}$ is the component if the image of $|^p$.*

Proof. The first statement follows since ϵ_{tot} is a bialgebraic unit. The second statement follows, from the bialgebra compatibility condition. □

More generally,

Proposition 2.46. *Let $\check{\mathcal{O}}$ be a cooperad with multiplication and a pointed bialgebraic counit on $\mathscr{B}$, then*

$$\begin{aligned} \Delta(|) &= | \otimes | + \bar{\Delta}(|) \text{ with} \\ \bar{\Delta}(|) &\in \bar{\check{\mathcal{O}}}(1) \otimes \bar{\check{\mathcal{O}}}(1) \end{aligned} \tag{2.24}$$

$$\begin{aligned} \Delta(|^p) &= |^p \otimes |^p + \bar{\Delta}(|^p) \text{ with} \\ \bar{\Delta}(|^p) &\in \bar{\check{\mathcal{O}}}(p) \otimes \bar{\check{\mathcal{O}}}(p) \end{aligned} \tag{2.25}$$

And for $a \in \bar{\check{\mathcal{O}}}(n) \cap F^{\geq p}$

$$\begin{aligned} \Delta(a) &= \sum_{k \geq p}^{n} |^k \otimes a_k + a \otimes |^n + \bar{\Delta}(a) \text{ with} \\ & a_k \in \bar{\check{\mathcal{O}}}(n), \bar{\Delta}(a) \in \bar{\mathscr{B}} \otimes \bar{\check{\mathcal{O}}}(n) \end{aligned} \tag{2.26}$$

with $a = \sum_{k \geq p}^{n} a_k$ and the a_k are as in Corollary 2.37.

Likewise, in the associated graded case, for $a \in \bar{\check{\mathcal{O}}}(n,p)$

$$\begin{aligned} \Delta(a) &= |^p \otimes a + a \otimes |^n + \bar{\Delta}(a) \text{ with} \\ & \bar{\Delta}(a) \in \bar{Gr} \otimes \bar{Gr} \end{aligned} \tag{2.27}$$

[3]Strictly speaking $1 = id_{\mathbb{1}}$ in the usual language for monoidal cateogories.

Again, if these equations hold having a bialgebraic coounit ϵ_{tot} is equivalent to ϵ_1 being a right cooperadic counit.

Proof. Using Corollary 2.37 and applying ϵ_{tot} on the left, we obtain the first term and applying ϵ_{tot} on the right, the second term. These are different if $a \neq |^k$ for some k. In the case $a = |^k$ the equation follows from the Lemma above. In general, the remaining terms lie in the reduced space. Replacing $\mathscr{B}$ with Gr proves the rest. □

We also get a practical criterion for a bialgebra counit.

Corollary 2.47. *Assume the equations in Propositions 2.46 hold, then having a bialgebraic coounit ϵ_{tot} is equivalent to ϵ_1 being a right cooperadic counit.*

Proof. By Lemma 2.30, we see that ϵ_k is the projection to the factor $|^k$ of $\check{\mathcal{O}}(k) = \mathbb{1} \oplus \bar{\check{\mathcal{O}}}(k)$ and on that factor it is $\epsilon_1^k \circ \mu^{k-1}$ and hence determined by ϵ_1. Now the second term of (2.26) is equivalent to ϵ_{tot} being a right bialgebra counit. Furthermore, since this is the term relevant for the right cooperad counit, we obtain the equivalence for the right bialgebra counit. Similarly, applying the given ϵ_{tot} as a potential left bialgebra counit, we see that having a left bialgebra counit is equivalent to $a = \sum_k a_k$, i.e. the first term in (2.26). □

2.6. **Hopf Structure.** In this section, unless otherwise stated, we will assume that $\check{\mathcal{O}}$ is a cooperad with multiplication and bialgebraic counit.

Assumption 2.48. We also assume that tensor and kernels commute. Under this assumption the notions of conilpotent and connected are equivalent.

For example this is the case if we are working in k–Vect.

Definition 2.49. We call a pointed cooperad $\check{\mathcal{O}}$ with bialgebraic counit ϵ_{tot} *almost connected* if

(1) The element $|$ is group–like: $\Delta(|) = | \otimes |$
(2) $(\check{\mathcal{O}}(1), \eta_1, \epsilon_1)$ is connected as a coalgebra in the sense of Quillen [Qui67] (see Appendix B).

Notice that a reduced $\check{\mathcal{O}}$ is automatically almost connected, but this is not a necessary condition.

Lemma 2.50. *Let $\check{\mathcal{O}}^{nc}$ be the free construction on the dual $\check{\mathcal{O}}$ of a split unital operad $\mathcal{O}$. Then it is almost connected if any element $a \in \mathcal{O}(1)$ is only represented by finite reduced words, that is any decomposition $a = \prod_{i \in I} a_i$ with all $\eta_1(a_i) = 0$, I is finite (or empty).*

Proof. Recall that the coproduct is dual to multiplication in the monoid, that is, it is decomposition. Being conilpotent then is just equivalent to the given finiteness condition. □

Example 2.51.

(1) If the unital operad $\mathcal{O}$ is reduced, that is $\mathcal{O}(1) \simeq \mathbb{1}$ its dual is also reduced. This is the case for the surjection and the simplex operads.

(2) More generally, if a split unital $\mathcal{O}$ is such that $\mathcal{O}(1) = \mathbb{1} \oplus \overline{\check{\mathcal{O}}}(1)$ and $\overline{\check{\mathcal{O}}}(1)$ is free of finite rank as a unital monoid, then $\check{\mathcal{O}}^{nc}$ is almost connected. This is the case for the operad of trees. $\overline{\check{\mathcal{O}}(1)}$ is free of rank 1 with the generator being the rooted corolla with one tail . The generator corresponding to the dual of the identity can be depicted as the degenerate "no vertex" corolla with one input and output $|$ and the other generator as ϕ.

This is linked to the considerations of [Moe01] in the rank 1 case and those of higher rank to [vdLM06a], see §5.1.2, where the generators can the thought of as ϕc, where c is a color index.

(3) However, if $\check{\mathcal{O}}(1)$ contains group-like elements except for the unit then $\check{\mathcal{O}}$ is *not* almost connected.

(4) In the free case, if $\mathcal{O}(1)$ contains any isomorphisms except for the unit, then $\check{\mathcal{O}}^{nc}$ is also *not* almost reduced. More precisely, if $\mathcal{O}(1)$ splits as $\mathbb{1} \oplus \bar{\mathcal{O}}(1)$, then $\bar{\mathcal{O}}(1)$ may not contain any invertible elements if $\check{\mathcal{O}}$ is to be almost connected. Indeed, if a is such an isomorphism it has representatives of infinite length.

Remark 2.52. Notice that for an almost connected $\check{\mathcal{O}}$ the bialgebra $\mathscr{B}'$ is not connected, since all powers $|^k$ are group like: $\Delta(|^k) = |^k \otimes |^k$, $\epsilon_{tot}(|^k) = 1$.

For a pointed $\check{\mathcal{O}}$, let $\mathcal{I}$ be the two-sided ideal spanned by $1 - |$. Set

$$\mathscr{H} := \mathscr{B}'/\mathcal{I} \tag{2.28}$$

Notice that in $\mathscr{H}$ we have that $|^k \equiv 1 \mod \mathcal{I}$ for all k.

Proposition 2.53. *If $\check{\mathcal{O}}$ is connected, then $\mathcal{I}$ is a coideal and hence $\mathscr{H}$ is a coalgebra. The unit η descends to a unit $\bar{\eta} : \mathbb{1} \to \mathscr{H}$ and the counit ϵ_{tot} factors as $\bar{\epsilon}$ to make $\mathscr{H}$ into a bialgebra.*

Proof. $\Delta(1 - |) = 1 \otimes 1 - | \otimes | = (1 - |) \otimes | + 1 \otimes (1 - |) \in \mathcal{I} \otimes \mathscr{B} + \mathscr{B} \otimes \mathcal{I}$ and $\epsilon_{tot}(1 - |) = 1 - 1 = 0$. □

Theorem 2.54. *If $\check{\mathcal{O}}$ is almost connected then $\mathscr{H}$ is conilpotent and hence admits a unique structure of Hopf algebra.*

Proof. Let $\pi = id - \bar{\epsilon} \circ \bar{\eta}$ be the projection $\mathscr{H} = \mathbb{1} \oplus \bar{\mathscr{H}} \to \bar{\mathscr{H}}$ to the augmentation ideal. We have to show that each element lies in the kernel of some $\pi^{\otimes m} \circ \Delta^m$. For $\mathbb{1}$ this is clear, for the image of $\check{\mathcal{O}}(1)$ this follows from the assumptions, from the Lemma above and the identification of 1 and $|$ in the quotient. Now we proceed by induction on n, namely, for $a \in \check{\mathcal{O}}(n)$, we have that $\Delta(a) \in \bigoplus_{k,n} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n)$. Since the coproduct is coassociative, we see that all summands with $k < n$ are taken care of by the induction assumption. This leaves the summands with $k = n$. Then the right hand side of the tensor product is the product of elements which are all in $\check{\mathcal{O}}(1)$. Since Δ is compatible with the multiplication, we are done by the assumption on $\check{\mathcal{O}}(1)$ and coassociativity.

□

2.7. **The Hopf algebra as a deformation.** Rather then taking the approach above, we can produce the Hopf algebra in two seprate steps. Without adding a unit, we will first mod out by the two-sided ideal $\mathcal{C}$ generated by $|a - a|$. This forces $|$ to lie in the centre. We denote the result by $\mathscr{H}_q := \mathscr{B}/\mathcal{C}$, where the image of $|$ under this quotient is denoted by q. This allows us to view q as a deformation parameter and view $\mathscr{H}$ as the classical limit $q \to 1$ of $\mathscr{H}_q$. In this section we assume that $\check{\mathcal{O}}$ is pointed.

Proposition 2.55. *$\mathcal{C}$ is a coideal and hence $\mathscr{H}_q$ with the induced unit and counit is a bialgebra.*

Proof.

$\Delta(|a{-}a|) = |a^{(1)}{\otimes}|a^{(2)}{-}a^{(1)}|{\otimes}a^{(2)}| = (|a^{(1)}{-}a^{(1)}|){\otimes}|a^{(2)}{+}a^{(1)}|{\otimes}(|a^{(2)}{-}a^{(2)}|) \in \mathcal{C}{\otimes}\mathscr{B}{+}\mathscr{B}{\otimes}\mathcal{C}$ using Sweedler notation. Furthermore $\epsilon(|a - a|) = \epsilon(a) - \epsilon(a) = 0$. □

Remark 2.56. If $\rho_|$ and $\lambda_|$ are right and left multiplication by $|$, then $\mathcal{C}$ is also a coequalizer in the sequence

$$\mathscr{B} \underset{\lambda_|}{\overset{\rho_|}{\rightrightarrows}} \mathscr{B} \longrightarrow \mathscr{H} \tag{2.29}$$

Notice that the image of $|^n$ is q^n and if we give q the degree 1, then the grading by operadic degree is preserved as well as the depth filtration and all other filtrations and gradings mentioned above.

By moving all the q's to the left the elements in $\mathscr{H}_q$ can be thought of as polynomials in q whose coefficients lie in $\bar{\mathscr{B}}$, i.e. $\mathscr{H}_q \subset \bar{\mathscr{B}}[q]$. The degree of a polynomial is the operadic degree plus the degree of q.

Proposition 2.57. *$\mathscr{H}_q$ is a deformation of $\mathscr{H}$ given by $q \to 1$.* □

2.7.1. **The $|$ filtration on $\mathscr{B}$.** Let $\mathscr{J}$ be the two-sided ideal of $\mathscr{B}$ spanned by $|$. Then there is an exhaustive filtration of $\mathscr{B}$ by the powers of $\mathscr{J}$. This filtration survives the quotient by $\mathcal{C}$ and gives a filtration in powers of q. Here we can then also view the filtration as a deformation over a formal disc, with the central fiber $z = 0, q = 1$ being the associated graded.

Example 2.58. For the free construction $\check{\mathcal{O}}^{nc}$, we have that as an algebra:

$$\mathscr{H}_q = \bigoplus_d \bigoplus_{n \leq d} q^{n-d} \check{\mathcal{O}}^{nc,red}(n) \simeq T\check{\mathcal{O}}^{red}[q] \tag{2.30}$$

where $\check{\mathcal{O}}^{nc,red}(n) = \bigoplus_k \bigoplus_{(n_1,\dots n_k):\sum n_i = n} \check{\mathcal{O}}^{red}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}^{red}(n_k)$ and $\check{\mathcal{O}}^{red} = ker(\epsilon_1)$. This is so, since the terms with $|$ only arise from products with elements from $ker(\epsilon_1) = \mathbb{1} \subset \check{\mathcal{O}}(1)$.

The associated graded with respect to $\mathscr{J}$ is isomorphic to $\mathscr{H}_q$.

2.8. **Infinitesimal version.** The filtration above also allows us to obtain the infinitesimal version of the Hopf algebra. This involves the use of pseudo-cooperads that we briefly review.

2.8.1. **Pseudo-cooperads $\check{\circ}_i$.** A right cooperadic counit allows one to write the dual operations to the $\circ_i$:

$$\check{\circ}_i(a) = (id \otimes \epsilon \otimes \cdots \otimes \epsilon \otimes id \otimes \epsilon \otimes \cdots \otimes \epsilon) \circ (\check{\gamma}(a)) \tag{2.31}$$

with id in the 1st and $i+1$-st place. Here we again implicitly use the structural isomorphism for the unit.

Dualizing the picture above, a pseudo–cooperad is a collection $\check{\mathcal{O}}(n)$ of Abelian groups (or objects of a symmetric monoidal category in general) each with an $\mathbb{S}_n$ action together with structure maps

$$\check{\circ}_i : \check{\mathcal{O}}(n) \to \bigoplus_{k=i}^{n} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n-k+1) \text{ for } 1 \leq i \leq n \tag{2.32}$$

2.8.2. **Copre-Lie $\check{\circ}$.** Summing over all the $\check{\circ}_i$ we get a map

$$\check{\circ} : \check{\mathcal{O}}(n) \to \bigoplus_{k=1}^{n} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n-k+1) \tag{2.33}$$

We call the projection onto the factor $\check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(n-k+1)$ the degree k part of $\check{\circ}_i$ and $\check{\circ}$ respectively.

2.8.3. **A type of bialgebra from cooperads with multiplication.**

Definition 2.59. A *pseudo-cooperad with multiplication* μ is a pseudo-cooperad $\check{\mathcal{O}}$ with a family of maps, $n, m \geq 0$,

$$\mu_{n,m} : \check{\mathcal{O}}(n) \otimes \check{\mathcal{O}}(m) \to \check{\mathcal{O}}(n+m)$$

which together with the comultiplication $\delta := \check{\circ}$ satisfies the equation

$$\delta \circ \mu \quad = \quad (\mathrm{id} \otimes \mu)(\delta \otimes \mathrm{id}) + (\mu \otimes \mathrm{id})(\mathrm{id} \otimes \delta) \tag{2.34}$$

Remark 2.60. Although equation (2.34) is the same equation as that for an infinitesimal bialgebra our δ is not coassociative; just like $\circ$ is not associative, but only pre-Lie. What we do have is what one could call a copre-Lie bialgebra.

Proposition 2.61. *If $\check{\mathcal{O}}$ is a non-Σ cooperad with multiplication and multiplicative right co-operadic counit. Then the multiplication is also compatible with the non-Σ pseudo-cooperad structure.*

Proof. Straightforward. □

Remark 2.62. In the example of Connes and Kreimer, this corresponds to making a single cut. In simplicial terms, the dual defines the $\cup_1$ product. See also §5.2.2.

Remark 2.63. If the cooperad is pointed, one can reconstruct the cooperad structure from the pseudo–cooperad structure. This fact is used with great skill in [Bro17, Bro15, Bro12b].

After passing to the Hopf quotient, the factors of $|$ are identified with 1. In the case of Brown [Bro12a] this gives the operators D_r determining the coaction. In general, it is easy to see that:

Proposition 2.64. *If $\mathscr{B}$ is almost connected, then in the Hopf quotient, the copreLie structure induces a coLie algebra structure on the indecomposables $\mathscr{H}_>/\mathscr{H}_>\mathscr{H}_>$, where $\mathscr{H}_>$ is reduced version of $\mathscr{H}$.* □

Example 2.65. In the free case $\check{\mathcal{O}}^{nc}$, the indecomposables are precisely given by $\check{\mathcal{O}}$ and the copreLie structure is $\check{\circ}$. If $\check{\mathcal{O}}$ is the dual of $\mathcal{O}$ then the coLie structure corresponds dually to the usual Lie structure of Gerstenhaber.

2.8.4. **Infinitesimal part as the degree q^{k-1} part in the free case.** We have seen above that we can extract the infinitesimal version using counits. Furthermore using the construction of the double quotient, first identifying $|$ with q gives credence to the name infinitesimal.

Lemma 2.66. *Consider the free case $\check{\mathcal{O}}^{nc}$ then $\Delta(|^n) = |^n \otimes |^n$ and if $a \in \check{\mathcal{O}}^{nc}(n,p)$ and $\epsilon_{tot}(a) = 0$, then*

$$\begin{aligned} \Delta(a) &= |^p \otimes a + a \otimes |^n + \bar{\Delta}(a) \text{ with} \\ \bar{\Delta}(a) &= \sum_{k=p}^{n}\sum_{i=1}^{k-1} a_k^{(i,1)} \otimes |^i a_{n-k+1}^{(i,2)} |^{k-i-1} + \sum_{k=p}^{n} \mathscr{B}^{red} \otimes \mathscr{J}^{<k-1} \end{aligned}$$

and setting

$$\check{\circ}_i(a) = \sum_{k=p}^{n} a_k^{(i,1)} \otimes a_{n-k+1}^{(i,2)} \tag{2.35}$$

defines the pseudo-cooperad structure, where the $a_k^{(i,1)} \in \bar{\check{\mathcal{O}}}(k)$ and the $a_{n-k+1}^{(i,2)} \in \bar{\check{\mathcal{O}}}(n-k+1)$.

Proof. The first statement follows from the bialgebra structure. The second statement follows from the fact that ϵ is a left and right counit. In general one can count the factors of $|$ that may occur in Δ_k. Since we are in the free case, factors of $|$ can only come from factors of $\check{\mathcal{O}}(1)$. Applying formula (2.31) then gives the last statement. □

Proposition 2.67. *The degree q^{k-1} part of $\bar{\Delta}_k$ gives the degree k part of the pseudo cooperad structure.*

Proof. This follows from from equation (2.35). If the q degree of a is 0, we see that indeed the needed term has is the degree q^{k-1} term. If the degree q degree of a is larger then one, this appears in both the left and the right hand side as $\Delta(|) = | \otimes |$ and thus this power of q cancels out. □

2.9. **Coinvariants: commutative version.** We now assume that the cooperad is symmetric. To pass to invariants, it will be instructive to first consider operads in arbitrary sets, see e.g. [MSS02]. This means that for any finite set S we have an $\mathcal{O}(S)$ and any isomorphism $\sigma : S \to S'$ an isomorphism $\mathcal{O}(S) \to \mathcal{O}(S')$. The composition is then defined for any map $f : S \to T$ as a morphism $\mathcal{O}(T) \otimes \bigotimes_{t\in T} \mathcal{O}(f^{-1}(t)) \to \mathcal{O}(S)$ which is equivariant for any diagram of the form

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ {\scriptstyle\sigma'}\downarrow & & \downarrow{\scriptstyle\sigma} \\ S' & \xrightarrow{f'} & T' \end{array} \tag{2.36}$$

Recall that if we are only given the $\check{\mathcal{O}}(n)$ then the extension to finite sets is given by $\check{\mathcal{O}}(S) := \mathrm{colim}_{f:S\leftrightarrow \underline{n}}\check{\mathcal{O}}(n)$. Where $\underline{n} = \{1,\dots,n\}$. This actually yields an equivalence of categories between finite sets and its skeleton of consisting of the $\bar{n}$ and maps between them. Notice that $\bar{0} = \varnothing$ and we can restrict the considerations to non–empty sets and surjections with the skeleton consisting of $\underline{n}, n \geq 1$ and surjections.

Let $|S| = |S'| = k$, $n_t = |f^{-1}(t)| = |f'^{-1}(\sigma(t))|$ and $\sigma'_t : f^{-1}(t) \to (f')^{-1}(\sigma(t))$ be the restriction. Then for any pair of isomoprhisms (σ', σ) given above, the outer square of the diagram below commutes.

$$\begin{array}{ccccccc}
 & \mathcal{O}(T)\otimes\bigotimes_T\mathcal{O}(f^{-1}(t)) & & \xrightarrow{\quad\gamma_f\quad} & & \mathcal{O}(S) & \\
 & \uparrow{\scriptstyle i} & & & & \uparrow & \\
\sigma\otimes\bigotimes_T\sigma'_t & (\mathcal{O}(k)\otimes\bigotimes_{i=1}^k\mathcal{O}(n_i)^{\mathbb{S}(n_i)})^{\mathbb{S}_k} & \xleftarrow{\imath} & \mathcal{O}(k)^{\mathbb{S}_k}\otimes Symm(\bigotimes_{i=1}^k\mathcal{O}(n_i)^{\mathbb{S}(n_i)}) & \xrightarrow{\bar{\gamma}} & \mathcal{O}(n)^{\mathbb{S}_n} & \sigma' \\
 & {\scriptstyle i}\downarrow & & & & \downarrow & \\
 & \mathcal{O}(T')\otimes\bigotimes_{T'}\mathcal{O}(f'^{-1}(t')) & & \xrightarrow[\quad\gamma_{f'}\quad]{} & & \mathcal{O}(S') &
\end{array} \tag{2.37}$$

To explain the other morphisms, let $Iso(n,k)$ be the category with objects the surjections $S \to T$ with $|S| = n$ and $|T| = k$ and morphisms the commutative diagrams of the type (2.36) with σ, σ' bijections and f, f' surjections, and $Iso(n)$ the category with objects S, with $|S| = n$ and bijections. Then

(1) $lim_{Iso(n)}\mathcal{O} = \mathcal{O}(n)^{\mathbb{S}_n}$ are the invariants.
(2) $lim_{Iso(n.k)}\mathcal{O} = \bigoplus_{(n_1,\dots,n_k):\sum n_i = n}\left(\mathcal{O}(k)\otimes\bigotimes_{i=1}^k\mathcal{O}(n_i)^{\mathbb{S}(n_i)}\right)^{\mathbb{S}_k}$ where on $\mathcal{O}(f : S \to T) = \mathcal{O}(T)\otimes_T\bigotimes_T\mathcal{O}(f^{-1}(t))$ $Aut(T) \simeq \mathbb{S}_k$ acts anti–diagonally as $\sigma\otimes\sigma^{-1}$.
(3) These contain the invariants under the full $\mathbb{S}_k\times\mathbb{S}_k$ action using the anti–diagonal embedding $\mathbb{S}_k \subset \mathbb{S}_k\times\mathbb{S}_k$. The invariants under $\mathbb{S}_k\times\mathbb{S}_k$ are $\mathcal{O}(k)^{\mathbb{S}_k}\otimes Symm(\bigotimes_{i=1}^k\mathcal{O}(n_i)^{\mathbb{S}(n_i)})$, where $Symm$ is the subspace of symmetric tensors,
(4) The map $\bar{\gamma}$exists by the universal property of limits applied to $\mathcal{O}(n)^{\mathbb{S}_n}$ and the cone given by the γ_f precomposed with the inclusion of the invariants, i.e. $\gamma_f\circ i\circ\imath$.

Remark 2.68. These are exactly universal operations in the sense of [KW17]. In order to establish this, we recall that any operad under the equivalence established in [KW17][Example 4.12] can be thought of either an enrichment of the Feynman category of sets and surjections or as a functor from the Feynman category for operads to a target category. As the latter, we obtain universal operations through colimits, see paragraph §6 of [KW17]. On the other hand via the construction in paragraph §4 below.

Dualizing these diagrams we obtain the diagrams

$$
\begin{array}{ccccc}
\check{\mathcal{O}}(S) & \xrightarrow{\check{\gamma}_f} & & & \check{\mathcal{O}}(T) \otimes \bigotimes_T \check{\mathcal{O}}(f^{-1}(t)) \\
\downarrow & & & & \downarrow p \\
\check{\mathcal{O}}(n)_{\mathbb{S}_n} & \xrightarrow{\bar{\gamma}} & \check{\mathcal{O}}(k)_{\mathbb{S}_k} \otimes \bigodot_{i=1}^k \mathcal{O}(n_k)_{\mathbb{S}(k)} & \xleftarrow{\pi} & \check{\mathcal{O}}(k) \otimes_{\mathbb{S}_k} \bigotimes_{i=1}^k \check{\mathcal{O}}(n_k) \\
\uparrow & & & & \uparrow p \\
\check{\mathcal{O}}(S') & \xrightarrow{\check{\gamma}_{f'}} & & & \check{\mathcal{O}}(S') \otimes \bigotimes_{T'} \check{\mathcal{O}}(f')^{-1}(t'))
\end{array}
\qquad \sigma' : \check{\mathcal{O}}(S) \to \check{\mathcal{O}}(S'),\quad \sigma \otimes \bigotimes_T \sigma'_t
\tag{2.38}
$$

where

(1) $colim_{Iso(n)}\check{\mathcal{O}} = \mathcal{O}(n)_{\mathbb{S}_n}$ are the coinvariants.
(2) $colim_{Iso(n.k)}\check{\mathcal{O}} = (\check{\mathcal{O}}(k) \otimes_{\mathbb{S}_k} \bigotimes_{i=1}^k \check{\mathcal{O}}(n_k)$, where $\mathbb{S}(k)$ acting anti–diagonally yields the relative tensor product.
(3) These project to the full coinvariants under the $\mathbb{S}(k) \times \mathbb{S}(k)$ action: $\check{\mathcal{O}}(k)_{\mathbb{S}_k} \otimes \bigodot_{i=1}^k \check{\mathcal{O}}(n_k)_{\mathbb{S}(k)}$ where $\bigodot$ denotes the symmetric tensor product,
(4) The map $\check{\gamma}$ exists by the universal property of colimits applied to $\check{\mathcal{O}}(n)_{\mathbb{S}_k}$ and the cocone given by the $\pi \circ p \circ \gamma_f$.

Remark 2.69. There is the intermediate possibility to keep the "comultiplication" as a morphism $\check{\gamma} : \check{\mathcal{O}}(n)_{\mathbb{S}(n)} \to \check{\mathcal{O}}(k) \otimes_{\mathbb{S}_k} \bigotimes_{i=1}^k \check{\mathcal{O}}(n_k)_{\mathbb{S}(k)}$. This is an interesting structure that has appeared for instance in [DEEFG16].

Definition 2.70. A cooperad with multiplication in finite sets, is a cooperad in finite sets with multiplications $\mu_{S,T} : \check{\mathcal{O}}(S) \otimes \check{\mathcal{O}}(T) \to \check{\mathcal{O}}(S \sqcup T)$, such that the following diagram commutes.

$$
\begin{array}{ccc}
\check{\mathcal{O}}(S) \otimes \check{\mathcal{O}}(T) & \xrightarrow{\mu_{S,T}} & \check{\mathcal{O}}(S \sqcup T) \\
\downarrow{\scriptstyle \sigma\otimes\sigma'} & & \downarrow{\scriptstyle \sigma\sqcup\sigma'} \\
\check{\mathcal{O}}(S') \otimes \check{\mathcal{O}}(T') & \xrightarrow{\mu_{S',T'}} & \check{\mathcal{O}}(S' \sqcup T')
\end{array}
\tag{2.39}
$$

and the analogue of (2.13) holds equivariantly.

Due to the diagram above and the equivariance:

Lemma 2.71. *For a cooperad with multiplication in finite sets the cooperad structure and the multiplication descend to the coinvariants.* □

Set $\mathscr{B}_{\mathbb{S}} = \bigoplus_n \check{\mathcal{O}}(n)_{\mathbb{S}_n}$. A bialgebraic counit ϵ is called invariant if for all $a_S \in \check{\mathcal{O}}(S)$ and any isomorphism $\sigma : S \to S'$, $\epsilon \circ \sigma = \epsilon$.

Proposition 2.72. *With the assumption above, $\mathscr{B}_{\mathbb{S}}$ is a non-unital, non-counital, bialgebra. If we furthermore assume that an invariant bialgebraic counit for $\mathscr{B}$ exists then $\mathscr{B}'_{\mathbb{S}} = k \oplus \mathscr{B}_{\mathbb{S}}$ is a unital and counital bialgebra and $\mathscr{H} := \mathscr{B}/\bar{\mathcal{I}}$, where $\bar{\mathcal{I}}$ is the image of $\mathcal{I}$ in $\mathscr{B}'_{\mathbb{S}}$ is a connected commutative Hopf algebra*

2.9.1. **The free example.** In the free example, starting with a symmetric operad, we do not only have to take the sum, but also induce the representation to $\mathbb{S}_n$ in order to obtain a symmetric cooperad with multiplication. Let

$$\check{\mathcal{O}}^{symnc}(n) = \bigoplus_k \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} Ind^{\mathbb{S}_n}_{(\mathbb{S}(n_1)\times\cdots\times\mathbb{S}(n_k))\wr\mathbb{S}(k)} \check{\mathcal{O}}(n_1) \otimes \cdots \otimes \check{\mathcal{O}}(n_k) \tag{2.40}$$

Remark 2.73. When taking coinvariants, this induction step is cancelled and we only have to take coinvariants with respect to $\mathbb{S}(n_1) \times \cdots \times \mathbb{S}(n_k) \times \mathbb{S}(k)$. That is

$$\mathscr{B}_{\mathbb{S}} = \bigoplus \check{\mathcal{O}}^{symnc}(n)_{\mathbb{S}_n} = \bigoplus_k \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} (\check{\mathcal{O}}(n_1)_{\mathbb{S}_{n_1}} \otimes \cdots \otimes \check{\mathcal{O}}(n_k)_{\mathbb{S}_{n_k}})_{\mathbb{S}_k} = \bigoplus_{(n_1,\dots,n_k):\sum_i n_i = n} \check{\mathcal{O}}(n_1)_{\mathbb{S}_{n_1}} \odot \cdots \odot \check{\mathcal{O}}(n_k)_{\mathbb{S}_{n_k}} \tag{2.41}$$

where $\odot$ is the symmetric product.

Proposition 2.74. *The $\check{\mathcal{O}}^{symnc}(n)$ form a symmetric cooperad with mutiplication and $\mathscr{B} = \bigoplus \check{\mathcal{O}}^{symnc}(n)_{\mathbb{S}_n}$ forms a bialgebra, and if $\check{\mathcal{O}}$ has an operadic counit, then $\mathscr{B}'$ is a unital an non–unital bialgebra. Furthermore if $\check{\mathcal{O}}(1)$ is almost connected, then the quotient $\mathscr{B}'/\mathcal{I}$ is a Hopf algebra.*

Proof. It is clear that the free multiplication then also satisfies (2.39) and the equivariant version of (2.13) holds. A counit for a symmetric cooperad is by definition a morphism $\check{\mathcal{O}}(\{s\}) \to k$ that is invariant under isomorphism, hence so is its extension. The rest of the statements are proved analogously to the non–symmetric case.

□

Remark 2.75. Since $\bar{\mathscr{H}}$ is commutative its dual is cocommutative and $\bar{\mathscr{H}}^* = U(Prim(\mathscr{H}^*))$ by the Cartier–Milnor–Moore theorem. This relates to the considerations of [Kau07, CL01]. We leave the complete analysis for further study.

Example 2.76. Reconsidering the examples in this new fashion, we see that:

(1) For the ordered surjections, in the symmetric version, we get all the surjections, since the permutation action induces any order. These are pictorially represented by forests of nonplanar corollas. Taking coinvariants makes these forests unlabelled.
(2) For the leaf labelled trees in the symmetric version, the planar trees become non–planar. Taking coinvariants kills the labelling of the leaves. The coproduct is then given by cutting edges with admissible cuts as described originally by Connes and Kreimer. See Figures 4 and 5.
(3) This carries on to the graded case like in Baues.

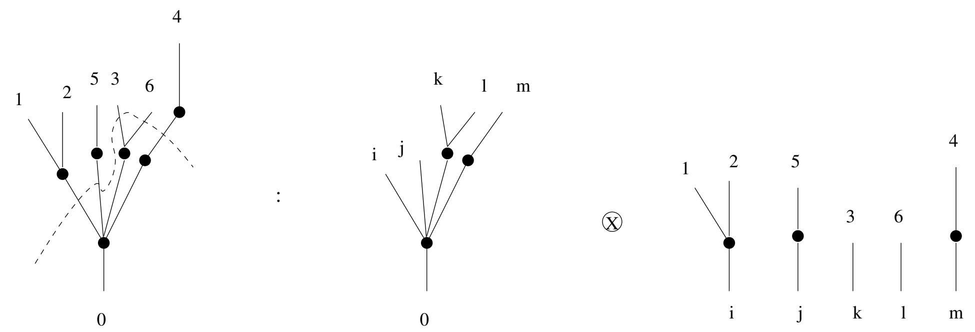


FIGURE 4. One term of the coproduct corresponding to the indicated cut in the labelled case.

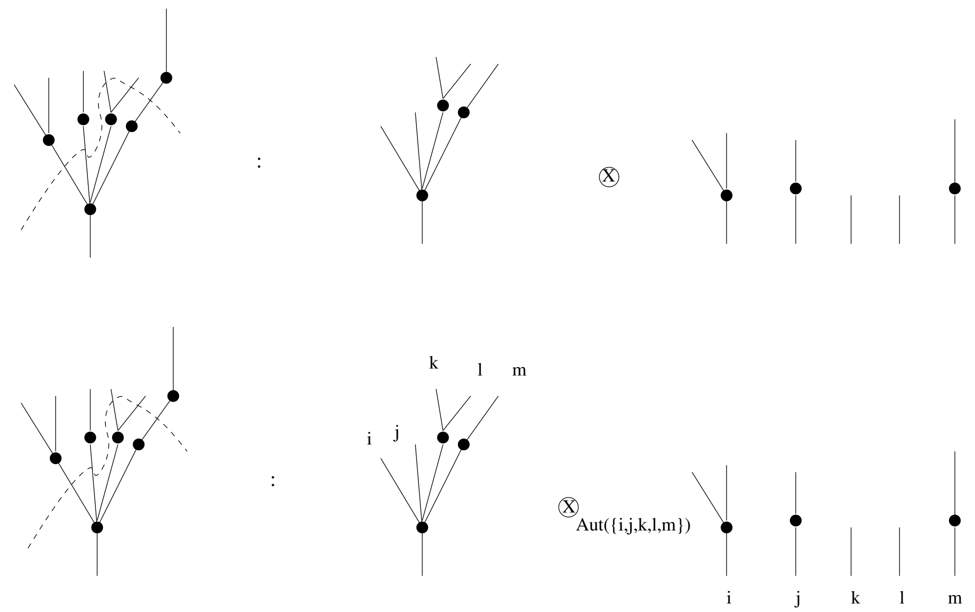


FIGURE 5. The first example corresponds to the same cut on the full coinvariants, viz. the unlabeled version where on the right side, the forest is a symmetric product. Alternatively, it can be seen as a term in the non–symmetric planar case. Then the right side is then an ordered tensor product. The second example depicts the results for the coinvarants yielding the relative tensor product according to Remark 2.68 part (2).

2.9.2. **Connes–Kreimer quotient.** To obtain the Hopf algebra of Connes and Kreimer on the nose, we have to take one more quotient and make one more assumption.

Definition 2.77. A (non–Σ) cooperad with multiplication has a clipping or amputation structure, if it has a cosemisimplical structure compatible with the structure of a cooperad with multiplication. That is

(1) there are maps $\sigma_i : \check{\mathcal{O}}(n) \to \check{\mathcal{O}}(n-1)$, and for $i \le j : \sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$
(2) For all n, and partition $(n_1, \dots, n_k)$ of n and each $1 \le i \le n$, with i in the n_j component of the partition and i_j its the position within this block:

$$\check{\gamma}_{n_1,\dots,n_j-1,\dots,n_k} \circ \sigma_i = (id \otimes id \otimes \cdots \otimes \sigma_{i_j} \otimes id \otimes \cdots \otimes id) \circ \check{\gamma}_{n_1,\dots,n_j,\dots,n_k} \tag{2.42}$$

(3) For a symmetric cooperad, we also demand compatibility with the permutation group actions.
(4) The σ_j are compatible with the multiplication.

with the factor σ_{i_j} in the j–th position in the biased definition. In the unbiased situation if $s \in S$, $\sigma_s : \mathcal{O}(S) \to \mathcal{O}(S \setminus s)$ given a partition $S_1, \dots, S_k$ of S with $i \in S_j$

$$\check{\gamma}_{S_1,\dots,S_j\setminus s,\dots,S_k} \circ \sigma_s = \sigma_s \circ \check{\gamma}_{S_1,\dots,S_k} \tag{2.43}$$

Using the short hand σ_i in this sense, it follows that $(id \otimes \sigma_i) \circ \Delta = \Delta \circ \sigma_i$.

In this case, we can take the colimit over the directed system given by the σ_i to obtain $\mathscr{B}^{amp} = \mathrm{colim}_\sigma \mathcal{O}$. Let $\mathscr{H}^{amp} = colim_\sigma \mathcal{O}$ and $\bar{\mathscr{H}}^{amp}$ be the respective quotients. The following is then straightforward from the compatibility demanded above.

Proposition 2.78. *$\mathscr{H}^{amp}$ is a Hopf algebra, and $\bar{\mathscr{H}}^{amp}$ is a commutative Hopf algebra.* □

2.9.3. **Adding a formal $\check{\mathcal{O}}(0)$ to compute $\mathscr{H}^{amp}$.** It is actually convenient to think of the elements representing the colimits as living in $\check{\mathcal{O}}(0)$ by extending the directed system of the σs by $\sigma_1 : \mathcal{O}(1) \to \mathcal{O}(0)$. In this case $\check{\mathcal{O}}(0)$ becomes a final object and the colimits are more easily computed. Notice that $\check{\mathcal{O}}(1)$ is not a final object for the clipping structure, as there are n morphisms from $\check{\mathcal{O}}(n)$ to $\check{\mathcal{O}}(1)$ "forgetting" all but one i. Considering the colimits to lie in an additional $\check{\mathcal{O}}(0)$ yields an equivalent formulation whose construction is more involved, but whose pictorial representations are more obvious.

Thus, we now allow $\check{\mathcal{O}}(0)$ and consider $\sigma_1 : \mathcal{O}(1) \to \mathcal{O}(0)$ with the conditions that

(1) $\check{\mathcal{O}}(0)$ only contains elements that are images of σ_i from higher $\check{\mathcal{O}}(n)$.
(2) Define a coproduct and product structure on $\check{\mathcal{O}}(0)$ by using representatives in $\check{\mathcal{O}}(n), n \ge 1$ and the cooperad and multiplication structure for the $\check{\mathcal{O}}(n), n \ge 1$.
(3) Identify $\sigma_1(|) = 1$ as the unit of the product on $\check{\mathcal{O}}(0)$.

It is clear then, that $\check{\mathcal{O}}(0)$ is just the colimit $\mathscr{B}^{amp}/\mathcal{I} = \mathscr{H}^{amp}$ with the induced structures.

Proposition 2.79. *Enlarging $\check{\mathcal{O}}$ in this way and taking the colimit directly yields $\check{\mathcal{O}}(0) = \mathscr{H}^{amp}$ and $\bar{\mathscr{H}}^{amp}$ as the symmetric quotient.* □

Example 2.80. The main example is the bialgebra of forests. Here a σ_i removes the i-th tail. The dual of the identity $|$ in $\check{\mathcal{O}}(1)$ is identified with $1 \in k$. The operation σ_i is to forget the i–th flag. This is either the i-th tail in the planar order or the tail labelled by i. What is left in the colimit are representatives which are trees without tails, sometimes called amputated trees [Kre99, BBM13].

The equation (2.43) then just states that it does not matter whether one first removes a tail and then cuts an internal edge, or vice–versa. Notice that if the tail itself is cut, the condition that $\sigma(|) = 1$ together with (2.43) says that in the results $|$ gets converted to 1.

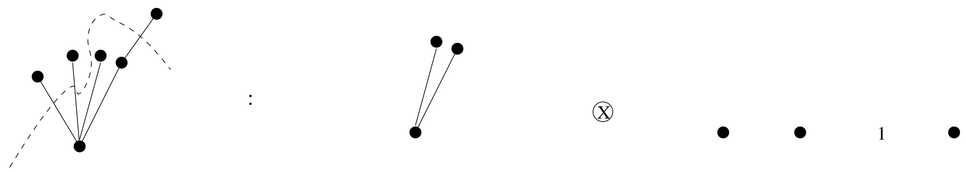


FIGURE 6. Coproduct for the amputated version. The same example for the amputated version: First all tails are removed. After cutting all newly formed tails are amputated and empty trees/forests are represented by $1 = 1_K$. Notice that indeed || from Figure 5 is set to 1_K as is done in the Hopf quotient.

This correpsonds to cuts "above" a leaf vertex or "below" the root vertex in the conventions of Connes and Kreimer and hence the coproduct is exactly that of Connes and Kreimer, both in the commutative and noncommutative case. For the latter one considers planar trees, which need not have labels on the flags, since they come in a fixed order.

A pictorial representation is given in Figure 6.

The computation in the colimit version can be made using the formalism of Appendix A, §A.4.

2.10. **Coaction.**

2.10.1. $\check{\mathcal{O}}(0)$ **and coaction.** As we have seen in the last section, it sometimes does make sense to include $\check{\mathcal{O}}(0)$. The reason why we did not consider it before, was that this would be a potential hindrance to being conilpotent and may cause issues for summing over the Δ_k.

There is a remedy in which $\check{\mathcal{O}}(0)$ can be viewed as a coalgebra over a cooperad and then as a comodule over the Hopf algebra.

For this consider a cooperad with multiplication and $\check{\mathcal{O}}(0)$. Then set $\mathscr{B} = \bigoplus_{n\geq 1} \check{\mathcal{O}}(n)$ as before, omitting the zero summand, and using the cooperad structure which is defined by using only the terms of the coproduct of the form $\check{\gamma}_{n_1,\dots,n_k}$ with k and $n_i > 0$. $\check{\mathcal{O}}(0)$ then becomes a coalgebra over the cooperad via $\check{\gamma}_{0^k} := \check{\gamma}_{0,\dots,0} : \check{\mathcal{O}}(0) \to \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(0)^{\otimes k}$, $k > 0$. We will assume that this coaction is locally finite, that is for any $a \in \check{\mathcal{O}}(0)$ there sum over all $\check{\gamma}_{0^k}(a)$ is finite. Notice that $\mu : \check{\mathcal{O}}(0) \otimes \check{\mathcal{O}}(0) \to \check{\mathcal{O}}(0)$ which makes $\check{\mathcal{O}}(0)$ into an algebra whose multiplication is compatible with the $\check{\gamma}_{0^k}$ by definition. Summing over all the $\check{\gamma}_{0^k}$ and post–composing with the multiplication on the factors of $\check{\mathcal{O}}(0)^{\otimes k}$ we get a coalgebra map

$$\check{\rho} : C \to \mathscr{B} \otimes C \tag{2.44}$$

where $C = \check{\mathcal{O}}(0)$

2.10.2. **Motivating examples.** Consider $\mathcal{O}$ with $\mathcal{O}(0)$ then $\mathcal{O}(0)$ is an algebra over $\mathcal{O}^{\oplus} = \bigoplus_{n\geq 1} \mathcal{O}(n)$ via: $\gamma : \mathcal{O}(k) \otimes \mathcal{O}(0)^{\otimes k} \to \mathcal{O}(0)$. Dually, we see that $\check{\mathcal{O}}(0)$ is a coalgebra over the cooperad $\check{\mathcal{O}}(n)$. This construction extends to $\check{\mathcal{O}}^{nc}$, where $\check{\mathcal{O}}^{nc}(0) = T\check{\mathcal{O}}(0) =$

$\bigoplus_n \check{\mathcal{O}}(0)^{\otimes n}$. Now, we do have a well defined cooperad with multiplication structure on $\mathscr{B} = \bigoplus_{n\geq 1} \check{\mathcal{O}}^{nc}(n)$ and by restriction, we have a comodule given by extending

$$\check{\mathcal{O}}(0) \xrightarrow{\check{\rho}} \bigoplus_{k\geq 1} \check{\mathcal{O}}(k) \otimes \check{\mathcal{O}}(0)^{\otimes k} \tag{2.45}$$

Example 2.81. In the case of trees, we can also consider trees without tails. These will have leaf vertices, i.e. unary non-root vertices. Declaring that the admissible cuts are only those that leave a trunk that has only tails and no leaf vertices, and branches that only have leaf vertices and no tails, we get a natural coalgebra structure. This is precisely the coaction (2.45).

The construction is related to §2.9.3 in that the latter differs from the former only in applying the colimit deleting the tails on the left hand side as well.

2.10.3. **Coalgebras.** We can start with any coalgebra C over a cooperad $\check{\mathcal{O}}$. Such a coalgebra by definition has $\check{\rho}^n : C \to \check{\mathcal{O}}(n) \otimes C^{\otimes n}, n > 0$. We now add the datum of an associative algebra structure for C: $\mu_C : C \otimes C \to C$, which is compatible with the coalgebra over the cooperad $\check{\mathcal{O}}$ in the usual way. We furthermore assume that the $\check{\rho}^n$ are locally finite. Then we can define coalgebra maps $\check{\rho} : C \to \mathscr{B} \otimes C$ by setting $\mathscr{B} = \bigoplus_{n\geq 1} \check{\mathcal{O}}(n)$ as usual and defining the coction by $\check{\rho} = \sum_n \mu^{\otimes n-1} \circ \check{\rho}^n$.

2.11. **Grading.** Before taking the Hopf quotient there was the grading by $n - p$ for the graded case and filtered accordingly in the general case. Now, $| - 1$ has degree 0, so the grading descends to the quotient $\mathscr{H} = \mathscr{B}/\mathcal{I}$. This works in the symmetric and the non–symmetric case. In the amputation construction, this grading will not prevail as the colimit kills the operadic grading. However, in the case that $\mathscr{H}$ is indeed a connected Hopf algebra, there is the additional filtration by the coradical degree r. We can lift this coradical filtration to $\mathscr{B}$ and $\mathscr{H}_q$.

Lemma 2.82. *For an almost connected cooperad with multiplication, the coradical filtration lifts to $\mathscr{B}$ and $\mathscr{H}_q$. The lifted coradical filtration $\mathcal{R}$ is compatible with multiplication and comultiplication and in particular satisfies. $\bar{\Delta}(\mathcal{R}^d) \subset \mathcal{R}^{d-1} \otimes \mathcal{R}^{d-1}$. This filtration descends to $\mathscr{H}^{amp}$ and $\bar{\mathscr{H}}^{amp}$ respectively.*

Proof. The coradical degree of an element a is given by its reduction, in which any occurrence of $|$ is replaced by 1. Since the lift or $|$ will lie in $\mathcal{R}^0$ and both 1 and $|$ are group like due to the bi–algebra equation the filtration is compatible with the multiplication and the comultiplication. Due to the form of Δ in Proposition 2.46, see (2.26), we see that the first term of Δ descends as the only term of the type to $1 \otimes a + a \otimes 1$ and hence $\bar{\Delta}$ descends to $\bar{\Delta}$ on $\mathscr{H}$. This shows the claimed property of $\bar{\Delta}$ on $\mathscr{B}$. The fact that the filtration descends through amputation is clear. □

Lemma 2.83. *For a almost connected cooperad with multiplication the depth filtration descends to $\mathscr{H}_q$ and $\mathscr{H}$ and satisfies $\bar{\Delta}_n(F^{\geq p}) \subset \bigoplus F^{\geq p} \otimes F^{\geq 1}$. The depth of $1 \in \mathscr{H}$ is 0. This filtration descends to $\mathscr{H}^{amp}$ and $\bar{\mathscr{H}}^{amp}$ respectively.*

Proof. Since $|$ is grouplike $\Delta(a|) = \Delta(a)(|\otimes|)$, it is clear that the depth filtration descends to $\mathscr{H}_q$. Now any lift of $a \in \mathscr{H}$ to $\mathscr{H}_q$ is of the form aq^k. We define the depth of $a \in \mathscr{B}$ to be the minimal depth of a lift or equivalently for any lift the difference between the depth and the q degree. This give 1 depth 0. The relation then follows from Proposition 2.23 and Lemma 2.66. The fact that the filtration descends through amputation is clear. □

Definition 2.84. We call the coradical filtration of $\mathscr{B}$ and consequentially of $\mathscr{H}_q, \mathscr{H}$ well behaved, if

$$\bar{\Delta}(\mathcal{R}^i) \subset \bigoplus_{p+q=i} \mathcal{R}^p \otimes \mathcal{R}^q \tag{2.46}$$

We will use the same terminology for all the cases $\mathscr{H}$,$\bar{\mathscr{H}}$, $\mathscr{H}^{amp}$ and $\bar{\mathscr{H}}^{amp}$.

Since μ is always additive in the coradical filtration due to the bi–algebra equation, we have that the coradical filtration respects both multiplication and comultiplication.

Proposition 2.85. *If the coradical filtration is well behaved, then $\mathscr{H}$,$\bar{\mathscr{H}}$, $\mathscr{H}^{amp}$ and $\bar{H}^{amp}$ is graded by the coradical degree.* □

Remark 2.86. In the free case $\check{\mathcal{O}}^{nc}$, if $\check{\mathcal{O}}(1)$ is reduced, then the maximal coradical degree for an element in $\check{\mathcal{O}}(n,p)$ is indeed $n-p$. To see this consider $\check{\mathcal{O}}(n)$, applying $\bar{\Delta}$ we generically get a term $\check{\mathcal{O}}(n-1)\otimes\check{\mathcal{O}}(2)\otimes\check{\mathcal{O}}(1)^{\otimes n-2}$, repeating this procedure and "peeling off" an $\check{\mathcal{O}}(2)$ then the maximal coradical degree will be $n-1$. Both n and p are additive under μ which finishes the argument.

Example 2.87. Goncharov's and Baues' Hopf algberas are examples where this maximum is attained. Indeed any surjection $\underline{n} \twoheadrightarrow \underline{1}$ factors as $(\pi \amalg id \amalg \cdots \amalg id)\circ\cdots\circ(\pi \amalg id)\circ\pi : \underline{n} \twoheadrightarrow \underline{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \underline{1}$ where $\pi : \underline{2} \twoheadrightarrow \underline{1}$ is the unique surjection. Using Joyal duality, see Appendix C.1, the same holds true for base–point preserving injections.

Example 2.88. Another instructive example is the Connes–Kreimer Hopf–algebra of rooted forests with tails. Here the coradical degree of a tree is simply $E+1 = V$, where E is the number of edges and V is the number of vertices. This is so, since each application of $\bar{\Delta}$ will cut at least one edge and cutting just one edge is possible. Since we are dealing with a tree $E+1 = V$. For a forest with p trees, this it is $E+p = V$

This is the grading that descends to $\mathscr{H}^{amp}$. The same reasoning holds for the symmetric and the non–symmetric case.

Now there are two different gradings. The coradical degree and the original grading by $n-p$. There is a nice relationship here. Notice that n is the number of tails, p is the number of roots thus for a forest the number of flags $F = n+p+2E = n-p+2(E-p) = n-p+2V$ and this is a third grading that is preserved. It is important to note that in the Hopf algebra $| = 1$ and does not count as a flag.

Vice–versa since the flag grading and the $n-p$ grading are preserved it follows that the coradical grading is preserved, giving an alternative explanation of it.

Proposition 2.89. *The free construction $\check{\mathcal{O}}^{nc}$ on the dual $\check{\mathcal{O}}$ of a unital operad $\mathcal{O}$ with an almost connected $\check{\mathcal{O}}(1)$ has a well behaved coradical grading.*

Proof. Fix an element $\check{a}$ dual to $a \in \mathcal{O}(n)$. Due to the conditions there are finitely many iterated γ compositions that result in a. Each of these can be presented by a level tree whose vertices v are decorated by elements of $\mathcal{O}(|v|-1)$. If we delete the nodes decorated by the identities, we remain with trees with vertices decorated by non–identity operad elements, see [KW17][2.2.1] for details about this construction. The number of edges of the tree then represent the number of operadic concatenations, and dually the number of cooperad and Δ operations that are necessary to reach the decomposition. It follows that the coradical degree is equal to the maximal value of $E+1=V$. This is easily seen to be additive under μ and preserved under Δ. The computations is parallel to the one above for the Connes–Kreimer algebra of trees, only that now the vertices are decorated by operad elements. In this picture, V is also the word length of the expression of an element as an itereated application of $\circ_i$ operations. □

This proposition also reconciles the two examples, Goncharov and Connes–Kreimer. Futhermore it explains the "lift" of Goncharov to the Hopf algebra of trivalent forests. Indeed the expression in Example 2.87 is word of length $n-1$ represented by a binary rooted tree. See also Example §4.55.

3. Cooperads from Simplicial Objects

In this section, we present an important (but accessible) construction of some cooperads with multiplication. This construction is best expressed in the language of simplical objects, and so we will first recall some of the basic notions. Some of the examples, however, can be understood with no simplicial background. For an arbitrary set S, we will see that the set X of all sequences or words in S has the structure of a cooperad, and Goncharov's Hopf algebra may be obtained from the case $S=\{0,1\}$. Elements of X can be understood as strings of consecutive edges in the complete graph (with vertex loops) K_S, and further geometric intuition can be obtained by considering also strings of triangles or more generally n-simplices. The way to encode this construction is to think of the graph K_S as defining a groupoid $G(S)$, i.e. a category whose morphisms are invertible. The set of objects is S and for any pair of objects there is a unique invertible morphism between them. The transition to the simplicial setting is then made by considering the nerve of this category.

In fact, our construction defines a cooperad with multiplication, and hence a bialgebra (or Hopf algebra) for any (reduced) simplicial set X, see Proposition 3.8. In this guise, we also recover the Hopf algebra of Baues.

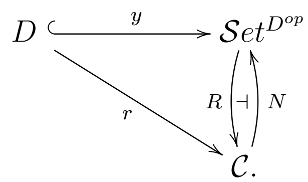
3.1. **Recollections: the simplicial category and simplicial objects.** Let Δ be the small category whose objects are the finite non-empty ordinals $[n]=\{0<1<\cdots<n\}$ and whose morphisms are the order-preserving functions between them. Of course, each $[n]$ can itself be regarded as a small category, with objects $0,1,\dots,n$ and a (unique) arrow $i\to j$ iff $i\le j$, and order preserving functions are just functors. Thus Δ is a full subcategory of the category of small categories.

Among the order-preserving functions $[m]\to[n]$ one considers the following generators: the injections $\partial^i:[n-1]\to[n]$ which omit the value i, termed coface maps, and the surjections $\sigma^i : [n+1] \to [n]$ which repeat the value i, termed codegeneracy maps. These maps satisfy certain obvious cosimplicial relations.

For D a small category, and $\mathcal{C}$ any category, we can consider the contravariant functors or the covariant functors X from D to $\mathcal{C}$. For $D = \Delta$ these are termed the simplicial and the cosimplicial objects in $\mathcal{C}$. A functor $D^{op} \to \mathcal{S}et$ is *representable* if it is $\hom_D(-, d)$ for some object d. In general, such functors are also called pre–sheaves on $\mathcal{D}$. If $\mathcal{D}$ is monoidal then so is the category of pre=sheaves, with the product given by Day convolution. The Yoneda Lemma gives a bijection between the set of natural transformations $\hom_D(-, d) \to X$ and the set $X(d)$, and in particular $d \mapsto \hom_D(-, d)$ defines a full embedding y of D into the functor category $\mathcal{S}et^{D^{op}}$. This category together with the embedding y is also called the cocompletion and has the universal property that any functor from D to a cocomplete category (one that contains all colimits) factors through it.

The following result is central to the classical theory and in particular for us it will yield the construction of a nerve of a small category.

Lemma 3.1. *Let D be a small category and $\mathcal{C}$ a cocomplete category. Any functor $r : D \to \mathcal{C}$ has a unique extension along the Yoneda embedding to a functor $R : \mathcal{S}et^{D^{op}} \to C$ with a right adjoint N,*



If $r : D \to \mathcal{C}$ is a monoidal functor between monoidal categories, then R sends monoidal functors $D^{op} \to \mathcal{S}et$ to monoids in $\mathcal{C}$.

The functor R is sometimes denoted $(-) \otimes_D r$, where the tensor over D is thought of as giving an object of D for every pair of D^{op}- and D-objects in $\mathcal{C}$, analogously to the language of tensoring left and right modules or algebras over a ring. The right adjoint N is termed the *nerve*, and is given on objects by

$$N(C) = \hom_{\mathcal{C}}(r(\text{-}), C).$$

Now a simplicial object is determined by the sequence of objects X_n, and the face and degeneracy maps $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$, given by the images of $[n]$, and ∂^i and σ^i, and dually for cosimplicial objects. Maps $X \to Y$ of (co)simplicial objects, that is, natural transformations, are just families of maps $X_n \to Y_n$ that commute with the (co)face and (co)degeneracy maps.

We write $\Delta[n]$ for the representable simplicial set $\hom_\Delta(-, [n])$ so, by Yoneda, simplicial maps $\Delta[n] \to X$ are just elements of X_n and maps $\Delta[m] \to \Delta[n]$ are just order preserving maps $[m] \to [n]$. For such a map α we use the notation $\alpha^* = X(\alpha) : X_n \to X_m$ and

$$x_{(\alpha_0, \dots, \alpha_m)} \in X_m$$

to denote the image under α^* of an n-simplex x in a simplicial set X.

If $D = \Delta$ and X is a simplicial set then $R(X)$ is usually called the *realization* of a simplicial set with respect to the models r. Considering for example the embedding $r : \Delta \to \mathcal{C}at$ we obtain the notion of the simplicial nerve of a category: for C a small category, there is a natural bijection between the functors from $[n]$ to C and the n-simplices of the nerve NC,

$$N(C)_n = \hom_{\mathcal{C}at}([n], C).$$

Example 3.2. Let S be a set, and let $X(S)$ be the simplicial set given by the nerve of the *contractible $G(S)$ with object set S,*

$$X(S) = NG(S).$$

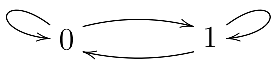
If $S = [n]$, for example, we may identify $G(S)$ with the fundamental groupoid of $\Delta[n]$, and

$$X([n]) \cong N\pi_1\Delta[n].$$

Giving a functor from $[n]$ to the contractible groupoid $G(S)$ is the same as giving the function on the objects, so an n-simplex of $X(S)$ is just a sequence of $n+1$ elements of S,

$$X(S)_n \;=\; S^{n+1} \;=\; \{\,(a_0; a_1, a_2, \ldots, a_{n-1}; a_n)\;:\; a_i \in S\,\}.$$

In the case $S = \{0, 1\}$, the groupoid $G(S)$ is



and the n-simplices of X are words of length $n+1$ in the alphabet $\{0, 1\}$.

3.2. **The operad of little ordinals.** The category of small categories, and the category of simplicial sets, can be regarded as monoidal categories with the disjoint union playing the role of the tensor product, and the initial object $\varnothing$ the neutral object. In this context, we have the following result, compare for example [DK12, Example 3.6.4].

Proposition 3.3. *The sequence of finite nonempty ordinals $([n])_{n\geq 0}$ forms an operad in the category of small categories. For any partition $n = m_1 + m_2 + \cdots + m_k$, consider the subset $\{0 = n_0 < n_1 < n_2 < \cdots < n_k = n\}$ of $[n]$ given by $n_r = m_1 + \cdots + m_r$. Then the structure map*

$$\gamma_{m_1,\ldots,m_k} = (\gamma^0, \gamma^1, \ldots, \gamma^k) : [k] \cup [m_1] \cup \cdots \cup [m_k] \to [n]$$

is defined by

$$\gamma^0(i) = n_i \quad (0 \leq i \leq k) \text{ and } \gamma^r(j) = n_r + j \quad (0 \leq j \leq m_r, 1 \leq r \leq k).$$

This operad clearly has a unit $u : \varnothing \to [1]$.

This construction is related, via Joyal duality (see Appendix C), to the factorisations of maps $\underline{n} \to \underline{1}$ into order preserving surjections $\underline{n} \to \underline{k} \to \underline{1}$, where $\underline{n} = \{1, \ldots, n\}$. Under the Joyal duality between end-point preserving ordered maps —see Appendix C— $[k] \to [n]$ and ordered maps $\underline{n} \to \underline{k}$, the injection $\gamma^0 : [k] \to [n]$ defined in the Proposition corresponds to the order preserving surjection $\underline{n} \to \underline{k}$ whose fibres over each i have cardinality m_i (see Figure 7).

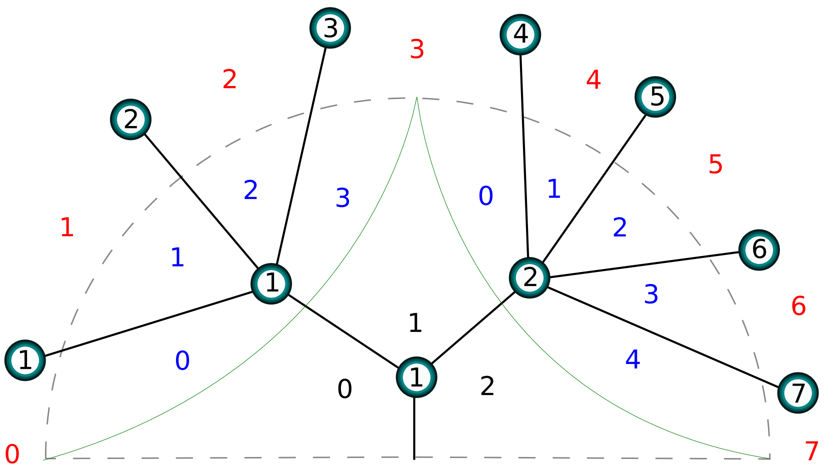


FIGURE 7. An example of a factorization $\underline{7} \to \underline{2} \to \underline{1}$ of order preserving surjections and, reading outwards from the root to the leaves, the corresponding operad structure map $\gamma_{3,4} : [2] \cup [3] \cup [4] \to [7]$.

The image of the operad structure in Proposition 3.3 under the Yoneda embedding gives:

Corollary 3.4. *The collection of representable simplicial sets $(\Delta[n])_{n\geq 0}$ forms a unital operad in the category of simplicial sets.*

If X is a simplicial set, then the unital operad structure on the sequence $\Delta[n]$, $n \geq 0$, induces a counital cooperad structure on the sequence $X_n = \hom(\Delta[n], X)$. That is, the sequence $(X_n)_{n\geq 0}$ forms a counital cooperad with

$$\begin{array}{ccccccccc} X_n & \xrightarrow{\check{\gamma}_{m_1,\dots,m_k}} & X_k & \times & X_{m_1} & \times & \dots & \times & X_{m_k} \\ x & \longmapsto & \big(x_{(n_0,n_1,\dots,n_k)}, & & x_{(n_0,n_0+1,\dots,n_1)}, & & \dots, & & x_{(n_{k-1},n_{k-1}+1,\dots,n_k)})\big) \end{array} \tag{3.1}$$

where $0 = n_0 < n_1 < n_2 < \cdots < n_k = n$ are given by $n_r = m_1 + \cdots + m_r$ as usual. The counit is given by the unique map

$$X_1 \to \{*\}.$$

More generally:

Corollary 3.5. *Let X be a simplicial object in a category $\mathcal{C}$ with finite products. Then the sequence $(X_n)_{n\geq 0}$ forms a counital cooperad in $\mathcal{C}$.*

Example 3.6. The set of all words in a given alphabet S is naturally a simplicial set (see Example 3.2 above) and so by Corollary 3.5 it forms a counital cooperad X in the category of sets. The elements of arity n in this cooperad are the words of length $n+1$ in S,

$$X_n \quad = \quad S^{n+1} \quad = \quad \{\, (a_0; a_1, a_2, \dots, a_{n-1}; a_n) \;:\; a_i \in S \,\}$$

and the operation $\check{\gamma}_{m_1,\dots,m_k}$ sends such an element $(a_0; a_1, a_2, \dots, a_{n-1}; a_n)$ to

$$\big((a_{n_0}; a_{n_1}, \dots; a_{n_k}),\ (a_{n_0}; a_{n_0+1}, \dots; a_{n_1}), \dots, (a_{n_{k-1}}; a_{n_{k-1}+1}, \dots; a_{n_k})\big)$$

where $n_0 = 0$, $n_k = n$ and $n_r - n_{r-1} = m_r$.

This construction can also be carried out in an algebraic setting.

Proposition 3.7. *Let X be a simplicial set, and let $\check{\mathcal{O}}(n)$ be the free abelian group on the set X_n, for each $n \geq 0$. Then $\check{\mathcal{O}}$ forms a counital cooperad in the category of abelian groups, with the cooperadic structure given by*

$$\begin{array}{ccccccccc} \check{\mathcal{O}}(n) & \xrightarrow{\check{\gamma}} & \check{\mathcal{O}}(k) & \otimes & \check{\mathcal{O}}(m_1) & \otimes & \dots & \otimes & \check{\mathcal{O}}(m_k) \\ x & \longmapsto & x_{(n_0,n_1,\dots,n_k)} & \otimes & x_{(n_0,n_0+1,\dots,n_1)} & \otimes & \dots & \otimes & x_{(n_{k-1},n_{k-1}+1,\dots,n_k)}) \end{array}$$

and the counit given by the augmentation

$$\check{\mathcal{O}}(1) \longrightarrow \mathbb{Z}.$$

Proof. This follows by applying free abelian group functor (which carries finite cartesian products of sets to tensor products) to the cooperad structure considered in (3.1). □

From section 2.2.3 we therefore have

Proposition 3.8. *Let X be a simplicial set. The cooperad structure $\check{\mathcal{O}}$ on $(\mathbb{Z}X_n)_{n\geq 1}$ of the previous proposition extends to a structure of a cooperad with (free) multiplication, and hence to a graded bialgebra structure, on the free tensor algebra*

$$\mathscr{B}(X) = \bigoplus_n \check{\mathcal{O}}^{nc}(X)(n) = \bigoplus_{n_1,n_2,\dots\geq 1} \bigotimes_i \mathbb{Z}X_{n_i}$$

generated by X, where elements of $\mathbb{Z}X_n$ have degree $n-1$.

3.2.1. **Goncharov's first Hopf algebra.** Let S be the set $\{0, 1\}$. We considered in Example 3.2 the contractible groupoid $G(S)$ with object set S, and the simplicial set $X = X(S)$ given by its simplicial nerve. If we denote the simplices of X_n by tuples $(a_0; a_1, \dots, a_{n-1}; a_n)$ as in Example 3.6 and apply Proposition 3.8 we obtain a graded bialgebra

$$\mathscr{B}(X) = \mathbb{Z}[(a_0; a_1, \dots, a_{n-1}; a_n); a_i \in \{0, 1\}]$$

with the coproduct that sends a polynomial generator $(a_0; a_1, \dots, a_{n-1}; a_n)$ in degree $n-1$ to

$$\sum_{0=n_0<n_1<\dots<n_k=n} (a_{n_0}; a_{n_1}, \dots; a_{n_k}) \otimes \prod_{i=0}^{k-1} (a_{n_i}; a_{n_i+1}, \dots; a_{n_{i+1}})$$

When we identify all generators in degree 0 we obtain Goncharov's connected graded Hopf algebra $\mathscr{H}_G$, as in Theorem 1.2.

For any simplicial set X, let $C_n(X)$ be the free abelian group on the n-simplices X_n. This defines a chain complex $(C(X), d_X)$ where

$$d_X(x) = \sum_{i=0}^{n} (-1)^i d_i x.$$

Diagonal approximation makes CX a differential graded coalgebra,

$$C(X) \longrightarrow C(X \times X) \longrightarrow CX \otimes CX$$

whose classical cobar construction is the tensor algebra on the desuspension of the reduced coalgebra

$$\Omega CX = (T\Sigma^{-1}\overline{C}X, d_\Omega)$$

where the differential d_Ω is formed from d_X and the coproduct. For the moment, however, we merely observe that if one takes the symmetric rather than the tensor algebra then the underlying graded abelian group is isomorphic to Goncharov's $\mathscr{H}_G$.

3.3. **Simplicial strings.** For $(D, \otimes)$ a strict monoidal category, consider $(\Omega' D, \boxtimes)$ the strict monoidal category generated by D together with morphisms $a \boxtimes b \to a \otimes b$ for objects a, b of D, subject to the obvious naturality and associativity relations. In this way a strict monoidal functor on $\Omega' D$ is exactly a (strictly unital) *lax monoidal* functor on D: a functor F on D together with maps $Fa \otimes Fb \to F(a \otimes b)$ satisfying appropriate naturality and associativity conditions.

Definition 3.9. Let $\Delta_{*,*}$ be the strict monoidal category given as the subcategory of Δ containing just the generic (that is, end-point preserving) maps $[m] \to [n]$, with the monoidal structure $[p] \otimes [q] = [p+q]$ given by identifying $p \in [p]$ and $0 \in [q]$.

We define the category of *simplicial strings* $\Omega\Delta$ to be the strict monoidal category $\Omega' \Delta_{*,*}$.

This agrees with Baues' construction in [Bau80, Definition 2.7]. Now a contravariant monoidal functor on the category of simplicial strings is just an oplax monoidal functor on $\Delta^{op}_{*,*}$. Explicitly, if $\mathcal{C}$ is a category with the cartesian monoidal structure, then to give a monoidal functor $(\Omega\Delta)^{op} \to \mathcal{C}$ is to give a functor $X : \Delta^{op}_{*,*} \to \mathcal{C}$ together with associative natural transformations $\mu_{p,q} = (\lambda_{p,q}, \rho_{p,q}) : X_{p+q} \to X_p \times X_q$. Note that X becomes a simplicial object, if we define outer face maps $X_n \to X_{n-1}$ by $d_0 = \rho_{1,n-1}$ and $d_n = \lambda_{n-1,1}$. Moreover these determine all maps $\rho_{p,q}$ and $\lambda_{p,q}$ via the naturality conditions $(d_1^{p-1} \times \mathrm{id})\mu_{p,q} = \mu_{1,q} d_1^{p-1}$ and $(\mathrm{id} \times d_1^{q-1})\mu_{p,q} = \mu_{p,1} d_{p+1}^{q-1}$. Thus we have:

Proposition 3.10. *Let $\mathcal{C}$ be a cartesian monoidal category. Then the following categories are equivalent:*

- *The category of simplicial objects in $\mathcal{C}$,*
- *The category of oplax monoidal functors $\Delta^{op}_{*,*} \to \mathcal{C}$,*
- *The category of monoidal functors $(\Omega\Delta)^{op} \to \mathcal{C}$.*

Given a simplicial object X, the corresponding oplax monoidal functor is given by the restriction of X to the endpoint preserving maps, with the structure map

$$(d^q_{p+1}, d^p_0) : X_{p+q} \to X_p \times X_q.$$

Definition 3.11. An *interval object* [BT97] (or a *segment* [BM06]) in a monoidal category $(\mathcal{D}, \otimes, \mathbb{1})$ is an augmented monoid $(L, L^{\otimes 2} \xrightarrow{\mu} L, \mathbb{1} \xrightarrow{\eta} L, L \xrightarrow{\varepsilon} \mathbb{1})$ together with an *absorbing object*, that is, $\overline{\eta} : \mathbb{1} \to L$ satisfying $\mu(\mathrm{id}_L \otimes \overline{\eta}) = \overline{\eta}\varepsilon = \mu(\overline{\eta} \otimes \mathrm{id}_L)$, $\varepsilon\overline{\eta} = \mathrm{id}_I$.

To any augmented monoid L one associates a simplicial object or, under Joyal duality, a covariant functor $L^\bullet$ on $\Delta_{*,*}$ with $L^0 = L^1 = \mathbb{1}$, $L^n = L^{\otimes(n-1)}$,

$$s^0 = \varepsilon \otimes \mathrm{id},\ s^n = \mathrm{id} \otimes \varepsilon,\ s^i = \mathrm{id} \otimes \mu \otimes \mathrm{id} : L^{\otimes n} \to L^{\otimes(n-1)},$$
$$d^i = \mathrm{id} \otimes \eta \otimes \mathrm{id} : L^{\otimes(n-2)} \to L^{\otimes(n-1)},$$

If in addition L has an absorbing object then $L^\bullet$ has a lax monoidal structure

$$\mathrm{id} \otimes \overline{\eta} \otimes \mathrm{id} : L^{\otimes(p-1)} \otimes L^{\otimes(q-1)} \to L^{\otimes(p+q-1)}$$

so we obtain a monoidal functor $L^\bullet : \Omega\Delta \to \mathcal{D}$.

Definition 3.12. Let X be a simplicial set, or the corresponding contravariant monoidal functor on the category of simplicial strings (Proposition 3.10). Baues' *geometric cobar construction* $\Omega_L X$ with respect to an interval object L in a cocomplete monoidal category $\mathcal{D}$ is defined as the monoid object in $\mathcal{D}$ given by the realisation functor (see Lemma 3.1),

$$\Omega_L(X) = X \otimes_{\Omega\Delta} L^\bullet$$

We have four fundamental examples:

(1) Let $L = [0, 1]$ be the unit interval in the category of CW complexes, with unit and absorbing objects $0, 1 : \{*\} \to [0, 1]$, and multiplication given by $\max : [0, 1]^2 \to [0, 1]$. Then the geometric cobar construction on a 1-reduced simplicial set is homotopy equivalent to the loop space of the realisation of X.

(2) Taking the cellular chains on the previous interval object we gives an interval object L in the category of chain complexes. In this case $\Omega_L(X)$ coincides with Adams' cobar construction, which has the same homology as the loop space on X, if X is 1-reduced.

(3) If we forget the boundary maps in example (2) we obtain an interval object L in the category of graded abelian groups, and $\Omega_L(X)$ coincides as an algebra with the object $\mathscr{B}(X)$ of Proposition 3.8: it is just the free tensor algebra whose generators in dimension n are the $n + 1$-simplices of X.

(4) Let $L = \Delta[1]$ in the category of simplicial sets, with unit and absorbing object d^1 and $d^0 : \Delta[0] \to \Delta[1]$, and multiplication $\mu : \Delta[1]^2 \to \Delta[1]$ defined by

$$\mu_n([n] \xrightarrow{x} [1], [n] \xrightarrow{x'} [1]) = (i \mapsto \max(x_i, x'_i)).$$

Berger has observed that, up to group completion, $\Omega_L X$ has the same homotopy type as the simplicial loop group GX of Kan.

Note that the CW complex given by the simplicial realisation of $\Delta[1]^2$ does not have the same cellular structure as $[0, 1]^2$: to relate examples (1–3) with (4) requires appropriate diagonal approximation and shuffle maps.

In example (3) the multiplication is free, and we have seen that the cooperad structure $\check{\gamma}$ on the simplicial set X gives a comultiplication and hence a bialgebra structure on $\Omega_L(X) = \mathscr{B}(X)$. Baues showed that essentially the same coproduct gives a differential graded bialgebra structure on $\Omega_L(X)$ in example (2), and used this to iterate the classical cobar construction to obtain an algebraic model of the double loop space. In example (4) we remain in the category of simplicial sets, and we have the following result:

Proposition 3.13. *Let X be a simplicial set, and $\Omega_L(X)$ the simplicial monoid given by the geometric cobar construction on X with respect to the interval object $L = \Delta[1]$. Then the cooperad structure $\check{\gamma}$ on X induces a map*

$$\Omega_L(X)_n \longrightarrow \prod_{m_1+\cdots+m_k=n} \Omega_L(X)_{k-1} \times \Omega_L(X)_{n-k}$$

for each $n, k \geq 1$

Proof. Let $Y = \Omega_L(X)$. For each partition $m_1 + \cdots + m_k = n$ the cooperad structure map $\check{\gamma}_{m_1,\dots,m_k}$ of (3.1) induces a map $Y_{n-1} \longrightarrow Y_{k-1} \times Y_{n-k}$ as follows. The map $\gamma_{m_1,\dots,m_k}$ of Proposition 3.3 restricts to give a bijection $\underline{k-1} \cup \underline{m_1-1} \cup \cdots \cup \underline{m_k-1} \to \underline{n-1}$ and hence an isomorphism

$$\Delta[1]^{n-1} \longrightarrow \Delta[1]^{k-1} \times \Delta[1]^{m_1-1} \times \cdots \times \Delta[1]^{m_k-1}.$$

Together with the map $\check{\gamma}_{m_1,\dots,m_k}$ of (3.1) this defines a map

$$X_n \times \Delta[1]^{n-1} \longrightarrow X_k \times \Delta[1]^{k-1} \times (X_{m_1} \times \Delta[1]^{m_1-1} \times \cdots \times X_{m_k} \times \Delta[1]^{m_k-1})$$

which induces the map on Y as required. □

3.4. **Comparison with Goncharov's second Hopf algebra.** We have seen above that Goncharov's first Hopf algebra $\mathscr{H}_G$ and Baues Hopf algebra $\Omega_L(X)$ are closely related. The differences between Baues' and Goncharov's algebras are as follows

- Baues' Hopf algebra has a differential, and the underlying graded abelian group $\mathscr{B}(X)$ is the free tensor algebra, that is, a free associative algebra. No differential is given on Goncharov's algebra, which is a free polynomial algebra, that is, a free commutative and associative algebra.
- To obtain a model for the double loop space Baues requires X to have trivial 2-skeleton (only one vertex, one degenerate edge, and one degenerate 2-simplex), but to construct Goncharov's bialgebra we take X to be 0-coskeletal (a unique n-simplex for any $(n+1)$-tuple of vertices). In the latter construction, however, one may still impose the relations $x \sim 1$ and $x \sim 0$ for 1- and 2-simplices x *after* taking the polynomial algebra (compare (1.3) and (1.10) respectively).

For Goncharov's second Hopf algebra $\tilde{\mathscr{H}}_G$, and the variants due to Brown, one imposes extra relations such as the shuffle formula (1.5). This has the following natural expression in the language of the cobar construction. Let $X = X(S)$, the 0-coskeletal simplicial set with vertex set $X_0 = S$. The cobar construction $\Omega_L X$ is a colimit of copies of $C(x_{n+1}) = L^{\otimes n}$ for each $(n+1)$-simplex $x_{n+1} = (s; w_n; s')$, where w_n is a word of length n in the alphabet S. In a symmetric monoidal category each $(p, n-p)$-shuffle corresponds to a natural isomorphism $L^{\otimes p} \otimes L^{\otimes(n-p)} \to L^{\otimes n}$ and the content of the shuffle relation is that this isomorphism i s also obtained from the shuffle of the letters of a word w_p with a word w_{n-p} to obtain a word w_n.

3.5. **Cubical structure.** Baues' and Goncharov's comultiplications come from path or loop spaces and may be seen having natural cubical structure. The space of paths P from 0 to n in the n-simplex $|\Delta[n]|$ is a cell complex homeomorphic to the $(n-1)$-dimensional cube.

Cubical complexes have a natural diagonal approximation,

$$\delta : P = [0,1]^{n-1} \xrightarrow{\cong} \bigcup_{K \cup L = \{1,\dots,n-1\}} \partial_K^-[0,1]^{n-1} \times \partial_L^+[0,1]^{n-1} \xrightarrow{\subset} P \times P$$

One can identify faces ∂_i^- of the cube P as the spaces of paths through the faces $x_{(0,\dots,\widehat{i},\dots,n)}$ of the n-simplex x. Faces ∂_i^+ are products of a $(i-1)$-cube and $(n-i-1)$-cube: the spaces of paths through $x_{(0,\dots,i)}$ and through $x_{(i,\dots,n)}$.

The term for $L = \{i_1, \dots, i_{k-1}\}$ under this identification is

$$x_{(0,i_1,\dots,i_{k-1},n)} \times x_{(0,1,\dots,i_1)}\, x_{(i_1,i_1+1,\dots,i_2)} \cdots x_{(i_{k-1},i_{k-1}+1,\dots,n)}.$$

which reproduces the summands of the coproduct.

The cubical stucture is illustrated for the case of Δ^3 in Figure 8

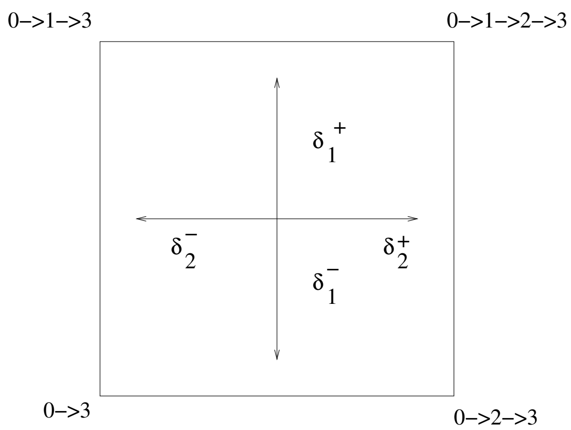


FIGURE 8. The cubical structure in the case of $n = 3$

To get into this analysis, we can choose two other alternative presentations. The first is given by a self–explanatory bar notation and the second is a parametrized notation. For the latter, we use $0 \xrightarrow{a} 1 \xrightarrow{b} 2 \xrightarrow{c} 3$. Then s, t are formal parameters. At $t = 1$ an edge disappears, while for $t = 0$ the morphisms are composed. The latter also explains the shuffles very nicely. Indeed in the usual diagonal approximation there is a shuffle of inner degeneracies. The degeneracies are the composition and the square modulo the symmetric group action yields the simplex. Lifting this yields the terms in the shuffle product.

The cubical stucture is also related to Cutkosky rules [Blo15, BK15, Kre16] Outer Space [CV03]. This natural appearance of cubical chains can be understood using decorated Feynman categories [KL16] and the W–construction [KW17], as explained in [BK18].

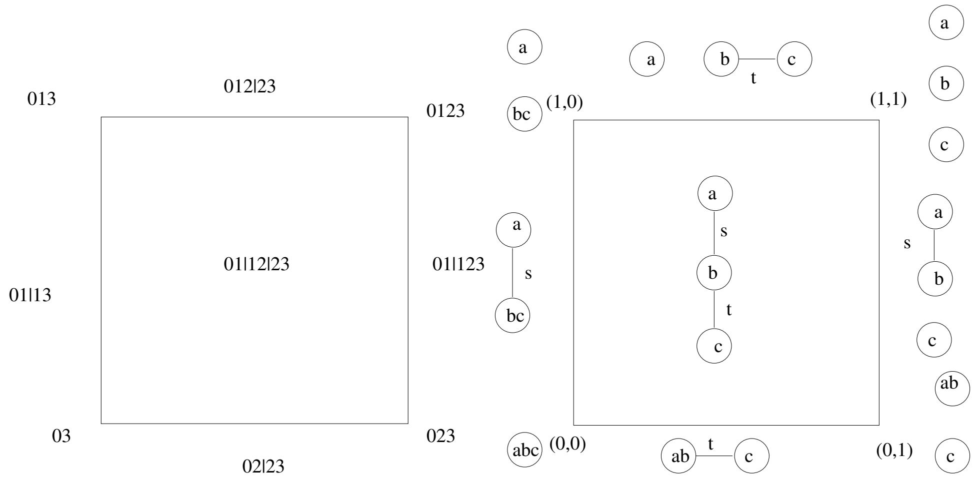


FIGURE 9. Two other renderings of the same square

4. Feynman Categories

4.1. **Definition of a Feynman category.** Consider the following data:

(1) $\mathcal{V}$ a groupoid, with $\mathcal{V}^{\otimes}$ the free symmetric monoidal category on $\mathcal{V}$.
(2) $\mathcal{F}$ a symmetric monoidal category, with monoidal structure denoted by $\otimes$.
(3) $\imath : \mathcal{V} \to \mathcal{F}$ a functor, which by freeness extends to a monoidal functor $\iota^{\otimes}$ on $\mathcal{V}^{\otimes}$,

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\imath} & \mathcal{F} \\ {\scriptstyle \jmath}\downarrow & \overset{\imath^{\otimes}}{\nearrow} & \uparrow \\ \mathcal{V}^{\otimes} & \longrightarrow & Iso(\mathcal{F}) \end{array}$$

where $Iso(\mathcal{F})$ is the maximal (symmetric monoidal) subgroupoid of $\mathcal{F}$.

Consider the comma categories $(\mathcal{F} \downarrow \mathcal{F})$ and $(\mathcal{F} \downarrow \mathcal{V})$ defined by $(id_{\mathcal{F}}, id_{\mathcal{F}})$ and $(id_{\mathcal{F}}, \imath)$.

Definition 4.1. A triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ as above is called a *Feynman category* if

(i) $\imath^{\otimes}$ induces an equivalence of symmetric monoidal groupoids between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.
(ii) $\imath$ and $\imath^{\otimes}$ induce an equivalence of symmetric monoidal groupoids $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.
(iii) For any object $*_v$ of $\mathcal{V}$, $(\mathcal{F} \downarrow *_v)$ is essentially small.

The first condition says that $\mathcal{V}$ knows all about the isomorphisms. The third condition is technical to guarantee that certain colimits exist. The second condition, also called the *hereditary condition*, is the key condition. It can be understood as follows:

(1) For any morphism $\phi : X \to X'$, if we choose $X' \simeq \bigotimes_{v\in I} \imath(*_v)$ by (i), there are X_v and $\phi_v : X_v \to \imath(*_v)$ in $\mathcal{F}$ such that ϕ is isomorphic to $\bigotimes_{v\in I}\phi_v$,

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ \downarrow \simeq & & \downarrow \simeq \\ \bigotimes_{v\in I} X_v & \xrightarrow{\bigotimes_{v\in I}\phi_v} & \bigotimes_{v\in I} \imath(*_v). \end{array} \tag{4.1}$$

(2) For any two such decompositions $\bigotimes_{v\in I}\phi_v$ and $\bigotimes_{v'\in I'}\phi'_{v'}$ there is a bijection $\psi : I \to I'$ and isomorphisms $\sigma_v : X_v \to X'_{\psi(v)}$ such that $P \circ \bigotimes_v \phi_v = \bigotimes_v(\phi'_{\psi(v)} \circ \sigma_v)$ where P is the permutation corresponding to ψ.
(3) These are the only isomorphisms between morphisms.

We call a Feynman category *strict* if the monoidal structure on $\mathcal{F}$ is strict, ι is an inclusion, and $\mathcal{V}^{\otimes} = Iso(\mathcal{F})$ where we insist on using the strict free monoidal category, see e.g. [Kau17] for a thorough discussion. Up to equivalence in $\mathcal{V}$, $\mathcal{F}$ and in $\mathfrak{F}$ this can always be achieved.

Note that the equivalence in (ii) cannot be assumed to be an equality. One can assume the left vertical arrow in (4.1) is an identity, but the right vertical arrow may be a non-trivial symmetry isomorphism in the free symmetric monoidal category $\mathcal{V}^{\otimes} = Iso(\mathcal{F})$, which will not decompose as a tensor of maps in $\mathcal{V}$.

4.1.1. **Non-symmetric version.** Now let $(\mathcal{V}, \mathcal{F}, \imath)$ be as above with the exception that $\mathcal{F}$ is only a monoidal category, $\mathcal{V}^{\otimes}$ the free monoidal category, and $\imath^{\otimes}$ is the corresponding morphism of monoidal groupoids.

Definition 4.2. A non-symmetric triple $\mathfrak{F} = (\mathcal{V}, \mathcal{F}, \imath)$ as above is called a *non-Σ Feynman category* if

(i) $\imath^{\otimes}$ induces an equivalence of monoidal groupoids between $\mathcal{V}^{\otimes}$ and $Iso(\mathcal{F})$.
(ii) $\imath$ and $\imath^{\otimes}$ induce an equivalence of monoidal groupoids $Iso(\mathcal{F} \downarrow \mathcal{V})^{\otimes}$ and $Iso(\mathcal{F} \downarrow \mathcal{F})$.
(iii) For any object $*_v$ in $\mathcal{V}$, $(\mathcal{F} \downarrow *_v)$ is essentially small.

Remark 4.3. If a Feynman category is strict then for any morphism ϕ we have $\phi = P \circ \bigotimes \phi_v$. Here $\phi_v : X_v \to \iota(*_v)$ and P is a symmetry isomorphism in $\mathcal{V}^{\otimes} = Iso(\mathcal{F})$, or is trivial in the non-Σ case.

4.1.2. **Native Length.** Notice that due to (i) every object X has a unique length $|X|$: the tensor word length of the object of $\mathcal{V}^{\otimes}$ representing it. We define the length decrease (or just length) of a morphism $\phi : X \to Y$ as $|\phi| = |X| - |Y|$. This is additive under composition and tensor. Isomorphic objects have the same length, so isomorphisms have length zero. Morphisms can also increase length, that is, have negative length (decrease), as one may have a morphism $\mathbb{1} \to \imath(*)$ which increases length by one and hence has length -1, see [KW17].

We call a Feynman category *non-negative* or *non-positive* if all morphisms have non-negative or non-positive length respectively and all morphisms of degree 0 are either invertible or do not have any left or right inverse. In either case, we call the Feynman category *definite*. One has extra structure in the definite case, see Lemma 4.46.

Remark 4.4. Sometimes it is enough to use native length, but other times different length functions are useful, see below §4.2.3. If we use such a refined length, then we will use the terms above according to this length. A more general example is that of a (proper) degree function in [KW17, Definition 7.2.1].

4.2. **Examples.**

4.2.1. **Basic Examples: $\mathfrak{S}urj, \mathfrak{S}urj_<, FI, FI_<, \mathcal{F}in\mathcal{S}et, \mathcal{F}in\mathcal{S}et_<, \Delta_+$.** Let $\mathcal{V} = \underline{*}$ be the trivial category with one object $*$ and its identity morphism id_*. $\mathcal{V}^\otimes$ will have the natural numbers $\mathbf{N}_0$ as objects $n = *^{\otimes n}$. In the non–symmetric case, the $\mathcal{V}^\otimes = \mathbf{N}_0$ will be discrete, while in the non–symmetric case $\mathcal{V}^\otimes = \mathbb{S}$ is the skeletal groupoid with $Hom(n,n) = \mathbb{S}_n$, the symmetric group. This category called $\mathbb{S}$ or Σ. For more details, see [Kau17].

The first Feynman category with trivial $\mathcal{V}$ is $\mathcal{F} = \mathcal{F}in\mathcal{S}et$, here $F = FinSet$ the category of finite sets. The inclusion of $*$ to the atom $\{*\}$ and the monoidal structure of disjoing union $\amalg$. The equivalence between $\mathbb{S}$ and $Iso(FinSet)$ is clear as $\mathbb{S}$ is the skeleton of $Iso(FinSet)$. Condition (iii) holds as well. Given any morphisms $S \to T$ between finite sets, we can decompose it using fibers as.

$$\begin{array}{ccc} S & \xrightarrow{\quad f \quad} & T \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle =} \\ \amalg_{t\in T} f^{-1}(t) & \xrightarrow{\quad \amalg f_t \quad} & \amalg_{t\in T}\{*\} \end{array} \tag{4.2}$$

where f_t is the unique map $f^{-1}(t) \to \{*\}$. Note that this map exists even if $f^{-1}(t) = \varnothing$. This shows the condition (ii), since any isomorphisms of this decomposition must preserve the fibers.

$\mathfrak{S}urj, FI$ are now the Feynman subcategories, where the maps are restricted to be surjections resp. injections. This means the none of the fibers is empty or all of the fibers are empty.

In the non–Σ case, $\mathcal{F} = \mathcal{F}in\mathcal{S}et_<$ has as $\mathcal{F}$ the category of ordered finite sets with order preserving maps and with $\amalg$ as monoidal structure. The image of n will be the set $\underline{n}$ with its natural order. Here $0 = \varnothing$. Viewing and order on S as a bijection to $\{1, \dots, |S|\}$, we see that $\mathbf{N}_0$ is the skeleton of $Iso(\mathcal{F}in\mathcal{S}et_<)$. The diagram (4.2) translates to this situation. And we obtain a non–Σ Feynman category.

Restricting to order preserving surjections and injections, we obtain the FCs $(\mathfrak{S}urj_<, FI_<)$. We can also restrict the skeleton of $\mathcal{F}in\mathcal{S}et_<$ given by Δ_+ and the subcategory of order preserving surjections and injections.

4.2.2. **Examples from graphs.** To give new examples beyond the cooperadic structure, motivated by questions from number theory of physics like the Feynman category of graphs of Connes and Kreimer, we use the language of [KW17, BM08]. The essential idea is that although there is a general category of graphs [BM08], one should consider graphs as morphisms not as objects. More precisely, morphisms from aggregates of corollas to aggregates of corollas and are indexed by graphs called ghost graphs. An aggregate of corollas is a graph without edges or loops, thus it is a collection of vertices with tails. Composition of morphisms induces the composition of the underlying graphs. This is given by inserting graphs into vertices. This is not to be confused with gluing at tails (see the appendix and [KW17]). Different varieties of Feynman categories are then given by restricting or decorating graphs in a manner respected by composition (see the appendix and the examples in §5).

A first new example is that of collections of 1-PI graphs, which we call the Broadhurst–Connes–Kreimer Feynman category.

Recall that a connected 1–PI graph is a connected graph that stays connected, when one severs any edge. A 1–PI graph is then a graph whose every component is 1–PI.

A nice way to write this is as follows [Bro17]. Let $b_1(\Gamma)$ be the first Betti number of the graph Γ. Then a graph is 1–PI if for any subgraph $\gamma \subset \Gamma$: $b_1(\gamma) < b_1(\Gamma)$. This means that 1-PI for non-connected graphs any edge cut decreases the first Betti (or loop) number by one.

Now, indeed, blowing up a vertex of a 1-PI graphs into a 1-PI graph leaves the defining property (namely connectivity) invariant which is easey to check with the above definition. Our second new example is Brown's Hopf algebra of motic graphs, see below.

4.2.3. **Refined length.** The basic morphisms for graphs are simple edge contractions, simple loop contractions and simple mergers, see [KW17][5.1]. Basically, a simple edge contraction glues two flags from two different vertices together to form an edge and then contracts the edge. A simple loop contraction does the same with the exception that the two flags come from the same vertex. A simple merger identifies two distinct vertices. Any morphism can be factored into these three types of morphisms [BM08, KW17]. The ghost graph keeps track of which flags have been glued together to form edges that are subsequently contracted. The native length of edge contractions and mergers is 1. while that for loop contractions is 0.

In the case of graphs of higher genus ($b_1 > 0$), loop contractions are of native length 0. It is more natural, to have a different grading, in which both loop and edge contractions have length of better degree 1 and mergers have degree 0. This makes the relations homogeneous.

From the Hopf point of view, the most natural grading is the one in which all simple morphisms have degree 1 and thus consider the degree to be the word length of a decomposition. Then there is a minimal word length and a maximal word length. The minimal one is given by first contracting edges and then merging, which the maximal one is given by first merging and then contracting only loops, see [KW17][5.1].

In most practical examples, mergers are excluded, making life simple. (This however excludes PROPs and other "disconnected" types.)

4.2.4. **Enriched versions.** There are also enriched versions of Feynman categories whose definition is a more involved and we refer to [KW17] for details. In principle, one needs to deal with indexed limits [Kel82] throughout and, if the monoidal structure is not Cartesian, fix the definition of a groupoid. In the case of k vector spaces, it still means that the decomposition is unique up to factors which yield the same tensor product, and that all the isomorphisms are indexed by an underlying groupoid. The relevant examples here are actually of a particular type given *a priori* as an enrichment of an underlying Feynman category, see §4.11.1 for the relevant facts.

We need enrichment to reproduce the example corresponding to the free construction coming from operads in a monoidal category, see §2.2.3 and the following paragraph.

Example 4.5 (Examples from operads)**.** This is the construction relevant for the previous parts of the paper. Operads naturally form the enrichments for $\mathfrak{S}urj$, see [KW17].

We will give a short disucssion to match up with the previous results. Let $\mathcal{F}$ be an enrichment of $\mathfrak{S}urj$ this mean that we can specify new spaces of morphisms. By condition (ii), up to equivalence all morphisms in $\mathcal{F}$ are fixed by the fixing the spaces of morphisms $Hom(\underline{n}, \underline{1}) =: \mathcal{O}(n)$. Given an operad $\mathcal{O}$ with $\mathcal{O}(1) \simeq \mathbb{1}$ we denote the corresponding Feynman category by $\mathfrak{S}urj_{\mathcal{O}}$. These are exactly the enrichments of $\mathfrak{S}urj_{<}$ in the non Σ case respectively $\mathfrak{S}urj$ in the symmetric case.

Under this correspondence the $\mathbb{S}_n$ action comes from precomposing and the composition of morphisms corresponds to γ. If the elements of $\mathcal{O}(n)$ are rooted leaf labelled graphs, the morphisms of $\mathcal{F}$ are given by disjoint union of graphs (a.k.a. forests) and the composition is given by gluing leaves to roots.

We can also enrich $Hom(\underline{1}, \underline{1}) = \mathcal{O}(1)$. The isomorphism condition then says that the only invertible element of $\mathcal{O}(1)$ is the identity. If $\mathcal{O}(1)$ has nontrivial isomorphisms then we just enlarge $\mathcal{V}$. Both can be done with a decoration.

Notice that the elements in $\mathcal{O}(n)$ have native length $n-1$. The length of a morphism in $Hom(\underline{n}, \underline{p})$ is $n-p$. first part.

For the moment, we will stick to Feynman categories over $\mathcal{S}et$ and return to enrichment later.

Given a monoidal category $\mathcal{F}$ satisfying appropriate finiteness conditions there is a natural decomposition coalgebra structure on the vector space spanned by the arrows (generalising the classical incidence coalgebra of a locally finite poset) and also an algebra structure induced by the tensor product. We will see in the following sections that if $\mathcal{F}$ is a Feynman category then the hereditary condition ensures that that these coalgebra and algebra structures will be compatible and define a bialgebra.

4.3. The three main examples.

4.3.1. **The operad of surjections.** Again $\mathcal{V}$ is trivial. In the non-Σ version $\mathcal{F} = \mathfrak{S}urj_{<}$, which is the wide subcategory of order preserving surjections inside the augmented simplicial category Δ_+. In the symmetric version $\mathcal{F} = \mathfrak{S}urj$, the skeleton of the category of finite sets and surjections, which is the wide subcategory of surjections inside the augmented crossed simplicial group $\Sigma\Delta_+$.

In both cases $\mathcal{V}$ is trivial and we will write $\mathfrak{S}urj_<$ or $\mathfrak{S}urj$ for the Feynman categories.

4.3.2. **The Feynman category of simplices, Intervals and the Joyal dual of $\mathfrak{S}urj_<$.** As stated previously, There is a very interesting and useful contravariant duality [Joy97] of subcategories of Δ_+ between Δ and the category of intervals, which are the end-point preserving morphisms in Δ. It maps surjections in Δ to double base point preserving injections $\mathcal{I}nj_{*,*}$, see Appendix C.1. Thus the category $\mathcal{I}nj_{*,*}^{op}$ is again a Feynman category with trivial $\mathcal{V}$.

But, surprisingly, $\mathcal{I}nj_{*,*}$ is also a Feynman category itself. Another interesting fact is that $\mathcal{I}nj_{*,*}$ also gives rise to a Feynman category $\mathfrak{I}nj_{*,*}$ with trivial $\mathcal{V}$. This is parallel to the discussion in [KW17, §2.10.3], although we need to tweak the construction slightly. As stated $\mathcal{V}$ for $\mathcal{I}nj_{*,*}$ is trivial and the underlying objects of $\mathcal{F}$ are the natural numbers. To each n we associate $[n+1]$. We take the identity in $Hom(1,1)$ and its tensor powers give the identities in $Hom_{\mathcal{F}}(n,n)$. Now we add one morphism in $Hom_{\mathcal{F}}(\mathbb{1}=0,1)$ which we will call special. Any double–base point preserving injection from $[n+1]$ to $[m+1]$ is then represented by a tensor product of identities and special maps. This gives a representation of the Feynman category in terms of generators and relations [KW17, Chapter 5].

This fact gives rise to an interesting interpretations, see Example 4.60 and §5.3.

4.3.3. **The operad of leaf–labelled rooted trees.** Let $\mathfrak{F}_{CK}$ be the Feynman category with trivial $\mathcal{V}$, $\mathcal{F}$ having objects $\mathbb{N}_0$ and morphisms given by rooted forests: $Hom(\underline{n},\underline{m})$ is the set of n-labelled rooted forests with m roots. The composition is given by gluing the roots to the leaves. This is the twist of $\mathfrak{S}urj$ by the operad of leaf-labelled rooted trees. In the non-Σ version, one uses planar forests/trees and omits labels or equivalently uses orders on the sets of labels. Here this is the twist by the non–sigma operad of planar forests of $\mathfrak{S}urj_<$.

Here there is non–trivial $\mathcal{O}(1)$. This is basically the difference of the $+$ and the *hyp* construction, see §4.11.1. The grading $n-p$ is the native grading and the coradical length is the word length of a morphism and is given by the number of vertices.

4.4. **Algebra and coalgebra structures for Feynman categories.** We will now introduce the main algebra and coalgebra operations. In order to proceed further, we will need some assumptions, which are natural when regarding the coproduct of the identity, which is what we turn to right after the definitions.

4.4.1. **Algebra from the morphisms of a Feynman category.** Given a Feynman category consider the free abelian group $\mathscr{B}$ on the set of all morphisms of $\mathcal{F}$, and its subgroup generated by the one–comma generators, viz. morphisms from $\imath^{\otimes}(X) \to \imath(*)$.

$$\mathscr{B} = \mathbb{Z}[Mor(\mathcal{F})],$$

Assumption 4.6. Since in the following we will be interested in fixing $\mathcal{V}$ and fixing $\mathcal{F}$ only up to equivalence, we will assume (after using MacLane's coherence theorem [ML98]) that $\mathcal{F}$ is a strict monoidal category, that is, that the associativity and unit constraints are all identities. In this current situation, we also want to specify that $V^{\otimes}$ in the definition of a Feynman category is taken to be the free *strict* symmetric monoidal catogory.

Without this assumption, our (co)algebra structures will all be weak. Given for example $\phi_i : X_i \to Y_i$ in $\mathcal{F}$ for $i = 1, 2, 3$ we have $(\phi_1 \otimes \phi_2) \otimes \phi_3 = A(\phi_1 \otimes (\phi_2 \otimes \phi_3))$ in $\mathscr{B}$ where $A : \mathscr{B} \cong \mathscr{B}$ is induced by pre- and post-composing with associativity isomorphisms a_{X_1,X_2,X_3} and $a^{-1}_{Y_1,Y_2,Y_3}$.

With the assumption above, $\mathscr{B}$ has a unital associative product induced by the monoidal product $\otimes_{\mathcal{F}}$ of $\mathcal{F}$ with the unit $id_{\mathbb{1}}$, the identity morphism on the monoidal unit $\mathbb{1}$ of $\mathcal{F}$. This is the (free) $\mathbb{Z}$-algebra on the unital monoid $(Mor(\mathcal{F}), \otimes, id_{\mathbb{1}})$. Indeed $X \otimes \mathbb{1}_{\mathcal{F}} = X$ and hence $\phi \otimes id_{\mathbb{1}} = \phi$

Note that if we are working in the enriched version $Hom(\mathbb{1}, \mathbb{1}) = K$ will play the role of a ground ring.

Remark 4.7. One can enlarge the setting to the situation in which the sets of morphisms are graded and composition preserves the grading. In this case, one only needs degreewise composition finite. This will be the case for any graded Feynman category [KW17].

4.4.2. **The decomposition coproduct.** Suppose that $\mathcal{F}$ is decomposition finite. This means that for each morphism ϕ of $\mathcal{F}$ the set $\{(\phi_1, \phi_0) : \phi = \phi_1 \circ \phi_0\}$ is finite. Then $\mathscr{B}$ carries a coassociative coproduct given by the dual of the composition. On generators it is given by:

$$\Delta(\phi) = \sum_{\{(\phi_1,\phi_0):\phi=\phi_1\circ\phi_0\}} \phi_1 \otimes \phi_0. \tag{4.3}$$

where we have abused notation to denote by ϕ the morphism $\delta_\phi(\psi)$ that evaluates to 1 on ϕ and zero on all other generators.

A counit is defined on the generators by:

$$\epsilon(\phi) = \begin{cases} 1 & \text{if for some object } X : \phi = id_X \\ 0 & \text{else} \end{cases} \tag{4.4}$$

Remark 4.8. We realized with hindsight that the coproduct we constructed on indecomposables, see below, is equivalent to the coproduct above. A little bilbliographical sleuthing revealed that the the coproduct for any finite decomposition category appeared already in [Ler75] and was picked up later in [JR79].

4.4.3. **Coproduct of the identity morphisms.** Any factorization of id_X is of the form $id_X : X \stackrel{\phi_R}{\to} X' \stackrel{\phi_L}{\to} X$ with $\phi_L \circ \phi_R = id_X$. This mean that each ϕ_L has a right inverse ϕ_R and each ϕ_R has a left inverse ϕ_L.

$$\Delta(id_X) = \sum_{(\phi_L,\phi_R):\phi_L\circ\phi_R=id_X} \phi_L \otimes \phi_R \tag{4.5}$$

Note that $|\phi_R| + |\phi_L| = |id_X| = 0$.

Lemma 4.9. *In a decomposition finite category the automorphism groups $Aut(X)$ are finite for all objects X, as are the classes $Iso(X)$ of objects isomorphic to X.*

Proof. For each automorphism ϕ of X and for each isomorphism $\phi : X \to X'$ there is a factorisation $id_X = \phi^{-1} \circ \phi$, and there are only finitely many such factorisations. □

Definition 4.10. A Feynman category has *almost group–like* identities if each of the ϕ_L and hence each of the ϕ_R appearing in a factorization is an isomorphism.

If $\mathfrak{F}$ is decomposition finite and has almost group like identities then:

$$\Delta(id_X) = \sum_{X', \sigma \in Iso(\mathcal{F})(X,X')} \sigma \otimes \sigma^{-1} \tag{4.6}$$

If $\mathcal{V}$ is discrete, then the id_X are group–like

Example 4.11. A counter–example, that is a Feynman category that does not have group–like identities, is the indefinite Δ_+. In this case, the category is also not decomposition finite. The reason is that each $id : \underline{n} \to \underline{n}$ factors as $\underline{n} \hookrightarrow \underline{m} \twoheadrightarrow \underline{n}$ for all $m \geq n$.

The assumption of almost group like identities is, however, very natural and is often automatic. The example above is symptomatic.

Lemma 4.12. *Given a factorization $id_X : X \stackrel{\phi_R}{\to} Y \stackrel{\phi_L}{\to} X$ it follows that $|\phi_R| \leq 0$.*

Proof. Decomposing the morphisms for $X = \bigotimes_v *_v$ according to (ii) we end up with sequences

$$*_v \stackrel{\phi_{R,v}}{\to} Y_v \stackrel{\phi_{L,v}}{\to} *_v$$

with $\phi_{L,v} \circ \phi_{R,v} = id_{*_v}$. This follows from decomposing ϕ_L and ϕ_R and then comparing to the decomposition of the isomorphism ϕ. We see that $|Y_v| \geq 1$ since there are no morphisms from any X of length greater or equal to one to $\mathbb{1}$. Thus $|\phi_{R_v}| \leq 0$ and hence $|\phi| = \sum_v |\phi_v| \leq 0$. □

Corollary 4.13. *If $\mathfrak{F}$ is definite, decomposition finite and the only morphisms of $\mathfrak{F}$ with length 0, which have one–sided inverses are isomorphisms, then $\mathfrak{F}$ has almost group like identities.* □

Proof. In the definite case, the only factorizations that are possible are those in which both ϕ_R and ϕ_L have degree 0. By the second assumption, any morphism with a one–sided inverse is invertible and hence the statement follows. □

This will be the case in all the examples.

Lemma 4.14. *Moreover, if $\mathfrak{F}$ is decomposition finite, then*

1. *The identity of any object X does not have a factorization $id_X : X \stackrel{\phi_R}{\to} X \otimes Y \stackrel{\phi_L}{\to} X$ with $|\phi_R| < 0$.*
2. *The identity of an object does not have factorization $id_X : X \stackrel{\phi_R}{\to} X \stackrel{\phi_L}{\to} X$ for which ϕ_L does not have left inverse or equivalently ϕ_R does not have right inverse.*

Proof. In the case (1),

we define $\phi_R^1 = \phi_R$ and for $n \geq 2$: $\phi_R^n = \phi_R \otimes id \circ \phi_R^{n-1} : X \to X \otimes Y^{\otimes n}$ and likewise set $\phi_L^1 = \phi_L$ and for $n \geq 1$: $\phi_L^n = \phi_L^{\otimes n-1} \circ \phi_L \otimes id : X \otimes Y^n \to X$. These satisfy $\phi_L^n \circ \phi_R^n = id_X$ and there will be infinitely many possible decompositions of id_X, one for each n and hence we arrive at a contradiction.

In the second case, we can simply use the powers of ϕ_L and ϕ_R, we will get infinitely many different decompositions unless $\phi_L^n = \phi_L^m$ for some $m > n$ which is impossible. Indeed from that equality it would follow that $id = \phi_L^n \circ \phi_R^n = \phi_L^m \circ \phi_R^n = \phi_L^{m-n} = \phi^{m-n-1} \circ \phi_L$ and hence ϕ_L would have a left inverse. □

Corollary 4.15. *In a strict decomposition finite Feynman category $\Delta(id_{\mathbb{1}})$ is group like, i.e.: $\Delta(id_{\mathbb{1}}) = id_{\mathbb{1}} \otimes id_{\mathbb{1}}$*

Proof. In a strict $\mathcal{F}$, $\mathbb{1} = \varnothing$ is the empty word and the unique object of length 0 by condition (i). Since there are no maps from $X \to \varnothing$ by condition (ii) for $|X| \geq 1$ the only factorization of the identity morphism of $\mathbb{1}$ factor trough $\mathbb{1}$. $id_{\mathbb{1}} : \mathbb{1} \stackrel{\phi_R}{\to} \mathbb{1} \stackrel{\phi_L}{\to} \mathbb{1}$, By the previous Lemma, in any such factorization ϕ_R and ϕ_L are isomorphisms, because of the decomposition finiteness assumption. Finally, $id_{\mathbb{1}}$ is the only invertible element in $Hom_{\mathcal{F}}(\mathbb{1}, \mathbb{1})$ due to condition (i) of a Feynman category. Hence $\Delta(id_{\mathbb{1}})$ only has one summand corresponding to $id_{\mathbb{1}} \otimes id_{\mathbb{1}}$. □

4.5. **Bi-algebra structures in the non–symmetric case.**

Lemma 4.16. *In a strict non–Σ Feynman category, the counit is multiplicative $\epsilon(\phi \otimes \psi) = \epsilon(\phi)\epsilon(\psi)$.*

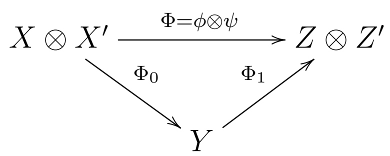
Proof. First, $id_X \otimes id_Y = id_{X \otimes Y}$, since $\mathcal{F}$ is strict monoidal. Because of axiom (i) this is then the unique decomposition of $id_{X \otimes Y}$, and hence both sides are either zero or $\phi = n\, id_X$ and $\psi = m\, id_Y$ in which case both sides equal to nm. □

4.5.1. **Bialgebra structure.** The product and coproduct above would actually work in any strict monoidal category with finite decomposition. However the compatibility axiom of a bialgebra *does not hold in general* for all monoidal category with finite decomposition. Indeed one needs to check the bialgebra axiom

$$\Delta \circ \mu = (\mu \otimes \mu) \circ \pi_{2,3} \circ (\Delta \otimes \Delta)$$

where $\pi_{2,3}$ switches the 2nd and 3rd tensor factors. Each side of the equation is represented by a sum over diagrams.

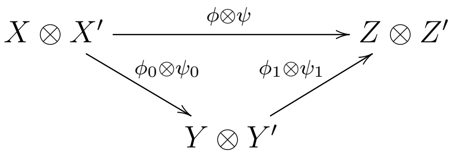
For $\Delta \circ \mu$ the sum is over diagrams of the type



(4.7)

where $\Phi = \Phi_1 \circ \Phi_0$.

When considering $(\mu \otimes \mu) \circ \pi_{23} \circ (\Delta \otimes \Delta)$ the diagrams are of the type



(4.8)

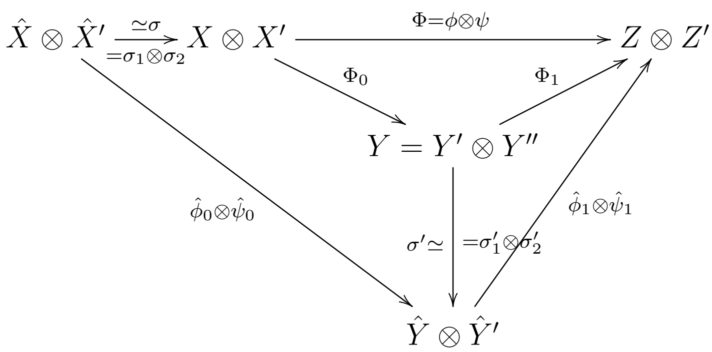
where $\phi = \phi_1 \circ \phi_0$ and $\psi = \psi_1 \circ \psi_0$. And there is no reason for there to be a bijection of such diagrams.

The compatibility *does hold* when dealing with strict non–symmetric Feynman categories due to the hereditary condition.

Theorem 4.17. *For any strictly monoidal, finite decomposition, non-Σ Feynman category $\mathfrak{F}$ the tuple $(\mathscr{B}, \otimes, \Delta, \epsilon, \eta)$ defines a bialgebra over $\mathbb{Z}$.*

Proof. We check the compatibility axioms. The axioms for the unit and counit follow from Corollary 4.15 and Lemma 4.15.

In order to prove that Δ is an algebra morphism, we consider the two sums over the diagrams (4.7) and (4.8) above and show that they coincide. First, it is clear that all diagrams of the second type appear in the first sum. Vice–versa, given a diagram of the first type, we know that $Y \simeq \hat{Y} \otimes \hat{Y}'$, since Φ_1 has to factor by axiom (ii) and the Feynman category is strict. Then again by axiom (ii) Φ_0 must factor. We see that we obtain a diagram:



(4.9)

Now since the Feynman category is strict and non-symmetric, the two isomorphisms also decompose as $\sigma = \sigma_1 \otimes \sigma_2$, and $\sigma' = \sigma'_1 \otimes \sigma'_2$, for a splitting $Y = Y' \otimes Y''$ so that $\Phi_0 = \sigma_1'^{-1} \circ \hat{\phi}_0 \circ \sigma_1^{-1} \otimes \sigma_2'^{-1} \circ \hat{\psi}_0 \circ \sigma_2^{-1}$ and $\Phi_1 = \hat{\phi}_1 \circ \sigma'_1 \otimes \hat{\psi}_2 \circ \sigma'_2 : Y = Y' \otimes Y'' \to Z \otimes Z'$ and one obtains that both diagram sums agree.

□

Remark 4.18. We could also already start with a Feynman category augmented over a tensor category $\mathcal{E}$ where $\mathcal{E}$ has a faithful functor to $\mathcal{A}b$, e.g. k-Vect. In this case one should work over the ring $K = Hom_{\mathcal{F}}(\mathbb{1}, \mathbb{1})$, see [KW17] for details.

4.5.2. **Bi-algebra structure induced from indecomposables.** For a strict Feynman category $Mor(\mathcal{F}) = Obj(\imath^{\otimes} \downarrow \imath)^{\otimes}$ and hence $\mathscr{B}$ is the strictly associative free monoid on

$\mathscr{B}_1 = \mathbb{Z}[Ob(\imath^{\otimes} \downarrow \imath)] \subset \mathscr{B}$ with additional symmetries possibly given by the commutativity constraints induced by $\mathcal{F}$.

Lemma 4.19. *If $\mathcal{F}$ is strict and non–Σ, $\mathscr{B}_1$ is the set of indecomposables.*

Proof. By axiom (ii) any morphism with target of length greater or equal to 2 is decomposable. If the target of a morphism ϕ has length 1, it can only decompose as $\phi = \hat{\phi} \otimes_{\mathbb{Z}} \lambda$ with $\lambda \in \mathbb{Z}[Hom(\mathbb{1}, \mathbb{1})] = \mathbb{Z}id_{\mathbb{1}}$, since the only object of length 0 is unit $\mathbb{1}$ and $\mathfrak{F}$ was taken to be strict. Hence $\lambda = \pm id_{\mathbb{1}}$ is itself a unit in the algebra and $\phi = \pm\hat{\phi}$. □

We now suppose that $\mathscr{B}_1$ is decomposition finite, which means that the sum in (4.10) is finite. Consider the one–comma generators $\mathscr{B}_1$ and define

$$\Delta_{indec}(\phi) = \sum_{\{(\phi_1,\phi_0):\phi=\phi_1\circ\phi_0\}} \phi_1 \otimes \phi_0 \tag{4.10}$$

here $\phi_1 \in \mathscr{B}_1$ and $\phi_0 = \bigotimes_{v\in V} \phi_v$ for $\phi_v \in \mathscr{B}_1$. We extend the definition of Δ_{indec} to all of $\mathscr{B}$ via the bi-algebra equation.

$$\Delta_{indec}(\phi \otimes \psi) := \sum (\phi_1 \otimes \psi_1) \otimes (\phi_0 \otimes \psi_0) \tag{4.11}$$

where we used Sweedler notation.

$$\epsilon(\phi) = \begin{cases} 1 & \text{if } \phi = id_X \\ 0 & \text{else} \end{cases}$$

In this case there is a direct proof of the bi–algebra structure. A posteriori using Lemma 4.19 it follows that this bialgebra structure coincides with the decomposition bialgebra structure.

Proposition 4.20. *With the assumptions on $\mathcal{F}$ as above and that $\mathscr{B}_1$ is decomposition finite, the tuple $(\mathscr{B}, \otimes_{\mathcal{F}}, \Delta_{indec}, \mathbb{1}, \epsilon)$ is a bi–algebra. A posteriori $\Delta = \Delta_{indec}$.*

Proof. The multiplication is unital and associative. That the coproduct is coassociative and ϵ is a counit is a straightforward check. The latter follows from the decomposition $id_X = \otimes_v id_{*_v}$ if $X = \otimes_v *_v$. The fact that the bi–algebra equation holds, follows from the fact that all elements in $\mathscr{B}_1$ are indecompsable with respect to this product. For the coassociativity, we notice that in both iterations we get sum over decomposition diagrams $\phi = \phi''' \circ \phi'' \circ \phi'$.

$$\begin{array}{ccccc} X = \bigotimes_v \bigotimes_{w\in V_v} X_w & = & \bigotimes_v \bigotimes_{w\in V_v} \bigotimes_{u\in V_w} *_u & \xrightarrow{\phi=\bigotimes_u \phi_u} & * \\ \Big\downarrow{\scriptstyle \phi'=\otimes_w \phi_w} & & & & \Big\uparrow{\scriptstyle \phi'''=\bigotimes \phi'''_v} \\ Z_1 = \bigotimes_v \bigotimes_{w\in V_v} *_w & = & \bigotimes_v Z_v & \xrightarrow{\phi''=\bigotimes_v \phi''_v} & Z_2 = \bigotimes_v *_v \end{array} \tag{4.12}$$

where the order of the factors is fixed and the sum is over the possible morphisms and bracketings. That $\Delta = \Delta_{basic}$ follows from the equality of the coproducts on indecomposables for the bi–algebra which by Lemma 4.19 are precisely $\mathscr{B}_1$. □

Remark 4.21. This two step process corresponds to the free construction $\check{\mathcal{O}}^{nc}$ in Chapter 1. A prime example is the bi–algebra of rooted planar trees aka. bialgebra of forests of Connes and Kreimer [CK98]. The usual way this is defined is to give the coproduct on indecomposable, viz. trees, and then extend using the bi–algebra equation.

4.6. **Coinvariants and isomorphisms.** In order to treat the symmetric case, but also in the non–Σ situation, one may and does pass to isomorphisms classes. This is a bit more sublte than expected.

4.6.1. **Iso and Automorphisms.** By the conditions of a Feynman category for $X = \bigotimes_{i=1}^{k} *_i$. In the non–symmetric case, any automorphism factors, so

$$Aut(X) \simeq Aut(*_1) \times \cdots \times Aut(*_k) \text{ in the non–symmetric case}$$

In the symmetric case its automorphisms group is the wreath product

$$Aut(X) \simeq (Aut(*_1) \times \cdots \times Aut(*_k)) \wr \mathbb{S}_k \text{ in the non–symmetric case}$$

The set $Hom(X,Y)$ has a natural action of $Aut(X) \times Aut(Y)$ $\phi \mapsto \sigma_Y \circ \phi \circ \sigma_X^{-1}$. We let $Aut(\phi) \subset Aut(X) \times Aut(Y)$ be the stabilizer group of ϕ.

There is also an action on decompositions. There is an action of $Aut(Z)$ on $Hom(Z,Y) \times Hom(X,Z)$ given by $\rho(\sigma)(\phi_1,\phi_0) = (\phi_1 \circ \sigma^{-1}, \sigma \circ \phi_0)$. Notice that the $Aut(Z)$ action leaves the composition map invariant: $\phi_1 \circ \phi_0 = \phi_1 \circ \sigma^{-1} \circ \sigma \circ \phi_0$.

4.6.2. $\mathscr{B}_{iso}$. Loosely, $\mathscr{B}_{iso} = \mathscr{B}/\sim$ where $\sim$ is the equivalence relation on morphisms given by isomorphisms in $(\mathcal{F} \downarrow \mathcal{F})$. In particular, the equivalence relation $\sim$, which exists on any category, means that for given f and g: $f \sim g$ if there is a commutative diagram with isomorphisms as vertical morphisms.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \simeq\sigma}\downarrow & & \downarrow{\scriptstyle \simeq\sigma'} \\ X' & \xrightarrow{g} & Y' \end{array}$$

i.e.: $f = \sigma'^{-1} \circ g \circ \sigma$. Plugging in $f = id_X$ we obtain:

Lemma 4.22. *$id_X \sim g$ if and only if $g : X' \to Y'$ is an isomorphism and $X \simeq X' \simeq Y'$.*

□

Remark 4.23. Notice that this equivalence is coarser than the equivalence studied in [JR79] for the standard reduced incidence category.

Remark 4.24. The morphisms of $\mathcal{F}$ together with these isormophisms are also precisely the groupoid of vertices $\mathcal{V}'$ of the iterated Feynman cagegroy $\mathfrak{F}'$, cf. [KW17, §3.4].

Theorem 4.25. *Let $\mathcal{F}$ be a decomposition finite Feynman category set $\mathscr{B} = \mathbb{Q}[Mor(\mathcal{F})]$ with the product induced by tensor product and the coproduct (4.3). Let $\mathcal{C}$ be the ideal generated by elements $f - g$ with $f \sim g$. Then*

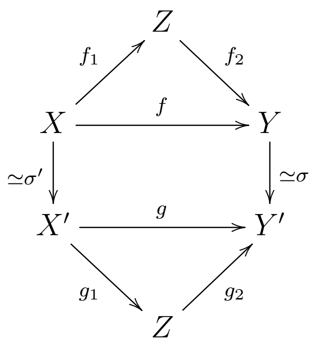
$$\Delta(\mathcal{C}) \subset \mathscr{B} \otimes \mathcal{C} + \mathcal{C} \otimes \mathscr{B} \tag{4.13}$$

and hence the product, unit and coproduct descend to $\mathscr{B}_{iso} := \mathscr{B}/\mathcal{C}$. *Furthermore there is a counit on* $\mathscr{B}/\mathcal{C}$

$$\bar{\epsilon}([f]) := \begin{cases} \frac{1}{|Iso(X)||Aut(X)|} & \text{if } [f] = [id_X] \\ 0 & \text{else} \end{cases} \tag{4.14}$$

In the case that $\mathcal{F}$ *is a non-symmetric Feynman category* $\mathscr{B}_{iso}$*, together with all these structures is a bialgebra.*

Proof. To compute the coproduct, we break up the sum over the factorizations of f and g with $f \sim g$ into the pieces that correspond to a factorization through a fixed space Z.



(4.15)

Now the term in Δ_{f-g} corresponding to Z is $\sum_i f_2^i \otimes f_1^i - \sum_j g_2^j \otimes g_1^j$. Resumming using the identification $g_1^i := f_1^i \circ \sigma'^{-1}$ and $g_2^i := \sigma \circ f_2^i$ this equals to $\sum_i (f_2^i \otimes f_1^i - g_2^i \otimes g_1^i) = \sum_i (f_2^i - g_2^i) \otimes g_1^i + \sum_i f_2^i \otimes (f_1^i - g_1^i)$. This proves the first claim.

For the counit, notice that $\Delta([f]) = [\Delta(f)]$ is a sum of terms factoring through an intermediate space Z. If $Z \not\simeq X, Y$ then these terms are killed by $\bar{\epsilon}$ on either side, since there will be no isomorphism in the decomposition. If $Z \simeq X$, then any factorization $f \circ \sigma^{-1} \otimes \sigma$ with $\sigma \in Iso(X, Z)$ descends to $[f \circ \sigma^{-1}] \otimes [\sigma] = [f] \otimes [id_X]$. Since $Iso(X, Z)$ is a left $Aut(X)$ torsor, there are exactly $|Aut(X)||Iso(X)|$ of these terms and $\bar{\epsilon} \otimes id$ evaluates to $1 \otimes [f]$ on their sum. By Lemma 4.22, all other decompositions will evaluate to 0 and we obtain that $\bar{\epsilon}$ is a left counit. Likewise ϵ is a right counit by considering the terms which factor through $Y' \in Iso(Y)$.

In the non–symmetric case, the compatibility of product and coproduct descend as does the compatibility of the unit. For the counit, we notice that $\epsilon([\phi \otimes \psi])$ as well as $\epsilon([\phi])\epsilon([\psi])$ are 0 unless $[\phi] = \lambda[id_X]$ and $[\psi] = \mu[id_Y]$. By the conditions of a non–symmetric Feynman cateogory $|Aut(X)||Aut(Y)| = |Aut(X \otimes Y)|$ as well as $|Iso(X)||Iso(Y)| = |Iso(X \otimes Y)|$. □

Remark 4.26. Note that $\mathcal{C}$ is not a coideal in general, since for any automorphism $\sigma_X \in Aut(X) : [\sigma_X] = [id_X]$ and hence $\epsilon(\mathcal{C}) \not\subseteq ker(\epsilon)$. Likewise if $X \simeq Y \stackrel{\sim\phi}{\to} Y'$ then $[id_X] = [\phi]$ from Lemma 4.22. This is why we need a new definition for the counit. If there are no automorphisms and the underlying category is skeletal, then ϵ descends as claimed in [JR79].

Remark 4.27. Extending scalars from $\mathbb{Z}$ to $\mathbb{Q}$ may not be necessary; we only need that $|Iso(X)|$ and $|Aut(X)|$ are invertible for all X. Although in the symmetric case, the automorphisms groups will contain all $\mathbb{S}_n$ and hence $\mathbb{Q}$ is necessary.

4.6.3. **Skeletal version.** One can get rid of the terms $X' \in Iso(X)$ in $\Delta(id_X)$ and the factor $|Iso(X)|$ by considering a skeletal version. Recall that skeletal means that there is only one object per isomorphism class. Replacing $\mathcal{V}$ by its skeleton $\mathcal{V}^{sk}$ we can replace the the strictly associative monoid $\mathcal{V}^{\otimes}$ by words in the skeletal letters $(\mathcal{V}^{sk})^{\otimes}$. In the non–symmetric case this is indeed is skeletal. $(\mathcal{V}^{sk})^{\otimes} = (\mathcal{V}^{\otimes})^{sk}$.

In the symmetric case, the skeleton of $(\mathcal{V}^{sk})^{\otimes}$ are the commutative words in the letters V^{sk} with symmetry groups reduced to wreath products of automorphism groups of the letters with permutations of equal letters.

In both cases, we define $\mathcal{F}^{sk}$ to be the skeleton of $(\mathcal{V}^{sk})^{\otimes}$. The skeletal version of $\mathfrak{F}$ be $(\mathcal{V}^{sk}, \mathcal{F}^{sk}, \imath_{sk})$ with the appropriate inclusion functor.

We set $\mathscr{B}^{sk}(\mathcal{F}) = \mathbb{Z}[Mor(\mathcal{F}^{sk})]$. Note that by axiom (ii) $\mathscr{B}^{sk}(\mathcal{F})$ is Morita equivalent to the monoid on all morphisms.

Example 4.28. It is instructive to study two examples.

The first is a Feynman category $\mathfrak{F}_{\mathcal{O}}$. $\mathscr{B}^{sk}$ in this case is given by the formula for $\check{\mathcal{O}}^{nc}$ (2.17), while the full $\mathscr{B}$ is provided by formula (2.40). Finally

$$\mathscr{B}_{iso} = \bigoplus_n \bigoplus_{n_1 \leq \cdots \leq n_k, \sum n_i = n} \bigodot \check{\mathcal{O}}(n_i)_{\mathbb{S}_i}$$

regardless.

The second type of examples are the case of connected or 1-PI graphs. Here $\mathscr{B}_{iso}$ is given by the free monoid of the isomorphisms classes of connected or 1–PI ghost graphs, i.e. graphs with unlabelled vertices and flags. $\mathscr{B}$ is given by the morphisms, that is, ghost graphs with all the additional data, and $\mathscr{B}^{sk}$ is given by the ghost graphs, where now the source morphism is to be picked up to an induction from the wreath product to the full symmetric group.

In general, we have an injection $\mathscr{B}^{sk} \hookrightarrow \mathscr{B}$ and a surjection $\mathscr{B} \twoheadrightarrow \mathscr{B}_{iso}$. Now picking an inverse functor for the equivalence of categories between the skeleton of $\mathcal{F}$ and $\mathcal{F}$, we obtain an inverse to inclusion map of $\mathscr{B}^{sk}$ and $\mathscr{B}$ the quotient map to $\mathscr{B}_{iso}$ factors trough this map.

4.6.4. **Symmetric version.** In the symmetric version there are two relevant constructions. The first involves quotienting by isomorphisms and the second uses cocycles. The reason for the complication is that $Aut(X) \times Aut(X') \subset Aut(X \otimes X')$ is a proper subset due to the permutation symmetries.

There is a third alternative, which is to use representations, in the spirit they appear in fusion rules in physics, but we will not delve into this furhter technical complication at this point.

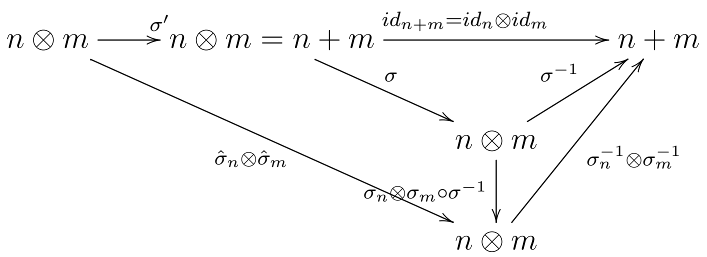
Example 4.29. In the case of $\mathcal{V} = \{1\}$, we have $Aut(n) \times Aut(m) = \mathbb{S}_n \times \mathbb{S}_m \subset \mathbb{S}_{n+m} = Aut(n+m)$.

This means that the bi–algebra equation does not hold in the symmetric case directly. It does hold when we pass to coinvariants either. It is still true that any tensor decomposition $\phi_0 \otimes \psi_0, \phi_1 \otimes \psi_1$, $\phi = \phi_1 \circ \phi_0, \psi = \psi_1 \circ \psi_0$ which are the terms in $\mu \otimes \mu \circ \pi_{23} \circ \Delta \otimes \Delta$ appears as a term in $\Delta \circ \mu$. But, not all terms of $\Delta \circ \mu$ appear in this way. However, the missing terms are equivalent to ones that do appear.

This also means that the formula for the counit in Theorem 4.25 is not multiplicative. There are two ways to remedy this (1) sum only over representative or (2) alter the product structure by using cocyles. A third route would be to use group representations, whose details we will not go into due to space considerations.

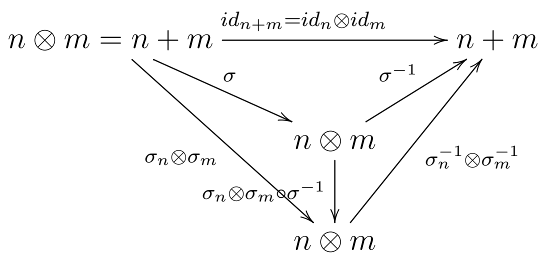
To see what the issue is, we consider the following instructive example.

Example 4.30. Let us consider $\mathcal{V} = 1$ and $\mathcal{F} = \mathbb{S}$, the skeletal version of $\mathcal{V}^{\otimes}$, which has the natural numbers as objects and only isomorphisms as morphisms, where $Hom(n, n) = Aut(n, n) = \mathbb{S}_n$. We will consider $\Delta(id_n \otimes id_m) = \Delta(id_{n+m}) = \sum_{\sigma \in \mathbb{S}_{n+m}} \sigma \otimes \sigma^{-1}$. We analyze the possible diagrams (4.9) for the summand $\sigma \otimes \sigma^{-1}$ in the proof of Theorem 4.17.



(4.16)

And we see that $\sigma' = \sigma_n^{-1} \otimes \sigma_m^{-1} \circ \hat\sigma_n \otimes \hat\sigma_m = \sigma_n^{-1} \circ \hat\sigma_n \otimes \sigma_m^{-1} \circ \hat\sigma_m$ absorbing this block isomorphism into $\hat\sigma_n \otimes \hat\sigma_m$, we get the diagram.



(4.17)

If σ is of the form $\sigma_n \otimes \sigma_m$, then the term appears in $\Delta(id_n) \otimes \Delta(id_m)$. Otherwise, the action of $Aut(Y)$ on $Hom(X, Y) \otimes Hom(Y, Z)$ with $X = Y = Z = n + m$, on the decompositions appearing in $\Delta(id_n) \otimes \Delta(id_m)$ and moreover, picking representatives σ^r of $Aut(n + m)/(Aut(n) \times Aut(m))$ and summing over their action, we get an equality

$$\Delta(id_n \otimes id_m) = \sum_{\sigma^r \in \mathbb{S}_{n+m}/(\mathbb{S}_n \otimes \mathbb{S}_m)} \rho(\sigma^r)\Delta(id_n) \otimes \Delta(id_m)$$

In particular for equivalence classes, we get

$$\Delta([id_n] \otimes [id_m]) = \frac{(n+m)!}{n!m!}\Delta([id_n]) \otimes \Delta([id_m])$$

Alternatively the difference can be absorbed by a cocycle or by using isomorphism classes in the following sense. Set $\beta(\sigma_n, \sigma_n^{-1}) = \frac{1}{|Aut(n)|} = \frac{1}{n!}$. Define a new comultiplication: $\Delta_\beta(id_n) = \beta(\sigma_n, \sigma_n^{-1})\, \sigma_n \otimes \sigma_n^{-1}$ then $\otimes$ and Δ_β on $\mathscr{B}_{iso}$ satisfy the bialgebra equation. Here

In general, the situation is as follows:

Lemma 4.31. *For any factorization of $\Phi = \phi \otimes \psi : X \times X' \to Z \otimes Z'$ as $\Phi_1 \circ \Phi_0 : X \times X' \to Y \to Z \otimes Z'$ there exists a decomposition $\sigma' : Y \simeq \hat{Y} \otimes \hat{Y}'$ and a factorization $(\phi_0 \otimes \psi_0, \phi_1 \otimes \psi_1)$ factoring through $\hat{Y} \otimes \hat{Y}'$ such that $(\Phi_1, \Phi_0) = \sigma'(\phi_1 \otimes \psi_1, \phi_0 \otimes \psi_0) = (\phi_1 \otimes \psi_1 \circ \sigma'^{-1}, \sigma' \circ \phi_0 \otimes \psi_0)$. And furthermore, all such factorizations are given by the cosets $Iso(Y, \hat{Y} \otimes \hat{Y}')/Aut(\hat{Y}) \times Aut(\hat{Y}')$.*

Proof. Given a decomposition of Φ as (Φ_0, Φ_1), we can follow the argument of the proof of Theorem 4.17 up until the discussion of the isomorphisms σ and σ'.

In the symmetric case, there could be permutations involved for σ and σ'. This is not the case for σ, and hence we can absorb it to get decompositions of Φ. More precisely, the isomorphism σ has to be a block isomorphism as axiom (ii) applies to the two decompositions $\Phi = \phi \otimes \psi$ and $\Phi \simeq \hat{\phi}_1 \circ \hat{\psi}_1 \otimes \hat{\phi}_0 \circ \hat{\psi}_0$. This means that σ in (4.9) is uniquely a tensor product of isomorphisms $\sigma = \sigma_1 \otimes \sigma_2$, since both decompositions have the same target decomposition $Z \otimes Z'$. By pre–composing, we get the tensor decomposition $\Phi = (\hat{\phi}_1 \otimes \hat{\psi}_1) \circ (\hat{\phi}_0 \otimes \hat{\psi}_0) \circ (\sigma_1^{-1} \otimes \sigma_2^{-1})$.

Continuing with the decomposition of this form, we turn to σ'. We know that by (ii) that σ' can be written as a tensor product decomposition preceded by a permutation. If $\sigma' = \sigma'_1 \otimes \sigma'_2$, we have that $Y = Y_1 \otimes Y_2$ and (Φ_1, Φ_0) appears as a tensor product. Again absorbing the tensor decomposition means that the remaining terms corresponding to non–tensor decomposable permutations, and hence to a sum over the respective cosets. □

Notice that fixing any isomorphism in $Iso(Y, \hat{Y} \otimes \hat{Y}')$ identifies it with $Aut(\hat{Y} \otimes \hat{Y}')$ so that the quotient group $Iso(Y, \hat{Y} \otimes \hat{Y}')/[Aut(\hat{Y}) \times Aut(\hat{Y}')]$ becomes identified with $Aut(\hat{Y} \otimes \hat{Y}')/[Aut(\hat{Y}) \times Aut(\hat{Y}')]$

Corollary 4.32. *If $\mathcal{F}$ is a Feynman category, then in the proof of Theorem 4.17 the sets of diagrams agree only up to a choice of cosets of isomorphisms of σ' in (4.9). More explicitely the difference in the count of diagrams will result from the cosets $Aut(\hat{Y} \otimes \hat{Y}')/(Aut(\hat{Y}) \times Aut(\hat{Y}'))$. More precisely, splitting the sum $\Delta \circ \mu$ into subsums over a fixed space Y, $\Delta \circ \mu = \sum_Y \Delta \circ \mu_Y$, we have*

$$\sum_Y (\Delta \circ \mu)_Y = \sum_{[\sigma'] \in Aut(\hat{Y} \otimes \hat{Y}')/(Aut(\hat{Y}) \times Aut(\hat{Y}'))} \rho(\sigma') \left(\mu \otimes \mu \circ \pi_{23} \circ \Delta \otimes \Delta\right)_{\hat{Y} \otimes \hat{Y}'} \tag{4.18}$$

where we have used the identification above.

Thus we see that the bi–algebra equation will not hold in general in the symmetric case. Moreover, we do see how it fails. To remedy the situation, we should look at isomorphisms classes, which will be done under the name of channels.

4.6.5. **Isomorphism classes and decomposition channels.** We call a pair (ϕ_1, ϕ_0) of morphisms weakly composable, if there is an isomorphism σ, such that $\phi_1 \circ \sigma \circ \phi_0$ is composable.

A weak decomposition of a morphism ϕ is a pair of morphisms (ϕ_1, ϕ_0) for which there exist isomorphisms $\sigma, \sigma', \sigma''$ such that $\phi = \sigma \circ \phi_1 \circ \sigma' \circ \phi_0 \circ \sigma''$. In particular (ϕ_1, ϕ_0) is weakly composable.

We introduce an equivalence relation on weakly composable morphisms, which says that $(\phi_1, \phi_0) \sim (\psi_1, \psi_0)$ if they are weak decompositions of the same morphism. An equivalence class of weak decompositions will be called a decomposition channel.

For an element/equivalence class $[\phi] \in B^{iso}$.

$$\Delta^{sym}([\phi]) = \sum_{[(\psi,\phi_0)]:[\phi]=[\psi\circ\phi_0]} [\phi_0] \otimes [\psi] \tag{4.19}$$

where the sum is over a complete system of decomposition channels that is a set of representatives of almost composable classes.

Lemma 4.33. *Δ^{sym} is a coproduct with a counit on generators given by*

$$\epsilon^{sym}([\phi]) = \begin{cases} 1 & \text{if } [\phi] = [id_X] \\ 0 & \text{else} \end{cases} \tag{4.20}$$

Proof. All that need to be checked is associativity. This follows readily from considering double compositions $X \xrightarrow{\phi_0} Z_1 \xrightarrow{\phi_1} Z_2 \xrightarrow{\phi_2} Y$. In particular consider the equivalence relations on triples $(\phi_0, \phi_1, \phi_2) \sim (\phi'_0, \phi'_1, \phi'_2)$ is given by the existence of $\sigma_X \in Aut(X), \sigma_Y \in Aut(Y), \sigma_i \in Aut(Z_i), i = 1, 2$ $\phi_2 \circ \phi_1 \circ \phi_0 = \sigma_Y \circ \phi_2 \circ \sigma_2 \circ \phi_1 \circ \sigma_1 \circ \phi_0 \circ \sigma_X$. Then it is a straightforward check that the equivalence classes of both iterations of Δ^{sym} are a sum over the equivalence classes of triples. The counit is a simple computation. □

Remark 4.34. This coproduct actually corresponds to the category $\mathcal{F}'_{\mathcal{V}'}$ of universal operations [KW17, §6]. Here all channels with $[\phi_1] = [\psi]$ corresponds to the class of morphisms in $Hom_{\mathcal{F}'}(\phi, \psi)$. That means that each class of such a morphism under isomorphism corresponds to a channel and contributes a term to the sum. The associativity of the coproduct is then just the associativity of the composition.

Theorem 4.35. *Given a decomposition finite Feynman category $\mathfrak{F}$, $\mathscr{B}^{iso}$ with the induced $\otimes, \eta$ and $\Delta^{sym}, \epsilon^{sym}$ is a biaglebra.*

Proof. What remains to be shown is that the compatibility equations hold. For the unit and counit these are simple computations. The bi-algebra equation itself follows from Corollary 4.32. □

Remark 4.36. Alternatively this also either follows from the fact that $\mathfrak{F}''_{\mathcal{V}}$ is a Feynman category and composition is well defined, or more from the consideration presented in §4.12

Remark 4.37. Notice that there are many ways in which two weakly composable morphisms are composable and hence may yield different compositions. In the skeletal case, one has that different composition of weakly composable morphisms may differ by an isomorphism on the middle space. This is essentially the reason that the composition in (2.37) is on invariants. A similar phenomenon is known in physics, when composing graphs [Kre06].

4.6.6. **Connection to the cooperad case.** Indeed in $\mathfrak{S}urj$ decomposing $\pi_S : S \twoheadrightarrow \{*\}$ yields the sum $S \stackrel{f}{\twoheadrightarrow} T \stackrel{\pi_T}{\twoheadrightarrow} \{*\}$. This is a typical morphism in $\mathfrak{S}urj'$ from π_S to π_T.

We work in the framework of twisted Feynman categories, specifically $\mathfrak{S}urj_{\mathcal{O}}$, see §4.11.1. In this language, the diagrams (2.38) identify certain summands in the coproduct and on the coinvariants one is left with the channels.

The composition operation on the twisted $\mathfrak{S}urj_{\mathcal{O}}$: $\gamma_f : \mathcal{O}(f) \otimes \mathcal{O}(T) \to \mathcal{O}(S)$, corresponding to the composition $\pi_T \circ f = \pi_S$ cf. 4.11.1. Dually, there is one summand of this type $\check{\gamma}_f$ in the coproduct. We identify two such summands in the coproduct under the action of the automorphism groups. This corresponds to the diagrams 2.36 which are the isomorphisms in $\mathfrak{S}urj'$. Effectively, this means that fixing the size of S and T there is only one channel per partition of $S = S_1 \amalg \cdots \amalg S_k$ into fibers of f.

4.6.7. **Actions and cocycles.** Another interesting aspect is the possiblility to twist the comultiplication by a cocycle. In certain cases this leads to the reduced coproduct. Recall that there is an $Aut(Z)$ on $Hom(Z, Y) \times Hom(X, Z)$ given by $\rho(\sigma)(\phi_1, \phi_0) = (\phi_1 \circ \sigma^{-1}, \sigma \circ f_0)$.

By a twisting cocycle for the coproduct, we mean a morphism $\mathcal{B} \to Hom(\mathcal{B} \otimes \mathcal{B}, K)$ that is a linear collection of bilinear morphisms β_ϕ, s.t. $\Delta_\beta(\phi) = \sum_{\phi_1,\phi_0} \beta_\phi(\phi_1, \phi_0)\phi_1 \otimes \phi_0$ is still coassociative. Such a cocylce is called multiplicative if $\beta_{\phi\otimes\psi} = \beta_\phi\beta_\psi$ on decomposables. β is called counital, if there exists a counit ϵ_β.

In particular, if the cocycle is multiplicative, and the bi-algebra equation holds for Δ, then it holds for Δ_β. Furthermore, if it is counital, the bialgebra inherits a counit.

Assume for simplicity that we are in the skeletal case.

Lemma 4.38. *If the $Aut(Z)$ action is free on all decompositions, then we can define a reduced coalgebra structure on $\mathcal{B}/\mathcal{C}$ via the reduced coproduct and counit on $\mathcal{B}/\mathcal{C}$ defined by*

$$\Delta^{red}(f) = \sum_Z \sum_{i_r} [\phi_1^{i_r}] \otimes [\phi_0^{i_r}] \tag{4.21}$$

where the sum runs over representatives of the $Aut(Z)$ action.

$$\epsilon^{red}([f]) = \begin{cases} 1 & \text{if } [\phi] = [id_X] \\ 0 & \text{else} \end{cases} \tag{4.22}$$

This corresponds to modifying the coproduct by a multiplicative cocycle, which is given by $\beta(\phi_0, \phi_1) = \frac{1}{|Aut(Z)|}$. Here $\epsilon^{red} = \epsilon_\beta$ This bi–algebra structure descends to the one on $\mathcal{B}_{iso}$ given by Δ^{sym}

Proof. One calculates:

$$\begin{aligned}\Delta([f]) = [\Delta(f)] &= \sum_Z \sum_i [f_1^i \otimes f_0^i] \\ &= \sum_Z \sum_{i_r} \sum_{\sigma \in Aut(Z)} [f_1^{i_r} \circ \sigma^{-1}] \otimes [\sigma \circ f_0^{i_r}] \\ &= \sum_Z \sum_{i_r} |Aut(Z)| [f_1^{i_r}] \otimes [f_0^{i_r}]\end{aligned}$$

The fact that this is the modification by the given cocycle is a straightforward calculation given that action is free and the $Aut(Z_1)$ and $Aut(Z_2)$ actions on decompositions $X \to Z_1 \to Z_2 \to Y$ commute.

□

4.6.8. **Balanced actions.** More generally, one could define the putative cocycle $\beta(\phi_1^i, \phi_0^i) = \frac{1}{|Or(\phi_1,\phi_0)|}$ where $Or(\phi_0, f\phi 1)$ is the orbit under the $Aut(Z)$ action. If this is indeed a cocycle then we say that $\mathfrak{F}$ has a *balanced* action by automorphisms. The trivial and free actions are balanced. We conjecture that this is always the case, but leave the analysis for the future.

Proposition 4.39. *If $\mathfrak{F}$ is non–symmetric, skeletal in the above sense, and decomposition finite with balanced actions as above then tuple $(\mathscr{B}, \otimes, \Delta^{red}, \eta, \epsilon^{red})$ is also a bialgebra.*

Proof. The fact that we have an algebra remains unchanged. For the coalgebra, we have to check coassociativity, which is clear due to the assumtption that the action is balanced. The bi–algebra equation still holds, since the cocycle is multiplicative: $\beta(\phi_1 \otimes \psi_1, \phi_0 \otimes \psi_0) = \beta(\phi_1, \psi_1)\beta(\phi_0, \psi_0)$. This follows from the fact that in the non–Σ case: $Aut(Z \otimes Z') = Aut(Z) \otimes Aut(Z')$. □

Note, this reduced structure is available for the non–skeletal version. Here, for instance in the free action case, one obtains factors $|Iso(Z)||Aut(Z)|$ which again constitutes a multiplicative cocycle.

4.6.9. **Balanced actions in the symmetric case.** In the symmetric case, there is are the additional problems that the bi–algebra equation does not hold and that the cocycle above is not multiplicative. It turns out that these two deficiencies cancel each other out.

Proposition 4.40. *If $\mathfrak{F}$ is a decomposition finite Feynman category with a free action by the automorphism groups, then Δ_β and ϵ^{red} provide a bi–algebra structure on $\mathscr{B}_{iso}$.*

Proof. Inspecting the proof of Corollary 4.32, we get an additional factor of $\frac{1}{|Aut(Y)|}$ for each summand in $\Delta \circ \mu$ while on the other side of the equation the factor is $\frac{1}{|Aut(\hat{Y})||Aut(\hat{Y}')|}$ which cancel with the additional factor of $\frac{|Aut(Y)|}{|Aut(\hat{Y})||Aut(\hat{Y}')|}$ in (4.18). □

We conjecture that this is true in the balanced case and even in general.

Remark 4.41. It seems that the two bialgebra structure Δ^{red} and Δ^{iso} may differ. The first is akin to the relative tensor product and the latter to the full coinvariants.

The difference is that on $Hom(X,Z)\times Hom(Z,Y)$ there is actually an $Aut(Z)\times Aut(Z)$ action, which is used to go to the full invariants. The relative calculation is then via the diagonal embedding $Aut(Z)\stackrel{\Delta}{\hookrightarrow} Aut(Z)\times Aut(Z)$

In the free case these coincide.

4.6.10. **Summary.** Since there are many constructions at work here, we will collect the results for the bialgebras in an overview theorem:

Theorem 4.42. *Fix a composition finite Feynman category, let $\mathscr{B}$ and $\mathscr{B}^{sk}$ as given above considered as algebras with $\otimes$ as product and $id_{\mathbb{1}}$ as the unit. Let $\mathcal{C}$ be the ideal generated by $\sim$ in $\mathscr{B}$ and $\mathcal{C}^{sk}$ the respective ideal in $\mathscr{B}^{sk}$. Set $\mathscr{B}_{iso}=\mathscr{B}/\mathcal{C}$ and set $\mathscr{B}^{sk}_{iso}=\mathscr{B}^{sk}/\mathcal{C}^{sk}$.*

1. *$\mathscr{B}$ and $\mathscr{B}^{sk}$ are Morita equivalent as algebras and furthermore $\mathscr{B}_{iso}\simeq\mathscr{B}^{sk}_{iso}$.*
2. *Both $\mathscr{B}$ and $\mathscr{B}^{sk}$ are coalgebras with respect to the deconcatenation coproduct with counit ϵ. Furthemore $\mathscr{B}$ and $\mathscr{B}^{sk}$ are unital, counital bialgebras.*
3. *In the non–symmetric case, after extending scalars, so that all $|Aut(X)|$ are invertible, $\mathscr{B}^{sk}_{iso}$ is a unital counital bialgebra with counit $\bar\epsilon$ and if the $|Iso(X)|$ are also invertible $\mathscr{B}_{iso}$ is unital counital bialgebra.*
4. *In the symmetric case, there is a bi–algebra stucture on $\mathscr{B}_{iso}$ given by Δ^{iso}.*
5. *If the action of $Aut(Z)$ on $Hom(X,Z)\times Hom(Z,Y)$ is balanced for all X,Y,Z, then $\mathscr{B}_{iso}$ is a bialgebra with respect to $(\otimes,\Delta^{red},\eta,\epsilon^{red})$ in the non–symmetric case, which is a twist of the original bi–algebra structure.*
6. *In the symmetric case with free action by automorphisms, the twisted coproduct satisfies the bi–algebra equation on $\mathscr{B}_{iso}$ and Δ^{red} coincides with Δ^{iso}.*
7. *All the structures above are graded by the length of a morphism.*

□

4.7. **Hopf algebras from Feynman categories.** The above bialgebras are usually not connected. There are two obstructions. Each isomorphism class of an object X gives a unit and, unless $\mathcal{V}$ is discrete, there are isomorphisms which are not conilpotent and which prevent the identities of the different X from being group-like elements. We will now formalize this.

4.7.1. **Hopf algebras from almost connected Feynman catgories.** We define the ideal $\bar{\mathscr{I}}=\langle|Aut(X)||Iso(X)|[id_X]-|Aut(Y)||Iso(Y)|[id_Y]\rangle$ of $\mathscr{B}_{iso}$, and then consider $\mathscr{H}=\mathscr{B}_{iso}/\bar{\mathscr{I}}$. If $\mathcal{V}$ is discrete, one is effectively quotienting $\mathscr{B}$ by the ideal $\mathscr{I}=\langle id_X-id_Y\rangle$.

Proposition 4.43. *Assume that $\mathfrak{F}$ is decomposition finite and has almost group like identities. If $\mathcal{V}$ is discrete, then $\mathscr{I}$ is a coideal in $\mathscr{B}$ and $\mathscr{H}=\mathscr{B}/\mathscr{I}$ is a bialgebra with counit induced by ϵ and unit $\eta(1)=[\mathrm{id}_{\mathbb{1}_{\mathcal{F}}}]$. In general, $\bar{\mathscr{I}}$ is a coideal in $\mathscr{B}_{iso}$ and $\mathscr{H}=\mathscr{B}_{iso}/\bar{\mathscr{I}}$ is a bialgebra with counit induced by $\bar\epsilon$ and unit $\bar\eta(1)=[\mathrm{id}_{\mathbb{1}_{\mathcal{F}}}]$,* *Proof.* In the discrete case, we have $\Delta(id_X - id_Y) = id_X \otimes id_X - id_Y \otimes id_Y = (id_X - id_Y) \otimes id_X + id_y \otimes (id_X - id_Y)$ and $\epsilon(id_X - id_Y) = 0$, so that $\mathscr{J}$ is a coideal. Fixing $\eta(1) = [id_{\mathbb{1}_{\mathcal{F}}}]$ gives a unit and defines a split counit for the coalgebra structure.

In the case of non–discrete $\mathcal{V}$ in $\mathscr{B}_{iso}$, (4.6) reads $\Delta([id_X]) = |Aut(X)||Iso(X)|[id_X] \otimes [id_X]$, so that

$$
\begin{aligned}
&\Delta(|Aut(X)||Iso(X)|[id_X]) - |Aut(Y)||Iso(Y)|[id_Y] \\
&\qquad = (|Aut(X)||Iso(X)|)^2[id_X] \otimes [id_X] - (|Aut(Y)||Iso(Y)|)^2[id_Y] \\
&= (|Aut(X)||Iso(X)|[id_X] - |Aut(Y)||Iso(Y)|[id_Y] \otimes |Aut(X)||Iso(X)|[id_X] + \\
&\qquad |Aut(Y)||Iso(Y)|[id_Y] \otimes (|Aut(X)||Iso(X)|[id_X] - |Aut(Y)||Iso(Y)|[id_Y])
\end{aligned}
$$

Hence, the ideal $\bar{\mathscr{J}}$ is generated by elements $|Aut(X)||Iso(X)|[id_X] - |Aut(Y)||Iso(Y)|[id_Y]$ is also a coideal, as these also satisfy $\bar{\epsilon}(|Aut(X)||Iso(X)|[id_X] - |Aut(Y)||Iso(Y)|[id_Y]) = 1 - 1 = 0$. Again $\bar{\eta}$ yields a split counit □

Definition 4.44. $\mathfrak{F}$ is called Hopf, if it is decomposition finite, has group like identities and the bialgebra $\mathscr{H} := \mathscr{B}_{iso}/\bar{\mathscr{J}}$, admits an antipode.

Theorem 4.45. *Any Hopf Feynman category yields a Hopf algebra $\mathscr{H} := \mathscr{B}_{iso}/\bar{\mathscr{J}}$, both in the symmetric and non–symmetric case. $\mathscr{H}$ is in general not cocommutative. It is commutative in the symmetric case and not necessarily commutative in the non–symmetric case.*

Proof. The only new claim is the commutativity in the symmetric case. This is due to the fact that the commutativity constraints are isomorphisms and these become identities already in $\mathscr{B}_{iso}$. □

In general, the existence of an antipode is complicated. We do know that for graded connected bialgebras an antipode exists. In terms of Feynman categories this situation can be achieved by looking at definite Feynman categories.

4.8. **Connectedness in the definite case.** We can reduce the question of the existence of an antipode further in the case of a definite Feynman category to the connectedness of the length 0 morphisms.

All three main examples are definite. For example, Feynman categories from operads are precisely non–negative, if there is no $\mathcal{O}(0)$; recall the length of elements of $\mathcal{O}(n)$ is $n-1$. Surjections are also non–negative. Dually, regarding only injections is an example of a non–positive Feynman category. All graph examples —without extra morphisms, see [KW17]— are also non–negative.

If the Feynman category is definite, $\mathscr{B}$ is already non–negatively, or non–positively graded. In the latter case, it is also non–negatively graded by the negative length.

4.8.1. **Morphisms of length 0.** Let $\mathscr{B}_0$ be the set of morphisms of length 0. In general $\mathscr{B}_0$ is not a sub–bialgebra, but it is in the definite case.

Furthermore, in the definite case the morphisms $Hom_{\mathcal{F}}(\imath(\mathcal{V}), \imath(\mathcal{V}))$ together with the counit ϵ and the unit η form a pointed coalgebra $\mathscr{B}_{\mathcal{V}}$, which generates $\mathscr{B}_0$ as an algebra with given commutatitity constraints as we show below. The elements of $\mathscr{B}_{\mathcal{V}}$ split according to whether they are isomorphisms or not. That is, whether or not they lie in $Mor(\mathcal{V})$. Basically all the conditions on $\mathscr{B}_0$ can then be checked on $\mathscr{B}_{\mathcal{V}}$ in the definite case.

Lemma 4.46. *In a strict definite Feynman category a morphism of length 0 has a decomposition into morphisms of $\mathscr{B}_{\mathcal{V}}$. Also, any morphisms with a left or right inverse is in $\mathscr{B}_0$. $\mathscr{B}_0$ is a sub–bialgebra and $\mathscr{B}_0 = Hom_{\mathcal{F}}(\imath(\mathcal{V}), \imath(\mathcal{V}))^{\otimes} = \mathscr{B}_{\mathcal{V}}^{\otimes}$ (again, this includes the permutations given by the commutativity constraints in the symmetric case). Finally, $\mathscr{B}_{\mathcal{V}}$ together with the counit ϵ and the unit η form a pointed coalgebra.*

Proof. Suppose $\phi : X \stackrel{\phi_R}{\to} Z \stackrel{\phi_L}{\to} Y$ has length 0, then $|\phi_R| + |\phi_L| = |\phi| = 0$. In the definite case this implies $|\phi_R| = |\phi_L| = 0$, which shows that $\mathscr{B}_0$ is a subcoalgebra and since $\otimes$ has is additive in length, $\mathscr{B}_0$ is a subalgebra. Finally the counit restricts and the unit is of length 0. Also if $|\phi| = 0$ then $\phi \simeq \bigotimes_{v\in I} \phi_v$ with $|\phi_v| \geq 0$ (or ≤ 0) and $\sum_{v\in I} |\phi_v| = 0$, which means that $\forall v \in I : |\phi_v| = 0$. In particular, if $|Y| = 1$, we see that $|X| = 1$ and $\mathscr{B}_{\mathcal{V}}$ is a subcoalgebra. The $\phi = id_X$ are in $\mathscr{B}_{\mathcal{V}}$, the counit restricts and the unit η lands in $\mathscr{B}_{\mathcal{V}}$. □

By induction, one can see that what can keep things from being connected is $\mathscr{B}_0$ or better $\mathscr{B}_{\mathcal{V}}$. This is analogous to the situation for cooperads with multiplication, where, $\mathcal{V}$ is trivial and $\mathscr{B}_{\mathcal{V}} = \mathcal{O}(1)$ is the pointed coalgebra as in Defintion 2.49.

Let $\mathscr{B}_{0,iso} := (\mathscr{B}_0/\sim)$ and $\mathscr{B}_{\mathcal{V},iso} := (\mathscr{B}_{\mathcal{V}}/\sim)$. Set $\bar{\mathscr{J}}_{\mathcal{V}}$ be the ideal and coideal $\bar{\mathscr{J}}$ restricted to $\mathscr{B}_{\mathcal{V},iso}$ and $\bar{\mathscr{J}}_0$ its restriction to $\mathscr{B}_0$.

Definition 4.47. We call $\mathscr{B}_{\mathcal{V}}$ almost connected, if all the $id_{\imath(*)}, * \in \mathcal{V}$ are almost grouplike and $\mathscr{B}_{\mathcal{V},iso}/\bar{\mathscr{J}}_{\mathcal{V}}$ is connected. Likewise $\mathscr{B}_0$ is almost connected, if all the identities are almost grouplike and $\mathscr{B}_{0,iso}/\bar{\mathscr{J}}_0$ is connected.

Lemma 4.48. *If $\mathscr{B}_{\mathcal{V}}$ is almost connected, then $\mathscr{B}_0$ is as well.*

Proof. First, note that by Lemma 4.46 and (ii) almost connected implies that all the identities are group like. Namely any factorization up to permutation is a factorization tensor factors of $id_{\imath(*)}$. Similarly, any decomposition of a morphism ϕ of length 0 factors (up to permutation) into tensor product of decompositions in $\mathscr{B}_{\mathcal{V}}$. Hence, due to the bialgbra equation, the connectedness follows. □

Theorem 4.49. *Assume that $\mathfrak{F}$ is decomposition finite, definite and $\mathscr{B}_{\mathcal{V}}$ is almost connected, then $\mathfrak{F}$ is Hopf. That is $\mathscr{B}_{iso}/\bar{\mathscr{J}}$ is a connected bialgebra, and hence a Hopf algebra.*

The same holds true for the skeletal version. Furthermore in the free case, the analogous statement holds for the reduced coproduct and counit.

Proof. We show that $\mathscr{H}$ is conilpotent and hence connected. The reason is that any decomposition which has a morphism on the left or the right that is not of length 0 has a shorter length. The terms with length 0 factors are taken care of by the almost connectedness of $\mathscr{B}_0$ which is guaranteed by the Lemma above. The skeletal and the reduced case are analogous. □

Remark 4.50. Any morphism $\phi : X \to Y$ satisfies $\Delta(\phi) = id_X \otimes \phi + \phi \otimes id_Y + \dots$, id_X (suitably normalized) are group like elements in $\mathscr{B}_{iso}$. Hence it is interesting to study the coradical filtration and the $([id_X], [id_Y])$–primitive elements. They correspond to the generators for morphisms in Feynman categories [KW17]. In the main examples they are all tensors of elements of length 1.

4.9. **Functoriality.** Let $F : \mathfrak{F} \to \mathfrak{F}'$ be a morphism of Feynman categories. In the strict case, this is basically a strict monoidal functor from $F : \mathcal{F} \to \mathcal{F}'$ compatible with all the structures, see [KW17, Chapter 1.5]. For a morphism $\phi \in Mor(\mathcal{F}')$ thought of as a characteristic function $\phi(\psi) = \delta_{\phi,\psi}$. We see that

$$F^*(\phi) := \phi \circ F = \sum_{\hat{\phi} \in Mor(\mathcal{F}) : F(\hat{\phi}) = \phi} \hat{\phi}$$

Proposition 4.51. *F^* induces a morphism of unital algebras. If F is injective on objects, then F^* induces a morphism of coalgebras. If F^* is bijective on objects, it induces a morphism of counital coalgebras. If furthermore $F|_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}'$ is surjective on morphisms then F^* induces a map of bialgberas $\mathscr{B}^{\mathfrak{F}'}_{iso} \to \mathscr{B}^{\mathfrak{F}}_{iso}$.*

Proof. We have to check the multiplication, but since $\mathcal{F}$ is strictly monoidal, we get $(F^* \otimes F^*)(\phi \otimes \psi) = (\phi \circ F) \otimes (\psi \circ F) = (\phi \otimes \psi) \circ F$. For the coproduct, we get

$$\Delta(F^*\phi) = \sum_{\hat{\phi} \in Mor(\mathcal{F}) : F(\hat{\phi}) = \phi} \sum_{(\hat{\phi}_0, \hat{\phi}_1) : \hat{\phi}_1 \circ \hat{\phi}_0 = \hat{\phi}} \hat{\phi}_0 \otimes \hat{\phi}_1 \tag{4.23}$$

$$(F^* \otimes F^*)\Delta(\phi) = \sum_{(\phi_0, \phi_1) : \phi_1 \circ \phi_0 = \phi} \sum_{\hat{\phi}_0, \hat{\phi}_1 \in Mor(\mathcal{F}) : F(\hat{\phi}_0) = \phi_0, F(\hat{\phi}_1) = \phi_1} \hat{\phi}_0 \otimes \hat{\phi}_1 \tag{4.24}$$

We now check that the sums coincide. Certainly for any term in the first sum corresponding to decomposition $\hat{\phi} = \hat{\phi}_1 \circ \hat{\phi}_0$ appears in the second sum, since F is a functor: $F(\hat{\phi}_1) \circ F(\hat{\phi}_0) = F(\hat{\phi}_0 \circ \hat{\phi}_1) = F(\hat{\phi}) = \phi$. The second sum might be larger, since the lifts need not be composable. If, however, F is injective on objects, then all lifts of a composition are composable and the two sums agree. The unit agrees, because of the injectivity and uniqueness of the unit object and the triviality of $Hom(\mathbb{1}, \mathbb{1})$. For the counit, we need bijectivity. In this case $\hat{id_X} = id_{\hat{X}} + T$, with $\epsilon(T) = 0$, so that $\epsilon(F^*\phi) = \epsilon\phi$, since $F(id_{\hat{X}}) = id_X$ and hence if $\phi \neq id_X$, then there is no $id_{\hat{X}}$ in the fiber. If the functor is not injective, we might have more objects in the fiber and if it is not surjective $F^*(id_X)$ can be 0.

We also have to check that these structures descend to $\mathscr{B}_{iso}$. This is clear for the multiplication, since F is a functor. For the comultiplication, we have to show that F^* sends $\mathcal{I}'$ to $\mathcal{I}$. For this, we regard the fibers of two functions $f \sim g$. Since $f \sim g$ we have isomorphisms such that $g = \sigma' \circ f \circ \sigma^{-1}$. Since F restricted to $\mathcal{V}$ is surjective onto $\mathcal{V}'$, we have invertible lifts $\hat{\sigma}$ and $\hat{\sigma}'$ of σ and σ'. Now $F^{-1}(f) = \hat{\sigma} \circ F^{-1}(g) \circ \hat{\sigma}'$. The inclusions follow from the fact that F is a functor and the lifts are invertible. □

Recall that in order to get a Hopf algebra, we needed to quotient by the ideal $\bar{\mathscr{J}}$ defined in §4.7

Definition 4.52. We call a functor F as above Hopf compatible if it satisfies the conditions of Proposition 4.51 and $F^*(\bar{\mathscr{J}}_{\mathfrak{F}'}) \subset \bar{\mathscr{J}}_{\mathfrak{F}}$.

The following is straightforward.

Proposition 4.53. *If $\mathfrak{F}$ and $\mathfrak{F}'$ are almost Hopf, a Hopf compatible functor induces a morphism of Hopf algebras $\mathscr{H}_{\mathfrak{F}'} \to \mathscr{H}_{\mathfrak{F}}$.* □

The following is a useful criterion:

Proposition 4.54. *If in addition to the conidtions of Proposition 4.51 $F|_{\mathcal{V}}$, F does not send any non–invertible elements of $Mor(\mathcal{F})$ to invertible elements in $Mor(\mathcal{F}')$, then F is Hopf compatible.*

Proof. Suppose the conditions are true and let $* \in \mathcal{V}'$ with $F^{-1}(*) = \hat{*}$. Set $H = Aut_{\mathcal{V}'}(*) = Aut_{\mathcal{F}'}(*)$, $G = Aut_{\mathcal{V}}(\hat{*}) = Aut_{\mathcal{F}}(\hat{*})$ and $K = ker(F|_G)$. then $F^*([id_*]) = \sum_{\phi_k \in K}[\phi_k] = |K|[\mathrm{id}_{\hat{*}}]$ and by the condition (i) of Feynman categories the same holds for $*$ replaced by X and $\hat{*}$ replaced by $\hat{X} = F^{-1}(X)$. Due to the bijectiveness and third condition $|Iso(X)| = |Iso(\hat{X})|$ and the statement follows from the orbit formula $|G| = |K||H|$. □

These criteria reflect that the Hopf algebras are very sensitive to invertible elements. It says that that we can identify isomorphisms and are allowed to identify morphisms, but only in each class separately.

Example 4.55. An example is provided by the map of operads: rooted 3–regular forests → rooted corollas. This give a functor of Feynman categories enriching $\mathfrak{Surj}$ or in the planar version of $\mathfrak{Surj}_<$. This functor is Hopf compatible thus induces a map of Hopf algebras which is the morphism considered by Goncharov in [Gon05].

Example 4.56. Another example is given by the map of rooted forests with no binary vertices → corollas. The corresponding morphisms of Feynman categories is again Hopf compatible.

However, if we consider the functor of Feynman categories induced by rooted trees → rooted corollas is not Hopf compatible. It sends all morphisms corresponding to binary trees to the identity morphism of the corolla with one input. Thus is maps non–invertible elements to invertible elements. The presence of these extra morphisms in $\mathscr{H}_{CK}$ is what makes it especially interesting. They also correspond to a universal property [Moe01] and Example 2.51.

4.10. **Opposite Feynman category yields the coopposite bialgebra.** Notice that usually the opposite category of a Feynman category is not a Feynman category, but it still defines a bialgebra. Namely, the constructions above just yield the coopposite bialgebra structure $\mathscr{B}^{coop}$ and Hopf algebra structure $\mathscr{H}^{coop}$ if the extra conditions are met.

This means, the multiplication is unchanged but the comultiplication is switched. That is $\Delta(\phi^{op}) = \sum_{\phi_1 \circ \phi^0 = \phi} \phi_1^{op} \otimes \phi_0^{op}$.

4.11. **Constructions on Feynman categories.** There are three constructions on Feynman categories that are relevant to these examples.

4.11.1. **Enrichments, plus construction and hyper category $\mathfrak{F}^{hyp}$.** The first construction is the plus construction $\mathfrak{F}^+$ and its quotient $\mathfrak{F}^{hyp}$ and its equivalent reduced version $\mathfrak{F}^{hyp,rd}$, see [KW17]. The main result of [KW17, Lemma 4.5] says that for any Feynman category $\mathfrak{F}$ there exists a Feynman category $\mathfrak{F}^{hyp}$ and the set of monoidal functors $\mathcal{O} : \mathcal{F}^{hyp} \to \mathcal{E}$ is in 1–1 correspondence with enrichments $\mathcal{F}_\mathcal{O}$ of $\mathcal{F}$ over $\mathcal{E}$.

For such an enrichment, one has

$$Hom_{\mathcal{F}_\mathcal{O}}(X,Y) = \coprod_{\phi \in Hom_\mathcal{F}(X,Y)} \mathcal{O}(\phi) \tag{4.25}$$

And that if ϕ is an isomorphism, then $\mathcal{O}(\phi) \simeq \mathbb{1}_\mathcal{E}$ This generalizes the notion of hyperoperads of [GK98], whence the superscript *hyp*.

The compositions in $\mathcal{F}$ then give rise to compositions in $\mathcal{F}_\mathcal{O}$ for instance for the composition $\phi = \phi_1 \circ \phi_0$, we get:

$$\mathcal{O}(\phi_1) \otimes \mathcal{O}(\phi_0) \to \mathcal{O}(\phi) \tag{4.26}$$

Therrise to compositions in $\mathcal{F}$e is an equivalent, but slightly smaller category $\mathfrak{F}^{hpy,rd}$, we can alternatively use. The relevant example is that $\mathfrak{S}urj^{hyp,rd} = \mathfrak{F}_{operads,0}$ that is, operads whose $\mathcal{O}(1)$ contains only 1 as an invertible element, we will call these operads almost pointed. Thus any such operad $\mathcal{O} : \mathcal{F}_{operads,0} \to \mathcal{E}$ gives rise to a Feynman cateogory $\mathfrak{S}urj_\mathcal{O}$ whose morphisms are detemined by

$$Hom_{\mathfrak{S}urj_\mathcal{O}}(n,1) = \mathcal{O}(n) \tag{4.27}$$

In particular, if $f : S \twoheadrightarrow T$ then $\mathcal{O}(f) = \bigotimes_{t \in T} \mathcal{O}(f^{-1}(t))$ since f decomposes as one–comma generators $f_t : f^{-1}(t) \twoheadrightarrow \{t\}$.

If $\mathcal{O}(1)$ has more invertible elements, one has to enlarge $\mathfrak{S}urj$ by choosing the appropriate $\mathcal{V}$. In the case of Cartesian $\mathcal{E}$ this is $Hom_{\mathcal{V}'}(1,1) = \mathcal{O}(1)^\times$.

This gives rise to extra isomorphisms and/or a K–collection, see [KW17, 2.6.4]. This means in particular that any operad gives rise to morphisms of a Feynman category. The dual of the morphisms are then cooperads and the cooperadic and Feynman categorical construction coincide.

The non-Σ case is similar. For this one uses $\mathfrak{F}_{surj,<}$ and then obtains enrichments by non–Σ operads. Thus again the cooperadic methods apply and yield the same results as the Feynman category constructions.

Proposition 4.57. *In both the symmetric and non–symmetric case, any pair of an $\mathfrak{F}$ and an $\mathfrak{F}^{hyp}$–$\mathcal{O}p$ $\mathcal{O}$ gives rise to a unital, counital bialgebra by regarding the morphisms of $\mathfrak{F}_\mathcal{O}$. If its quotient by the ideal generated by the $\mathcal{O}(id_X) \simeq \mathbb{1}$ corresponding to $\bar{\mathcal{J}}$ is connected, in which case we call $\mathfrak{F}_\mathcal{O}$ almost Hopf, we obtain a Hopf algebra.* □

Remark 4.58. Applying the constructions of this chapter to $\mathfrak{F}_{surj,<,\mathcal{O}}$ is equivalent to the construction of Chapter 2 in the free case, see 2.2.3. The symmetric case is then equivalent to considering $\mathfrak{F}_{surj,\mathcal{O}}$.

The condition of being almost connected then coincides with the definition of almost connected Definition 2.49.

Remark 4.59. The construction of identifying all invertible elements in $\check{\mathcal{O}}(1)$ is exactly the passage from $\mathfrak{F}^+$–$\mathcal{O}ps$ to $\mathfrak{F}^{hyp}$–$\mathcal{O}ps$.

4.11.2. **Decoration $\mathfrak{F}_{dec\mathcal{O}}$.** This type of modification is discussed in [KL16]. It gives a new Feynman category $\mathfrak{F}_{dec\mathcal{O}}$ from a pair $(\mathfrak{F},\mathcal{O})$ of a Feynman category $\mathfrak{F}$ and a strong monoidal functor $\mathcal{O}:\mathcal{F}\to\mathcal{C}$. The objects of $\mathfrak{F}_{dec\mathcal{O}}$ are pairs $(X,a_X), a_X\in\mathcal{O}(X)$ ($a_x\in Hom_{\mathcal{E}}(\mathbb{1},\mathcal{O}(X))$ for the fastidious reader). The morphisms from (X,a_X) to (Y,a_y) are those $\phi\in Hom_{\mathcal{F}}(X,Y)$ for which $\mathcal{O}(\phi)(a_X)=a_y$.

Looking at the free Abelian group generated by the morphisms turns the operad into a Feynman category and one can apply the results of this chapter.

This construction explains the constructions of chapter 3 as discussed below.

4.11.3. **Universal operations.** It is shown that $\mathfrak{F}_{\mathcal{V}}$, which is given by $\mathcal{F}_{\mathcal{V}}=\text{colim}_{\mathcal{V}}\imath$, yields a Feynman category with trivial $\mathcal{V}_{\mathcal{V}}$. This generalizes the Meta–Operad structure of [Kau07]. The result is again a Feynman category whose morphisms define an operad and hence the free Abelian group yields a cooperad.

Moreover in many situations, see [KW17] the morphisms of the category are weakly generated by a simple Feynman category obtained by "forgetting tails". The action is then via a foliation operator as introduced in [Kau07]. In fact there is a poly–simplicial structure hidden here, as can be inferred from [BB09].

4.11.4. **Enrichment over $\mathcal{C}^{op}$ and opposite Feynman category.** Notice that we can regard functors $\mathfrak{F}\to\mathcal{C}^{\mathcal{O}p}$ as coversions of operads, etc.. In particular if we have a functor $\mathfrak{F}^{hyp}\to\mathcal{C}^{op}$, we get a Feynman category $\mathfrak{F}_{\mathcal{O}}$ enriched over $\mathcal{C}^{op}$. This means that $\mathfrak{F}^{op}_{\mathcal{O}}$ is enriched over $\mathcal{C}$.

Example 4.60. In particular, if $\mathcal{O}:\mathfrak{S}urj^{hyp}=\mathfrak{F}_{operads,0}\to\mathcal{C}^{op}$ that is dually a pointed almost connected cooperad in $\mathcal{C}$. Then twisting with $\mathcal{O}$ gives us $\mathfrak{S}urj_{<,\mathcal{O}}$ which is enriched in $\mathcal{C}^{op}$. Taking the opposite we get $\mathfrak{S}urj^{op}_{<,\mathcal{O}}$. The underlying category is $\mathcal{I}nj_{*,*}$ enriched by $\check{\mathcal{O}}$, where $\check{\mathcal{O}}$ is the cooperad in $\mathcal{C}$ corresponding to the operad in $\mathcal{C}^{op}$. This means that the objects are the natural numbers n and the morphisms are $Hom(1,n)=\check{\mathcal{O}}(n)$. This is the enrichment in which the unique map in $Hom_{\mathcal{I}nj_{*,*}}([1],[n])$ is assigned $\check{\mathcal{O}}(n)$ in the overlying enriched category $(\mathcal{I}nj_{*,*})_{\check{\mathcal{O}}}$.

Putting all the pieces together then yields the following:

Theorem 4.61. *Given a cooperad $\check{\mathcal{O}}$ that is given by a functor $\mathcal{O}:\mathfrak{F}_{operads,0}\to\mathcal{C}^{op}$. Let $\mathscr{B}_{\check{\mathcal{O}}^{nc}}$ be the bialgebra of Example 2.2.3. And let $\mathscr{B}_{\mathfrak{S}urj^{op}_{<,\mathcal{O}}}$ be the bialgebra of the Feynman category discussed above then these two bialgebra coincide.*

Moreover if $\mathfrak{S}urj_{<,\mathcal{O}}$ is almost connected, the so is $\check{\mathcal{O}}$ and the corresponding Hopf algebras coincide. □

4.12. **From Feynman categories to bialgebras of groupoids.** A Feynman category without finiteness conditions has an associated *monoidal decomposition groupoid* which, using the machinery of decomposition spaces, induces a bialgebra structure in a category of 'linear functors' of comma categories of groupoids. From this, classical bialgebras may be recovered when finiteness conditions are imposed.

Consider the 'fat nerve' $\mathcal{X} = \mathcal{X}(\mathcal{F})$ of any category $\mathcal{F}$, the simplicial groupoid with $\mathcal{X}_n$ the groupoid of n-chains

$$\alpha_n = (X_0 \to X_1 \to \cdots \to X_n) \text{ in } \mathcal{F}$$

and the isomorphisms between such chains, and $\mathcal{X}_0 = Iso(\mathcal{F})$. The simplicial operator $d_1 : \mathcal{X}_2 \to \mathcal{X}_1$ is composition in $\mathcal{F}$. Its homotopy fiber over an object $\phi : X \to X'$ in $\mathcal{X}_1$ is thus the groupoid $\mathrm{Fact}(\phi)$ of factorizations $\phi \simeq \phi_1 \circ \phi_2$.

Suppose $\mathcal{F}$ is any Feynman category such that the factorisations of the identity on the monoidal unit form a contractible groupoid. Then it can be shown that in fact $\mathcal{X}(\mathcal{F})$ is a *symmetric monoidal decomposition groupoid* [GCKT15a, §9]. The tensor and unit of $\mathcal{F}$ clearly define $\eta : * \to \mathcal{X}$, $\mu : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$, but it is the key hereditary condition of a Feynman category that shows that tensor and composition are compatible: they form a homotopy pullback square

$$\begin{array}{ccccc}
\mathrm{Fact}(\phi) \times \mathrm{Fact}(\phi') & \longrightarrow & \mathcal{X}_2 \times \mathcal{X}_2 & \xrightarrow{\circ\times\circ} & \mathcal{X}_1 \times \mathcal{X}_1 \ \ni (\phi, \phi') \\
{\scriptstyle\otimes}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\otimes}\downarrow \lrcorner & & \downarrow{\scriptstyle\otimes} \\
\mathrm{Fact}(\phi \otimes \phi') & \longrightarrow & \mathcal{X}_2 & \xrightarrow[\circ]{} & \mathcal{X}_1 \ \ni \phi \otimes \phi',
\end{array}$$

for all $\phi : X \to Y$ and $\phi' : X' \to Y'$, that is, $\otimes : \mathrm{Fact}(\phi) \times \mathrm{Fact}(\phi') \to \mathrm{Fact}(\phi \otimes \phi')$ is a groupoid equivalence.

From [GCKT15a, Theorem 7.2 and §9] we see that $\mathcal{X}(\mathcal{F})$ induces a bialgebra in the symmetric monoidal category of comma categories of groupoids and linear functors between them, and in [GCKT15b] the finiteness conditions necessary and sufficient to pass to bialgebras in the category of $\mathbb{Q}$-vector spaces are studied.

A related construction appears in [KW17, §3.3] where the iterated Feynman categories $\mathfrak{F}', \dots \mathfrak{F}^{(n)}, \dots$ define a simplicial object. The associated maximal subgroupoids $\mathcal{V}'^{\otimes}, \dots$ are the 'fat nerve' above: objects at level n are factorizations of morphisms into n chains, with the isomorphisms between these chains. In operad theory this type of groupoid explicitly appeared earlier in [GK98] in the context of (twisted) modular operads, cf. also [MSS02].

5. Discussion of the Three Special Cases and Further Examples

We will now illustrate the different concepts and constructions by considering the three main cases as well as a few more instructive examples.

5.1. Connes–Kreimer and other graphs.

5.1.1. **Leaf labelled and planar version.** First, we can look at the operad $\mathcal{O}$ of leaf labelled rooted trees or planar planted trees. This gives a Feynman category by §4.11.1 and hence a bialgebra. Here $\mathcal{O}(1)$ has two generators id_1 which we denote by $|$ and $\bullet\!\!\!|$, the rooted tree with one binary non–root vertex. Now composing $\bullet\!\!\!|$ with itself will result in $\bullet\!\!\!|n$, the rooted tree with n binary non–root vertices, aka. a ladder. We also identify $\bullet\!\!\!|0 = |$. Taking the dual, either as the free Abelian group of morphisms, or simply the dual as a cooperad, we obtain a cooperad and the multiplication is either $\otimes$ from the Feynman category or $\otimes$ from the free construction. That these two coincide follows from condition (ii) of a Feynman category. η is given by $| = id_1$. The Feynman category and the cooperad are almost connected, since $\Delta(\phi n) = \sum_{(n_1,n_2):n_1,n_1\geq 0,n_1+n_2=n} \phi n_1 \otimes \phi n_2$ and hence the reduced coproduct is given by $\bar{\Delta}(\phi n) = \sum_{(n_1,n_2):n_1,n_1\geq 1,n_1+n_2=n} \phi n_1 \otimes \phi n_2$ whence $\check{\mathcal{O}}(1)$ is nilpotent.

If we take planar trees, there are no automorphisms and we obtain the first Hopf algebra of planted planar labelled forests. Notice that in the quotient $[|] = [||\dots|] = [1]$ which says that there is only one empty forest.

If we are in the non–planar case, we obtain a Hopf algebra of rooted forests, with labelled leaves. One uses $\mathcal{V}$ as finite subsets of $\mathbb{N}$ with isomorphisms.

These structures are also discussed in [Foi02b],[Foi02a] and [EFK05].

5.1.2. **Algebra over the operad description for Connes–Kreimer.** If one considers the algebras over the operad $\mathcal{O}$, then for a given algebra ρ, V, $\rho(\phi) \in Hom(V,V)$ is a "marked" endomorphism. This is the basis of the constructions of [Moe01]. One can also add more extra morphisms, say ϕc for $c \in C$ where C is some indexing set of colors. This was considered in [vdLM06b]. In general one can include such marked morphisms into Feynman categories (see [KW17][2.7]) as morphisms of $\varnothing \to *_{[1]}$.

5.1.3. **Half–infinite chains, coalgbera.** One interesting algebra comes from adding $\phi\infty$, representing a half–infinite rooted chain, with $\Delta(\phi\infty) = \sum_{n\geq 0} \phi n \otimes \phi\infty$. This is an example where there is a bi–grading in which the coproduct is finite in each bi–degree, the degrees lying in $\mathbf{N_0} \cup \{\infty\}$. With $s = \sum_{n\geq 0} \phi n$, we see from the associativity that $\Delta^n(\phi\infty) = s^{\otimes n-1} \otimes \phi\infty$ and s is a group like element. This fact leads to interesting physics, [Kre08].

We can also treat the half–infinite chain as a coalgebra $C = span(\phi\infty)$, and $\mathscr{H}$ being the graded Hopf algebra of trees, graded by the number of vertices or its subalgebra of finite linear trees, where $\check{\rho}(\phi\infty) = \sum_{n\geq 0} \phi n \otimes \phi\infty$. Everything is finite in each degree.

Lastly, we can condsider the larger coalgebra is spanned by Dirac–trees that is rooted trees with semi–infinite chains as leaves. The coaction is to cut the semi–infinite tree with a cut that leaves a finite base tree and Foch–tree branches.

Having infinite chains is not that easy, but this will be considered elsewhere.

5.1.4. **Unlabelled and symmetric version.** In the non–planar case, we have the action of the symmetric groups. In this case, we can use the symmetric construction or mod out by the automorphisms.

We then obtain the commutative Hopf algebra of rooted forests with non–labelled tails.

Alternatively, from the universal construction §4.11.3 on $\mathfrak{F}_{operads}$ one directly obtains the structure of a Hopf algebra of non–labelled rooted forests with leaves. The action of the automorphisms is free and hence there is also the reduced version of the co and Hopf algebras.

5.1.5. **No tail version.** For this particular operad, there is the construction of forgetting tails and we can use the construction of §2.9.2. In this case, we obtain the Hopf algebras of planted planar forests without tails or the commutative Hopf algebra of rooted forests, which is called $\mathscr{H}_{CK}$.

Finally, one can amputate the tails in the universal construction. One then obtains the cooperad dual to the pre–Lie operad [CL01, Kau07]. That is $\mathscr{H}_{amp}$ is realized naturally from a weakly generating suboperad, in the nomenclature of [KW17, §6.4].

5.1.6. **Graph version.** If we look at the Feynman category $\mathfrak{G} = (\mathcal{C}rl, \mathcal{A}gg, \imath)$ then, we obtain the Hopf algebra of graphs of Connes and Kreimer [CK98]. For this, we notice that the structure of composition in the Feynman category is given by grafting graphs into compatible vertices, i.e. those that have the correct structure of external legs; see Appendix A and [KW17]. Thus the coproduct gives a sum over subgraphs in a graph.

Taking the various quotients, we obtain the symmetric graph Hopf algebra, either with or without automorphism factors.

The main observation is that in the connected case, the ghost graph of a morphism fixes an *isomorphism class*, see [KW17, §2.1]. This means that in the coproduct in $\mathscr{B}_{iso}$ one looks at factorizations.

$$\begin{array}{ccc} X & \xrightarrow{\Gamma} & * \\ {\scriptstyle\gamma}\downarrow & \nearrow_{\Gamma/\gamma} & \\ Y & & \end{array} \tag{5.1}$$

where γ is a subgraph, Γ/γ is the cograph and $*$ is the so–called residue in the physics nomenclature.

As for the grading, one should take the refined grading. Usually there will be no mergers involved, so edge contractions and loop contractions get degree 1 and the coradical grading is by word length in the elementary morphisms which coincides with the number of edges.

5.1.7. **1–PI graph version.** It is easy to see that the property of being 1–PI is preserved under composition in $\mathfrak{G}$ and hence, we obtain the Hopf algebra of 1–PI graphs. In this formulation the condition is also easily checked.

5.1.8. **Other graphs.** The constructions works for any of the Feynman categories built on graphs and their decorations mentioned in [KW17, KL16]. The key thing is that the extra structures respect the concatenation of morphisms, which boils down to plugging graphs into vertices. Examples of this type furnish bi– and Hopf algebras of modular graphs, non–Σ modular graphs, trees, planar trees, etc..

5.1.9. **Brown's motic Hopf algebras.** In [Bro17] a generalization of 1–PI graphs is given. In this case there are the decorations of (ghost) edges of the morphisms by masses and the momenta; that is, maps $m : E(\Gamma) \to \mathbb{R}$ and $q : T(\Gamma) \to \mathbb{R}^d \cup \{\varnothing\}$. Notice that these are decorations in the technical sense of [KL16] as well. The masses carry over onto the new edges upon insertion. For the tails the composition rule is as follows: the tails that are labelled by $\varnothing$ become half of an edge on insertion and the tails that are labelled otherwise remain tails and keep their decoration.

A subgraph γ of a graph Γ is called momentum and mass spanning (m.m.) if it contains all the tails and all the edges with non–zero mass.

A graph Γ is called motic if for any m.m. subgraph γ: $b_1(\gamma) < b_1(\Gamma)$.

This condition is again stable under composition, i.e. gluing graphs into vertices as shown in [Bro17, Theorem 3.6]. After taking the quotient, we see that the one vertex ghost graph becomes identified with the empty graph and we obtain the Hopf algebra structure of [Bro17, Theorem 4.2] from this Feynman category after amputating the tails marked by $\varnothing$.

5.2. **(Semi)–Simplicial case.**

5.2.1. **With $\mathbb{Z}$ coefficients.** This is the case of a decorated Feynman category. Given a semi–simplicial set $X_\bullet$ then $C^*(X_\bullet)$ can be made into a functor from $\mathfrak{S}urj_<$. Namely, we assign to each n the set $C^*(X_\bullet)^{\otimes n} \simeq C^*(X_\bullet^{\times n})$ and to the unique map $n \to 1$ the iterated cup product $\cup^{n-1}$. This is just the fact that $C^*(X_\bullet)$ is an algebra. In other words $X_\bullet$ can be thought of as a functor $\mathfrak{S}urj_< \to \mathcal{C}$ and we can decorate with it. After decorating, the objects become collections of cochains, and there is a unique map with source an n–collection of cochains and target a single cochain, which is the iterated cup product. Thus, one can identify the morphisms of this type with the objects. Futhermore, the set of morphisms then posesses a natural structure of Abelian group. Dualizing this Abelian group, we get the cooperad structure on $C_*(X_\bullet)$ and the cooperad structure with multiplication on $C_*(X_\bullet)^\otimes$ that coincides with the one considered in chapter §3.

The bialgebra is almost connected if the 1–skeleton of $X_\bullet$ is connected. And after quotienting we obtain the same Hopf algebra structure from both constructions.

5.2.2. **Relation to $\cup_i$ products.** It is here that we find the similarity to the $\cup_i$ products also noticed by JDS Jones. Namely, in order to apply $\cup^{n-1}$ to a simplex, we first use the Joyal dual map $[1] \to [n]$ on the simplex. This is the map that is also used for the $\cup_i$ product. The only difference is that instead of using n cochains, one only uses two. To formalize this one needs a surjection that is not in Δ, but uses a permutation, and hence lives in $S\Delta_+$. Here the surjection $\mathfrak{S}urj$ gives rise to what is alternatively called the sequence operad. Joyal duality is then the fact that one uses sequences and overlapping sequences in the language of [MS03]. The pictorial realizations and associated representations can be found in [Kau08] and [Kau09]. This is also related to the notion of discs in Joyal [Joy97]. This connection will be investigated in the future.

In the Hopf algebra situation, we see that the terms of the iterated $\cup_1$ product coincide with the second factor of the coproduct Δ. Compare Figure 11.

5.2.3. **Over *Set*: Special case of the nerve of a category, colored operad structure.** In general there is no operad structure on $X_\bullet$ itself. By the operad structure on simplices, we can try to put an operad structure on $X_\bullet$ by composing an n simplex and an m simplex if the respective images of i and $i+1$ agree. This simplex need not exist, but it does if the simplicial set is the nerve of a category. In particular, if $X_0 \stackrel{\phi_1}{\to} \cdots \stackrel{\phi_n}{\to} X_n$ is an

n simplex and $X_{i-1} = Y_0 \stackrel{\psi_1}{\to} \cdots \stackrel{\psi_m}{\to} Y_m = X_i$, with $\psi_m \circ \cdots \circ \psi_1 = \phi_i$, then we can compose to

$$X_0 \stackrel{\phi_1}{\to} \cdots \stackrel{\phi_{i-1}}{\to} X_{i-1} = Y_0 \stackrel{\psi_1}{\to} \cdots \stackrel{\psi_m}{\to} Y_m = X_{i+1} \stackrel{\phi_{i+1}}{\to} \cdots \stackrel{\phi_n}{\to} X_n$$

In the Feynman category language, $\mathcal{V}$ is discrete, but not trivial, in particular $\mathcal{V} = \{X_0 \to X_1\}$ is the set of one–simplices. The morphisms the yield a colored operad structure over the set $Ob(\mathcal{V})$. Each morphism/n–simplex $X_0 \stackrel{\phi_1}{\to} \cdots \stackrel{\phi_n}{\to} X_n$ is a morphism from $\phi_1 \otimes \cdots \otimes \phi_n \to \phi = \phi_n \circ \cdots \circ \phi_1$.

5.2.4. **Over *Set*: Special case of the nerve of a complete groupoid.** If the underlying category is a complete groupoid, so that there is exactly one morphism per pair of objects, then any n–simplex can simply be replaced by the word $X_0 \cdots X_n$ of its sources and targets. Notice that $\mathcal{V} = \{X_0 X_1\}$ is the set of words of length 2 not 1. This explains the constructions of Goncharov involving multiple zeta values, but also polylogarithms [Gon05], and the subsequent construction of Brown. This matches our discussion in §3 and §5.1.9.

5.3. **Semi-simplicial objects and links to Chapter 3.** By definition a semi-simplicial object in $\mathcal{C}$ is a functor $X_\bullet : \mathfrak{Surj}_<^{op} \to \mathcal{C}$, and rewriting this, we see that this is equivalent either to a functor $\mathfrak{Surj}_< \to \mathcal{C}^{op}$ or to a functor $\mathcal{I}nj_{*,*} \to \mathcal{C}$. Our constructions of §3 actually work with the last interpretation.

The second and third descriptions open this up for a description in terms of Feynman categories. Notice that in this intepretation $X_\bullet$ is a functor from $\mathfrak{Surj}_<$, but it is not monoidal. In [KW17, Chapter 3.1], a free monoidal Feynman category $\mathfrak{F}^{\boxtimes}$ is constructed, such that $\mathfrak{F}^{\boxtimes}$–$\mathcal{O}ps_{\mathcal{C}}$ is equal to $Fun(\mathcal{F}, \mathcal{C})$, that is all functors, not neccessarily monoidal ones. So we could decorate $\mathfrak{Surj}_<^{\boxtimes}$ with the semi–simplical set $X_\bullet$, and then regard the decorated $\mathfrak{Surj}^{\boxtimes}_{<,decX_\bullet}$.

What is more pertinent however, is that since there is the oplax monoidal structure §3.3, induced by $X_{p+q} \to X_p \times X_q$ in the Feynman category language means that we get a morphism from the non–connected version $\mathfrak{Surj}_<^{nc}$ of $\mathfrak{Surj}_<$. The cubical realization of this using the functors L of §3.3 in the more general context will be the subject of further investigation.

5.3.1. **Goncharov multiple zeta values and polylogarithms.** Taking the contractible groupoid on $0, 1$ we obtain the construction of $\mathscr{H}_{Gon}$ for the multi–zeta values. If we take that with objects z_i, we obtain Goncharov's Hopf algebra for polylogarithms [Gon05].

5.3.2. **Baues.** This is the case of a general simplicial set, which however is 1-connected. We note that since we are dealing with graded objects, one has to specify that one is in the usual monoidal category of graded $\mathbb{Z}$ modules whose tensor product is given by the Koszul or super sign.

5.4. **Boot–strap.** There is the following nice observation. The simplest Feynman category is given by $\mathfrak{F}_{triv} = (\mathcal{V} = triv, \mathcal{F} = \mathcal{V}^{\otimes}, \imath)$ and $\mathfrak{F}^{+}_{triv} = \mathfrak{F}_{surj}$ [KW17, Example 3.6.1]. Going further, $\mathfrak{F}^{+}_{surj} = \mathfrak{F}_{May\ operads}$ [Example 3.6.2]. Adding units gives $\mathfrak{F}_{operads}$ and then $\mathfrak{F}_{\mathcal{V}}$ gives $\mathfrak{F}_{surj,\mathcal{O}=leaf\ labelled\ trees}$. Decorating by simplicial sets, we obtain the three original examples from these constructions.

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Appendix

Appendix A. Graph Glossary

A.1. **The category of graphs.** Interesting examples of Feynman categories used in operad–like theories are indexed over a Feynman category built from graphs. It is important to note that although we will first introduce a category of graphs $\mathcal{G}raphs$, the relevant Feynman category is given by a full subcategory $\mathcal{A}gg$ whose objects are disjoint unions or aggregates of corollas. The corollas themselves play the role of $\mathcal{V}$.

Before giving more examples in terms of graphs it will be useful to recall some terminology. A very useful presentation is given in [BM08] which we follow here.

A.1.1. **Abstract graphs.** An abstract graph Γ is a quadruple $(V_\Gamma, F_\Gamma, i_\Gamma, \partial_\Gamma)$ of a finite set of vertices V_Γ, a finite set of half edges or flags F_Γ, an involution on flags $i_\Gamma\colon F_\Gamma \to F_\Gamma; i_\Gamma^2 = id$ and a map $\partial_\Gamma\colon F_\Gamma \to V_\Gamma$. We will omit the subscript Γ if no confusion arises.

Since the map i is an involution, it has orbits of order one or two. We will call the flags in an orbit of order one *tails* and denote the set of tails by T_Γ. We will call an orbit of order two an *edge* and denote the set of edges by E_Γ. The flags of an edge are its elements. The function ∂ gives the vertex a flag is incident to. It is clear that the set of vertices and edges form a 1-dimensional CW complex. The realization of a graph is the realization of this CW complex.

A graph is (simply) connected if and only if its realization is. Notice that the graphs do not need to be connected. Lone vertices, that is, vertices with no incident flags, are also possible.

We also allow the empty graph $\mathbb{1}_\varnothing$, that is, the unique graph with $V = \varnothing$. It will serve as the monoidal unit.

Example A.1. A graph with one vertex and no edges is called a *corolla*. Such a graph only has tails. For any set S the corolla $*_{p,S}$ is the unique graph with $V = \{p\}$ a singleton and $F = S$.

We fix the short hand notation $*_S$ for the corolla with $V = \{*\}$ and $F = S$.

Given a vertex v of a graph, we set $F_v = \partial^{-1}(v)$ and call it *the flags incident to* v. This set naturally gives rise to a corolla. The *tails* at v is the subset of tails of F_v.

As remarked above, F_v defines a corolla $*_v = *_{\{v\},F_v}$.

Remark A.2. The way things are set up, we are talking about (finite) sets, so changing the sets even by bijection changes the graphs.

Remark A.3. As the graphs do not need to be connected, given two graphs Γ and Γ' we can form their disjoint union:

$$\Gamma \sqcup \Gamma' = (F_\Gamma \sqcup F_{\Gamma'}, V_\Gamma \sqcup V_{\Gamma'}, i_\Gamma \sqcup i_{\Gamma'}, \partial_\Gamma \sqcup \partial_{\Gamma'})$$

One actually needs to be a bit careful about how disjoint unions are defined. Although one tends to think that the disjoint union $X \sqcup Y$ is strictly symmetric, this is not the case. This becomes apparent if $X \cap Y \neq \varnothing$. Of course there is a bijection $X \sqcup Y \stackrel{1-1}{\longleftrightarrow} Y \sqcup X$. Thus the categories here are symmetric monoidal, but not strict symmetric monoidal. This is important, since we consider functors into other not necessarily strict monoidal categories.

Using MacLane's theorem it is however possible to make a technical construction that makes the monoidal structure (on both sides) into a strict symmetric monoidal structure

Example A.4. An *aggregate of corollas* or aggregate for short is a finite disjoint union of corollas, that is, a graph with no edges.

Notice that if one looks at $X = \coprod_{v \in I} *_{S_v}$ for some finite index set I and some finite sets of flags S_v, then the set of flags is automatically the disjoint union of the sets S_v. We will just say just say $s \in F_X$ if s is in some S_v.

A.1.2. **Category structure; Morphisms of Graphs.**

Definition A.5. [BM08] Given two graphs Γ and Γ', consider a triple (ϕ^F, ϕ_V, i_ϕ) where

(i) $\phi^F \colon F_{\Gamma'} \hookrightarrow F_\Gamma$ is an injection,
(ii) $\phi_V \colon V_\Gamma \twoheadrightarrow V_{\Gamma'}$ and i_ϕ is a surjection and
(iii) i_ϕ is a fixed point free involution on the tails of Γ not in the image of ϕ^F.

One calls the edges and flags that are not in the image of ϕ the contracted edges and flags. The orbits of i_ϕ are called *ghost edges* and denoted by $E_{ghost}(\phi)$.

Such a triple is *a morphism of graphs* $\phi\colon \Gamma \to \Gamma'$ if

(1) The involutions are compatible:
   (a) An edge of Γ is either a subset of the image of ϕ^F or not contained in it.
   (b) If an edge is in the image of ϕ^F then its pre–image is also an edge.
(2) ϕ^F and ϕ_V are compatible with the maps ∂:
   (a) Compatibility with ∂ on the image of ϕ^F:
   If $f = \phi^F(f')$ then $\phi_V(\partial f) = \partial f'$
   (b) Compatibility with ∂ on the complement of the image of ϕ^F:
   The two vertices of a ghost edge in Γ map to the same vertex in Γ' under ϕ_V.

If the image of an edge under ϕ^F is not an edge, we say that ϕ grafts the two flags.

The composition $\phi' \circ \phi\colon \Gamma \to \Gamma''$ of two morphisms $\phi\colon \Gamma \to \Gamma'$ and $\phi'\colon \Gamma' \to \Gamma''$ is defined to be $(\phi^F \circ \phi'^F, \phi'_V \circ \phi_V, i)$ where i is defined by its orbits viz. the ghost edges. Both maps ϕ^F and ϕ'^F are injective, so that the complement of their concatenation is in bijection with the disjoint union of the complements of the two maps. We take i to be the involution whose orbits are the union of the ghost edges of ϕ and ϕ' under this identification.

Remark A.6. A *naïve morphism* of graphs $\psi\colon \Gamma \to \Gamma'$ is given by a pair of maps $(\psi_F\colon F_\Gamma \to F_{\Gamma'}, \psi_V\colon V_\Gamma \to V_{\Gamma'})$ compatible with the maps i and ∂ in the obvious fashion. This notion is good to define subgraphs and automorphisms.

It turns out that this data *is not enough* to capture all the needed aspects for composing along graphs. For instance it is not possible to contract edges with such a map or graft two flags into one edge. The basic operations of composition in an operad viewed in graphs is however exactly grafting two flags and then contracting.

For this and other more subtle aspects one needs the more involved definition above which we will use.

Definition A.7. We let $\mathcal{G}raphs$ be the category whose objects are abstract graphs and whose morphisms are the morphisms described in Definition A.5. We consider it to be a monoidal category with monoidal product $\sqcup$ (see Remark A.3).

A.1.3. **Decomposition of morphisms.** Given a morphism $\phi\colon X \to Y$ where $X = \bigsqcup_{w\in V_X} *_w$ and $Y = \bigsqcup_{v\in V_Y} *_v$ are two aggregates, we can decompose $\phi = \bigsqcup \phi_v$ with $\phi_v\colon X_v \to *_v$ where X_v is the subaggregate $\bigsqcup_{\phi_V(w)=v} *_w$, and $\bigsqcup_v X_v = X$. Here $(\phi_v)_V$ is the restriction of ϕ_V to V_{X_v}. Likewise ϕ_v^F is the restriction of ϕ^F to $(\phi^F)^{-1}(F_{X_v} \cap \phi^F(F_Y))$. This is still injective. Finally i_{ϕ_v} is the restriction of i_ϕ to $F_{X_v} \setminus \phi^F(F_Y)$. These restrictions are possible due to the condition (2) above.

A.1.4. **Ghost graph of a morphism.** The following definition from [KW17] is essential. The underlying ghost graph of a morphism of graphs $\phi\colon \Gamma \to \Gamma'$ is the graph $\mathbb{\Gamma}(\phi) = (V(\Gamma), F_\Gamma, \hat{\imath}_\phi)$ where $\hat{\imath}_\phi$ is i_ϕ on the complement of $\phi^F(\Gamma')$ and identity on the image of flags of Γ' under ϕ^F. The edges of $\mathbb{\Gamma}(\phi)$ are called the ghost edges of ϕ.

A.2. **Extra structures.**

A.2.1. **Glossary.** This section is intended as a reference section. All the following definitions are standard.

Recall that an order of a finite set S is a bijection $S \to \{1, \dots, |S|\}$. Thus the group $\mathbb{S}_{|S|} = Aut\{1, \dots, n\}$ acts on all orders. An orientation of a finite set S is an equivalence class of orders, where two orders are equivalent if they are obtained from each other by an even permutation.

A tree	is a connected, simply connected graph.
A directed graph Γ	is a graph together with a map $F_\Gamma \to \{in, out\}$ such that the two flags of each edge are mapped to different values.
A rooted tree	is a directed tree such that each vertex has exactly one "out" flag.
A ribbon or fat graph	is a graph together with a cyclic order on each of the sets F_v.
A planar graph	is a a ribbon graph that can be embedded into the plane such that the induced cyclic orders of the sets F_v from the orientation of the plane coincide with the chosen cyclic orders.
A planted planar tree	is a rooted planar tree together with a linear order on the set of flags incident to the root.
An oriented graph	is a graph with an orientation on the set of its edges.
An ordered graph	is a graph with an order on the set of its edges.
A γ labelled graph	is a graph together with a map $\gamma : V_\Gamma \to \mathbb{N}_0$.
A b/w graph	is a graph Γ with a map $V_\Gamma \to \{black, white\}$.
A bipartite graph	is a b/w graph whose edges connect only black to white vertices.
A c colored graph	for a set c is a graph Γ together with a map $F_\Gamma \to c$ s.t. each edge has flags of the same color.
A connected 1–PI graph	is a connected graph that stays connected, when one severs any edge.
A 1–PI graph	is a graph whose every component is 1–PI.

A.2.2. **Remarks and language.**

(1) In a directed graph one speaks about the "in" and the "out" edges, flags or tails at a vertex. For the edges this means the one flag of the edges is an "in" flag at the vertex. In pictorial versions the direction is indicated by an arrow. A flag is an "in" flag if the arrow points to the vertex.

(2) As usual there are edge paths on a graph and the natural notion of an oriented edge path. An edge path is a (oriented) cycle if it starts and stops at the same vertex and all the edges are pairwise distinct. It is called simple if each vertex on the cycle has exactly one incoming flag and one outgoing flag belonging to the cycle. An oriented simple cycle will be called a *wheel*. An edge whose two vertices coincide is called a *(small) loop*.

(3) There is a notion of a the genus of a graph, which is the minimal dimension of the surface it can be embedded on. A ribbon graph is planar if this genus is 0.
(4) For any graph, its Euler characteristic is given by

$$\chi(\Gamma) = b_0(\Gamma) - b_1(\Gamma) = |V_\Gamma| - |E_\Gamma|;$$

where b_0, b_1 are the Betti numbers of the (realization of) Γ. Given a γ labelled graph, we define the total γ as

$$\gamma(\Gamma) = 1 - \chi(\Gamma) + \sum_{v \text{ vertex of } \Gamma} \gamma(v) \tag{A.1}$$

If Γ is *connected*, that is $b_0(\Gamma) = 1$ then a γ labeled graph is traditionally called a genus labeled graph and

$$\gamma(\Gamma) = \sum_{v \in V_\Gamma} \gamma(v) + b_1(\Gamma) \tag{A.2}$$

is called the genus of Γ. This is actually not the genus of the underlying graph, but the genus of a connected Riemann surface with possible double points whose dual graph is the genus labelled graph.

A genus labelled graph is called *stable* if each vertex with genus labeling 0 has at least 3 flags and each vertex with genus label 1 has at leas one edge.
(5) A planted planar tree induces a linear order on all sets F_v, by declaring the first flag to be the unique outgoing one. Moreover, there is a natural order on the edges, vertices and flags given by its planar embedding.
(6) A rooted tree is usually taken to be a tree with a marked vertex. Note that necessarily a rooted tree as described above has exactly one "out" tail. The unique vertex whose "out" flag is not a part of an edge is the root vertex. The usual picture is obtained by deleting this unique "out" tail.

A.2.3. **Category of directed/ordered/oriented graphs.**

(1) Define the category of directed graphs $\mathcal{G}raphs^{dir}$ to be the category whose objects are directed graphs. Morphisms are morphisms ϕ of the underlying graphs, which additionally satisfy that ϕ^F preserves orientation of the flags and the i_ϕ also only has orbits consisting of one "in" and one "out" flag, that is the ghost graph is also directed.
(2) The category of edge ordered graphs $\mathcal{G}raphs^{or}$ has as objects graphs with an order on the edges. A morphism is a morphism together with an order *ord* on all of the edges of the ghost graph.

The composition of orders on the ghost edges is as follows. $(\phi, ord) \circ \bigsqcup_{v \in V} (\phi_v, ord_v) := (\phi \circ \bigsqcup_{v \in V} \phi_v, ord \circ \bigsqcup_{v \in V} ord_v)$ where the order on the set of all ghost edges, that is $E_{ghost}(\phi) \sqcup \bigsqcup_v E_{ghost}(\phi_v)$, is given by first enumerating the elements of $E_{ghost}(\phi_v)$ in the order ord_v where the order of the sets $E(\phi_v)$ is given by the order on V, i.e. given by the explicit ordering of the tensor product in $Y = \coprod_v *_v$.[4] and then enumerating the edges of $E_{ghost}(\phi)$ in their order *ord*.

(3) The oriented version $\mathcal{G}raphs^{or}$ is then obtained by passing from orders to equivalence classes.

A.2.4. **Category of planar aggregates and tree morphisms.** Although it is hard to write down a consistent theory of planar graphs with planar morphisms, if not impossible, there does exist a planar version of special subcategory of $\mathcal{G}raphs$.

We let $\mathcal{C}rl^{pl}$ have as objects planar corollas — which simply means that there is a cyclic order on the flags — and as morphisms isomorphisms of these, that is isomorphisms of graphs, which preserve the cyclic order. The automorphisms of a corolla $*_S$ are then isomorphic to $C_{|S|}$, the cyclic group of order $|S|$. Let $\mathfrak{C}^{pl}$ be the full subcategory of aggregates of planar corollas whose morphisms are morphisms of the underlying corollas, for which the ghost graphs in their planar structure induced by the source is compatible with the planar structure on the target via ϕ^F. For this we use the fact that the tails of a planar tree have a cyclic order.

Let $\mathcal{C}rl^{pl,dir}$ be directed planar corollas with one output and let $\mathfrak{O}^{pl}$ be the subcategory of $\mathcal{A}gg^{pl,dir}$ of aggregates of corollas of the type just mentioned, whose morphisms are morphisms of the underlying directed corollas such that their associated ghost graphs are compatible with the planar structures as above.

A.3. Flag killing and leaf operators; insertion operations.

A.3.1. **Killing tails.** We define the operator *trun*, which removes all tails from a graph. Technically, $trun(\Gamma) = (V_\Gamma, F_\Gamma \setminus T_\Gamma, \partial_\Gamma|_{F_\Gamma \setminus T_\Gamma}, \imath_\Gamma|_{F_\Gamma \setminus T_\Gamma})$.

A.3.2. **Adding tails.** Inversely, we define the formal expression *leaf* which associates to each Γ without tails the formal sum $\sum_n \sum_{\Gamma': trun(\Gamma')=\Gamma; F(\Gamma')=F(\Gamma')\sqcup \underline{n}} \Gamma'$, that is all possible additions of tails where these tails are a standard set, to avoid isomorphic duplication. To make this well defined, we can consider the series as a power series in t: $leaf(\Gamma) = \sum_n \sum_{\Gamma': trun(\Gamma')=\Gamma; F(\Gamma')=F(\Gamma')\sqcup \bar{n}} \Gamma' t^n$

This is the foliage operator of [KS00, Kau07] which was rediscovered in [BBM13].

A.3.3. **Insertion.** Given graphs, Γ,Γ', a vertex $v \in V_\Gamma$ and an isomorphism ϕ: $F_v \mapsto T_{\Gamma'}$ we define $\Gamma \circ_v \Gamma'$ to be the graph obtained by deleting v and identifying the flags of v with the tails of Γ' via ϕ. Notice that if Γ and Γ' are ghost graphs of a morphism then it is just the composition of ghost graphs, with the morphisms at the other vertices being the identity.

A.3.4. **Unlabelled insertion.** If we are considering graphs with unlabelled tails, that is, classes $[\Gamma]$ and $[\Gamma']$ of coinvariants under the action of permutation of tails. The insertion naturally lifts as $[\Gamma]\circ[\Gamma'] := [\sum_\phi \Gamma \circ_v \Gamma']$ where ϕ runs through all the possible isomorphisms of two fixed lifts.

[4] Now we are working with ordered tensor products. Alternatively one can just index the outer order by the set V by using [Del90]

A.3.5. **No–tail insertion.** If Γ and Γ' are graphs without tails and v a vertex of v, then we define $\Gamma \circ_v \Gamma' = \Gamma \circ_v \text{coeff}(leaf(\Gamma'), t^{|F_v|})$, the (formal) sum of graphs where ϕ is one fixed identification of F_v with $\overline{|F_v|}$. In other words one deletes v and grafts all the tails to all possible positions on Γ'. Alternatively one can sum over all $\partial : F_\Gamma \sqcup F_{\Gamma'} \to V_\Gamma \setminus v \sqcup V_{\Gamma'}$ where ∂ is ∂_G when restricted to $F_w, w \in V_\Gamma$ and $\partial_{\Gamma'}$ when restricted to $F_{v'}, v' \in V_{\Gamma'}$.

A.3.6. **Compatibility.** Let Γ and Γ' be two graphs without flags, then for any vertex v of Γ $leaf(\Gamma \circ_v \Gamma') = leaf(\Gamma) \circ_v leaf(\Gamma')$.

A.4. **Graphs with tails and without tails.** There are two equivalent pictures one can use for the (co)operad structure underlying the Connes–Kreimer Hopf algebra of rooted trees. One can either work with tails that are flags, or with tail vertices. These two concepts are of course equivalent in the setting where if one allows flag tails, disallows vertices with valence one and vice–versa if one disallows tails, one allows one–valenced vertices called tail vertices. In [CK98] graphs without tails are used. Here we collect some combinatorial facts which represent this equivalence as a useful dictionary.

There are the obvious two maps which either add a vertex at each the end of each tail, or, in the other direction, simply delete each valence one vertex and its unique incident flag, but what is relevant for the Connes–Kreimer example is another set of maps. The first takes a graph with no flag tails to the tree which to every vertex, we *add* a tail, we will denote this map by $\sharp$ and we add one extra (outgoing) flag to the root, which will be called the root flag.

The second map $\flat$ simply deletes all tails. We see that $\flat \circ \sharp = id$. But $\flat$ is not the double sided inverse, since $\sharp \circ \flat$ replaces any number of tails at a given vertex by one tail. It is the identity on the image of $\sharp$, which we call single tail graphs.

Notice that $\sharp$ is well defined on leaf labelled trees by just transfering the labels as sets. Likewise $\flat$ is well defined on single tail trees again by transfering the labels. This means that each vertex will be labelled.

There are the following degenerate graphs which are allowed in the two setups: the empty graph $\varnothing$ and the graph with one flag and no vertices $|$. We declare that

$$\varnothing^\sharp = | \text{ and vice–versa } |^\flat = \varnothing \tag{A.3}$$

A.4.1. **Planted vs. rooted.** A planted tree is a rooted tree whose root has valence 1. One can plant a rooted tree τ to obtain a new planted rooted tree $\tau^\downarrow$, by adding a new vertex which will be the root of $\tau^\downarrow$ and adding one edge between the new vertex and the old root. Vice–versa, given a planted rooted tree τ, we let $\tau^\uparrow$ be the uprooted tree that is obtained from τ by deleting the root vertex and its unique incident edge, while declaring the other vertex of that edge to be the root.

A.5. **Operad structures on rooted/planted trees.** There are several operad structures on leaf–labelled trees which appear.

For rooted trees without tails and labelled vertices, we define

(1) $\tau \circ_i \tau'$ is the tree where the i-th vertex of τ is identified with the root of τ'. The root of the resulting tree being the image of the root of τ.

(2) $\tau \circ_i^+ \tau'$ is the tree where the i-th vertex of τ is joined to the root of τ' by a new edge, with the root of the resulting tree is then the image of the root of τ.

It is actually the second operad structure that underlies the Connes-Kreimer Hopf algebra.

One can now easily check that

$$\tau \circ_i^+ \tau' = \tau \circ_i \tau'^{\downarrow} = (\tau^{\downarrow} \circ_i \tau'^{\downarrow})^{\uparrow} \tag{A.4}$$

These constructions also allow us to relate the compositions of trees with and without tails as follows

$$(\tau^{\sharp} \circ_i \tau'^{\sharp})^{\flat} = \tau \circ_i^+ \tau' \tag{A.5}$$

where the $\circ_i$ operation on the left is the one connecting the ith flag to the root flag.

A.5.1. **Planar case: marking angles.** In the case of planar trees, we have to redefine $\sharp$ by adding a flag to every *angle* of a planar tree. The labels are then not on the vertices, but rather the angles. The analogous equations hold as above. Notice that to give a root to a planar tree actually means to specify a vertex and an angle on it. Planting it connects a new vertex into that angle.

This angle marking is directly to the angle marking in Joyal duality, see below and Figures 10 and 13. This also explains the appearance of angle markings in [Gon05].

Appendix B. Coalgebras and Hopf Algebras

A good source for this material is [Car07].

Definition B.1. A coalgebra with a split counit is a triple $(\mathscr{H}, \epsilon, \eta)$, where $(\mathscr{H}, \epsilon)$ is a cogebra and $\eta : \mathbb{1} \to \mathscr{H}$ is a section of η, such that if $| := \eta(1)$, $\Delta(|) = | \otimes |$.

Using η, we split $\mathscr{H} = \mathbb{1} \oplus \bar{\mathscr{H}}$ where $\bar{\mathscr{H}} := ker(\epsilon)$

Following Quillen [Qui67], one defines $\bar{\Delta}(a) := \Delta(a) - | \otimes a - a \otimes |$ where $| := \eta(1)$

If $(\mathscr{H}, \mu, \Delta, \eta, \epsilon)$ is a bialgebra then the restriction $(\mathscr{H}, \Delta, \epsilon)$ is a coalgebra with split counit.

A coalgebra with split counit $\mathscr{H}$ is said to be conilpotent if for all $a \in \bar{\mathscr{H}}$ there is an n such that $\bar{\Delta}^n(a) = 0$ or equivalently if for some $m : a \in ker(pr^{\otimes m+1} \circ \Delta^m)$.

Quillen defined the following filtered object.

$$F^0 = \mathbb{1}; F^m = \{a : \bar{\Delta}a \in F^{m-1} \otimes F^{m-1}\} \tag{B.1}$$

$\mathscr{H}$ is said to be connected, if $\mathscr{H} = \bigcup_m F^m$. If $\mathscr{H}$ is connected, then it is nilpotent, and conversely if taking kernels and the tensor product commute then conilpotent implies connected where $F^m = ker(pr^{\otimes m+1} \circ \Delta^m)$.

For a conilpotent bialgebra algebra there is a unique formula for a possible antipode given by:

$$S(x) = \sum_{n \geq 0} (-1)^{n+1} \mu^n \circ \bar{\Delta}^n(x) \tag{B.2}$$

where $\bar{\Delta}^n : \mathscr{H} \to \mathscr{H}^{\otimes n}$ is the $n-1$-st iterate of $\bar{\Delta}$ that is unique due to coassociativity and $\mu^n : \mathscr{H}^n \to \mathscr{H}$ is the $n-1$-st iterate of the multiplication μ that is unique due to associativity.

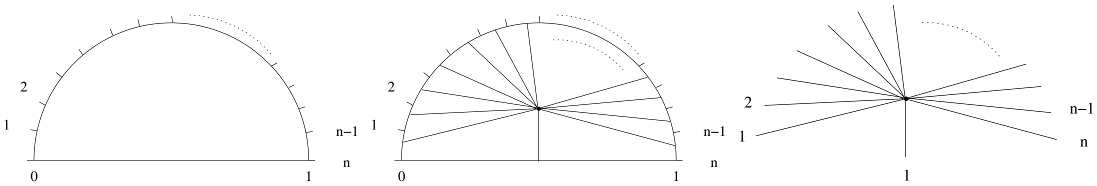

FIGURE 10. The interval injection $[1] \to [n]$ on the left, the surjection $\underline{n} \to \underline{1}$ on the right and and Joyal duality in the middle. Here reading the morphism upwards yields the double base point preserving injection, while reading it downward the surjection.

APPENDIX C. JOYAL DUALITY, SURJECTIONS, INJECTIONS AND LEAF VS. ANGLE MARKINGS

C.1. **Joyal duality.** There is a well known duality [Joy97] of two subcategories of Δ_+. This history of this duality can be traced back to [Str80]. Here we review this operation and show how it can be graphically interpreted. The graphical notation we present in turn connects to the graphical notation in [Gon05] and [GGL09].

The first of the two subcategories of Δ_+ is Δ and the second is the category of intervals. Since we will be dealing with both Δ and Δ_+, we will use the notation $\underline{n}$ for the small category $1 \to \cdots \to n$ in Δ and $[n]$ for $0 \to 1 \to \cdots \to n$ in Δ_+. The subcategory of intervals is then the wide subcategory of Δ_+ whose morphisms preserve both the beginning and the end point. We will denote these maps by $Hom_{*,*}([m],[n])$. Explicitly $\phi \in Hom_{*,*}([m],[n])$ is $\phi(0) = 0$ and $\phi(m) = n$.

The contravariant duality is then given by the association $Hom_{*,*}([m],[n]) \simeq Hom(\underline{n}, \underline{m})$ defined by $\phi \overset{1-1}{\leftrightarrow} \psi$ given by

$$\psi(i) = \min\{j : \phi(j) \geq i\} - 1, \quad \phi(j) = \max\{i : \phi(i) < j\} + 1.$$

This identification is contravariant.

C.2. **Semi–simplicial objects.** We will mostly be interested in the subcategory $\mathfrak{S}urj_<$ of Δ consisting of order preserving surjections. Notice that $Fun(\mathfrak{S}urj_<^{op}, \mathcal{C})$ are the semi–simplicial objects in $\mathcal{C}$. The Joyal dual of $\mathfrak{S}urj_<^{op}$ is the subcategory $\mathcal{I}nj_{*,*}$ of order preserving maps of intervals. In other words semi–simplicial objects are $Fun(\mathcal{I}nj_{*,*}, \mathcal{C})$

Just as the surjections are generated by the unique maps $\underline{n} \to \underline{1}$ so dually are the injections by the unique maps $[1] \to [n] \in Hom_{*,*}([1],[n])$. Pictorially the surjection is naturally depicted by a corolla while the injection is nicely captured by drawing an injection as a half circle. The duality can then be seen by superimposing the two graphical images. This duality is also that of dual graphs on bordered surfaces. This is summarized in Figure 10. Notice that in this duality, the elements of $[n]$ correspond to the angles of the corolla and the elements of $\underline{n}$ label the leaves of the corolla.

This also explains the adding and subtraction of 1 in the correspondence (C.1).

For general surjections, the picture is the a forest of corollas and a collection of half circles. The composition then is given by composing corollas to corollas and by gluing

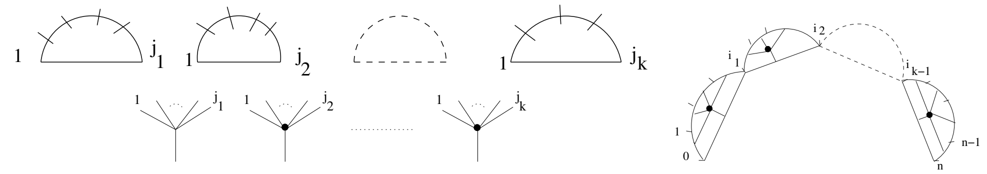

FIGURE 11. The first step of the composition is to assemble a collection of half discs or a forest into one morphism. This is pictured on the right. The j and i are related by $i_l = j_1 + \dots j_k$. Notice that in the half disc assembly is glued at the i_l essentially repeating them, while the forest assembly does not repeat. This also corresponds to an iterated cup product.

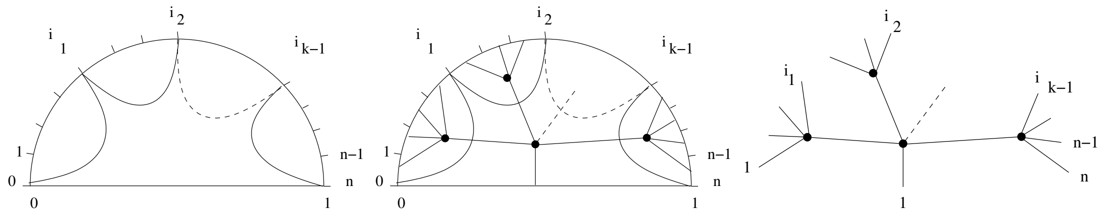

FIGURE 12. The second step of composition. For half circles on the left, where we have deformed the half circles such that the outer boundary is now a half circle, corollas on the right and the duality in the middle. is done in Figure 11. The result of the composition is after the third step, which erases the inner curves or sements and in the corrola picture contracts the edges. The result is in Figure 10.

on the half circles to the half circles by identifying the beginning and endpoints. This is exactly the map of combining simplicial strings. The prevalent picture for this in the literature on multi–zetas and polylogs is by adding line segments as the base for the arc segments. This is pictured in Figure 11. The composition is then given by contracting the internal edges or dually erasing the internal lines. This is depicted in Figure 12.

We have chosen here the traditional way of using half circles. Another equivalent way would be to use polygons with a fixed base side. Finally, if one includes both the tree and the half circle, one can modify the picture into a more pleasing aesthetic by deforming the line segments into arcs as is done in §3, where also an explicit composition is given in all details, see Figure 7.

C.2.1. **Marking angles by morphisms.** A particularly nice example of the duality between marking angles vs. marking tails is given by considering the simplicial object given by the nerve of a category $N_\bullet(\mathcal{C})$. An n–simplex $X_0 \xrightarrow{\phi_1} X_1 \cdots \xrightarrow{\phi_n} X_n$ naturally gives rise to a decorated corolla, where the leaves are decorated by the objects and the angles are decorated by the morphisms, see Figure 13. The operation that the corolla represents is the the composition of all of the morphisms to get a morphism $\phi = \phi_n \circ \cdots \circ \phi_0 : X_0 \to X_n$.

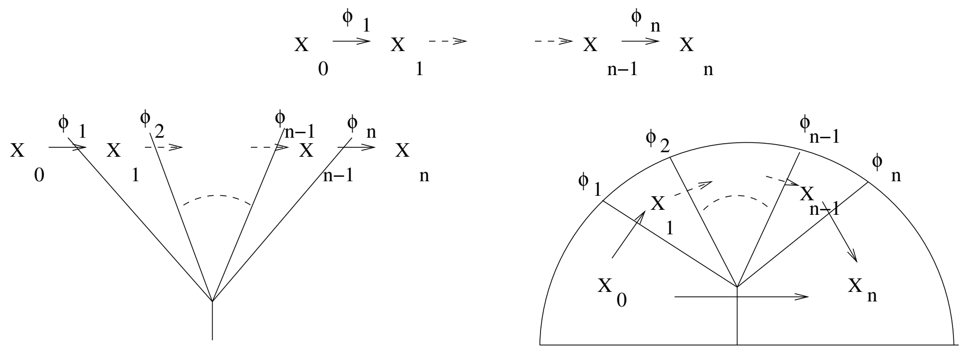

FIGURE 13. Marking a corolla by a simplex in $N_\bullet(\mathcal{C})$. The morphisms decorate the ends of the tree, while the objects decorate the angles which correspond to the marks on the half circle

If there is a single morphism between any two objects either one of the markings, tail or angle, will suffice to give a simplex. In the general case, one actually needs both the markings.

E-mail address: m.immaculada.galvez@upc.edu

DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT POLITÈCNICA DE CATALUNYA, ESEIAAT, CARRER COLOM, 1. 08222 TERRASSA, SPAIN.

E-mail address: rkaufman@math.purdue.edu

PURDUE UNIVERSITY DEPARTMENT OF MATHEMATICS, WEST LAFAYETTE, IN 47907

E-mail address: apt12@le.ac.uk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LEICESTER, UNIVERSITY ROAD, LEICESTER, LE1 7RH, UK.