Some aspects of topological Galois theory

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Septembre 2018

IHES/M/18/09

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August 31, 2018

Abstract

We establish a number of results on the subject of the first author's topos-theoretic generalization of Grothendieck's Galois formalism. In particular, we generalize in this context the existence theorem of algebraic closures, we give a concrete description of the atomic completion of a small category whose opposite satisfies the amalgamation property, and we explore to which extent a model of a Galois-type theory is determined by its symmetries.

Contents

1 Introduction

In this paper we explore a number of aspects of the general topos-theoretic framework for building topological 'Galois-type' theories introduced in [9].

The starting point of [9] was the observation that, given a (not necessarily finite-dimensional) Galois extension $F \subseteq L$, the classical Galois equivalence

$$\mathcal{L}_F^{L\,\mathrm{op}} \simeq \mathbf{Cont}_t(\mathrm{Aut}_F(K))$$

between the opposite of the category $\mathcal{L}_F^L$ of finite intermediate extensions and the category $\mathbf{Cont}_t(\mathrm{Aut}_F(K))$ of non-emtpy transitive actions on discrete sets of the Galois group $Aut_F(K)$ can be obtained as a restriction of an equivalence of toposes

$$\mathbf{Sh}(\mathcal{L}_F^{L\,\mathrm{op}}, J_{\mathrm{at}}) \simeq \mathbf{Cont}(\mathrm{Aut}_F(K))$$

where J_{at} is the *atomic topology* on $\mathcal{L}_F^{L\,\mathrm{op}}$ (that is, the Grothendieck topology whose covering sieves are exactly the non-empty ones). In fact, for any topological group G, the topos $\mathbf{Cont}(G)$ of continuous actions of G on discrete sets can be represented as the topos $\mathbf{Sh}(\mathbf{Cont}_t(G), J_{\mathrm{at}})$ of sheaves on the full subcategory $\mathbf{Cont}_t(G)$ of $\mathbf{Cont}(G)$ on the non-empty transitive actions with respect to the atomic topology on it. The opposites of the categories of the form $\mathbf{Cont}_t(G)$ notably satisfy the amalgamation and joint embedding properties.

Conversely, one can wonder when a topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ of sheaves with respect to the atomic topology J_{at} on the opposite of an essentially small category $\mathcal{C}$ whose opposite satisfies the amalgamation property can be represented as the topos $\mathbf{Cont}(G)$ for a topological group G. The central result of [9] is a representation theorem for these toposes $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$: if $\mathcal{C}$ also satisfies the joint embedding property and the ind-completion of $\mathcal{C}$ contains an object u satisfying some special properties (that of being $\mathcal{C}$-universal and

$\mathcal{C}$-ultrahomogeneous) then the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ is equivalent to the topos $\mathbf{Cont}(\mathrm{Aut}(u))$ of continuous actions on discrete sets of the automorphism group $\mathrm{Aut}(u)$ of u, topologized in such a way that the subgroups of the form $\{f : u \cong u \mid f \circ \chi = \chi\}$, where $\chi : c \to u$ is an arrow of Ind-$\mathcal{C}$, form a basis of open neighbourhoods of the identity. From this result many consequences follow; in particular, as it was shown in [9], one can characterize the categories which can be embedded as full dense subcategories of categories the form $\mathbf{Cont}_t(G)$ (and those which are equivalent to them), thus obtaining natural analogues of classical topological Galois theory in a great variety of different mathematical contexts.

The contents of this paper can be summarized as follows.

After reviewing the basic notions and the precise statement of the abovementioned theorem - to which we shall refer as 'the representation theorem' - and proving a few related results, we investigate in section 4 the existence of universal and (ultra)homogeneous objects expressible as colimits of chains of objects of a category satisfying the amalgamation and joint embedding properties, which may thus serve as points u of atomic toposes such as the ones involved in the representation theorem. More precisely, we identify some natural conditions under which the categorical theorem of [6] generalizing Fraïssé's construction applies, thus yielding such objects. Interestingly, this analysis allows to regard the classical construction of the algebraic (or separable) closure of a field as a particular instance of application of (the generalized) Fraïssé method for building ultrahomogeneous structures.

Next, in section 5, we establish, by functorializing the Morita-equivalence provided by the main representation theorem, an adjunction between a category of topological groups endowed with an algebraic base and a category of pairs consisting of a category and an object of its ind-completion satisfying the hypotheses of the theorem. This adjunction restricts to a duality between the category of (totally discontinuous) complete groups and a category of pairs whose underlying category is 'atomically complete'. We investigate in detail the notion of complete group and establish an explicit characterization for these groups also in terms of algebraic bases for them. We also describe the natural behavior of the Morita-equivalence of the main representation theorem with respect to localizations by an object coming from the atomic site.

In section 6, we apply the 'bridge' technique of [10] to the Morita-equivalence of the representation theorem to obtain a number of insights on the associated Galois theory. For instance, we obtain necessary and sufficient conditions for two pairs $(\mathcal{C}, u)$ and $(\mathcal{C}', u')$ satisfying the hypotheses of the theorem to give rise to Morita-equivalent topological groups $\mathrm{Aut}(u)$ and $\mathrm{Aut}(u')$. Then we provide a very concrete description of the 'atomic completion' of a small category satisfying the dual of the amalgamation property. This construction, which can be seen as a form of completion by the addition of 'imaginaries' (in the model-theoretic sense), was originally introduced in [9] (and an alternative description of it was given therein) as a means for making such a category $\mathcal{C}^{\text{op}}$, in presence of an equivalence $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Cont}(\text{Aut}(u))$ provided by the representation theorem, equivalent to the category of non-empty transitive actions of the group $\text{Aut}(u)$; indeed, the atomic completion of $\mathcal{C}^{\text{op}}$ is equivalent to the full subcategory of the topos $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{\text{at}})$ on its atoms. Next, we consider other topos-theoretic invariants from the points of view of the two sides of the Morita-equivalence of the representation theorem, notably including the notions of irreducible object, of Galois object and of coherent topos. We show in particular that the concept of irreducible generator of a topos, which admits natural characterizations both from the categorical side and from the group-theoretic one, allows one to capture the Galois theories that are discrete (up to Morita-equivalence), while the concept of Galois object, which is also shown to admit natural site characterizations, allows one to identify the fixator subgroups that are normal (in terms of a categorical condition that the objects corresponding to them should satisfy) as well as the Galois theories whose Galois groups are prodiscrete (through the invariant property of having enough Galois objects).

In the final section of the paper, we apply the logical interpretation of the representation theorem already established in [9] to the study of the relationships between special models of atomic and complete theories and the associated automorphism groups; in particular, we investigate conditions for a continuous homomorphism between the automorphism groups of two such structures to be induced by an interpretation of one structure into the other. Our results on this subject are shown to improve and generalize the classical model-theoretic ones. Our topos-theoretic perspective also allows us to understand the existence of different special models for a given atomic and complete theory in terms of the existence of non-trivial automorphisms of its classifying topos.

2 Review of topological Galois theory

Recall that a topological group is a group G with a topology such that the group operation and the inverse operation are continuous with respect to it; for basic background on topological groups we refer the reader to [13].

The following well-known result allows to make a given group into a topological group starting from a collection of subsets of the group satisfying particular properties:

Lemma 2.1. *Let G be a group and $\mathcal{B}$ be a collection of subsets N of G containing the neutral element e. Then there exists exactly one topology τ on G having $\mathcal{B}$ as a neighbourhood basis of e and making (G, τ) into a topological group if and only if all the following conditions are satisfied:*

(i) For any $N, M \in \mathcal{B}$ there exists $P \in \mathcal{B}$ such that $P \subseteq N \cap M$;

(ii) For any $N \in \mathcal{B}$ there exists $M \in \mathcal{B}$ such that $M^2 \subseteq N$;

(iii) For any $N \in \mathcal{B}$ there exists $M \in \mathcal{B}$ such that $M \subseteq N^{-1}$;

(iv) For any $N \in \mathcal{B}$ and any $a \in G$ there exists $M \in \mathcal{B}$ such that $M \subseteq aNa^{-1}$.

Notice that if all the subsets in the family $\mathcal{B}$ are subgroups of G then conditions (ii) and (iii) in the statement of the lemma are automatically satisfied. We shall say that a collection $\mathcal{B}$ of subgroups of G is an *algebraic base* for G if it is a basis of neighbourhoods of e, any finite intersection of subgroups in $\mathcal{B}$ contains a subgroup in $\mathcal{B}$, and any conjugate of a subgroup in $\mathcal{B}$ lies in $\mathcal{B}$.

We shall denote the topology τ generated by an algebraic base $\mathcal{B}$ as in Lemma 2.1 by $\tau_{\mathcal{B}}^{G}$; the resulting topological group will be denoted by $G_{\mathcal{B}}$.

All the topological groups considered in this paper are *totally discontinuous*, that is their topology is generated by a family of open subgroups.

Recall from [6] and [9] the following categorical notions.

Definition 2.2. Let $\mathcal{C}$ be a small category.

- $\mathcal{C}$ is said to satisfy the *amalgamation property* (AP) if for every objects $a, b, c \in \mathcal{C}$ and morphisms $f : a \to b$, $g : a \to c$ in $\mathcal{C}$ there exist an object $d \in \mathcal{C}$ and morphisms $f' : b \to d$, $g' : c \to d$ in $\mathcal{C}$ such that $f' \circ f = g' \circ g$:

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} \\ c & \xrightarrow[g']{} & d \end{array}$$

- $\mathcal{C}$ is said to satisfy the *joint embedding property* (JEP) if for every pair of objects $a, b \in \mathcal{C}$ there exists an object $c \in \mathcal{C}$ and morphisms $f : a \to c$, $g : b \to c$ in $\mathcal{C}$:

$$\begin{array}{ccc} & & a \\ & & \downarrow{\scriptstyle f} \\ b & \xrightarrow[g]{} & c \end{array}$$

- Given a full embedding of categories $\mathcal{C} \hookrightarrow \mathcal{D}$, an object u of $\mathcal{D}$ is said to be $\mathcal{C}$-*homogeneous* if for every objects $a, b \in \mathcal{C}$ and arrows $j : a \to b$ in $\mathcal{C}$ and $\chi : a \to u$ in $\mathcal{D}$ there exists an arrow $\tilde{\chi} : b \to u$ such that $\tilde{\chi} \circ j = \chi$:

$$\begin{array}{ccc} a & \xrightarrow{\chi} & u \\ {\scriptstyle j}\downarrow & \nearrow_{\tilde{\chi}} & \\ b & & \end{array}$$

- Given a full embedding of categories $\mathcal{C} \hookrightarrow \mathcal{D}$, an object u of $\mathcal{D}$ is said to be $\mathcal{C}$-*ultrahomogeneous* if for every objects $a, b \in \mathcal{C}$ and arrows $j : a \to b$ in $\mathcal{C}$ and $\chi_1 : a \to u$, $\chi_2 : b \to u$ in $\mathcal{D}$ there exists an isomorphism $\check{j} : u \to u$ such that $\xi \circ \chi_1 = \chi_2 \circ j$:

$$\begin{array}{ccc} a & \xrightarrow{\chi_1} & u \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \xi} \\ b & \xrightarrow[\chi_2]{} & u \end{array}$$

- Given a full embedding of categories $\mathcal{C} \hookrightarrow \mathcal{D}$, an object u of $\mathcal{D}$ is said to be $\mathcal{C}$-*universal* if it is $\mathcal{C}$-cofinal, that is for every $a \in \mathcal{C}$ there exists an arrow $\chi : a \to u$ in $\mathcal{D}$:

$$a \overset{\chi}{\dashrightarrow} u$$

Remark 2.3. Any $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous object is $\mathcal{C}$-homogeneous.

Recall that on any small category $\mathcal{C}$ satisfying the dual of AP, one can put the *atomic topology* J_{at}, namely the Grothendieck topology on $\mathcal{C}$ whose covering sieves are exactly the non-empty ones.

For any topological group G, the category $\mathbf{Cont}(G)$ whose objects are the left continuous actions $G \times X \to X$ (where X is endowed with the discrete topology and $G \times X$ with the product topology) and whose arrows are the G-equivariant maps between them is a Grothendieck topos. Recall that a left action $\alpha : G \times X \to X$ is continuous if and only if for every $x \in X$ the isotropy subgroup $I_x := \{g \in G \mid \alpha(g, x) = x\}$ is open in G. The topos $\mathbf{Cont}(G)$ is atomic (recall that an atomic topos is a topos generated by its atoms, that is the objects which are non-zero and which do not have any proper subobjects); in fact, its atoms are precisely the non-empty transitive continuous actions, and $\mathbf{Cont}(G)$ can be represented as the topos of sheaves on the full subcategory $\mathbf{Cont}_t(G)$ on the non-empty transitive actions with respect to the atomic topology on it. Notice that a non-empty transitive action $\alpha : G \times X \to X$ can be identified with the canonical action $G \times G/I_x \to G/I_x$ of G on the set G/I_x of left cosets gI_x of the isotropy group I_x of α at any point $x \in X$; conversely, for any open subgroup U of G, the canonical action of G on the set G/U makes G/U into a non-empty transitive G-set.

Theorem 2.4 (Theorem 3.5 [9]). *Let $\mathcal{C}$ be a small non-empty category satisfying AP and JEP, and let u be a $\mathcal{C}$-universal and $\mathcal{C}$-ultrahomogeneous object in* Ind-$\mathcal{C}$*. Then the collection $\mathcal{I}_{\mathcal{C}}$ of sets of the form $\mathcal{I}_{\chi} := \{f : u \cong u \mid f \circ \chi = \chi\}$, for an arrow $\chi : c \to u$ from an object c of $\mathcal{C}$ to u, defines an algebraic base for the group of automorphisms of u in* Ind-$\mathcal{C}$*, and, denoting by* $\mathrm{Aut}(u)$ *the resulting topological group, we have an equivalence of toposes*

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}}) \simeq \mathbf{Cont}(\mathrm{Aut}(u))$$

induced by the functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cont}_t(\mathrm{Aut}(u))$ which sends any object c of $\mathcal{C}$ to the set $\mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u)$ (equipped with the obvious action by $\mathrm{Aut}(u)$) and any arrow $f : c \to d$ in $\mathcal{C}$ to the $\mathrm{Aut}(u)$-equivariant map $- \circ f : \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{C}}(d, u) \to \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u)$.

It was shown in [9] that the functor F of Theorem 2.4 is full and faithful if and only if every arrow $f : d \to c$ in $\mathcal{C}$ is a *strict monomorphism* (in the sense that for any arrow $g : e \to c$ such that $h \circ g = k \circ g$ whenever $h \circ f = k \circ f$, g factors uniquely through f).

3 Some basic results

As observed at page 271 of [6], the points of the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$, where $\mathcal{C}$ is an essentially small category satisfying AP, can be identified with the $\mathcal{C}$-homogeneous objects of Ind-$\mathcal{C}$.

Proposition 3.1. *Let $\mathcal{C}$ be a non-empty essentially small category satisfying AP and u be a $\mathcal{C}$-homogeneous object of* Ind-$\mathcal{C}$ *(that is, a point of the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$). Then u is $\mathcal{C}$-universal if and only if the category $\mathcal{C}$ satisfies JEP (equivalently, if and only if the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ is two-valued).*

Proof. If $\mathcal{E}$ is two-valued, the arrow from the image $\ell(c)$ in $\mathcal{E}$ of any object c of $\mathcal{C}$ to the final object of $\mathcal{E}$ is an epimorphism.

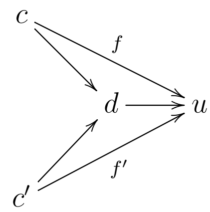
Therefore its inverse image $u^*(\ell(c)) = (u^* \circ \ell)(c) = \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{C}}(c, u)$ by the point u cannot be empty.

Conversely, let us suppose that u has this property.

For any objects c, c' of $\mathcal{C}$, there exist two arrows of Ind-$\mathcal{C}$

$$c \xrightarrow{f} u \quad \text{and} \quad c' \xrightarrow{f'} u \ .$$

As the category $\int F$ of elements of the functor $F = \mathrm{Hom}(-, u)$ is filtered, there exists an object d of $\mathcal{C}$ and a commutative diagram in Ind-$\mathcal{C}$ of the form:



A fortiori, the category $\mathcal{C}$ satisfies JEP (equivalently, by Theorem 3.6 and Lemma 3.7 [6], the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ is two-valued). □

By considering the invariant notion of point of a topos in the context of the Morita-equivalence of Theorem 2.4, and recalling that every limit-preserving (resp. colimit-preserving) functor between Grothendieck toposes has a left adjoint (resp. a right adjoint), we immediately obtain the following result:

Proposition 3.2. *Under the hypotheses of Theorem 2.4, any $\mathcal{C}$-homogeneous object of* Ind-$\mathcal{C}$*, regarded as a functor $\mathcal{C}^{\mathrm{op}} \to$ **Set**, can be extended via $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cont}(\mathrm{Aut}(u))$, uniquely up to isomorphism, to a cartesian colimit-preserving functor $\mathbf{Cont}(\mathrm{Aut}(u)) \to$ **Set**; conversely, any such functor restricts, via F, to a $\mathcal{C}$-homogeneous object of* Ind-$\mathcal{C}$.

□

The following proposition follows from 'bridges' (in the sense of [10]) arising from the fact that the key notions involved in the topological Galois theory of [9] can be formulated as topos-theoretic invariants. More precisely, we can define a point p of an atomic topos $\mathcal{E}$ to be *universal* if every set of the form $p^*(A)$ where A is an atom of $\mathcal{E}$ is non-empty, and to be *ultrahomogeneous* if the canonical action of the automorphism group $\mathrm{Aut}(p)$ on every set of the form $p^*(A)$ (where A is an atom of $\mathcal{E}$) is transitive. Since every arrow to an atom is an epimorphism and the inverse image p^* of any point p sends epimorphisms to surjections, it follows that one can equivalently require, in these definitions, A to vary among the atoms in a separating family $\mathcal{F}$ for the topos; for instance, if $\mathcal{E}$ is $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$, $\mathcal{F}$ can be the collection of the atoms of the form $l(c)$ where c is an object of $\mathcal{C}$. Recall also that the points of the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ are precisely the $\mathcal{C}$-homogeneous objects of Ind-$\mathcal{C}$.

It thus follows that, if we have a Galois-type equivalence $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}}) \simeq \mathbf{Cont}(\mathrm{Aut}(u))$ as in Theorem 2.4, a $\mathcal{C}$-homogeneous object u of Ind-$\mathcal{C}$ is $\mathcal{C}$-universal (resp. $\mathcal{C}$-ultrahomogeneous) if and only if the object $l(c)$ of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ is universal (resp. ultrahomogeneous).

Proposition 3.3. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two essentially small categories satisfying AP. Suppose that there exists an equivalence between the atomic toposes $\mathcal{E} = \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ and $\mathcal{E}' = \mathbf{Sh}(\mathcal{C}'^{\mathrm{op}}, J_{\mathrm{at}})$ associated to them (cf. Theorem 6.1). Then:*

(i) The equivalence $\mathcal{E} \simeq \mathcal{E}'$ induces an equivalence between the category of $\mathcal{C}$-homogeneous objects of Ind-$\mathcal{C}$ *towards that of $\mathcal{C}'$-homogeneous objects of* Ind-$\mathcal{C}'$.

(ii) If u and u' are two homogeneous objects respectively of Ind-$\mathcal{C}$ *and* Ind-$\mathcal{C}'$ *which correspond to each other as in point (i), u is $\mathcal{C}$-universal if and only if u' is $\mathcal{C}'$-universal.*

(iii) If u and u' are two homogeneous objects respectively of Ind-$\mathcal{C}$ *and* Ind-$\mathcal{C}'$ *which correspond to each other as in point (i), u is $\mathcal{C}$-ultrahomogeneous if and only if u' is $\mathcal{C}'$-ultrahomogeneous.*

□

4 Existence of points

In this section we shall prove a theorem which ensures the existence of homogeneous or ultrahomogeneous objects under some natural hypotheses. For this, we need the following definition:

Definition 4.1. In a category $\mathcal{C}$, a family $\mathcal{F}$ of arrows is said to be *dominant* if it satisfies the following properties:

(i) The family $\mathrm{Dom}\,(\mathcal{F})$ of domains of arrows of $\mathcal{F}$ is cofinal in $\mathcal{C}$. In other words, every object x of $\mathcal{C}$ admits an arrow

$$x \longrightarrow a$$

towards the domain a of an arrow $a \to b$ of $\mathcal{F}$.

(ii) For any object a of $\mathrm{Dom}\,(\mathcal{F})$ and any arrow $f : a \to x$ of $\mathcal{C}$, there exists an arrow $g : x \to b$ of $\mathcal{C}$ such that the composite

$$g \circ f : a \longrightarrow x \longrightarrow b$$

is an arrow of $\mathcal{F}$.

We have the following result:

Theorem 4.2. *Let $\mathcal{C}$ be a category whose arrows are all monomorphisms, κ an infinite regular cardinal and $\mathcal{D}$, $\mathcal{D}'$ the full subcategories of* Ind-$\mathcal{C}$ *on the objects which are colimits of objects of $\mathcal{C}$ indexed by filtered partially ordered sets of cardinality respectively $< \kappa$ et $\leq \kappa$. Let us suppose that:*

(1) *The category $\mathcal{D}$ satisfies AP and JEP.*

(2) *The category $\mathcal{D}$ admits a dominating family of cardinality $\leq \kappa$.*

Then:

(i) There exists in $\mathcal{D}'$ an object which is $\mathcal{D}$-homogeneous and $\mathcal{D}$-universal.

(ii) The $\mathcal{D}$-homogeneous and $\mathcal{D}$-universal objects of $\mathcal{D}'$ are automatically $\mathcal{D}$-ultrahomogeneous. Moreover, they are all isomorphic.

Proof. We apply Theorem 2.8 of [6] to the embedding $\mathcal{D} \hookrightarrow$ Ind-$\mathcal{C}$, showing that (using the notation of the theorem) $(\text{Ind-}\mathcal{C})_\kappa = (\text{Ind-}\mathcal{C})^c_\kappa = \mathcal{D}'$. For this, we check that its hypotheses are satisfied.

By hypothesis, the category $\mathcal{D}$ satisfies AP and JEP, and it admits a dominating family of cardinality $\leq \kappa$.

Let us now prove that the category $\mathcal{D}$ is closed with respect to colimits indexed by filtered partially ordered sets of cardinality $< \kappa$. This will imply in particular that the category $\mathcal{D}$ is κ-bounded (in the sense of Definition 2.5 of [6]).

We preliminarily notice that if all the arrows of $\mathcal{C}$ are monomorphisms then all the arrows of Ind-$\mathcal{C}$ are monomorphisms as well (apply Corollary 7.2.9 [10] to the theory of flat functors on the category $\mathcal{C}^{\text{op}}$), whence all the categories of the form $\int F$ for $F \in$ Ind-$\mathcal{C}$ are (filtered) preorders (and their skeleta are partially ordered sets).

Let us notice that the objects of Ind-$\mathcal{C}$ which can be expressed as colimits of objects of $\mathcal{C}$ indexed by filtered partially ordered sets of cardinality $\leq \lambda$ can be equivalently characterized as the objects of Ind-$\mathcal{C}$ whose category of elements contains a cofinal full subcategory of cardinality $\leq \lambda$ which is a partial order. Indeed, this is obvious in one direction, while in the other it suffices to observe that, since the objects of $\mathcal{C}$ are finitely presentable in Ind-$\mathcal{C}$, if an object a of Ind-$\mathcal{C}$ can be expressed as a colimit of objects b_j of $\mathcal{C}$ indexed by a filtered category $\mathcal{J}$ then for any object (c, x) of $\int a$, there are an object j of $\mathcal{J}$ and an arrow $c \to b_j$ in $\mathcal{C}$ with commutes with the canonical colimit arrows $c \to a$ and $\xi_j : b_j \to a$, in other words an arrow $(c, x) \to (b_j, \xi_j)$ in $\int a$.

Now, let d be an object of Ind-$\mathcal{C}$ which is given by the colimit of a diagram D with values in $\mathcal{D}$ defined on a filtered partially ordered set $\mathcal{I}$ of cardinality

$< \kappa$, say k'. The colimit arrows $\chi_i : D(i) \to d$ (for $i \in \mathcal{I}$) clearly induce functors $\int \chi_i : \int D(i) \to \int d$ which are jointly essentially surjective. Now, since each $D(i)$ is in $\mathcal{D}$, the category $\int D(i)$ contains a full cofinal subcategory of cardinality $< \kappa$. The full subcategory U of $\int d$ on the objects which belong to the union of the images of these subcategories under the functors $\int \chi_i$, is clearly cofinal in $\int d$ and filtered as all the $\int D(i)$ are. To deduce our claim, it therefore suffices to show that the cardinality of a skeleton of U is $< \kappa$; but this follows from the fact that a union indexed by a cardinal $< \kappa$ of sets whose cardinality is $< \kappa$ has cardinality $< \kappa$ since κ is regular.

By Remark 2.6 [6], the fact that $\mathcal{D}$ is closed with respect to colimits indexed by filtered partially ordered sets of cardinality $< \kappa$ implies that every object of $\mathcal{D}'$ can be expressed as the colimit of a continuous κ-chain with values in $\mathcal{D}$. Indeed, by definition of $\mathcal{D}'$, every object of $\mathcal{D}'$ can be expressed as the colimit of a λ-chain D with values in $\mathcal{D}$; if $\lambda = \kappa$ then we are done by Remark 2.6, while if $\lambda < \kappa$ then we can extend D to a κ-chain $\tilde{D}$ having the same colimit by setting $\tilde{D}(i) = \text{colim}(D)$ for every i such that $\lambda < i < \kappa$. Conversely, any colimit of a κ-chain of objects of $\mathcal{D}$ lies in $\mathcal{D}'$; this follows by an argument involving the categories of elements of the given objects similar to the one used for proving that $\mathcal{D}$ is closed with respect to colimits indexed by filtered partially ordered sets of cardinality $< \kappa$, noticing that a union indexed by κ of sets whose cardinality is $< \kappa$ has cardinality $\leq \kappa$ since κ is infinite. This shows that, by using the notation in Theorem 2.8 [6] (where we take $\mathcal{C}$ to be $\mathcal{D}$ and $\mathcal{D}$ to be Ind-$\mathcal{C}$), we have $(\text{Ind-}\mathcal{C})_\kappa = (\text{Ind-}\mathcal{C})^c_\kappa = \mathcal{D}'$.

To complete the verification of the hypotheses necessary for the application of Theorem 2.8 of [6] to the embedding $\mathcal{D} \hookrightarrow \text{Ind-}\mathcal{C}$, it remains to show that every object d of $\mathcal{D}$ is "κ-small" in Ind-$\mathcal{C}$ in the sense that the functor $\text{Hom}_{\text{Ind-}\mathcal{C}}(d, -) : \text{Ind-}\mathcal{C} \longrightarrow \mathbf{Set}$ preserves colimits of κ-chains. For this, since all the arrows of Ind-$\mathcal{C}$ are monic, it is enough to show that for every colimit representation $d' = \text{colim}(D)$, where $D : \kappa \to \text{Ind-}\mathcal{C}$, every arrow $f : d \to d'$ from an object d of $\mathcal{D}$ to d' factors (uniquely) through a colimit arrow $\xi_i : D(i) \to d'$. By definition of $\mathcal{D}$, d is the colimit of a λ-chain A with values in $\mathcal{C}$. For any $j \in \lambda$, the arrow $A(j) \to d'$ given by the composite of f with the canonical colimit arrow $a_j : A(j) \to d$ factors (uniquely) through a colimit arrow $\xi_{i(j)} : D(i(j)) \to d'$ since all the objects of $\mathcal{C}$ are finitely presentable in Ind-$\mathcal{C}$. This defines, by the axiom of choice, a function $\lambda \to \kappa$ given by the assignment $j \to i(j)$. Since the cardinal κ is regular, we have that $k' = \sup_{j \in \lambda} i(j) < \kappa$. So we have that all the arrows $f \circ a_j$ factor (uniquely) through $\xi_{k'} : D(k') \to d'$. Therefore, by the universal property of the colimit $d = \text{colim}(A)$, we obtain an arrow $z : d \to D(k')$ such that $\xi_{k'} \circ z = f$, as required. $\square$

Remarks 4.3. (a) An object of $\mathcal{D}'$ which is $\mathcal{D}$-homogeneous (resp. $\mathcal{D}$-universal, resp. $\mathcal{D}$-ultrahomogeneous) is *a fortiori* $\mathcal{C}$-homogeneous (resp. $\mathcal{C}$-universal, resp. $\mathcal{C}$-ultrahomogeneous).

(b) If $\kappa = \omega$, we have $\mathcal{D} = \mathcal{C}$. Conditions (1) et (2) thus rewrite as follows:

$$\begin{cases} (1) & \text{The category } \mathcal{C} \text{ satisfies AP and JEP.} \\ (2) & \text{It admits a countable dominating family.} \end{cases}$$

(c) The $\mathcal{C}$-universal and $\mathcal{C}$-homogeneous or $\mathcal{C}$-ultrahomogeneous objects of Ind-$\mathcal{C}$ are not all isomorphic in general. For example, the category $\mathcal{C}$ of finite sets and injections satisfies AP and JEP. Its ind-completion Ind-$\mathcal{C}$ identifies with the category of sets and injections. The $\mathcal{C}$-universal objects of Ind-$\mathcal{C}$ are the infinite sets; they are clearly $\mathcal{C}$-ultrahomogeneous and *a fortiori* $\mathcal{C}$-homogeneous. Now, any two infinite sets are isomorphic if and only if they have the same cardinality. The associated atomic topos is known under the name of *Schanuel topos*. It therefore admits as many Galois-type representations (of the kind specified in Theorem 2.4) as there exist non-isomorphic infinite sets.

(d) The requirement in Theorem 4.2 that the cardinal κ should be regular is not really restrictive since any cardinal can be replaced by its cofinality, which is always regular, without affecting the colimits of the chains defined on it.

Let us show that the existence of the algebraic closure of a field F can be deduced as a consequence of Theorem 4.2. Let us take $\mathcal{C}$ to be the category of finite separable field extensions of F; notice that Ind-$\mathcal{C}$ is the category of algebraic field extensions of $\mathcal{C}$. Take κ equal to the maximum of ω and the cardinality of F. The category $\mathcal{D}'$ coincides with Ind-$\mathcal{C}$ since every algebraic extension of F has cardinality κ and hence its representation as the colimit of its finite sub-extensions is indexed by κ. We have to show that the category $\mathcal{D}$ satisfies AP and JEP. Since $\mathcal{D}$ has an initial object (F itself), JEP follows from AP, so it is enough to verify the latter property. For this, we notice that if F' and F'' are separable field extensions of F, the quotient F''' of the tensor product of F' and F'' over F by a maximal ideal is a separable field extension of F which lies in $\mathcal{D}$ if F' and F'' do since if the representation of F' (resp. F'') as the colimit of its finite subextensions is indexed by a cardinal $\lambda < \kappa$ (resp. $\lambda' < \kappa$) then F''' can be expressed as the directed union of the finite field extensions generated by the union of a finite subextension of F' with a finite subextension of F'', and the cardinality of this union is $\leq \lambda \times \lambda' < \kappa$.

The category $\mathcal{D}$ admits a dominating family of cardinality $\leq \kappa$ since the collection $\mathcal{S}_F$ of separable polynomials with coefficients in F has cardinality κ and hence, by using an isomorphism $\mathcal{S}_F \cong \kappa$, one can define $\mathcal{F}$ to be the family obtained by choosing, for each $\lambda \leq \lambda' < \kappa$ an embedding of the splitting field S_λ of the polynomials indexed by the element $\leq \lambda$ into the splitting field $S_{\lambda'}$ of the polynomials indexed by the elements $\leq \lambda'$. For any $\lambda < \kappa$, the splitting fields of polynomials indexed by the elements $\leq \lambda$ actually belong to $\mathcal{D}$ since they can be expressed as the directed union of the (finite-dimensional) splitting fields of the finite subsets of such polynomials, which can be indexed by a finite cardinal if $\lambda < \omega$ and by λ if λ is infinite. The family $\mathcal{F}$ is dominating for $\mathcal{D}$ since any field k in $\mathcal{D}$ is the colimit indexed by a cardinal $\lambda < \kappa$ of a chain A with values in $\mathcal{C}$ and each $A(i)$ (for $i < \lambda$) is generated over F by a finite number of elements x_i^j. Each of these elements has a minimal polynomial P_i^j over F and, κ being regular, there is $\kappa' < \kappa$ such that all the P_i^j's are indexed by elements $\leq \kappa'$. So k, which is contained in the splitting field of the P_i^j's, maps into $S_{\kappa'}$, say via an arrow g, and for any arrow $f : S_l \to k$, at the cost of composing g with an automorphism of $S_{\kappa'}$, we can suppose $g \circ f$ to lie in $\mathcal{F}$.

The unique $\mathcal{D}$-universal and $\mathcal{D}$-ultrahomogeneous object of Ind-$\mathcal{C}$ is precisely the separable closure of F. Indeed, it is a separable extension of F being an object of Ind-$\mathcal{C}$, and it is separably closed since $\mathcal{D}$-universality implies that every separable polynomial with coefficients in a finite extension of F has a root in it.

If one does not want to invoke the notion of splitting field for a family of polynomials, it is possible to apply our theorem to construct the separable closure of F by choosing a cardinal bigger than κ, as follows. First, we notice that the category $\mathcal{C}$ is essentially small with a skeleton of cardinality κ. Since every functor $F : \mathcal{C} \to \mathbf{Set}$ in Ind-$\mathcal{C}$ actually takes values in the full subcategory of $\mathbf{Set}$ on finite sets, the functor F is isomorphic to a functor $F' : \mathcal{C} \to \mathcal{N}$ taking values in $\mathcal{N}$, where $\mathcal{N}$ is the full subcategory of $\mathbf{Set}$ on the finite cardinals. The number of such functors is therefore bounded by $|\mathrm{Arr}(\mathcal{C})|^\omega$. So the category $\mathcal{D}$ therefore admits a skeleton of cardinality $\leq |\mathrm{Arr}(\mathcal{C})|^\omega$, and this can be taken as a dominating family for it.

Similarly, by taking $\mathcal{C}$ to be the category of finite extensions of $\mathcal{C}$, Theorem 4.2 allows one to construct the algebraic closure of F.

5 Functorialization

We can functorialize the Morita-equivalence of Theorem 2.4 by means of a 'bridge' induced by the invariant notion of geometric morphism. For this, we need to recall the notion of a morphism of sites.

5.1 Morphisms of sites

Recall that for any (essentially) small site $(\mathcal{C}, J)$ and any Grothendieck topos $\mathcal{E}$, we have Diaconescu's equivalence

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$$

between the category $\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J))$ of geometric morphisms from $\mathcal{E}$ to $\mathbf{Sh}(\mathcal{C}, J)$ and the category $\mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$ of J-continuous flat functors from $\mathcal{C}$ to $\mathcal{E}$. In Chapter VII of [17] (cf. Definition 1 of section 8, page 394, and Theorem 1 of section 9, page 399), the authors established the following characterization of flat functors: a functor

$$A : \mathcal{C} \longrightarrow \mathcal{E}$$

from an (essentially) small category $\mathcal{C}$ to a Grothendieck topos $\mathcal{E}$ is flat if and only if it is filtering in the sense that it possesses the following three properties:

(1) If $1 = 1_{\mathcal{E}}$ is the terminal object of $\mathcal{E}$, the family of arrows

$$A(c) \longrightarrow 1$$

indexed by the objects c of $\mathcal{C}$ is jointly epimorphic.

(2) If c_1, c_2 are two objects of $\mathcal{C}$, the family of arrows

$$A(c) \longrightarrow A(c_1) \times A(c_2)$$

indexed by the diagrams

$$c_1 \longleftarrow c \longrightarrow c_2$$

of $\mathcal{C}$ is jointly epimorphic.

(3) If $f_1, f_2 : c \rightrightarrows d$ are two arrows of $\mathcal{C}$ and e is the subobject of $A(c)$ defined by the equation

$$A(f_1) = A(f_2)\,,$$

the family of arrows

$$A(b) \longrightarrow e$$

indexed by the arrows $h : b \to c$ de $\mathcal{C}$ such that

$$f_1 \circ h = f_2 \circ h$$

is jointly epimorphic.

The following result, giving a characterization of morphisms of toposes in terms of sites of definition, is an immediate consequence of Diaconescu's equivalence in light of the above-mentioned characterization of flat functors as filtering functors.

Corollary 5.1. *Let* $(\mathcal{C}, J)$ *and* $(\mathcal{C}', J')$ *be essentially small sites, and* $l : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$*,* $l' : \mathcal{C}' \to \mathbf{Sh}(\mathcal{C}', J')$ *be the canonical functors (given by the composite of the relevant Yoneda embedding with the associated sheaf functor). Then, given a functor* $A : \mathcal{C} \to \mathcal{C}'$*, the following conditions are equivalent:*

(i) A induces a geometric morphism $u : \mathbf{Sh}(\mathcal{C}', J') \to \mathbf{Sh}(\mathcal{C}, J)$ *making the following square commutative:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{A} & \mathcal{C}' \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle l'} \\ \mathbf{Sh}(\mathcal{C}, J) & \xrightarrow{u^*} & \mathbf{Sh}(\mathcal{C}', J'); \end{array}$$

(ii) The functor A is a morphism of sites *in the sense that it satisfies the following properties:*

(1) A sends every J-covering family in $\mathcal{C}$ *into a* J'*-covering family in* $\mathcal{C}'$*.*

(2) Every object c' *of* $\mathcal{C}'$ *admits a* J'*-covering family*

$$c'_i \longrightarrow c' , \quad i \in I ,$$

by objects c'_i *of* $\mathcal{C}'$ *which have morphisms*

$$c'_i \longrightarrow A(c_i)$$

to the images under A of objects c_i *of* $\mathcal{C}$*.*

(3) For any objects c_1, c_2 *of* $\mathcal{C}$ *and any pair of morphisms of* $\mathcal{C}'$

$$f'_1 : c' \longrightarrow A(c_1) , \quad f'_2 : c' \longrightarrow A(c_2) ,$$

there exists a J'*-covering family*

$$g'_i : c'_i \longrightarrow c' , \quad i \in I ,$$

and a family of pairs of morphisms of $\mathcal{C}$

$$f^i_1 : c_i \longrightarrow c_1 , \quad f^i_2 : c_i \to c_2 , \quad i \in I ,$$

and of morphisms of $\mathcal{C}'$

$$h'_i : c'_i \longrightarrow A(c_i)\,, \quad i \in I\,,$$

making the following squares commutative:

$$\begin{array}{ccc} c'_i & \xrightarrow{g'_i} & c' \\ {\scriptstyle h'_i}\downarrow & & \downarrow{\scriptstyle f'_1} \\ A(c_i) & \xrightarrow{A(f^i_1)} & A(c_1) \end{array} \qquad\qquad \begin{array}{ccc} c'_i & \xrightarrow{g'_i} & c' \\ {\scriptstyle h'_i}\downarrow & & \downarrow{\scriptstyle f'_2} \\ A(c_i) & \xrightarrow{A(f^i_2)} & A(c_2) \end{array}$$

(4) For any pair of arrows $f_1, f_2 : c \rightrightarrows d$ *of* $\mathcal{C}$ *and any arrow of* $\mathcal{C}'$

$$f' : b' \longrightarrow A(c)$$

satisfying

$$A(f_1) \circ f' = A(f_2) \circ f'\,,$$

there exist a J'*-covering family*

$$g'_i : b'_i \longrightarrow b'\,, \quad i \in I\,,$$

and a family of morphisms of $\mathcal{C}$

$$h_i : b_i \longrightarrow c\,, \quad i \in I\,,$$

satisfying

$$f_1 \circ h_i = f_2 \circ h_i\,, \quad \forall\, i \in I\,,$$

and of morphisms of $\mathcal{C}'$

$$h'_i : b'_i \longrightarrow A(b_i)\,, \quad i \in I\,,$$

making commutative the following squares:

$$\begin{array}{ccc} b'_i & \xrightarrow{g'_i} & b' \\ {\scriptstyle h'_i}\downarrow & & \downarrow{\scriptstyle f'} \\ A(b_i) & \xrightarrow{A(h_i)} & A(c) \end{array}$$

Remark 5.2. One can prove that the notion of morphism of sites appearing in Corollary 5.1 coincides with that of Definition 4.10 [20]; under this identification, Corollary 5.1 is subsumed by Proposition 11.14 [20].

Specializing the above to atomic sites, we obtain that a functor $F : \mathcal{C} \to \mathcal{C}'$ between categories satisfying the dual of the amalgamation property is a morphism of sites $(\mathcal{C}, J_{\text{at}}) \to (\mathcal{C}', J'_{\text{at}})$ if and only if it is *atomic* in the following sense:

Definition 5.3. A functor $F : \mathcal{C} \to \mathcal{C}'$ between categories satisfying the dual of the amalgamation property is said to be *atomic* if it satisfies the following properties:

(1) For any object c' of $\mathcal{C}'$, there exists an object c of $\mathcal{C}$ and an object b' of $\mathcal{C}'$ admitting two arrows

$$b' \longrightarrow c' \quad \text{and} \quad b' \longrightarrow A(c) \ .$$

(2) For any objects c_1, c_2 of $\mathcal{C}$ and any object c' of $\mathcal{C}'$ with a pair of morphisms

$$f'_1 : c' \longrightarrow A(c_1)\,, \quad f'_2 : c' \longrightarrow A(c_2)\,,$$

there exists an object b' of $\mathcal{C}'$, an arrow

$$g' : b' \longrightarrow c'\,,$$

a pair of morphisms of $\mathcal{C}$

$$f_1 : c \longrightarrow c_1\,, \quad f_2 : c \longrightarrow c_2\,,$$

and an arrow of $\mathcal{C}'$

$$h' : b' \longrightarrow A(c)$$

such that

$$A(f_1) \circ h' = f'_1 \circ g'\,, \quad A(f_2) \circ h' = f'_2 \circ g' \ .$$

(3) For any pair of arrows $f_1, f_2 : c \rightrightarrows d$ of $\mathcal{C}$ and any object c' of $\mathcal{C}'$ with a morphism

$$f' : c' \longrightarrow A(c)$$

satisfying

$$A(f_1) \circ f' = A(f_2) \circ f'\,,$$

there exist an object b' of $\mathcal{C}'$, an arrow

$$g' : b' \longrightarrow c'\,,$$

an arrow of $\mathcal{C}$

$$h : b \longrightarrow c$$

satisfying

$$f_1 \circ h = f_2 \circ h$$

and an arrow of $\mathcal{C}'$

$$h' : b' \longrightarrow A(b)$$

satisfying

$$A(h) \circ h' = f' \circ g' \ .$$

5.2 Localizations

Let us now discuss the behaviour of the equivalence of Theorem 2.4 with respect to taking slices.

Recall that for any (essentially) small site $(\mathcal{C}, J)$, the slice topos $\mathbf{Sh}(\mathcal{C}, J)/l(c)$ is equivalent to $\mathbf{Sh}(\mathcal{C}/c, J/c)$, where J/c is the Grothendieck topology canonically induced by J on the slice category $\mathcal{C}/c$ (see Proposition 5.4 in vol. 1 of [3]). If the opposite of a category $\mathcal{C}$ satisfies the amalgamation and joint embedding properties, then the opposite of the category $\mathcal{C}/c$ satisfies them as well. It thus follows that

$$\mathbf{Sh}(\mathcal{C}, J_{\mathrm{at}})/l(c) \simeq \mathbf{Sh}(\mathcal{C}/c, J_{\mathrm{at}})$$

for any object c of $\mathcal{C}$.

Lemma 5.4. *Let G be a topological group. Then, for any open subgroup Z of G, we have an equivalence of toposes*

$$\mathbf{Cont}(G)/(G/Z) \simeq \mathbf{Cont}(Z) \ .$$

Proof. In light of the above observation concerning the representation of slice toposes in terms of slice sites, it clearly suffices to exhibit an equivalence of categories $\mathbf{Cont}_t(G)/(G/Z) \simeq \mathbf{Cont}_t(Z)$, where $\mathbf{Cont}_t(G)$ and $\mathbf{Cont}_t(Z)$ are respectively the full subcategories of $\mathbf{Cont}(G)$ and $\mathbf{Cont}(Z)$ on the non-empty transitive actions. Let us consider the functor $\mathbf{Cont}_t(G)/(G/Z) \to \mathbf{Cont}_t(Z)$ sending any equivariant map $f : X \to G/Z$ in $\mathbf{Cont}_t(G)$ to the Z-set $f^{-1}([e])$, and acting on the arrows accordingly. The G-equivariance of f implies that the action of Z on $f^{-1}([e])$ is non-empty and transitive, so the functor is well-defined. It remains to show that it is full, faithful and essentially surjective. The essential surjectivity follows from the fact that, for any open subgroup V of Z, the Z-set Z/V is isomorphic to the image of the canonical G-equivariant map $G/V \to G/Z$ under our functor, so it remains to prove the fullness and faithfulness. But these properties immediately follow from the fact that, for any open subgroups U, U' of G

contained in Z, the G-equivariant maps $G/U \to G/U'$ compatible with the canonical projections $G/U \to G/Z$ and $G/U \to G/Z$ are precisely the maps given by right multiplication by an element z of Z such that $z^{-1}Uz \subseteq U'$. □

In light of Lemma 5.4, we thus obtain the following result:

Proposition 5.5. *For any pair* $(\mathcal{C}, u)$ *satisfying the hypotheses of Theorem 2.4 and any object* c *of* $\mathcal{C}$*, we have an equivalence*

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})/l(c) \simeq \mathbf{Cont}(\mathrm{Aut}(u)/I_{\chi}),$$

where χ *is any arrow* $c \to u$ *in* Ind-$\mathcal{C}$ *and* I_{χ} *is the open subgroup of* $\mathrm{Aut}(u)$ *given by the fixator of* χ*.*

□

The equivalences of Proposition 5.5 can be made functorial, as follows. For any arrow $f : c \to c'$ in $\mathcal{C}$, we have an arrow $l(f) : l(c') \to l(c)$ in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$, which induces a geometric morphism

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})/l(c') \to \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})/l(c)$$

with the special property that its inverse image has also a left adjoint. This morphism clearly corresponds, via equivalences as in Proposition 5.5, to the geometric morphism

$$\mathbf{Cont}(I_{\chi'}) \to \mathbf{Cont}(I_{\chi' \circ f})$$

induced by the continuous embedding of open subgroups $I_{\chi'} \subseteq I_{\chi' \circ f}$, for any arrow $\chi' : c' \to u$ in Ind-$\mathcal{C}$.

5.3 Algebraic bases and complete groups

Let us denote by **GTop** the category of totally discontinuous topological groups and continuous group homomorphisms between them. We can construct a category $\mathbf{GTop}_b$ of 'groups paired with algebraic bases' as follows: the objects of $\mathbf{GTop}_b$ are pairs $(G, \mathcal{B})$ consisting of a group G and an algebraic base $\mathcal{B}$ for it, while the arrows $(G, \mathcal{B}) \to (G', \mathcal{B}')$ in $\mathbf{GTop}_b$ are the group homomorphisms $f : G \to G'$ such that for any $V \in \mathcal{B}'$, $f^{-1}(V) \in \mathcal{B}$. We have a functor $F : \mathbf{GTop}_b \to \mathbf{GTop}$ sending to any object $(G, \mathcal{B})$ of $\mathbf{GTop}_b$ the topological group $(G, \tau^G_{\mathcal{B}})$ and acting accordingly on arrows. On the other hand, any topological group G has a canonical algebraic base C_G, namely the one consisting of all the open subgroups of it; this allows one to define a functor $G : \mathbf{GTop} \to \mathbf{GTop}_b$ sending G to (G, C_G) and acting on arrows in the obvious way. It is easily verified that G is left adjoint to F

and $F \circ G \cong 1_{\mathbf{GTop}}$, which allows us to regard **GTop** as a full subcategory of $\mathbf{GTop}_b$.

There is a natural link between algebraic bases for a topological group and dense subcategories of the associated topos of continuous actions. Indeed, as observed in Remark 2.2 [9], for any algebraic base for G, the G-sets of the form G/U for $U \in \mathcal{B}$ define a dense full subcategory of $\mathbf{Cont}_t(G)$ (in the sense that for any object of $\mathbf{Cont}_t(G)$ there exists an arrow from a G-set of this form to it) which is closed under isomorphisms. Conversely, any dense full subcategory of $\mathbf{Cont}_t(G)$ which is closed under isomorphisms gives rise to an algebraic base for G which is stable under conjugation, namely the base consisting of the open subgroups U of G such that G/U lies in the subcategory. The algebraic bases for G which are stable under conjugation can be thus identified with the dense full subcategories of $\mathbf{Cont}_t(G)$ which are closed under isomorphisms.

Proposition 5.6 (Proposition 2.3 [9]). *For any algebraic base $\mathcal{B}$ of a group G, the full subcategory $\mathbf{Cont}_{\mathcal{B}}(G)$ of $\mathbf{Cont}_t(G)$ on the objects of the form G/U for $U \in \mathcal{B}$ satisfies the dual of the amalgamation property and the dual of the joint embedding property (as defined in section 2).*

Notice that, since the subcategory $\mathbf{Cont}_{\mathcal{B}}(G)$ is dense in $\mathbf{Cont}_t(G)$ and hence in the topos $\mathbf{Cont}(G)$, Grothendieck's Comparison Lemma yields an equivalence

$$\mathbf{Sh}(\mathbf{Cont}_{\mathcal{B}}(G), J_{\text{at}}) \simeq \mathbf{Cont}(G),$$

where J_{at} is the atomic topology on $\mathbf{Cont}_{\mathcal{B}}(G)$.

The topos $\mathbf{Cont}(G)$ has a canonical point p_G, namely the geometric morphism $\mathbf{Set} \to \mathbf{Cont}(G)$ whose inverse image functor is the forgetful functor $\mathbf{Cont}(G) \to \mathbf{Set}$. Let us denote by $\text{Aut}(p_G)$ the group of automorphisms of p_G in the category of points of $\mathbf{Cont}(G)$. We have a canonical map $\xi_G : G \to \text{Aut}(p_G)$, sending any element $g \in G$ to the automorphism of p_G which acts at each component as multiplication by the element g (this is indeed an automorphism because the naturality conditions hold as the maps in $\mathbf{Cont}(G)$ are G-equivariant).

As shown in [19], for any topological group G, the group $\text{Aut}(p_G)$ can intrinsically be endowed with a pro-discrete topology (that is a topology which is a projective limit of discrete topologies) in which the open subgroups are those subgroups of the form $U_{(X,x)}$ for a continuous G-sets X and an element $x \in X$, where $U_{(X,x)}$ denotes the set of automorphisms $\alpha : p_G \cong p_G$ such that $\alpha(X)(x) = x$; the canonical map $\xi_G : G \to \text{Aut}(p_G)$ is continuous with respect to this topology.

It is natural to characterize the topological groups G for which the map ξ_G is a bijection (equivalently, a homeomorphism). Following the terminology of

[19], we shall call such groups *complete*, and we shall refer to the topological group $\mathrm{Aut}(p_G)$ as to the *completion* of G. For any complete group G with an algebraic base $\mathcal{B}$, we can alternatively describe the topology on $\mathrm{Aut}(p_G)$ induced by the topology on G via the bijection ξ_G as follows: a basis of open neighbourhoods of the identity is given by the sets of the form $\{\alpha : p_G \cong p_G \mid \alpha(G/U)(eU) = eU\}$ for $U \in \mathcal{B}$.

For any group G and algebraic base $\mathcal{B}$ for G, the collection of subsets of the form $\mathcal{I}_{U,x} := \{\alpha : p_G \cong p_G \mid \alpha(G/U)(x) = x\}$ for $x \in G/U$ and $U \in \mathcal{B}$ forms an algebraic base for the group $\mathrm{Aut}(p_G)$ of automorphisms of p_G, and, if we consider $\mathrm{Aut}(p_G)$ endowed with the resulting topology, the canonical map $\xi_G : G \to \mathrm{Aut}(p_G)$ becomes a homomorphism of topological groups which induces a Morita equivalence $\mathbf{Cont}(\xi_G) : \mathbf{Cont}(G) \simeq \mathbf{Cont}(\mathrm{Aut}(p_G))$ between them (cf. section 5.4 below).

For any (totally discontinous) topological group G, we have a canonical homomorphism

$$G \to \mathrm{End}(p_G)$$

towards the monoid $M = \mathrm{End}(p_G)$ of endomorphisms of the forgetful functor

$$p_G^* : \mathbf{Cont}(G) \to \mathbf{Set} .$$

Notice that this homomorphism is not necessarily surjective, nor the monoid $\mathrm{End}(p_G)$ is necessarily a group. Nonetheless, as shown by the following proposition (which corrects Proposition 2.4 of [9] - the first author wishes to thank Emmanuel Lepage for pointing out the mistake), we can describe M in terms of G and an algebraic base $\mathcal{B}$ for it:

Proposition 5.7. *Let G be a topological group with an algebraic base $\mathcal{B}$. Then*

(i) The endomorphisms of the point p_G can be identified with the element of the projective limit $M = \varprojlim_{U\in\mathcal{B}}(G/U)$ of the G/U for $U \in \mathcal{B}$; in particular, this projective limit has the structure of a monoid.

(ii) The automorphism group of the point p_G is isomorphic to the group $M^\times$ of invertible elements of the monoid $M = \varprojlim_{U\in\mathcal{B}}(G/U)$.

(iii) The group G is complete if and only if the canonical map from G to the set $M^\times$ of invertible elements of $\varprojlim_{U\in\mathcal{B}}(G/U)$ is an isomorphism.

(iv) More concretely, G is complete if and only if for any assignment $U \to a_U$ of an element $a_U \in G/U$ to any subset $U \in \mathcal{B}$ such that for any $U, V \in \mathcal{B}$ with $U \subseteq V$, $a_U \equiv a_V$ modulo V and there exist elements

$b_U \in G/U$ for $U \in \mathcal{B}$ such that $b_U \equiv b_V$ modulo V whenever $U, V \in \mathcal{B}$ with $U \subseteq V$ and $b_{a_U U a_U^{-1}} a_U \equiv e$, $a_{b_U U b_U^{-1}} b_U \equiv e$ modulo U for each U, there exists a unique $g \in G$ such that $a_U = gU$ for all $U \in \mathcal{B}$.

Proof Since the full subcategory $\mathbf{Cont}_{\mathcal{B}}(G)$ of the topos $\mathbf{Cont}(G)$ on the objects of the form G/U for $U \in \mathcal{B}$ is dense in $\mathbf{Cont}(G)$, the endomorphisms of p_G correspond exactly to the endomorphisms of the flat functor $F : \mathbf{Cont}_{\mathcal{B}}(G) \to \mathbf{Set}$ corresponding to p_G, that is of the forgetful functor. An endomorphism $\alpha : F \to F$ is uniquely determined by the elements $a_U := \alpha(G/U)(eU) \in G/U$ since the naturality condition for α with respect to the G-equivariant arrows $G/gUg^{-1} \to G/U$, $g' \to g'g$, sending $e(gUg^{-1})$ to gU forces $\alpha(G/U)(gU)$ to be equal to $a_{gUg^{-1}}gU$ for any $g \in G$:

$$\begin{array}{ccc} G/(gUg^{-1}) & \xrightarrow{\alpha(G/gUg^{-1})} & G/(gUg^{-1}) \\ \downarrow & & \downarrow \\ G/U & \xrightarrow[\alpha(G/U)]{} & G/U \end{array}$$

On the other hand, since any arrow in $\mathbf{Cont}_{\mathcal{B}}(G)$ can be factored as the composition of a canonical projection arrow of the form $G/U \to G/V$ for $U \subseteq V$ with a canonical isomorphism of the form $G/gWg^{-1} \to G/W$, any assignment $U \to a_U$ of an element $a_U \in G/U$ to any subset $U \in \mathcal{B}$ such that for any $U, V \in \mathcal{B}$ with $U \subseteq V$, $a_U \equiv a_V$ modulo V defines an endomorphism α of F by means of the formula $\alpha(G/U)(gU) = a_{gUg^{-1}}gU$. This proves the proposition. □

Remarks 5.8. (a) If g is a pro-group, that is if G admits an algebraic base $\mathcal{B}$ consisting of normal open subgroups then $M = \varprojlim_{U \in \mathcal{B}}(G/U)$ is a group; otherwise, it is not necessarily the case.

(b) We can endow $M = \mathrm{End}(p_G)$ with the least topology such that all the subsets

$$\{m \in M \mid mx = y\}$$

for $x, y \in G/U, U \in \mathcal{B}$ are open. Then the multiplication law $M \times M \to M$ is continuous, and G identifies with a subgroup of

$$M^{\times} = \{(m, m') \in M \times M \mid mm' = m'm = 1\}$$

endowed with the topology induced by that of $M^{\times}$, that is of $M \times M$. Denoting by $\mathbf{Cont}(M)$ the category of discrete sets endowed with a continuous action of M, the embedding $G \subseteq M$ induces a functor

$$\mathbf{Cont}(M) \to \mathbf{Cont}(G)$$

which is an equivalence. In other words, every continuous action of G on a discrete set naturally extends to a continuous action of M on it.

5.4 A general adjunction

Notice that, for any categories $\mathcal{C}$ and $\mathcal{C}'$ satisfying the amalgamation property, any morphism of sites (in the sense of Corollary 5.1(ii)) $F : (\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}}) \to (\mathcal{C}'^{\mathrm{op}}, J_{\mathrm{at}})$ induces a geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{C}'^{\mathrm{op}}, J_{\mathrm{at}}) \to \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$, which in turn yields, via Diaconescu's equivalence, a functor

$$\tilde{F} : \mathbf{Flat}_{J_{\mathrm{at}}}(\mathcal{C}'^{\mathrm{op}}, \mathbf{Set}) \to \mathbf{Flat}_{J_{\mathrm{at}}}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$$

which can be identified with $- \circ F$.

Let us define $\mathcal{G}$ to be the category whose objects are the pairs $(\mathcal{C}, u)$, where $\mathcal{C}$ is a small category satisfying AP and JEP and u is a $\mathcal{C}$-ultrahomogeneous and $\mathcal{C}$-universal object of Ind-$\mathcal{C}$, and whose arrows $(\mathcal{C}, u) \to (\mathcal{C}', u')$ are the atomic functors $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}'^{\mathrm{op}}$ (in the sense of Definition 5.3) such that $\tilde{F}(u') = u$ (notice that this is well-defined by Remark 2.3, as u is $\mathcal{C}$-ultrahomogeneous and $\mathcal{C}$-universal). Then we have a functor $A : \mathcal{G}^{\mathrm{op}} \to \mathbf{GTop}_b$ sending any pair $(\mathcal{C}, u)$ to the object $(\mathrm{Aut}(u), \mathcal{I}_{\mathcal{C}})$ of $\mathbf{GTop}_b$ and any arrow $F : (\mathcal{C}, u) \to (\mathcal{C}', u')$ to the arrow $\tilde{F} : \mathrm{Aut}(u') \to \mathrm{Aut}(u)$ in $\mathbf{GTop}_b$:

$$\begin{array}{ccc} & \begin{array}{ccc} \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}}) & \simeq & \mathbf{Cont}(\mathrm{Aut}(u)) \\ \uparrow & & \uparrow \\ \mathbf{Sh}(\mathcal{C}'^{\mathrm{op}}, J_{\mathrm{at}}) & \simeq & \mathbf{Cont}(\mathrm{Aut}(u')) \end{array} & \\ \begin{array}{c} (\mathcal{C}, u) \\ \downarrow{\scriptstyle F} \\ (\mathcal{C}', u') \end{array} & & \begin{array}{c} (\mathrm{Aut}(u), \mathcal{I}_{\mathcal{C}}) \\ {\scriptstyle \tilde{F}}\uparrow \\ (\mathrm{Aut}_{\mathcal{C}'}(u), \mathcal{I}_{\mathcal{C}'}) \end{array} \end{array}$$

This is well-defined since for any arrow $\chi : c \to u$ in Ind-$\mathcal{C}$ from an object c of $\mathcal{C}$ to u, $\tilde{F}^{-1}(\mathcal{I}_{\chi}) = \mathcal{I}_{\xi}$ for some arrow $\xi : F(c) \to u'$ in Ind-$\mathcal{C}'$. Indeed, if we denote by $\tilde{u} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and by $\tilde{u'} : \mathcal{C}'^{\mathrm{op}} \to \mathbf{Set}$ the flat functors corresponding to the objects u and u' respectively of Ind-$\mathcal{C}$ and of Ind-$\mathcal{C}'$, we have that $\tilde{u'} \circ F = \tilde{u}$ and hence the arrows $\chi : c \to u$ in Ind-$\mathcal{C}$ (i.e., by the Yoneda Lemma, the elements of $\tilde{u}(c) = \tilde{u'}(F(c))$) correspond exactly to the arrows $\xi : F(c) \to u'$ in Ind-$\mathcal{C}'$ (i.e., by Yoneda, the elements of $\tilde{u'}(F(c))$); we thus have $\tilde{F}^{-1}(\mathcal{I}_{\chi}) = \mathcal{I}_{\xi}$, where ξ is the arrow $F(c) \to u'$ in Ind-$\mathcal{C}'$ associated to $\chi : c \to u$ via this correspondence.

In the converse direction, we can define a functor $B : \mathbf{GTop}_b \to \mathcal{G}^{\mathrm{op}}$ sending any object $(G, \mathcal{B})$ of $\mathbf{GTop}_b$ to the pair $(\mathbf{Cont}_{\mathcal{B}}(G)^{\mathrm{op}}, p_{G_{\mathcal{B}}})$ (where

$p_{G_{\mathcal{B}}}$, namely the canonical point of the topos $\mathbf{Cont}(G_{\mathcal{B}})$, is regarded as an object of Ind-$\mathbf{Cont}_{\mathcal{B}}(G)^{\mathrm{op}}$ in the canonical way) and any arrow $f : (G, \mathcal{B}) \to (G', \mathcal{B}')$ in $\mathbf{GTop}_b$ to the arrow $\mathbf{Cont}(f)^{*\mathrm{op}}| : \mathbf{Cont}_{\mathcal{B}'}(G')^{\mathrm{op}} \to \mathbf{Cont}_{\mathcal{B}}(G)^{\mathrm{op}}$ (notice that this restriction is indeed well-defined since by our hypotheses the inverse image under f of any open subgroup of G' belonging to $\mathcal{B}'$ is an open subgroup of G belonging to $\mathcal{B}$). To prove that this functor is well-defined we observe that by Proposition 5.6 for any object $(G, \mathcal{B})$ of $\mathbf{GTop}_b$ the category $B(G, \mathcal{B})$ satisfies the amalgamation and joint embedding properties. On the other hand, $p_{G_{\mathcal{B}}}$ is a $\mathbf{Cont}_{\mathcal{B}}(G)^{\mathrm{op}}$-ultrahomogeneous and $\mathbf{Cont}_{\mathcal{B}}(G)^{\mathrm{op}}$-universal object, since for any object c of $\mathbf{Cont}_{\mathcal{B}}(G)^{\mathrm{op}}$,

$$\mathrm{Hom}_{\text{Ind-}\mathbf{Cont}_{\mathcal{B}}(G)^{\mathrm{op}}}(c, p_{G_{\mathcal{B}}}) \cong p_{G_{\mathcal{B}}}(c) \cong c,$$

which is a non-empty $\mathrm{Aut}(p_{G_{\mathcal{B}}})$-transitive set (this follows from the fact that it is a transitive $G_{\mathcal{B}}$-set and for any element g of $G_{\mathcal{B}}$, the action of g on c coincides with the component at c of the action on $p_{G_{\mathcal{B}}}$ of the image of g under the canonical map $G_{\mathcal{B}} \to \mathrm{Aut}(p_{G_{\mathcal{B}}})$). The fact that for any arrow $f : (G, \mathcal{B}) \to (G', \mathcal{B}')$ in $\mathbf{GTop}_b$, $B(f) : B(G', \mathcal{B}') \to B(G, \mathcal{B})$ is an arrow in $\mathcal{G}$ is immediate from the fact that $B(f)$ is the restriction to subcanonical sites of the inverse image functor of a geometric morphism.

We can visualize this as follows:

$$\begin{array}{ccccc}
 & \mathbf{Sh}(\mathbf{Cont}_{\mathcal{B}'}(G'), J_{\mathrm{at}}) & \simeq & \mathbf{Cont}(G_{\mathcal{B}'}) & \\
 & \uparrow & & \uparrow & \\
 & \mathbf{Sh}(\mathbf{Cont}_{\mathcal{B}}(G), J_{\mathrm{at}}) & \simeq & \mathbf{Cont}(G_{\mathcal{B}}) & \\
(\mathbf{Cont}_{\mathcal{B}'}(G')^{\mathrm{op}}, p_{G'_{\mathcal{B}'}}) & & & & (G', \mathcal{B}') \\
\downarrow {\scriptstyle \mathbf{Cont}(f)^{*\mathrm{op}}|} & & & & {\scriptstyle f}\uparrow \\
(\mathbf{Cont}_{\mathcal{B}}(G)^{\mathrm{op}}, p_{G_{\mathcal{B}}}) & & & & (G, \mathcal{B})
\end{array}$$

We have the following result:

Theorem 5.9. *The functor*

$$A : \mathcal{G}^{\mathrm{op}} \to \mathbf{GTop}_b$$

is right adjoint to the functor

$$B : \mathbf{GTop}_b \to \mathcal{G}^{\mathrm{op}} .$$

This adjunction restricts to a duality between the full subcategory $\mathcal{G}_{\mathrm{sm}}$ of $\mathcal{G}^{\mathrm{op}}$ on the objects $(\mathcal{C}, u)$ such that every morphism in $\mathcal{C}$ is a strict monomorphism and the full subcategory $\mathbf{GTop}_b^c$ of $\mathbf{GTop}_b$ on the objects $(G, \mathcal{B})$ such that the topological group $G_{\mathcal{B}}$ is complete.

Proof The counit $\epsilon : B \circ A \to \mathrm{id}_{\mathcal{G}^{\mathrm{op}}}$ of the adjunction is given, for any $(\mathcal{C}, u)$ in $\mathcal{G}$, by $\epsilon(\mathcal{C}, u) = F^{\mathrm{op}} := (\mathcal{C}, u) \to (\mathbf{Cont}_{\mathcal{I}_{\mathcal{C}}}(\mathrm{Aut}(u))^{\mathrm{op}}, p_{\mathrm{Aut}(u)_{\mathcal{I}_{\mathcal{C}}}})$, regarded as an arrow in $\mathcal{G}$, where F is the functor defined in the statement of Theorem 2.4, while the unit $\eta : \mathrm{id}_{\mathbf{GTop}_b} \to A \circ B$ is given, for any $(G, \mathcal{B}) \in \mathbf{GTop}_b$, by $\eta(G, \mathcal{B}) = \xi_{G_{\mathcal{B}}} := (G, \mathcal{B}) \to (\mathrm{Aut}_{\mathbf{Cont}_{\mathcal{B}}(G)^{\mathrm{op}}}(p_{G_{\mathcal{B}}}), \mathcal{I}_{\mathbf{Cont}_{\mathcal{B}}(G)^{\mathrm{op}}})$ (cf. section 5.3 for the definition of the canonical map ξ).

One easily verifies the naturality of ϵ and η and the fact that the induced maps

$$\mathrm{Hom}_{\mathbf{GTop}_b}((G, \mathcal{B}), A(\mathcal{C}, u)) \to \mathrm{Hom}_{\mathcal{G}^{\mathrm{op}}}(B(G, \mathcal{B}), (\mathcal{C}, u))$$

and

$$\mathrm{Hom}_{\mathcal{G}^{\mathrm{op}}}(B(G, \mathcal{B}), (\mathcal{C}, u)) \to \mathrm{Hom}_{\mathbf{GTop}_b}((G, \mathcal{B}), A(\mathcal{C}, u))$$

are inverse to each other.

Now, by Proposition 4.1 [9], $\epsilon(\mathcal{C}, u)$ is an isomorphism in $\mathcal{G}$ if and only if every arrow of $\mathcal{C}$ is a strict monomorphism, while $\eta(G, \mathcal{B})$ is an isomorphism in $\mathbf{GTop}_b$ if and only if $G_{\mathcal{B}}$ is complete. □

Remark 5.10. Up to Morita equivalence, the functors A and B defining the adjunction of Theorem 5.9 are inverse to each other. Indeed, for any $(\mathcal{C}, u)$ in $\mathcal{G}$, $\mathbf{Sh}(B(A(\mathcal{C}, u)), J_{\mathrm{at}}) \simeq \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$, while for any $(G, \mathcal{B}) \in \mathbf{GTop}_b$, $\mathbf{Cont}(G_{\mathcal{B}}) \simeq \mathbf{Cont}(G'_{\mathcal{B}'})$, where $A(B(G, \mathcal{B}) = (G', \mathcal{B}')$.

The following result is an immediate consequence of Theorem 5.9.

Corollary 5.11. *Let $(\mathcal{C}, u)$ and $(\mathcal{C}', u')$ be objects of $\mathcal{G}_{\mathrm{sm}}^{\mathrm{op}}$. Then a continuous group homomorphism $h : \mathrm{Aut}(u') \to \mathrm{Aut}(u)$ is induced by a (unique) functor $F : \mathcal{C} \to \mathcal{C}'$ such that $\tilde{F}(u') = u$ if and only if the inverse image under h of any open subgroup of the form $\mathcal{I}_{\chi}$ (where $\chi : c \to u$ is an arrow in Ind-$\mathcal{C}$) is of the form $\mathcal{I}_{\chi'}$ (where $\chi' : c' \to u'$ is an arrow in Ind-$\mathcal{C}'$).*

□

In section 7 we shall apply this corollary in a logical context.

As we have seen above, the category $\mathbf{GTop}$ can be identified with a full subcategory of the category $\mathbf{GTop}_b$, by choosing the canonical algebraic base associated to any topological group. It is thus natural to wonder whether it is possible to characterize the objects $(\mathcal{C}, u)$ of the category $\mathcal{G}$ which correspond to such objects under the adjunction of Theorem 5.9. To this end, we remark that the objects of $\mathbf{GTop}_b$ of the form (G, C_G) can be characterized as the objects $(G, \mathcal{B})$ such that the category $\mathbf{Cont}_{\mathcal{B}}(G)$ coincides with the full subcategory of $\mathbf{Cont}(G_{\mathcal{B}})$ on its atoms. Therefore, the pairs $(\mathcal{C}, u)$ of the form $B(G, C_G)$ for some topological group G satisfy the property that every atom of $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ has, up to isomorphism, the form

$l(c)$ for some object c of $\mathcal{C}$, where $l : \mathcal{C}^{\text{op}} \to \mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{\text{at}})$ is the composite of the Yoneda embedding $y : \mathcal{C}^{\text{op}} \to [\mathcal{C}, \mathbf{Set}]$ with the associated sheaf functor $a_{J_{\text{at}}} : [\mathcal{C}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{\text{at}})$. Conversely, if $\mathcal{C}$ satisfies this condition then $A(\mathcal{C}, u) = (\text{Aut}(u), \mathcal{I}_{\mathcal{C}})$ is of the form (G, C_G), since the category $\mathbf{Cont}_{\mathcal{I}_{\mathcal{C}}}(\text{Aut}(u))$ coincides with the full subcategory of $\mathbf{Cont}(\text{Aut}(u))$ on its atoms. An alternative characterization of the objects $(\mathcal{C}, u)$ of $\mathcal{G}$ such that $A(\mathcal{C}, u)$ is of the form (G, C_G) is the following: $A(\mathcal{C}, u)$ is of the form (G, C_G) if and only if every open subgroup of $\text{Aut}(u)$ is of the form $\mathcal{I}_{\chi}$ for some arrow $\chi : c \to u$.

Summarizing, we have the following result:

Proposition 5.12. *For any object $(\mathcal{C}, u)$ of $\mathcal{G}$, the following conditions are equivalent:*

(i) $A(\mathcal{C}, u)$ is, up to isomorphism in $\mathbf{GTop}_b$, of the form (G, C_G) for some topological group G.

(ii) Every open subgroup of $\text{Aut}(u)$ *is of the form $\mathcal{I}_{\chi}$ for some $\chi : c \to u$ in* Ind-$\mathcal{C}$.

(iii) Every atom of the topos $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{\text{at}})$ is, up to isomorphism, of the form $l(c)$ for some object $c \in \mathcal{C}$.

This motivates the following definition: we shall say that a category $\mathcal{C}$ is *atomically complete* if its opposite category $\mathcal{C}^{\text{op}}$ satisfies AP, the atomic topology on $\mathcal{C}^{\text{op}}$ is subcanonical and every atom of the topos $\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{\text{at}})$ is, up to isomorphism, of the form $l(c)$ for some object $c \in \mathcal{C}$ (see Theorem 4.17 [9], Corollary 6.8 or Remark 6.9 for explicit characterizations of this class of categories).

These results lead to the following duality theorem.

Theorem 5.13. *The functors A and B defined above restrict to a duality between the full subcategory of $\mathcal{G}$ on the objects of the form $(\mathcal{C}, u)$ for $\mathcal{C}^{\text{op}}$ atomically complete and the category of complete (totally discontinuous) topological groups.*

□

6 Other insights from the 'bridge' technique

In this section we shall consider the equivalence of classifying toposes provided by Theorem 2.4 in conjunction with appropriate topos-theoretic invariants to construct 'bridges' (in the sense of [10]) for connecting the two sides with each other:

$$\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}}) \simeq \mathbf{Cont}(\mathrm{Aut}(u))$$

$$\mathcal{C}^{\mathrm{op}} \qquad\qquad\qquad\qquad \mathrm{Aut}(u)$$

This will yield various insights on the corresponding Galois theories.

6.1 A criterion for Morita equivalence

Theorem 6.1. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two small categories satisfying the dual of the amalgamation property. Then the following conditions are equivalent:*

(i) The toposes $\mathbf{Sh}(\mathcal{C}, J_{\mathrm{at}})$ and $\mathbf{Sh}(\mathcal{C}', J_{\mathrm{at}})$ are equivalent.

(ii) There is a small category $\mathcal{A}$ and two functors $H : \mathcal{C} \to \mathcal{A}$ and $K : \mathcal{C}' \to \mathcal{A}$ such that

(a) for any object $a \in \mathcal{A}$, there exist objects c of $\mathcal{C}$ and c' of $\mathcal{C}'$ and arrows $H(c) \to a$ and $K(c') \to a$ in $\mathcal{A}$;

(b) for any objects c, d of $\mathcal{C}$ (resp. of $\mathcal{C}'$) and any arrow $\xi : H(c) \to H(d)$ (resp. $\xi : K(c) \to K(d)$) of $\mathcal{A}$ there exist an objet e of $\mathcal{C}$ (resp. of $\mathcal{C}'$) and arrows $f : e \to c$ and $g : e \to d$ of $\mathcal{C}$ (resp. of $\mathcal{C}'$) such that $\xi \circ H(f) = H(g)$ (resp. $\xi \circ K(f) = K(g)$);

(c) for any arrows $f, g : c \to d$ of $\mathcal{C}$ (resp. of $\mathcal{C}'$), if $H(f) = H(g)$ (resp. $K(f) = K(g)$) then there exists an arrow $h : a \to c$ of $\mathcal{C}$ (resp. of $\mathcal{C}'$) such that $f \circ h = g \circ h$.

Proof. Let us recall from [20] the following definition: a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a *dense morphism of sites* $(\mathcal{C}, J) \to (\mathcal{D}, K)$ if it satisfies the following properties:

(a) P is a covering family in $\mathcal{C}$ if and only if $F(P)$ is a covering family in $\mathcal{D}$;

(b) for any object d of $\mathcal{D}$ there exists a covering family of arrows $d_i \to d$ whose domains d_i are in the image of F;

(c) for every $x, y \in \mathcal{C}$ and any arrow $g : F(x) \to F(y)$ in $\mathcal{D}$, there exist a covering family of arrows $f_i : x_i \to x$ and a family of arrows $g_i : x_i \to y$ such that $g \circ F(f_i) = g_i$ for all i;

(d) for any arrows $h, k : x \to y$ in $\mathcal{C}$ such that $F(h) = F(k)$ there exists a covering family of arrows $f_i : x_i \to x$ such that $h \circ f_i = k \circ f_i$ for all i.

By Theorem 11.8 [20], if F is a dense morphism of sites then the associated geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ is an equivalence. One can prove (see [12]) that, conversely, if F is a morphism of sites (in the sense of Corollary 5.1(ii)) such that $(\mathcal{D}, K)$ is subcanonical and $\mathbf{Sh}(F)$ is an equivalence then F is a dense morphism of sites.

Recall that the following diagram, where the functor $l_{\mathcal{C}}$ (resp. $l_{\mathcal{D}}$) is the composite of the relevant Yoneda embedding with the associated sheaf functor, is commutative:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad F \quad} & \mathcal{D} \\ \Big\downarrow{\scriptstyle l_{\mathcal{C}}} & & \Big\downarrow{\scriptstyle l_{\mathcal{D}}} \\ \mathbf{Sh}(\mathcal{C}, J) & \xrightarrow{\mathbf{Sh}(F)^{*}} & \mathbf{Sh}(\mathcal{D}, K) \end{array}$$

Now, the enumerated conditions in the statement of the proposition amount precisely to the requirement that the functors H and K define dense morphisms of sites respectively $(\mathcal{C}, J_{\text{at}}) \to (\mathcal{A}, J_{\text{at}})$ and $(\mathcal{C}', J_{\text{at}}) \to (\mathcal{A}, J_{\text{at}})$ in the sense of Definition 11.1 [20]. To deduce the necessity of this condition for the toposes $\mathbf{Sh}(\mathcal{C}, J_{\text{at}})$ and $\mathbf{Sh}(\mathcal{C}', J_{\text{at}})$ to be equivalent, in light of the above discussion, it suffices to notice that, taking $\mathcal{A}$ to be the full subcategory of $\mathbf{Sh}(\mathcal{C}, J_{\text{at}})$ (resp. of $\mathbf{Sh}(\mathcal{C}', J_{\text{at}})$) on its atoms, we have an equivalence $\mathbf{Sh}(\mathcal{C}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{A}, J_{\text{at}})$ (resp. $\mathbf{Sh}(\mathcal{C}', J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{A}', J_{\text{at}})$) induced by a morphism of sites $(\mathcal{C}, J_{\text{at}}) \to (\mathcal{A}, J_{\text{at}})$ (resp. $(\mathcal{C}', J_{\text{at}}) \to (\mathcal{A}, J_{\text{at}})$). Conversely, if both $(\mathcal{C}, J_{\text{at}}) \to (\mathcal{A}, J_{\text{at}})$ and $(\mathcal{C}', J_{\text{at}}) \to (\mathcal{A}, J_{\text{at}})$ are dense morphisms of sites, then we have equivalences $\mathbf{Sh}(\mathcal{C}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{A}, J_{\text{at}})$ and $\mathbf{Sh}(\mathcal{C}', J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{A}', J_{\text{at}})$, which yield an equivalence $\mathbf{Sh}(\mathcal{C}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{C}', J_{\text{at}})$. □

Corollary 6.2. *Let $(\mathcal{C}, u)$ and $(\mathcal{C}', u')$ be pairs satisfying the hypotheses of Theorem 2.4. Then the following conditions are equivalent:*

(i) The topological groups $\mathrm{Aut}(u)$ *and* $\mathrm{Aut}(u')$ *are Morita-equivalent.*

(ii) There is a small category $\mathcal{A}$ and two functors $H : \mathcal{C} \to \mathcal{A}$ and $K : \mathcal{C}' \to \mathcal{A}$ such that

(a) for any object $a \in \mathcal{A}$, there exist objects c of $\mathcal{C}$, c' of $\mathcal{C}'$ and arrows $a \to H(c)$, $a \to K(c')$ in $\mathcal{A}$;

(b) for any objects c, d of $\mathcal{C}$ (resp. of $\mathcal{C}'$) and any arrow $\xi : H(d) \to H(c)$ (resp. $\xi : K(d) \to K(c)$) of $\mathcal{A}$ there exists an objet e of $\mathcal{C}$ (resp. of $\mathcal{C}'$) and arrows $f : c \to e$ and $g : d \to e$ of $\mathcal{C}$ (resp. of $\mathcal{C}'$) such that $H(f) \circ \xi = H(g)$ (resp. $K(f) \circ \xi = K(g)$);

(c) for any arrows $f, g : c \to d$ of $\mathcal{C}$ (resp. of $\mathcal{C}'$), if $H(f) = H(g)$ (resp. $K(f) = K(g)$) then there exists an arrow $h : d \to a$ of $\mathcal{C}$ (resp. of $\mathcal{C}'$) such that $h \circ f = h \circ g$.

□

6.2 Categories of imaginaries

Section 4.3 of [9] explicitly described a completion process for the opposite of an essentially small category $\mathcal{D}$ satisfying the amalgamation property, which was called the *atomic completion*, making it equivalent to the full subcategory of the associated topos $\mathbf{Sh}(\mathcal{D}^{\text{op}}, J_{\text{at}})$ on its atoms. As was observed in that context, the objects of the atomic completion can be thought of as 'imaginaries' (in the model-theoretic sense) as they are formal quotients of objects $l(d)$ coming from $\mathcal{D}$ by equivalence relations internal to the topos; recall that an equivalence relation on an object of the form $l(d)$ in the topos $\mathbf{Sh}(\mathcal{D}^{\text{op}}, J_{\text{at}})$ can be identified with a function which assigns to each object e of $\mathcal{D}$ an equivalence relation R_e on the set $\text{Hom}_{\mathcal{D}^{\text{op}}}(e, d)$ in such a way that for any arrow $h : e' \to e$ in $\mathcal{D}^{\text{op}}$ and any $(\chi, \xi) \in \text{Hom}_{\mathcal{D}^{\text{op}}}(e, d)^2$, $(\chi, \xi) \in R_e$ if and only if $(\chi \circ h, \xi \circ h) \in R_{e'}$.

In this section, we shall provide an alternative, more combinatorial but equivalent description of this construction; we shall denote the atomic completion of an essentially small category $\mathcal{C}$ satisfying the dual of the amalgamation property by $\mathcal{C}_{\text{at}}$. This description relies in particular on the following lemma, which shows how to reconstruct an atomic topos from its full subcategory on atoms:

Lemma 6.3. *Let $\mathcal{E}$ be an atomic topos and $\mathcal{E}^{\text{at}}$ its full subcategory on atoms. Then $\mathcal{E}$ is equivalent to the category whose*

- *objects are the families $(a_i)_{i \in I}$ of objects a_i of $\mathcal{E}^{\text{at}}$ indexed by a set I,*
- *arrows*

$$(a_i)_{i \in I} \longrightarrow (b_j)_{j \in J}$$

consist of a map $\alpha : I \to J$ and a family of arrows of $\mathcal{E}^{\text{at}}$

$$a_i \longrightarrow b_{\alpha(i)}\,, \quad i \in I\ .$$

The equivalence consists in associating

$$\left\{\begin{array}{l}\bullet \ \textit{to any family } (a_i)_{i\in I} \textit{ the coproduct in } \mathcal{E} \\ \qquad\qquad \coprod_{i\in I} a_i\,, \\ \bullet \ \textit{to any family of arrows } \big(a_i \to b_{\alpha(i)}\big)_{i\in I} \textit{ from } (a_i)_{i\in I} \textit{ to } (b_j)_{j\in J} \textit{ the arrow of } \mathcal{E} \\ \qquad\qquad \coprod_{i\in I} a_i \longrightarrow \coprod_{j\in J} b_j \\ \textit{defined by the collection of composite arrows} \\ \qquad\qquad a_i \longrightarrow b_{\alpha(i)} \longrightarrow \coprod_{j\in J} b_j\,, \quad i\in I\ . \end{array}\right.$$

Proof. By definition of an atomic topos, every object of $\mathcal{E}$ can be written as a coproduct

$$\coprod_{i\in I} a_i$$

of atoms a_i of $\mathcal{E}$ indexed by a set I.

Every arrow between any two such coproducts

$$\coprod_{i\in I} a_i \longrightarrow \coprod_{j\in J} b_j$$

consists of a collection of arrows

$$a_i \longrightarrow \coprod_{j\in J} b_j\,, \quad i\in I\ .$$

The b_j, $j\in J$, are pairwise disjoint subobjects of $\coprod_{j\in J} b_j = b$ and form a covering family of b. For each $i\in I$, their inverse images in a_i are pairwise disjoint subobjects of a_i which form a covering family of a_i. Therefore they are all zero with the exception of a single one, of index say $\alpha(i)\in J$, which is isomorphic to a_i. This means that the arrow

$$a_i \longrightarrow \coprod_{j\in J} b_j$$

is induced by a unique arrow

$$a_i \longrightarrow b_{\alpha(i)}\ .$$

□

Lemma 6.4. *Let $\mathcal{C}$ be an essentially small category whose opposite category $\mathcal{C}^{\mathrm{op}}$ has the amalgamation property (AP).*

(i) For any objects $c_1, \ldots, c_n$ of $\mathcal{C}$, we say that two families of arrows of $\mathcal{C}$

$$f_i : d \longrightarrow c_i\,, \quad 1 \leq i \leq n\,,$$

and

$$f_i' : d' \longrightarrow c_i\,, \quad 1 \leq i \leq n\,,$$

are equivalent *if there exist two arrows of $\mathcal{C}$*

$$f : e \longrightarrow d\,, \quad f' : e \longrightarrow d'\,,$$

such that

$$f_i \circ f = f_i' \circ f'\,, \quad 1 \leq i \leq n\ .$$

Then this relation between families of arrows $\left(d \xrightarrow{f_i} c_i\right)_{1 \leq i \leq n}$ is an equivalence relation.

Its classes form a set, which we call the set of components *of $c_1 \times \ldots \times c_n$.*

(ii) For any objets $c_1, \ldots, c_n$ of $\mathcal{C}$ and any map $\alpha : \{1, \ldots, k\} \to \{1, \ldots, n\}$, the maps

$$\left(d \xrightarrow{f_i} c_i\right)_{1 \leq i \leq n} \longmapsto \left(d \xrightarrow{f_{\alpha(j)}} c_{\alpha(j)}\right)_{1 \leq j \leq k}$$

define a function from the set of components of $c_1 \times \ldots \times c_n$ to that of components of $c_{\alpha(1)} \times \ldots \times c_{\alpha(k)}$.

(iii) For any arrows of $\mathcal{C}$

$$g_i : c_i \longrightarrow b_i\,, \quad 1 \leq i \leq n\,,$$

the composition

$$\left(d \xrightarrow{f_i} c_i\right)_{1 \leq i \leq n} \longmapsto \left(d \xrightarrow{g_i \circ f_i} b_i\right)_{1 \leq i \leq n}$$

defines a function from the set of components of $c_1 \times \ldots \times c_n$ to that of components of $b_1 \times \ldots \times b_n$.

Proof. (i) This relation is clearly reflexive and symmetric. It is transitive since the category $\mathcal{C}^{\mathrm{op}}$ satisfies AP.

Its equivalence classes form a set since the category $\mathcal{C}$ is assumed to be essentially small.

(ii) et (iii) are straightforward. □

This lemma allows us to give the following

Definition 6.5. Let $\mathcal{C}$ be, as above, an essentially small category whose opposite category $\mathcal{C}^{\mathrm{op}}$ satisfies the amalgamation property (AP).

(i) For any objects c_1, c_2 of $\mathcal{C}$, we call *relation of* c_1 *into* c_2 every subset R of the set of components of $c_1 \times c_2$.

We then call *opposite relation* R^{op} of R the relation of c_2 into c_1 which is obtained from R by permutation of c_1 and c_2.

(ii) For any objects c_1, c_2, c_3 of $\mathcal{C}$ and any relations R of c_1 into c_2 and R' of c_2 into c_3, we call *composite relation* $R' \circ R$ of R and R' the relation of c_1 into c_3 consisting of the components of $c_1 \times c_3$ which lift to a component of $c_1 \times c_2 \times c_3$ whose images in $c_1 \times c_2$ and in $c_2 \times c_3$ are elements of R and R'.

(iii) For any object c of $\mathcal{C}$, we call *equivalence relation* on c every relation R of c in c such that

- R is reflexive in the sense that it contains the diagonal component of $c \times c$,
- R is symmetric in the sense that $R^{\mathrm{op}} = R$,
- R is transitive in the sense that $R \circ R \subseteq R$.

The following lemma provides an explicit, site-level description of the atomic decomposition of finite products in the topos $\mathbf{Sh}(\mathcal{C}, J_{\mathrm{at}})$ of objects of the form $l(c)$.

Lemma 6.6. *Under the hypotheses of Lemma 6.4 and Proposition 6.7, let us consider the canonical functor*

$$\ell : \mathcal{C} \longrightarrow \mathcal{E}^{\mathrm{at}}$$

and associate with any family of arrows of $\mathcal{C}$

$$f_i : d \longrightarrow c_i \,, \quad 1 \leq i \leq n \,,$$

the image of the arrow of $\mathcal{E}$

$$\prod_{1 \leq i \leq n} \ell(f_i) : \ell(d) \longrightarrow \ell(c_1) \times \ldots \times \ell(c_n) \ .$$

Then:

(i) *For any objects $c_1, \dots, c_n$ de $\mathcal{C}$, this assignment defines a bijection from the set of "components" of $c_1 \times \dots \times c_n$ (in the sense of Lemma 6.4) to the set of subobjects of $\ell(c_1) \times \dots \times \ell(c_n)$ which are atoms of $\mathcal{E}$.*

(ii) *For any objects c_1, c_2 of $\mathcal{C}$, this map defines a bijection from the set of "relations" of c_1 into c_2 (in the sense of Definition 6.5(i)) to the set of subobjects of $\ell(c_1) \times \ell(c_2)$ in the topos $\mathcal{E}$.*

Proof. (i) As any $\ell(d)$ is an atom, the image in the atomic topos $\mathcal{E}$ of any arrow

$$\ell(d) \longrightarrow \ell(c_1) \times \dots \times \ell(c_n)$$

is an atom of $\mathcal{E}$.

As any arrow $\ell(e) \to \ell(d)$ is an epimorphism of $\mathcal{E}$, the image of $\ell(d) \to \ell(c_1) \times \dots \times \ell(c_n)$ does not depend on the representative chosen for each equivalence class of families of arrows $\left(d \xrightarrow{f_i} c_i\right)_{1 \leq i \leq n}$.

Conversely, let us consider an atom c of a product of image objects $\ell(c_i)$ of objects $c_1, \dots, c_n$ of $\mathcal{C}$. There exists an object d of $\mathcal{C}$ admitting an epimorphism $\ell(d) \to c$ whence c is the image of an arrow of $\mathcal{E}$ of the form

$$\ell(d) \longrightarrow \ell(c_1) \times \dots \times \ell(c_n) \ .$$

For each i, $1 \leq i \leq n$, the contravariant functor on $\mathcal{C}$

$$d' \longmapsto \mathrm{Hom}_{\mathcal{E}}(\ell(d'), \ell(c_i))$$

is the sheafification for the atomic topology J_{at} of the presheaf

$$d' \longmapsto \mathrm{Hom}_{\mathcal{C}}(d', c_i) \ .$$

Therefore, there exists an arrow of $\mathcal{C}$

$$d' \longrightarrow d$$

such that the composite arrow

$$\ell(d') \longrightarrow \ell(d) \longrightarrow \ell(c_1) \times \dots \times \ell(c_n)$$

is induced by a family of arrows of $\mathcal{C}$

$$d' \longrightarrow c_i \,, \quad 1 \leq i \leq n \ .$$

Lastly, let us consider two families of arrows of $\mathcal{C}$

$$(f_i : d \longrightarrow c_i)_{1 \leq i \leq n} \quad \text{and} \quad (f'_i : d' \longrightarrow c_i)_{1 \leq i \leq n}$$

such that the induced arrows of $\mathcal{E}$

$$\ell(d) \longrightarrow \ell(c_1) \times \ldots \times \ell(c_n) \quad \text{and} \quad \ell(d') \longrightarrow \ell(c_1) \times \ldots \times \ell(c_n)$$

have as image the same atom.

Then the pullback in the topos $\mathcal{E}$

$$\ell(d) \times_{\ell(c_1) \times \ldots \times \ell(c_n)} \ell(d')$$

is non-zero and there exists an object e of $\mathcal{C}$ and two arrows of $\mathcal{E}$

$$\ell(e) \longrightarrow \ell(d) \quad \text{and} \quad \ell(e) \longrightarrow \ell(d')$$

which make the following square commutative:

$$\begin{array}{ccc} \ell(e) & \longrightarrow & \ell(d) \\ \downarrow & & \downarrow \\ \ell(d') & \longrightarrow & \ell(c_1) \times \ldots \times \ell(c_n) \end{array}$$

By replacing, if necessary, e by a covering $e' \to e$ in $\mathcal{C}$, we can suppose that the arrows $\ell(e) \to \ell(d)$ and $\ell(e) \to \ell(d')$ are the images under ℓ of two arrows of $\mathcal{C}$

$$f : e \longrightarrow d \quad \text{and} \quad f' : e \longrightarrow d'$$

and that these arrows satisfy the relations

$$f_i \circ f = f'_i \circ f' \,, \quad 1 \leq i \leq n \ .$$

(ii) is an immediate consequence of (i).

□

We can now prove the following

Proposition 6.7. *Let $\mathcal{C}$ be an essentially small category whose opposite category $\mathcal{C}^{\text{op}}$ satisfies the amalgamation property (AP), $\mathcal{E} = \mathbf{Sh}(\mathcal{C}, J_{\text{at}})$ be the atomic topos of sheaves on the category $\mathcal{C}$ with respect to the atomic topology J_{at}, and $\mathcal{C}_{\text{at}}$ be the atomic completion of $\mathcal{C}$.*

Then $\mathcal{C}_{\text{at}}$ is equivalent to the category defined as follows:

- *The objects are the pairs (c, R) consisting of an object c of $\mathcal{C}$ and an equivalence relation R on c.*

- *The arrows*

$$(c_1, R_1) \longrightarrow (c_2, R_2)$$

are the relations R of c_1 into c_2 such that

$$\begin{aligned} R \circ R_1 &\subseteq R\,, \\ R_2 \circ R &\subseteq R\,, \\ R \circ R^{\mathrm{op}} &\subseteq R_2\,, \\ R^{\mathrm{op}} \circ R &\supseteq R_1\,. \end{aligned}$$

Proof. It follows from Lemma 6.6(ii) that, for any object c of $\mathcal{C}$, giving an equivalence relation on c in the sense of Definition 6.5(iii) is the same thing as giving an equivalence relation on the object $\ell(c)$ of the topos $\mathcal{E}$. Now, any such equivalence relation on $\ell(c)$ defines in $\mathcal{E}$ a quotient object which is necessarily an atom.

Conversely, there exists for each atom a of $\mathcal{E}$ an objet c of $\mathcal{C}$ together with an arrow $\ell(c) \to a$, whence a identifies with the quotient of $\ell(c)$ by the equivalence relation $\ell(c) \times_a \ell(c) \rightarrowtail \ell(c) \times \ell(c)$.

Lastly, let us consider two atoms a_1 and a_2 of $\mathcal{E}$ regarded as quotients of two objects of the form $\ell(c_1)$ and $\ell(c_2)$, $c_1, c_2 \in \mathrm{Ob}(\mathcal{C})$ by equivalence relations R_1 and R_2.

Giving an arrow $a_1 \to a_2$ in $\mathcal{E}$ is equivalent to giving its graph as a subobject of $a_1 \times a_2$ or, equivalently, the inverse image R of this graph by the epimorphism

$$\ell(c_1) \times \ell(c_2) \longrightarrow a_1 \times a_2\ .$$

Conversely, a subobject R of $\ell(c_1) \times \ell(c_2)$ is the inverse image of the graph of an arrow $a_1 \to a_2$ if and only if it satisfies the four conditions

$$\begin{aligned} R \circ R_1 &\subseteq R\,, \\ R_2 \circ R &\subseteq R\,, \\ R \circ R^{\mathrm{op}} &\subseteq R_2\,, \\ R^{\mathrm{op}} \circ R &\supseteq R_1\,. \end{aligned}$$

□

In section 4.3 of [9] (cf. Theorem 4.17) the categories which are atomically complete (i.e. equivalent to their atomic completions) were explicitly characterized. Proposition 6.7 allows us to obtain an alternative characterization for them, as provided by the following

Corollary 6.8. *Let $\mathcal{C}$ be an essentially small category whose opposite $\mathcal{C}^{\text{op}}$ satisfies the amalgamation property (AP). Then $\mathcal{C}$ is atomically complete if and only if every arrow of $\mathcal{C}$ is a strict epimorphism and for any equivalence relation R on an object c of $\mathcal{C}$ (in the sense of Definition 6.5) there exist an object d and a quotient arrow $q : c \to d$ characterized by the property that for any arrows $f, g : a \to c$ in $\mathcal{C}$, $q \circ f = q \circ g$ if and only if $(f, g) \in R$.*

Proof If suffices to recall that the atomic topology J_{at} on $\mathcal{C}$ is subcanonical, that is the canonical functor from $\mathcal{C}$ to $\mathbf{Sh}(\mathcal{C}, J_{\text{at}})$ is full and faithful, if and only if every arrow of $\mathcal{C}$ is a strict epimorphism. If this condition is verified then $\mathcal{C}$ is equivalent to its atomic completion if and only if every atom of the topos $\mathbf{Sh}(\mathcal{C}, J_{\text{at}})$ is, up to isomorphism, of the form $l(d)$ for an object d of $\mathcal{C}$. Now, by the explicit description of the atomic completion given by Proposition 6.7, the latter condition is equivalent to the requirement that for any object c of $\mathcal{C}$ and equivalence relation R on c, the quotient $l(c) \twoheadrightarrow l(c)/R$ should be the image under l of an arrow $q : c \to d$ in $\mathcal{C}$; but $l(q)$ is isomorphic over $l(c)$ to the quotient arrow $l(c) \twoheadrightarrow l(c)/R$ if and only if for any arrows $f, g : a \to c$ in $\mathcal{C}$, $q \circ f = q \circ g$ if and only if $(f, g) \in R$. □

Remark 6.9. In light of Proposition 6.7, atomically complete categories can alternatively be characterized as the essentially small categories $\mathcal{A}$ satisfying the following properties:

- the opposite $\mathcal{A}^{\text{op}}$ of $\mathcal{A}$ has the amalgamation property (AP);
- every equivalence relation R on an object a of $\mathcal{A}$ defines a quotient arrow $q : a \to a_R$ of $\mathcal{A}$ characterized by the property that the arrows

  $$a_R \longrightarrow b$$

  correspond bijectively, via composition with q, to the arrows

  $$f : a \longrightarrow b$$

  such that for every element $\left(c \xrightarrow{g} a, c \xrightarrow{g'} a\right)$ of R, the identity $f \circ g = f \circ g'$ holds;
- for any object b together with an equivalence relation R and any object a with a relation S of a into b such that

  $$R \circ S \subseteq S\,, \quad S \circ S^{\text{op}} \subseteq R \quad \text{and} \quad S^{\text{op}} \circ S \supseteq \Delta_a \quad \text{(the diagonal of } a\text{)}$$

there exists in $\mathcal{A}$ a unique arrow

$$f : a \longrightarrow b_R$$

the inverse image of whose graph in $a \times b$ is S, and every arrow

$$a \longrightarrow b_R$$

in $\mathcal{A}$ is of this form.

Indeed, these conditions ensure that there exists a full and faithful functor $\mathcal{A}^{\text{at}} \to \mathcal{A}$ whose composite with the canonical functor $\mathcal{A} \to \mathcal{A}^{\text{at}}$ is isomorphic to the identity functor on $\mathcal{A}$, and which is therefore essentially surjective.

Notice also that a quotient arrow in the sense of the second of the above conditions is the same as a quotient arrow in the sense of Corollary 6.8 (with respect to the same relation) if $\mathcal{A}$ is atomically complete, but not in general. Indeed, by the characterization of epimorphisms in a topos in terms of their kernel pairs, an arrow of $\mathcal{C}$ is a quotient by an equivalence relation R on an object c in the sense of Corollary 6.8 if and only if its image in $\mathbf{Sh}(\mathcal{C}, J_{\text{at}})$ is a quotient of R, regarded as an equivalence relation on $l(c)$ internal to $\mathbf{Sh}(\mathcal{C}, J_{\text{at}})$. On the other hand, the quotient in a topos $\mathcal{E}$ of an object X by an equivalence relation R on it can be realized as the coequalizer of the pair of canonical arrows $R \to X$, and if $\mathcal{E}$ is the topos $\mathbf{Sh}(\mathcal{C}, J_{\text{at}})$ for an atomically complete category $\mathcal{C}$ and $X = l(c)$ for an object $c \in \mathcal{C}$ then it suffices to check the coequalizer property with respect to epimorphisms with domain $l(c)$ since the existence property follows from the fact that every arrow in $\mathcal{E}$ can be factored as a epimorphism followed by a monomorphism and the uniqueness property follows from the fact that every quotient arrow is an epimorphism.

The following result gives a characterization of atomic completions by a universal property:

Corollary 6.10. *Let $\mathcal{C}$ be an essentially small category whose opposite $\mathcal{C}^{\text{op}}$ satisfies the amalgamation property (AP), $\mathcal{C}_{\text{at}}$ its atomic completion and $\ell : \mathcal{C} \to \mathcal{C}_{\text{at}}$ the canonical functor.*

Let $F : \mathcal{C} \to \mathcal{A}$ be an atomic functor (in the sense of Definition 5.3) from $\mathcal{C}$ to an atomically complete category $\mathcal{A}$. Then there exists a unique (up to isomorphism) atomic functor

$$\widetilde{F} : \mathcal{C}_{\text{at}} \longrightarrow \mathcal{A}$$

such that the functors

$$F, \widetilde{F} \circ \ell : \mathcal{C} \longrightarrow \mathcal{A}$$

are isomorphic.

In other words, the atomic completion operation defines a reflection of the category of essentially small categories whose opposite satisfies the amalgamation property and atomic functors between them into the full subcategory on the atomically complete categories.

Proof. This result arises from a 'bridge' based on the equivalence

$$\mathbf{Sh}(\mathcal{C}, J_{\text{at}}) \simeq \mathbf{Sh}(\mathcal{C}^{\text{at}}, J_{\text{at}})$$

by choosing as invariant the notion of geometric morphism from a topos of the form $\mathbf{Sh}(\mathcal{A}, J_{\text{at}})$, where $\mathcal{A}$ is an atomically complete category, whose inverse image sends atoms to atoms. By Diaconescu's equivalence, the geometric morphisms $\mathbf{Sh}(\mathcal{A}, J_{\text{at}}) \to \mathbf{Sh}(\mathcal{C}, J_{\text{at}})$ which send atoms to atoms correspond precisely to the morphisms of sites $(\mathcal{C}, J_{\text{at}}) \to (\mathcal{A}, J_{\text{at}})$ (since every atom of the topos $\mathbf{Sh}(\mathcal{C}, J_{\text{at}})$ is a quotient of an atom of the form $l(c)$ for $c \in \mathcal{C}$), that is to the atomic functors $\mathcal{C} \to \mathcal{A}$, while the geometric morphisms $\mathbf{Sh}(\mathcal{A}, J_{\text{at}}) \to \mathbf{Sh}(\mathcal{C}^{\text{at}}, J_{\text{at}})$ which send atoms to atoms correspond to the atomic functors $\mathcal{C}^{\text{at}} \to \mathcal{A}$. □

The description of the components of the $c_1 \times \ldots \times c_n$ arising in the construction of Proposition 6.7 simplifies when the category $\mathcal{C}$ has "multi-products" in the sense of the following lemma:

Lemma 6.11. *Let $\mathcal{C}$ be an essentially small category whose opposite $\mathcal{C}^{\text{op}}$ has the amalgamation property (AP).*

Suppose that $\mathcal{C}$ has "multi-products" *in the sense that, for any objects $c_1, \ldots, c_n$ of $\mathcal{C}$, there exists a family of objects d_i, $i \in I$, of $\mathcal{C}$ each of which endowed with arrows*

$$f_1^i : d_i \longrightarrow c_1, \ldots, f_n^i : d_i \longrightarrow c_n$$

and such that, for any object d of $\mathcal{C}$ endowed with arrows

$$f_1 : d \longrightarrow c, \ldots, f_n : d \longrightarrow c_n\,,$$

there exists a unique index $i \in I$ and a unique arrow

$$f : d \longrightarrow d_i$$

such that

$$f_1^i \circ f = f_1, \ldots, f_n^i \circ f = f_n \ .$$

Then, for any such objects $c_1, \ldots, c_n$ with multi-product $(d_i)_{i \in I}$, every component of $c_1 \times \ldots \times c_n$ in the sense of Lemma 6.4 is represented by a unique element d_i of this family, which therefore identifies with the set of components of $c_1 \times \ldots \times c_n$.

Remarks 6.12. (a) When the multi-product $(d_i)_{i\in I}$ of objects $c_1, \ldots, c_n$ of $\mathcal{C}$ exists, it is uniquely determined up to a unique family of isomorphisms.

(b) If $\mathcal{C}$ is the category of atoms $\mathcal{E}^{\text{at}}$ of an atomic topos $\mathcal{E}$, then it has multi-products: indeed, for any atoms $c_1, \ldots, c_n$, their multi-product consists in the subobjects of the product $c_1 \times \ldots \times c_n$ in the topos $\mathcal{E}$ which are atoms.

(c) If the families $(d_i)_{i\in I}$ are always finite, we say that the category $\mathcal{C}$ has *finite multi-products*. This is for instance the case if $\mathcal{C}^{\text{op}}$ is the category of finite separable extensions of a given field (see also Example 6.13(b) below).

Examples 6.13. (a) Let $\mathbf{Gp}^i_f$ be the category of finite groups and injective homomorphisms between them (a Galois theory for ${\mathbf{Gp}^i_f}^{\text{op}}$ was discussed in section 5.5 of [9]). Then ${\mathbf{Gp}^i_f}^{\text{op}}$ has multi-products that are not necessarily finite. These are obtained as follows. Given $c_1, \ldots, c_n \in \mathbf{Gp}^i_f$, the family $\mathcal{F}$ of finite groups q which are quotients of the free group generated by the c_i such that the induced homomorphisms $c_i \to q$ are injective is a multi-product of $c_1, \ldots, c_n$. Indeed, given a family of arrows $f_1 : d \to c, \ldots, f_n : d \to c_n$ in ${\mathbf{Gp}^i_f}^{\text{op}}$, there exist a unique element of $\mathcal{F}$ and arrows from it to the c_i through which the family uniquely factors, namely the group-theoretic image of the arrow from the free group generated by the c_i to d induced by the f_i. Notice that these multi-products are not finite in general.

(b) Let $\mathbf{Gp}^s_f$ be the category of finite groups and surjective homomorphisms between them. Then $\mathbf{Gp}^s_f$ has finite multi-products, which can be described as follows. Given $c_1, \ldots, c_n \in \mathbf{Gp}^s_f$, the family $\mathcal{G}$ of finite subgroups h of $c_1 \times \ldots \times c_n$ such that the induced homomorphisms $h \to c_i$ are surjective is a (finite) multi-product of $c_1, \ldots, c_n$. Indeed, given a family of arrows $f_1 : d \to c, \ldots, f_n : d \to c_n$ in $\mathbf{Gp}^s_f$, there exist a unique element of $\mathcal{G}$ and arrows from it to the c_i through which the family uniquely factors, namely the group-theoretic image of the arrow from d to $c_1 \times \ldots \times c_n$ induced by the f_i.

6.3 Irreducibility and discreteness

Let us analyze the notion of irreducible object in the context of the Galois-type equivalence

$$\mathbf{Sh}(\mathcal{C}^{\text{op}}, J_{\text{at}}) \simeq \mathbf{Cont}(\text{Aut}(u))$$

provided by Theorem 2.4.

Recall that an object of a Grothendieck topos is said to be irreducible if any epimorphic sieve on it contains the identity; in a Boolean topos, an object is irreducible if and only if it is an atom and every epimorphism to it is an isomorphism. For any topological group G, the irreducible objects of the topos $\mathbf{Cont}(G)$ are precisely the G-sets of the form G/U where U is an open subgroup which does not contain any proper open subgroup. Notice that any such subgroup is contained in any other open subgroup (otherwise their intersection would be a smaller open subgroup) and, in particular, is normal. Furthermore, $\mathbf{Cont}(G) \simeq \mathbf{Cont}(G/U)$.

The following proposition gives a site characterization of the irreducible objects of an atomic topos.

Proposition 6.14. *Let $\mathcal{C}$ be a small category satisfying the dual of the amalgamation property. Then the irreducible objects of the topos $\mathbf{Sh}(\mathcal{C}, J_{\text{at}})$ are all, up to isomorphism, of the form $l(c)$ for some $c \in \mathcal{C}$. Moreover:*

(a) For any $c \in \mathcal{C}$, the object $l(c)$ is irreducible if and only if for any arrow $f : d \to c$ in $\mathcal{C}$, if $f \circ z = f \circ w$ for some arrows z, w from a common domain to d then there exists an arrow ξ in $\mathcal{C}$ such that $z \circ \xi = w \circ \xi$.

(b) If the topology J_{at} is subcanonical then an object of the form $l(c)$ is irreducible if and only if every arrow in $\mathcal{C}$ with codomain c is an isomorphism.

Proof. Let A be an irreducible object of the topos $\mathbf{Sh}(\mathcal{C}, J_{\text{at}})$. Then there exists a split epimorphism from an object of the form $l(c)$ to A. But the splitting arrow $A \to l(c)$ must be an isomorphism, $l(c)$ being an atom, and hence an isomorphism.

First, let us show that $l(c)$ is irreducible if and only if for any arrow $f : d \to c$ in $\mathcal{C}$, the arrow $l(f)$ is an isomorphism. The object $l(c)$ being an atom, every arrow to it is an epimorphism and hence $l(c)$ is irreducible if and only if any arrow to $l(c)$, whose domain can be supposed to be of the form $l(d)$ without loss of generality, is a split epimorphism (equivalently, an isomorphism).

Let us show that, if for any arrow $f : d \to c$ in $\mathcal{C}$ the arrow $l(f)$ is an isomorphism then any arrow $\xi : l(d) \to l(c)$ is an isomorphism. There exists an arrow $h : a \to d$ in $\mathcal{C}$ and an arrow $k : a \to c$ such that $\xi \circ l(h) = l(k)$. Since $l(h)$ is an epimorphism (by definition of the atomic topology J_{at}), if $l(k)$ is an isomorphism then the identity $\xi \circ l(h) = l(k)$ implies that $l(h)$ is also a monomorphism and hence an isomorphism; so ξ is an isomorphism as well, as required.

Now, we can express the condition that for any arrow $f : d \to c$ in $\mathcal{C}$, the arrow $l(f)$ is an isomorphism as the requirement that $l(f)$ be a monomorphism, equivalently that the canonical monomorphism in the presheaf topos

$[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ from $y(d)$ to the kernel pair of $y(f)$ be J_{at}-dense. This rewrites explicitly as follows: if $f \circ z = f \circ w$ for some arrows z, w from a common domain to d then there exists an arrow ξ in $\mathcal{C}$ such that $z \circ \xi = w \circ \xi$.

Let us now suppose that J_{at} is subcanonical and show that $l(c)$ is irreducible if and only if every arrow $f : d \to c$ in $\mathcal{C}$ with codomain c is an isomorphism. The 'if' part follows from the general characterization established in point (a), so it remains to prove the 'only if' part. For this we observe that, since $l(c)$ is both an atom and an irreducible object and $l(d)$ is an atom, the arrow $l(f)$ is a split epimorphism, equivalently an isomorphism. But the Yoneda embedding reflects isomorphisms, it being full and faithful, whence our thesis follows. □

Every irreducible object A of an atomic topos $\mathcal{E}$ is a generator for it. Given an atom B of $\mathcal{E}$, consider the canonical projection $A \times B \to A$; this is an epimorphism (A being an atom) and hence admits a section, which provides an arrow $A \to B$; but this arrow is necessarily an epimorphism, B being an atom. Notice that every endomorphism of A is an isomorphism since, A being irreducible, it admits a section which is both a monomorphism and an epimorphism. So we have by Grothendieck's Comparison Lemma an equivalence $\mathcal{E} \simeq [\text{Aut}_{\mathcal{E}}(A)^{\text{op}}, \mathbf{Set}]$.

Notice that, if $(\mathcal{C}, u)$ is a pair satisying the hypotheses of Theorem 2.4 then it follows that the open subgroups of $\text{Aut}(u)$ which do not contain any proper open subgroup are precisely the ones of the form $\mathcal{I}_{\chi}$ where $dom(\chi)$ is an object satisfying the condition of Proposition 6.14

The following proposition, which summarizes the above discussion, relates the concept of irreducible generator in a topos and the property of discreteness of a topological group.

Proposition 6.15. *Under the hypotheses of Proposition 6.14, the following conditions are equivalent.*

(i) There exists an open subgroup of $\text{Aut}(u)$ *which does not contain any proper open subgroup.*

(ii) There exists an object c *of* $\mathcal{C}$ *satisfying the conditions of Proposition 6.14.*

(iii) The topological group $\text{Aut}(u)$ *is Morita-equivalent to a discrete group.*

□

6.4 Galois objects

Recall that a *Galois object* of a Grothendieck topos $\mathcal{E}$ is an object X of $\mathcal{E}$ such that the canonical arrow $\gamma_{\mathcal{E}}^{*}(\mathrm{Aut}_{\mathcal{E}}(X)) \times X \to X \times X$, where $\mathrm{Aut}_{\mathcal{E}}(X)$ is the set of automorphisms of X in $\mathcal{E}$ and $\gamma_{\mathcal{E}}^{*}$ is the inverse image of the unique geometric morphism $\gamma_{\mathcal{E}} : \mathcal{E} \to \mathbf{Set}$ (which sends any set S to the coproduct of $1_{\mathcal{E}}$ indexed by it), is an isomorphism.

In this section we shall describe the Galois objects of a topos as in Theorem 2.4 in terms of the two sites of definition.

Proposition 6.16. *(i) The Galois objects of a topos* $\mathbf{Cont}(G)$ *of continuous actions of a topological group* G *are precisely the objects isomorphic to one of the form* G/U *where* U *is a normal open subgroup of* G.

(ii) The Galois objects of an atomic topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$*, where* $\mathcal{C}$ *is an essentially small category satisfying the amalgamation property, are precisely the objects of the form* $l(c)$ *for an object* c *satisfying the following properties:*

For any arrows $f, g : c \to d$ *and arrows* $x, x' : d \to e$*, if* $x \circ f = x' \circ f$ *then there exists* $y : e \to e'$ *such that* $y \circ x \circ g = y \circ x' \circ g$.

If $\mathcal{C}$ *satisfies moreover the property that all its arrows are strict monomorphisms then the above condition on* c *is equivalent to the following condition: for any arrows* $f, g : c \to d$ *in* $\mathcal{C}$ *there exists exactly one automorphism* $\alpha : c \to c$ *such that there exists an arrow* $\gamma : d \to e$ *satisfying* $\gamma \circ g = \gamma \circ f \circ \alpha$.

(iii) Given a pair $(\mathcal{C}, u)$ *satisfying the hypotheses of Theorem 2.4 and such that all the arrows of* $\mathcal{C}$ *are strict monomorphisms, an object* $l(c)$ *of* $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ *is Galois if and only if it satisfies the following property: for any arrow* $\chi : c \to u$ *in* Ind-$\mathcal{C}$ *and any automorphism* $\alpha : u \to u$ *there exists exactly one automorphism* $\beta : c \to c$ *in* $\mathcal{C}$ *such that* $\alpha \circ \chi = \chi \circ \beta$*:*

$$\begin{array}{ccc} c & \xrightarrow{\chi} & u \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle \alpha} \\ c & \xrightarrow[\chi]{} & u \end{array}$$

Proof. (i) For a subgroup U of G, let $\simeq_U$ be the equivalence relation corresponding to it (given by $x \simeq_U y$ if and only if $x^{-1}y \in U$) and $[g]_U$ be the resulting equivalence classes. The automorphisms of G/U in $\mathbf{Cont}(G)$ can be identified with the equivalence classes $[h]_U$ of elements such that $hUh^{-1} = U$, via the assignment which sends an automorphism to the image under it of the equivalence class $[e]_U$ of the neutral element e. The condition that G/U is Galois can thus be reformulated as the requirement that the G-equivariant maps $\alpha_h : G/U \to G/U \times G/U$ corresponding to the elements h satisfying $hUh^{-1} = U$, which send an element $[g]_U$ to the pair $([g]_U, [gh]_U)$, are jointly surjective. This is equivalent to saying that for any elements $g_1, g_2 \in G$ there exists h satisfying $hUh^{-1} = U$ such that $[g_2]_U = [g_1 h]_U$. But this is clearly equivalent to saying that U is a normal subgroup of G.

(ii) Notice that the subobjects of an object $X \times X$ which are of the form $\langle 1_X, \alpha \rangle$, where α is an automorphism of X, can be identified with the subobjects $\langle r_1, r_2 \rangle$ of $X \times X$ such that both r_1 and r_2 are isomorphisms. By definition, an object of the form $l(c)$ is Galoisian in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ if and only if all the atomic subobject of $l(c) \times l(c)$ are of this form; see Lemma 6.6 for a desciption of them in terms of "components" of $c \times c$.

On the other hand, notice that an arrow $P \to Q$ in $[\mathcal{C}, \mathbf{Set}]$ is sent by the associated sheaf functor $a_{J_{\mathrm{at}}} : [\mathcal{C}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ to a monomorphism if and only if the canonical arrow $P \to P \times_Q P$ is a J_{at}-dense monomorphism.

Now, for a given pair of arrows $f, g : c \to d$ in $\mathcal{C}$, let us consider the image $A' \rightarrowtail yc \times yc$ in $[\mathcal{C}, \mathbf{Set}]$ of the arrow $\langle yf, yg \rangle : yd \to yc \times yc$. Since any arrow with codomain an object of the form $l(c)$ is an epimorphism, its image under the associated sheaf functor $a_{J_{\mathrm{at}}}$ is of the above form if and only if both the canonical arrows $\pi_1 : A' \to yc$ and $\pi_2 : A' \to yc$ are sent by $a_{J_{\mathrm{at}}}$ to a monomorphism. But the latter condition for A' follows from the first condition for the subobject A'' obtained from f and g considered in the exchanged order. The former condition is equivalent, by the above remark, to the condition that the diagonal subobject $k_1 : A' \rightarrowtail A' \times_{yc} A'$ of the kernel pair of π_1 is J_{at}-dense. Now, for any object e of $\mathcal{C}$, we have

$$A'(e) = \{(x \circ f, x \circ g) \mid \text{ for some } x : d \to e\} .$$

The property of k_1 to be J_{at}-dense can thus be reformulated as follows: for any arrows $x, x' : d \to e$, if $x \circ f = x' \circ f$, there exists $y : e \to e'$ such that $y \circ x \circ g = y \circ x' \circ g$.

If J_{at} is subcanonical, then for any object c of $\mathcal{C}$ the automorphisms of $l(c)$ in $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ are precisely the arrows of the form $l(\alpha)$ where α is an automorphism of c in $\mathcal{C}$. Therefore, in light of Lemma 6.6 (applied to the opposite category $\mathcal{C}^{\mathrm{op}}$), the object $l(c)$ is Galois if and only if every pair of arrows $f, g : c \to d$ is equivalent in $\mathcal{C}^{\mathrm{op}}$, in the sense of Lemma 6.4(i), to exactly one pair of the form $1_c, \alpha : c \to c$ where α is an automorphism of c in $\mathcal{C}$. But this amounts precisely to saying that there is exactly one automorphism $\alpha : c \to c$ in $\mathcal{C}$ such that there exists an arrow $\gamma : d \to e$ with $\gamma \circ g = \gamma \circ f \circ \alpha$.

(iii) Under the equivalence of Theorem 2.4, the object $l(c)$ corresponds to the set $\mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{C}}(c,u)$ with the canonical action of $\mathrm{Aut}(u)$, and, if the topology J_{at} is subcanonical, the "component" of $c \times c$ corresponding to an automorphism β of c as in the proof of (ii) is the $\mathrm{Aut}(u)$-equivariant map $\mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{C}}(c,u) \to \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{C}}(c,u) \times \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{C}}(c,u)$ sending any element $\chi \in \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{C}}(c,u)$ to the pair $(\chi, \chi \circ \beta)$. So the requirement for the action of $\mathrm{Aut}(u)$ on $\mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{C}}(c,u)$ to be Galois is equivalent to the condition that for any arrow $\chi : c \to u$ in Ind-$\mathcal{C}$ and any automorphism $\alpha : u \to u$ there exists exactly one automorphism $\beta : c \to c$ in $\mathcal{C}$ such that $\alpha \circ \chi = \chi \circ \beta$. □

The following 'bridge' result is an immediate corollary of Proposition 6.16 in light of the equivalence of Theorem 2.4 (noticing for point (i) that for any automorphism α of u, $\alpha I_{\chi} \alpha^{-1} = I_{\alpha \circ \chi}$).

Corollary 6.17. *Let $(\mathcal{C}, u)$ be a pair satisfying the hypotheses of Theorem 2.4. Then, for any object c of $\mathcal{C}$ and any arrow $\chi : c \to u$ in* Ind-$\mathcal{C}$*, the following conditions are equivalent:*

(i) The open subgroup I_{χ} of $\mathrm{Aut}(u)$ *is normal (equivalently, for any automorphism α of u, $I_{\chi} = I_{\alpha \circ \chi}$).*

(ii) The object c satisfies the properties in Proposition 6.16(ii).

If moreover all the arrows of $\mathcal{C}$ are strict monomorphisms, these conditions are also equivalent to any of the following ones:

(iii) For any arrows $f, g : c \to d$ in $\mathcal{C}$ there exists exactly one automorphism $\alpha : c \to c$ such that there exists an arrow $\gamma : d \to e$ satisfying $\gamma \circ g = \gamma \circ f \circ \alpha$.

(iv) For any arrow $\chi : c \to u$ in Ind-$\mathcal{C}$ *and any automorphism $\alpha : u \to u$ there exists exactly one automorphism $\beta : c \to c$ in $\mathcal{C}$ such that $\alpha \circ \chi = \chi \circ \beta$.*

□

6.5 Prodiscreteness

Recall that an object X of an atomic topos $\mathcal{E}$ is said to be *split* by an atom U if we have an isomorphism $X \times U \cong \gamma_{\mathcal{E}}^{*}(S) \times U$ over U, where S is a set and $\gamma_{\mathcal{E}}^{*}$ is the inverse image of the unique geometric morphism $\gamma_{\mathcal{E}} : \mathcal{E} \to \mathbf{Set}$. Notice that if U is a Galois object then X is split by U if and only if there exists an arrow $U \to X$. Indeed, one implication is clear, while in the converse direction, if $f : U \to X$ is an arrow then one can easily see, by considering the pullback of an atom of $X \times U$ along the epimorphism $f \times 1_U : U \times U \to X \times U$, that every atom of $X \times U$ is of the form $\langle f, \alpha \rangle : U \to X \times U$ where α is an automorphism of U.

For any Galois object U of an atomic topos $\mathcal{E}$, let $\mathrm{Split}(U)$ be the full subcategory of $\mathcal{E}$ on the atoms of $\mathcal{E}$ which are split by U.

Definition 6.18. An atomic topos is said to have *enough Galois objects* if the Galois objects form a separating set for it.

Remarks 6.19. (a) If $(\mathcal{E}, p)$ is a pointed atomic topos with enough Galois objects, the objects of the form (c, x) where c is a Galois object of $\mathcal{E}$ and $x \in p^*(c)$, are final in the category of elements of the functor p^*; indeed, if any atom e of $\mathcal{E}$ admits a morphism $f : c \to e$ from a Galois object c, which is necessarily an epimorphism, the map $p^*(f)$ is a surjection.

(b) If $\mathcal{E}$ has enough Galois objects then the category $\mathcal{E}_{\mathrm{at}}$ of atoms of $\mathcal{E}$ is the union of the categories $\mathrm{Split}(U)$ where U is a Galois object.

The next lemma will be instrumental in proving the main theorem of this section.

Recall that a functor $j : \mathcal{C} \to \mathcal{D}$ is said to be *final* if for every $d \in \mathcal{D}$ the comma category j/d (whose objects are the triplets (c, f) where a is an object of $\mathcal{C}$ and f is an arrow $j(a) \to d$ in $\mathcal{D}$ and whose arrows $(c, f) \to (c, f')$ are the arrows $k : c \to c'$ in $\mathcal{C}$ such that $f' \circ j(k) = f$) is non-empty and connected. If j is final then for any functor $F : \mathcal{D} \to \mathbf{Set}$, the limit of F is isomorphic to the limit of $F \circ j$.

Lemma 6.20. *Let $\mathcal{C}$ be a full subcategory of a filtered category $\mathcal{D}$. Then, if for any object d of $\mathcal{D}$ there exists an arrow $d \to c$ in $\mathcal{D}$ to an object c of $\mathcal{C}$, $\mathcal{C}$ is filtered and cofinal in $\mathcal{D}$.*

Proof To prove that $\mathcal{C}$ is filtered we observe that $\mathcal{C}$ is non-empty since $\mathcal{D}$ is non-empty and for any object d of $\mathcal{D}$ there exists an arrow $d \to c$ in $\mathcal{C}$ to an object c of $\mathcal{C}$. The fact that $\mathcal{C}$ satisfies the joint embedding property and the weak coequalizer property follows at once from the fact that $\mathcal{C}$ is full in $\mathcal{D}$ as these properties are satisfied by $\mathcal{D}$ by our hypotheses. It remains to prove that for any $d \in \mathcal{D}$ the comma category $d/\mathcal{C}$ is connected. We shall prove that it satisfies the joint embedding property, from which our thesis will clearly follow. Given any two objects $f : d \to c$ and $g : d \to c'$ in $d/\mathcal{C}$, since $\mathcal{D}$ satisfies the joint embedding property, there exist an object d' and two arrows $h : c \to d'$ and $k : c' \to d'$ in $\mathcal{D}$; now, the fact that $\mathcal{D}$ satisfies the weak coequalizer property and the fact that there exists an arrow $d' \to c''$ where c'' is an object of $\mathcal{C}$ imply that we can suppose without loss of generality that $h \circ f = k \circ g$ and that $d' \in \mathcal{C}$; the arrow $h \circ f = k \circ g : d \to d'$ thus defines an object of $(d/\mathcal{C})$ (as $\mathcal{C}$ is full in $\mathcal{D}$) and the arrows h and k define respectively arrows $f \to h \circ f$ and $g \to g \circ k$ in $d/\mathcal{C}$. □

Theorem 6.21. *Let $\mathcal{E}$ be an atomic two-valued topos with enough Galois objects. Then, for any point p of $\mathcal{E}$, the topological group $\mathrm{Aut}(p)$ is prodiscrete.*

- *For instance, $\mathrm{Aut}(p)$ is given by the projective limit of the following diagram $D : \mathcal{P}^{\mathrm{op}} \to \mathbf{Gp}$ of discrete groups: $\mathcal{P}$ is the poset of (isomorphism classes of) Galois objects of $\mathcal{E}$ with the order given by $U \leq V$ if and only if U is split by V, $D(U) = \mathrm{Aut}(p^*|_{\mathrm{Split}(U)})$ for any U and if $U \leq V$ then we have a canonical surjective homomorphism $D(V) = \mathrm{Aut}(p^*|_{\mathrm{Split}(V)}) \to \mathrm{Aut}(p^*|_{\mathrm{Split}(V)}) = D(U)$ given by restriction.*

- *Alternatively, $\mathrm{Aut}(p)$ can be represented as the projective limit of the diagram $D' : \mathcal{P}' \to \mathbf{Gp}$ defined as follows: $\mathcal{P}'$ is the full subcategory of the category $\int p^*|_{\mathcal{E}_{\mathrm{at}}}$ of elements of the functor $p^*|_{\mathcal{E}_{\mathrm{at}}} : \mathcal{E}_{\mathrm{at}} \to \mathbf{Set}$ on the objects of the form (e, x) where e is a Galois object and $x \in p^*(e)$ is an arbitrary element, and D assigns to any such object (e, x) the group of automorphisms $\mathrm{Aut}(e)$ of e in $\mathcal{E}$ and to any arrow $f : (e, x) \to (e', x')$ of $\int p^*$ the canonically induced group homomorphism $\mathrm{Aut}(e) \to \mathrm{Aut}(e')$.*

- *If $\mathcal{E}$ is the topos $\mathbf{Sh}(\mathcal{C}^{\mathrm{op}}, J_{\mathrm{at}})$ with a point u as in Theorem 2.4, every arrow of $\mathcal{C}$ is a strict monomorphism and every object of $\mathcal{C}$ admits an arrow to an object of $\mathcal{C}$ satisfying the condition in Proposition 6.16(iii) then the automorphism group $\mathrm{Aut}(u)$ is the projective limit of the diagram $D_{\mathcal{C}} : {\mathcal{P}_{\mathcal{C}}}^{\mathrm{op}} \to \mathbf{Gp}$ defined as follows: $\mathcal{P}_{\mathcal{C}}$ is the category having as objects the (isomorphism classes of) pairs (c, f), where c is such an object and $f : c \to u$ is an arrow in $\mathrm{Ind}\text{-}\mathcal{C}$, and whose arrows $(c, f) \to (d, g)$ are the arrows $l : c \to d$ in $\mathcal{C}$ such that $g \circ l = f$ in $\mathrm{Ind}\text{-}\mathcal{C}$, and the diagram $D_{\mathcal{C}}$ assigns to any object (c, f) the automorphism group $\mathrm{Aut}(c)$ of c in $\mathcal{C}$ and to any arrow $l : (c, f) \to (d, g)$ in $\mathcal{P}_{\mathcal{C}}$ the group homomorphism $\mathrm{Aut}(d) \to \mathrm{Aut}(c)$ sending any automorphism of d in $\mathcal{C}$ to its (unique) restriction $c \to c$ along l. Moreover, the colimit arrows (for $(c, f) \in \mathcal{P}_{\mathcal{C}}$) are precisely the maps $\mathrm{Aut}(u) \to \mathrm{Aut}(c)$ sending to any automorphism ξ of u the unique automorphism s of c such that $\xi \circ f = f \circ s$.*

Proof. Let us begin by establishing the first prolimit representation for $\mathrm{Aut}(p)$. If $U \leq V$ then, clearly, $\mathrm{Split}(U) \subseteq \mathrm{Split}(V)$ and hence we have a group homomorphism $\mathrm{Aut}(p^*|_{\mathrm{Split}(V)}) \to \mathrm{Aut}(p^*|_{\mathrm{Split}(V)})$ given by restriction. The diagram D is therefore well-defined. The fact that the category $\mathcal{P}$ is cofiltered follows at once from the fact that the category $\mathcal{E}_{\text{at}}$ satisfies JEP (the topos $\mathcal{E}$ being two-valued) and that, by our hypothesis, every atom of $\mathcal{E}$ admits an arrow from a Galois object to it. The fact that $\text{Aut}(p)$ is the limit of D is clear set-theoretically (in light of Remark 6.19(b)). Topologically, we have to verify that the topology on $\text{Aut}(p)$ is the coarsest making all the maps $r_U : \text{Aut}(p) \to \text{Aut}(p^*|_{\text{Split}(U)})$ continuous (equivalently, it is the smallest topology containing all the inverse images of the identity elements in a subgroup of the form $\text{Aut}(p^*|_{\text{Split}(U)})$). Recall that the topology on $\text{Aut}(p)$ has as basis of open neighbourhoods of the identity the subgroups of the form $U_{(e,x)} = \{\alpha : p^* \cong p^* \mid \alpha(e)(x)\}$ for $e \in \mathcal{E}$ and $x \in p^*(e)$. Notice that $r_U^{-1}(1_{\text{Split}(U)}) = \{\alpha : p^* \cong p^* \mid \alpha(U) = 1_{p^*(U)}\}$; indeed, if V is an object in $\text{Split}(U)$, that is admitting an arrow $f : U \to V$, then for any automorphism α of p, if $\alpha(U) = 1_{p^*(U)}$ then $\alpha(V) = 1_{p^*(V)}$ since $p^*(f)$ is an epimorphism. But $\{\alpha : p^* \cong p^* \mid \alpha(U) = 1_{p^*(U)}\} = U_{(U,x)}$ for any $x \in p^*(U)$, so our claim follows in light of the cofinality of Galois objects among the atoms of $\mathcal{E}$.

Let us now show how to derive from this prolimit representation of $\text{Aut}(p)$ the alternative one in terms of the diagram D'. First, let us notice that, since $(\mathcal{E}_{\text{at}}, J_{\text{at}})$ is a site of definition for $\mathcal{E}$, the functor $p^*|_{\mathcal{E}_{\text{at}}} : \mathcal{E}_{\text{at}} \to \textbf{Set}$ is flat and hence its category of elements is filtered. Therefore, the category $\mathcal{P}'$ is filtered by Lemma 6.20. Notice that, for any Galois object U of $\mathcal{E}$, the category $\text{Split}(U)$ satisfies the dual of the amalgamation property. Indeed, given two arrows $f : V' \to V$ and $g : V'' \to V$ in $\text{Split}(U)$, all the atoms of the fiber product $V' \times_V V''$ are split by U. Clearly, the category $\text{Split}(U)$ also satisfies the dual of the joint embedding property, and the object U is $\text{Split}(U)$-universal and $\text{Split}(U)$-ultrahomogeneous. So we have, by Theorem 2.4, an equivalence

$$\textbf{Sh}(\text{Split}(U), J_{\text{at}}) \simeq \textbf{Cont}(\text{Aut}(p^*|_{\text{Split}(U)})),$$

where the group $\text{Aut}(p^*|_{\text{Split}(U)})$ has the discrete topology. The geometric morphism

$$\begin{array}{ccc} \textbf{Sh}(\text{Split}(U), J_{\text{at}}) & \simeq & \textbf{Cont}(\text{Aut}(p^*|_{\text{Split}(U)})) \\ \uparrow & & \uparrow \\ \textbf{Sh}(\text{Split}(V), J_{\text{at}}) & \simeq & \textbf{Cont}(\text{Aut}(p^*|_{\text{Split}(V)})) \end{array} .$$

is actually induced by the group homomorphism

$$r_{U,V} : \text{Aut}(p^*|_{\text{Split}(V)}) \to \text{Aut}(p^*|_{\text{Split}(U)})$$

given by restriction along the embedding $\text{Split}(U) \subseteq \text{Split}(V)$. Let us now show that if U and V are related by a morphism $f : (V, x') \to (U, x)$ in the category of elements of the functor $p^*|_{\mathcal{E}_{\text{at}}}$ then we have group isomomorphisms $\xi_{x,U} : \text{Aut}(U) \to \text{Aut}(p^*|_{\text{Split}(U)})$ and $\xi_{x',V} : \text{Aut}(U) \to \text{Aut}(p^*|_{\text{Split}(U)})$ such that the group homomorphism $r_{U,V}$ corresponds to the group homomorphism $\rho^f_{U,V} : \text{Aut}(V) \to \text{Aut}(U)$ defined as follows: $\rho^f_{U,V}$ sends any automorphism $\beta : V \to V$ to the unique automorphism $\xi : U \to U$ such that $f \circ \xi = \beta \circ f$:

$$\begin{array}{ccc} V & \xrightarrow{\beta} & V \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[\xi]{} & U \end{array}$$

Recall from Proposition A.2.7 [14] that an object A of an atomic pointed two-valued topos $(\mathcal{E}, p)$ is Galois if and only if there exists an element $x \in p^*(A)$ such that the map $\phi_{A,x} : \text{Aut}(A) \to p^*(A)$ sending any automorphism χ of A to the element $p^*(\chi)(x)$ is a bijection. In fact, this property holds *for every* $x \in p^*(A)$ if A is Galois.

Let us first show that, under the identifications $\phi_{U,x} : \text{Aut}(U) \cong p^*(U)$ and $\phi_{V,x'} : \text{Aut}(V) \cong p^*(V)$ respectively provided by the elements x and x', our morphism $\rho^f_{U,V}$ corresponds to the arrow $p^*(f) : p^*(V) \to p^*(U)$. We have that for any element $\alpha \in \text{Aut}(V)$, $(p^*(f) \circ \phi_{V,x'})(\alpha) = p^*(f)(p^*(\alpha)(x')) = p^*(f \circ \alpha)(x') = p^*(\rho^f_{U,V}(\alpha) \circ f)(x') = p^*(\rho^f_{U,V}(\alpha))(x) = (\phi_{U,x} \circ \rho^f_{U,V})(\alpha)$, as required.

Next, we observe that for any Galois object U, the action of the group $\text{Aut}(p^*|_{\text{Split}(U)})$ on the set $p^*(U)$ is simply transitive. Indeed, we already know from the Galois representation $\mathbf{Sh}(\text{Split}(U), J_{\text{at}}) \simeq \mathbf{Cont}(\text{Aut}(p^*|_{\text{Split}(U)}))$ that the action is transitive and non-empty, so it remains to show that if an element $g \in \text{Aut}(p^*|_{\text{Split}(U)})$ acts identically on an element $x \in p^*(U)$ then it is the identical automorphism. Notice that, the object U being Galois, the fixators of all the points in $p^*(U)$ are the same (since they are all conjugate to each other and the fixator subgroup is normal), so if g fixes an element of $p^*(U)$ then it fixes the whole of $p^*(U)$, and hence also every $p^*(V)$ since if $V \in \text{Split}(U)$ then there is an epimorphism $U \to V$ in $\mathcal{E}$ whose image under p^* is a $\text{Aut}(p^*|_{\text{Split}(U)})$-equivariant surjective map. The action of $\text{Aut}(p^*|_{\text{Split}(U)})$ on $p^*(U)$ being simply transitive, we have a bijection $\psi_{U,x} : p^*(U) \to \text{Aut}(p^*|_{\text{Split}(U)})$ sending any element $z \in p^*(U)$ to the unique element $g \in \text{Aut}(p^*|_{\text{Split}(U)})$ such that $g(U)(x) = z$. Similarly, we have a bijection $\psi_{V,x'} : p^*(V) \to \text{Aut}(p^*|_{\text{Split}(V)})$ sending any element $y \in p^*(V)$ to the unique element $h \in \text{Aut}(p^*|_{\text{Split}(V)})$ such that $h(V)(x') = y$. Let us show that, under these bijections, the restriction homomorphism $r_{U,V} : \text{Aut}(p^*|_{\text{Split}(V)}) \to \text{Aut}(p^*|_{\text{Split}(U)})$ corresponds to the map $p^*(f)$. Given $y \in p^*(V)$, we want to prove that $(r_{U,V} \circ \psi_{V,x'})(y) = (\psi_{U,x} \circ p^*(f))(y)$. But

$(r_{U,V} \circ \psi_{V,x'})(y) = \psi_{V,x'}(y)(U)$, and $\psi_{V,x'}(y)(U)(x) = \psi_{V,x'}(y)(U)(p^*(f)(x')) = p^*(f)((\psi_{V,x'}(y)(V))(x')) = p^*(f)(y)$ (where the second equality follows from naturality), while on the other hand $(\psi_{U,x}(p^*(f)(y)))(U)(x) = p^*(f)(y)$, whence we have our desired equality by the simple transitivity of the action of $\mathrm{Aut}(p^*|_{\mathrm{Split}(U)})$ on $p^*(U)$.

Summarizing, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Aut}(V) & \xrightarrow{\phi_{V,x'}} & p^*(V) & \xrightarrow{\psi_{V,x'}} & \mathrm{Aut}(p^*|_{\mathrm{Split}(V)}) \\ \rho^f_{U,V} \downarrow & & p^*(f) \downarrow & & r_{U,V} \downarrow \\ \mathrm{Aut}(U) & \xrightarrow{\phi_{U,x}} & p^*(U) & \xrightarrow{\psi_{U,x}} & \mathrm{Aut}(p^*|_{\mathrm{Split}(U)}) \end{array}$$

The bijections

$$\xi_{x,U} : \mathrm{Aut}(U) \to \mathrm{Aut}(p^*|_{\mathrm{Split}(U)})$$

and

$$\xi_{x',V} : \mathrm{Aut}(U) \to \mathrm{Aut}(p^*|_{\mathrm{Split}(U)})$$

given by the horizontal composites in the above diagram are group isomorphisms. Indeed, $\xi_{x,U}$ is the homomorphism induced by the functor $p^*|_{\mathrm{Split}(U)} : \mathrm{Split}(U) \to \mathbf{Cont}(\mathrm{Aut}(p^*|_{\mathrm{Split}(U)}))$ modulo the group isomorphism

$$\mathrm{Aut}_{\mathbf{Cont}(\mathrm{Aut}(p^*|_{\mathrm{Split}(U)}))}(p^*(U)) \cong \mathrm{Aut}(p^*|_{\mathrm{Split}(U)})$$

given by the fact that the action of $\mathrm{Aut}(p^*|_{\mathrm{Split}(U)})$ on $p^*(U)$ is simply transitive, and similary for $\xi_{x',V}$.

Now, D' is the composite of D with the opposite of the canonical functor $\pi : \mathcal{P}' \to \mathcal{P}^{\mathrm{op}}$. To deduce the second prolimit representation from the first, it therefore suffices to prove that the functor π is final. But this follows immediately from the fact that $\mathcal{P}'$ satisfies the dual of the joint embedding property.

Finally, the third prolimit representation can be obtained from the second by observing that if every object of $\mathcal{C}$ admits an arrow to an object of $\mathcal{C}$ satisfying the condition in Proposition 6.16(iii) then the full subcategory of the category $\mathcal{P}'$ on the Galois objects of the form $l(c)$, which is precisely the opposite of the category $\mathcal{P}_{\mathcal{C}}$ defined in the statement of the theorem, is final in $\mathcal{P}'$ (by Remark 6.19(a)) and filtered (by Proposition 6.20) and hence the diagram $D_{\mathcal{C}}$, which is the composite of D with the inclusion functor ${\mathcal{P}_{\mathcal{C}}}^{\mathrm{op}} \hookrightarrow \mathcal{P}'$, has the same limit as D. □

Remarks 6.22. (a) The first prolimit representation of $\mathrm{Aut}(p)$ in Theorem 6.21 involves a simple indexing category but an abstract description of the diagram D, while the second involves a bigger indexing category but a more concrete description of the diagram D.

(b) Conversely to Theorem 6.21, if G is a prodiscrete group then the topos $\mathbf{Cont}(G)$ has enough Galois objects; indeed, if $G = \varprojlim_{i \in \mathcal{I}} G_i$ in the category of topological groups, where the G_i are discrete groups, the kernels H_i of the canonical projections $G \to G_i$ are open normal subgroups of G which generate its topology; so the cosets G/H_i (for $i \in \mathcal{I}$) form a separating set of Galois objects of the topos $\mathbf{Cont}(G)$ (cf. Proposition 6.16(i)).

6.6 Coherence

Recall from [5] that a (totally discontinuous) topological group G is said to be *coherent* if for any open subgroup H of G the number of subsets of the form HgH for $g \in G$ is finite; it is proved in [5] that a topological group G is coherent if and only if the topos $\mathbf{Cont}(G)$ is coherent. Recall that a topos is said to be coherent if it can be represented as the topos of sheaves on a site $(\mathcal{C}, J)$ where $\mathcal{C}$ is a small cartesian category and J is generated by finite covering families, equivalently if it is the classifying topos of a coherent theory.

A prodiscrete topological group is coherent if and only if it is profinite (cf. section D3.4 of [16]). For any topological group G, the coherent objects of the topos $\mathbf{Cont}(G)$ are exactly the compact objects, that is the actions with a finite number of orbits (cf. section D3.4 of [16]).

Proposition 6.23. *Given an atomic topos $\mathcal{E}$, the following conditions are equivalent:*

(i) $\mathcal{E}$ is coherent.

(ii) For every atoms $A_1, \ldots, A_n$ of $\mathcal{E}$, the object $A_1 \times \ldots \times A_n$ has a finite number of components (in the sense of Lemma 6.4).

(iii) There exist a separating family $\mathcal{S}$ of atoms of $\mathcal{E}$ such that for any $A_1, \ldots, A_n$ in $\mathcal{S}$, the object $A_1 \times \ldots \times A_n$ has a finite number of components.

If $\mathcal{E}$ is the topos of $\mathbf{Sh}(\mathcal{C}, J_{\mathrm{at}})$ on sheaves on an atomic site then $\mathcal{E}$ is coherent if and only if for any $c_1, \ldots, c_n$ in $\mathcal{C}$, $c_1 \times \ldots \times c_n$ has a finite number of components. In particular, if $\mathcal{C}$ has multi-products (in the sense of Lemma 6.11) then $\mathcal{E}$ is coherent if and only if they are finite.

Proof (i) $\Rightarrow$ (ii) Since $\mathcal{E}$ is coherent, every atom of it is covered by a coherent object. So we have epimorphisms $H_1 \to A_1$, ..., $H_n \to A_n$, which induce an epimorphism $H_1 \times \ldots \times H_n \to A_1 \times \ldots \times A_n$. But $H_1 \times \ldots \times H_n$ is a coherent object (since in any coherent topos the full subcategory on the coherent objects is closed under finite limits), whence it has a finite number of components (in the sense of Lemma 6.4). So $A_1 \times \ldots \times A_n$ has a finite number of components as well, as it is covered by it.

(ii) $\Rightarrow$ (iii) This follows at once from the fact that in every atomic topos the family of its atoms is separating for it.

(iii) $\Rightarrow$ (i) The full subcategory $\mathcal{C}$ of $\mathcal{E}$ consisting of the finite coproducts of atoms in $\mathcal{S}$ is closed under finite limits in $\mathcal{E}$. Indeed, it is closed under finite products since the latter commute with coproducts, and it is closed under equalizers since any subobject of a finite coproduct of atoms is a finite sub-coproduct of them. Since $\mathcal{C}$ is separating for $\mathcal{E}$ (as $\mathcal{S}$ is), $\mathcal{E}$ can be represented as the topos of sheaves $\mathbf{Sh}(\mathcal{C}, J_{\mathcal{E}}^{\text{can}}|_{\mathcal{C}})$ on $\mathcal{C}$ with respect to the Grothendieck topology on $\mathcal{C}$ induced by the canonical topology $J_{\mathcal{E}}^{\text{can}}$ on $\mathcal{E}$. But $\mathcal{C}$ is cartesian and $J_{\mathcal{E}}^{\text{can}}|_{\mathcal{C}}$ is of finite type, whence $\mathcal{E}$ is coherent.

The last statement of the proposition follows from the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) in light of the fact that the objects of the form $l(c)$ for $c \in \mathcal{C}$ form a separating set for $\mathcal{E} = \mathbf{Sh}(\mathcal{C}, J_{\text{at}})$. □

It is interesting to consider the invariant property of coherence in the context of a theory classified by an atomic two-valued topos admitting a representation as in Theorem 2.4.

Let $\mathbb{T}$ be an atomic and complete geometric theory with a special model M (see section 7 for the definition of these notions). Then $\mathbb{T}$ is syntactically equivalent to a coherent theory (over its signature) if and only if for any finite string of sorts $A_1, \ldots, A_n$ over the signature of $\mathbb{T}$, the action of $\text{Aut}(M)$ on $MA_1 \times \ldots \times MA_n$ has only a finite number of orbits (see section 7.3 and the criterion for a geometric theory to be coherent established in [8]).

From Theorem 7.1 it then follows that for any atomic and complete coherent theory $\mathbb{T}$ with a special model M, the topological group $\text{Aut}(M)$ is coherent; in particular, for any string of elements $a_1, \ldots, a_n$ of M there exists a finite number of automorphisms $f_1, \ldots, f_m$ of M such that every automorphism f of M can be written as gf_jh for some $j \in \{1, \ldots m\}$, where g and h are automorphisms which fix all the a_i. Anyway, this property holds more generally for any atomic and complete theory which is *Morita-equivalent* to a coherent (atomic and complete) theory with a special model M; for example, by Lemma 3.3 [6], $\mathbb{T}$ can be the theory of homogeneous $\mathbb{S}$-models where $\mathbb{S}$ is a theory of presheaf type with a universal and ultrahomogeneous model and such that its category of finitely presentable models satisfies AP and JEP and has all fc finite colimits (recall that a category is said to possess fc finite colimits if if every finite diagram D with values in it admits a finite set of cocones over it such that any other cocone over D factors through one in that family); notice that the latter condition is automatically satisfied if $\mathbb{S}$ is coherent, cf. [4].

7 Special models and their automorphism groups

Recall from [7] that *atomic and complete* theories are precisely the geometric theories classified by atomic and two-valued toposes. Given a geometric theory $\mathbb{T}$ over a signature Σ, a geometric formula-in-context $\phi(\vec{x})$ over Σ is said to be $\mathbb{T}$-*complete* if the sequent $(\phi \vdash_{\vec{x}} \bot)$ is not provable in $\mathbb{T}$, but for any geometric formula $\psi(\vec{x})$ over Σ in the same context, either $(\phi \vdash_{\vec{x}} \psi)$ is provable in $\mathbb{T}$ or $(\phi \wedge \psi \vdash_{\vec{x}} \bot)$ is provable in $\mathbb{T}$ (see section D3.4 of [16]). We denote by $\mathcal{C}_{\mathbb{T}}^{c}$ the full subcategory of the geometric syntactic category of $\mathbb{T}$ on the $\mathbb{T}$-complete formulae. The $\mathbb{T}$-complete formulae $\{\vec{x} \,.\, \phi\}$ are precisely the objects of $\mathcal{C}_{\mathbb{T}}$ which are sent by the canonical functor $y_{\mathbb{T}} : \mathcal{C}_{\mathbb{T}} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ to atoms of $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$.

In [9] a set-based model M of an atomic and complete theory is defined to be *special* if each $\mathbb{T}$-complete formula $\phi(\vec{x})$ is realized in M and for any $\vec{a}, \vec{b} \in [[\vec{x} \,.\, \phi]]_M$ there exists an automorphism f of M such that $f(\vec{a}) = \vec{b}$.

Theorem 7.1 (cf. Theorem 3.1 [9]). *Let $\mathbb{T}$ be an atomic and complete theory and M be a special model of $\mathbb{T}$. Then, if we denote by $\mathrm{Aut}(M)$ the group of ($\mathbb{T}$-model) automorphisms of M, we have that the sets of the form $\{f : M \cong M \;/\; f(\vec{a}) = \vec{a}\}$, where $\vec{a} \in [[\vec{x} \,.\, \phi]]_M$ for some $\mathbb{T}$-complete formula $\phi(\vec{x})$ form an algebraic base for $\mathrm{Aut}(M)$ and, if we endow $\mathrm{Aut}(M)$ with the resulting topology, we have an equivalence*

$$\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \simeq \mathbf{Cont}(\mathrm{Aut}(M))$$

between the classifying topos of $\mathbb{T}$ and the topos of continuous $\mathrm{Aut}(M)$-sets (where $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the geometric syntactic site of $\mathbb{T}$), which restricts to a functor

$$\mathcal{C}_{\mathbb{T}}^{c} \simeq \mathbf{Cont}_t(\mathrm{Aut}(M)) \ .$$

This functor sends any $\mathbb{T}$-complete formula $\phi(\vec{x})$ to the set $[[\vec{x} \,.\, \phi]]_M$ with the obvious $\mathrm{Aut}(M)$-action and any $\mathbb{T}$-provably functional formula θ from $\phi(\vec{x})$ to $\psi(\vec{y})$ to the $\mathrm{Aut}(M)$-equivariant map $[[\vec{x} \,.\, \phi]]_M \to [[\vec{y} \,.\, \psi]]_M$ whose graph is the interpretation $[[\vec{x}, \vec{y} \,.\, \theta]]_M$.

Remark 7.2. Given an atomic and complete theory $\mathbb{T}$, the Comparison Lemma yields an equivalence

$$\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{c}, J_{\mathrm{at}});$$

indeed, the fact that the images under the functor $y_{\mathbb{T}}$ of the objects of $\mathcal{C}_{\mathbb{T}}^{c}$ are atoms of $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ ensures that Grothendieck topology induced on $\mathcal{C}_{\mathbb{T}}^{c}$ by the canonical topology on $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ is the atomic topology on $\mathcal{C}_{\mathbb{T}}^{c}$. We thus have an equivalence between the $\mathbb{T}$-models and the J_{at}-continuous flat functors on $\mathcal{C}_{\mathbb{T}}^{c}$, which allows us to regard a set-based model of $\mathbb{T}$ as an object of the ind-completion of $\mathcal{C}_{\mathbb{T}}^{c}$.

For any theory $\mathbb{T}$ and model M satisfying the hypotheses of Theorem 7.1, the category $\mathcal{C}_{\mathbb{T}}^{c}$ and the model M, regarded as an object of Ind-$\mathcal{C}_{\mathbb{T}}^{c}$, satisfy the hypotheses of Theorem 2.4.

The following theorem provides an explicit description of 'the' universal model of a theory $\mathbb{T}$ as in Theorem 7.1 in its classifying topos $\mathbf{Cont}(\mathrm{Aut}(M))$.

Theorem 7.3. *Let $\mathbb{T}$ be an atomic and complete theory with a special model M. Then the model M, endowed with the (continuous) canonical action of* $\mathrm{Aut}(M)$*, is a universal model of $\mathbb{T}$ in the topos* $\mathbf{Cont}(\mathrm{Aut}(M))$*.*

Proof Let Σ be the signature of $\mathbb{T}$. Consider the Σ-structure $\widetilde{M}$ of $\mathbb{T}$ in the topos $\mathbf{Cont}(\mathrm{Aut}(M))$ given by the canonical (continuous) action of $\mathrm{Aut}(M)$ on M. Then the Σ-structure $\widetilde{M}$ is a model of $\mathbb{T}$ in $\mathbf{Cont}(\mathrm{Aut}(M))$; indeed, the forgetful functor $\mathbf{Cont}(\mathrm{Aut}(M)) \to \mathbf{Set}$ is faithful, and the image of $\widetilde{M}$ under this functor is isomorphic to M, which, by our hypothesis, is a model of $\mathbb{T}$ in $\mathbf{Set}$.

Let $\mathcal{C}_{\mathbb{T}}^{c}$ be the full subcategory of the geometric syntactic category $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ on the $\mathbb{T}$-complete formulae.

For any Grothendieck topos $\mathcal{E}$, since $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{c}, J_{\mathrm{at}})$ can be identified, via the equivalence $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{c}, J_{\mathrm{at}})$ of Remark 7.2, with the classifying topos for $\mathbb{T}$, we have a correspondence between the $\mathbb{T}$-models in $\mathcal{E}$ and the geometric morphisms $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{c}, J_{\mathrm{at}})$, which in turn can be identified with the flat J_{at}-continuous functors $\mathcal{C}_{\mathbb{T}}^{c} \to \mathcal{E}$; the flat functor corresponding to a model N of $\mathbb{T}$ in $\mathcal{E}$ is given by the functor assigning to any $\mathbb{T}$-complete formula $\phi(\vec{x})$ its interpretation $[[\vec{x}\ .\ \phi]]_N$ (and acting on the arrows accordingly). Now, the universal model of $\mathbb{T}$ in $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{c}, J_{\mathrm{at}})$ corresponds to the flat functor $l : \mathcal{C}_{\mathbb{T}}^{c} \to \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{c}, J_{\mathrm{at}})$ given by the Yoneda embedding, while the model $\widetilde{M}$ corresponds to the functor $\mathcal{C}_{\mathbb{T}}^{c} \to \mathbf{Cont}(\mathrm{Aut}(M))$ sending any $\mathbb{T}$-complete formula $\phi(\vec{x})$ to its interpretation $[[\vec{x}\ .\ \phi]]_M$ in M. Hence the two flat functors correspond to each other under the equivalence defined in the proof of Theorem 7.1; the Σ-structure $\widetilde{M}$ thus corresponds to the universal model of $\mathbb{T}$ in $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}^{c}, J_{\mathrm{at}})$ under this equivalence, and hence it is itself a universal model of $\mathbb{T}$ in the topos $\mathbf{Cont}(\mathrm{Aut}(M))$, as required. □

The fact that $\widetilde{M}$ is a universal model of $\mathbb{T}$ has several remarkable consequences, notably including the following ones.

Theorem 7.4 (Theorem 10.5.3 [10]). *Let $\mathbb{T}$ be an atomic and complete theory with a special model M. Then:*

(i) For any subset $S \subseteq MA_1 \times \ldots \times MA_n$ which is closed under the action of $\mathrm{Aut}(M)$, there exists a (unique up to $\mathbb{T}$-provable equivalence) geometric formula $\phi(\vec{x})$ over the signature of $\mathbb{T}$ (where $\vec{x} = (x_{A_1}, \ldots, x_{A_n})$) such that $S = [[\vec{x} \, . \, \phi]]_M$.

(ii) For any $\mathrm{Aut}(M)$-equivariant map $f : S \to T$ between invariant subsets S and T as in (i) there exists a (unique up to $\mathbb{T}$-provable equivalence) $\mathbb{T}$-provably functional geometric formula $\theta(\vec{x}, \vec{y})$ from $\phi(\vec{x})$ to $\psi(\vec{y})$, where $S = [[\vec{x} \, . \, \phi]]_M$ and $T = [[\vec{y} \, . \, \psi]]_M$, such that the graph of f coincides with $[[\vec{x}, \vec{y} \, . \, \theta]]_M$.

Proof This immediately follows from Theorem 7.1 and Theorem 2.2 [11]. □

Remark 7.5. It easily follows from the theorem that for any finite string $A_1, \ldots, A_n$ of sorts of the signature of the theory $\mathbb{T}$, the orbits of the action of $\mathrm{Aut}(M)$ on $MA_1 \times \ldots \times MA_n$ coincide precisely with the interpretations $[[\vec{x} \, . \, \phi]]_M$ of $\mathbb{T}$-complete formulae $\phi(\vec{x})$, where $\vec{x} = (x^{A_1}, \ldots, x^{A_n})$, that is they correspond exactly to the $\mathbb{T}$-provable equivalence classes of $\mathbb{T}$-complete formulae in the context $\vec{x}$.

It is interesting to investigate to which extent a structure is determined by its automorphism group. We can immediately deduce, from Remark 7.5, that if M and N are two special models of an atomic and complete theory then for any finite string $A_1, \ldots, A_n$ of sorts of the signature of the theory $\mathbb{T}$, the orbits of the action of $\mathrm{Aut}(M)$ on $MA_1 \times \ldots \times MA_n$ are in bijective correspondence with the orbits of the action of $\mathrm{Aut}(N)$ on $NA_1 \times \ldots \times NA_n$. The following result shows that, if the topological group $\mathrm{Aut}(M)$ is discrete then M is uniquely determined by it.

Proposition 7.6. *Let $\mathbb{T}$ be an atomic and complete theory with two special models M and N. If $\mathrm{Aut}(M)$ is a discrete group then $M \cong N$.*

Proof By Theorem 7.1, the classifying topos of $\mathbb{T}$ can be represented as $\mathbf{Cont}(\mathrm{Aut}(M))$. But if $\mathrm{Aut}(M)$ is discrete then the topos $\mathbf{Cont}(\mathrm{Aut}(M))$ has just one point, up to isomorphism, whence $M \cong N$, as required. □

It is interesting to apply Corollary 5.11 in the context of the investigation of the relationship between group homomorphisms between the automorphism groups of special models of two atomic and complete theories and interpretations which could induce them.

Recall from section 2.1.3 of [10] that there are various natural notions of interpretations between theories. For instance, it is natural to define an interpretation between geometric theories $\mathbb{T}$ and $\mathbb{T}'$ as a geometric functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}'}$ between the geometric syntactic categories of $\mathbb{T}$ and $\mathbb{T}'$, while for coherent theories there are two additional natural notions of interpretations directly inspired by classical Model Theory: we can define a *coherent interpretation* of a coherent theory $\mathbb{T}$ into a coherent theory $\mathbb{T}'$ as a coherent functor $\mathcal{C}_{\mathbb{T}}^{\text{coh}} \to \mathcal{C}_{\mathbb{T}'}^{\text{coh}}$, where $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$ and $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$ are respectively the coherent syntactic categories of $\mathbb{T}$ and of $\mathbb{T}'$, and a *generalized coherent interpretation* as a coherent functor $\mathcal{P}_{\mathbb{T}} \to \mathcal{P}_{\mathbb{T}'}$, where $\mathcal{P}_{\mathbb{T}}$ and $\mathcal{P}_{\mathbb{T}'}$ are respectively the pretopos completions of $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$ and of $\mathcal{C}_{\mathbb{T}}^{\text{coh}}$, that is the categories of model-theoretic coherent imaginaries of $\mathbb{T}$ and $\mathbb{T}'$. Notice incidentally that if the theories in question are Boolean then any first-order formula over their signature is provably equivalent in the theory to a coherent formula so that these categories coincide with the usual first-order syntactic categories or first-order categories of imaginaries arising in classical model theory.

Moreover, it is natural to define an *atomic interpretation* of an atomic theory $\mathbb{T}$ into an atomic theory $\mathbb{T}'$ to be a morphism of sites $(\mathcal{C}_{\mathbb{T}}^{c}, J_{\text{at}}) \to (\mathcal{C}_{\mathbb{T}'}^{c}, J_{\text{at}})$, where $\mathcal{C}_{\mathbb{T}}^{c}$ (resp. $\mathcal{C}_{\mathbb{T}'}^{c}$) is the full subcategory of the geometric syntactic category $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$ (resp. $\mathcal{C}_{\mathbb{T}'}$ of $\mathbb{T}'$) on the $\mathbb{T}$-complete (resp. the $\mathbb{T}'$-complete) formulae, and a *generalized atomic interpretation* of $\mathbb{T}$ into $\mathbb{T}'$ as a morphism of sites $(\mathcal{C}_{\mathbb{T}_{\text{at}}}^{c}, J_{\text{at}}) \to (\mathcal{C}_{\mathbb{T}'_{\text{at}}}^{c}, J_{\text{at}})$, where $(\mathcal{C}_{\mathbb{T}_{\text{at}}}^{c}$ (resp. $\mathcal{C}_{\mathbb{T}'_{\text{at}}}^{c}$) is the atomic completion of $\mathcal{C}_{\mathbb{T}}^{c}$ (resp. of $\mathcal{C}_{\mathbb{T}'}^{c}$), in the sense of section 6.2.

Proposition 7.7. *Let $\mathbb{T}$ and $\mathbb{T}'$ be atomic and complete theories with special models M and M' respectively. Then a continuous group homomorphism $h : \text{Aut}(M') \to \text{Aut}(M)$ (where the groups $\text{Aut}(M)$ and $\text{Aut}(M')$ are endowed with the topology of pointwise convergence) is induced by an atomic interpretation of $\mathbb{T}$ into $\mathbb{T}'$ sending M' to M (as in section 5.4) if and only if for any string $\vec{a}$ of elements of M there exists a string $\vec{b}$ of elements of M' such that $\{f : M' \cong M' \mid f(\vec{b}) = \vec{b}\} = \{f : M' \cong M' \mid h(f)(\vec{a}) = \vec{a}\}$. On the other hand, every continuous group homomorphism is induced by a generalized atomic interpretation of $\mathbb{T}$ into $\mathbb{T}'$.*

Proof Corollary 5.11 can be applied to the pairs of the form $(\mathcal{C}_{\mathbb{T}}^{c}, M)$, where $\mathbb{T}$ is a theory satisfying the hypotheses of Theorem 2.4 with respect to the model M (cf. Remark 7.2). From this the first assertion of the proposition follows at once. □

Proposition 7.8. *Let $\mathbb{T}$ and $\mathbb{T}'$ be atomic and complete theories with special models M and M' respectively. Then:*

(i) A continuous group homomorphism $h : \mathrm{Aut}(M') \to \mathrm{Aut}(M)$ (where the groups $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M')$ are endowed with the topology of pointwise convergence) is induced by a generalized atomic interpretation if and only if the image of h is dense in $\mathrm{Aut}(M)$.

(ii) If $\mathbb{T}$ and $\mathbb{T}'$ are coherent then a continuous group homomorphism $h : \mathrm{Aut}(M') \to \mathrm{Aut}(M)$ is induced by a generalized coherent interpretation of $\mathbb{T}$ into $\mathbb{T}'$ if and only if for every open subgroup H of $\mathrm{Aut}(M)$, the double quotient $\mathrm{Aut}(M')\backslash\mathrm{Aut}(M)/H$ is finite (notice that this condition is automatically satisfied if the image of h is of finite index in $\mathrm{Aut}(M)$).

Proof (i) The inverse image functor of the geometric morphism $\mathbf{Cont}(G) \to \mathbf{Cont}(G')$ induced by a continuous group homomorphism $h : G \to G'$ sends atoms to atoms if and only if the action of G on every quotient G'/H is transitive, that is if and only if the image of h is dense in G. Our thesis thus follows at once in light of the results of section 5.4.

(ii) The classifying pretopos of a coherent theory is equivalent to the full subcategory of its classifying topos on the coherent objects, and the coherent objects of a topos of actions of a topological group are precisely the actions with a finite number of orbits (cf. section 6.6). Our thesis then follows form the fact that the inverse image functor of the geometric morphism $\mathbf{Cont}(G) \to \mathbf{Cont}(G')$ induced by a continuous group homomorphism $h : G \to G'$ thus sends coherent objects to coherent objects if and only if for every open subgroup H of G', the double quotient $G\backslash G'/H$ is finite. $\square$

Remarks 7.9. (a) Given two atomic and complete geometric theories $\mathbb{T}$ and $\mathbb{T}'$ with special models M and N, we can define an *interpretation of M in N* as a generalized atomic interpretation $\mathcal{C}^{\mathrm{c}}_{\mathbb{T}_{\mathrm{at}}} \to \mathcal{C}^{\mathrm{c}}_{\mathbb{T}'_{\mathrm{at}}}$. This notion is stronger than the classical model-theoretic one (cf. [2]), since it implies that for any sort A of the signature of $\mathbb{T}$, MA can be represented in the form $[[\vec{y} \,.\, \psi]]_N/R$, where $\{\vec{y} \,.\, \psi\}$ is a $\mathbb{T}$-complete formula and R is a geometrically definable equivalence relation on $[[\vec{y} \,.\, \psi]]_N$. We shall say that M and N are bi-interpretable if there exist interpretations of M in N and of N in M which are mutually inverse to one another (up to isomorphism). Proposition 7.8(i) thus implies that $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$ are isomorphic as topological groups if and only if M and N are bi-interpretable (in our sense), strenghtening the classical result by Coquand-Ahlbrandt-Ziegler (Corollary 1.4(ii) [2]).

(b) Proposition 7.8(ii) generalizes Theorem 1.2 [2] (whose condition that the image of h should be of finite index is stronger than ours).

In view of Corollary 5.11, it is natural to wonder whether we can explicitly characterize, given any two atomic and complete theories $\mathbb{T}$ and $\mathbb{T}'$ with special models respectively M and N, the continuous homomorphisms $\mathrm{Aut}(M') \to \mathrm{Aut}(M)$ which are induced by an interpretation of $\mathbb{T}$ into $\mathbb{T}'$.

Thanks to Remark 7.5, we can characterize the continuous homomorphisms $h : \mathrm{Aut}(M') \to \mathrm{Aut}(M)$ induced by an interpretation $\mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}'}$ (via the equivalences

$$\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \simeq \mathbf{Cont}(\mathrm{Aut}(M))$$

and

$$\mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \simeq \mathbf{Cont}(\mathrm{Aut}(M'))$$

of Theorem 7.1): they are exactly the homomorphisms h such that for any sort A of the signature of $\mathbb{T}$ there exists a finite set of sorts $B_1, \ldots B_n$ of the signature of $\mathbb{T}'$ such that the action of $\mathrm{Aut}(M')$ via h on MA is isomorphic (in $\mathbf{Cont}(\mathrm{Aut}(M'))$) to the action of $\mathrm{Aut}(M')$ on a $\mathrm{Aut}(M')$-invariant subset of $M'B_1 \times \ldots \times M'B_n$. Indeed, giving an interpretation $I : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}'}$ amounts precisely to giving a $\mathbb{T}$-model inside $\mathcal{C}_{\mathbb{T}'}$, and we have a commutative diagram of the form

$$\begin{array}{ccc}
\mathcal{C}_{\mathbb{T}} & \xrightarrow{\quad I \quad} & \mathcal{C}_{\mathbb{T}'} \\
\big\downarrow{\scriptstyle y_{\mathcal{C}_{\mathbb{T}}}} & & \big\downarrow{\scriptstyle y_{\mathbb{T}'}} \\
\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) & \xrightarrow{\mathbf{Sh}(I)^*} & \mathbf{Sh}(\mathcal{C}_{\mathbb{T}'}, J_{\mathbb{T}'}) \\
\big\downarrow{\scriptstyle \tau_M} & & \big\downarrow{\scriptstyle \tau_{M'}} \\
\mathbf{Cont}(\mathrm{Aut}(M)) & \xrightarrow{\mathbf{Cont}(h)^*} & \mathbf{Cont}(\mathrm{Aut}(M'))
\end{array}$$

where $y_{\mathbb{T}}$ and $y_{\mathbb{T}'}$ are the Yoneda embeddings and τ_M and τ'_M are the equivalences of Theorem 7.1, if and only if for any sort A of the signature of $\mathbb{T}$ the object $\mathbf{Cont}(h)^*(\widetilde{M})A$ is the image under $\tau_{M'} \circ y_{\mathbb{T}'}$ of an object of $\mathcal{C}_{\mathbb{T}'}$, since by Theorem 7.4 every $\mathrm{Aut}(M')$-equivariant arrow between definable subsets, and every $\mathrm{Aut}(M')$-equivariant subset of a definable subset, is definable.

The analogue of this characterization for coherent theories is obtained by replacing the condition '$\mathrm{Aut}(M')$-equivariant' with '$\mathrm{Aut}(M')$-equivariant with a finite number of orbits'.

The following theorem shows a connection between endomorphisms of the classifying topos of a theory satisfying the hypotheses of Theorem 7.1 and homorphisms between special models for it. Before proving it, we need a lemma.

Lemma 7.10. *Let $\mathbb{T}$ be a geometric theory and G a topological group. Then a $\mathbb{T}$-model in the topos* $\mathbf{Cont}(G)$ *can be identified with a pair (M, f) consisting of a set-based $\mathbb{T}$-model M and a continuous group homomorphism $f : G \to \mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ is endowed with the topology of pointwise convergence.*

Proof Giving a $\mathbb{T}$-model in $\mathbf{Cont}(G)$ clearly amounts to giving a set-based model M of $\mathbb{T}$ together with, for each sort A of the signature of $\mathbb{T}$, a continuous action $\alpha_A : G \times MA \to MA$ of G on MA such that for each relation symbol $R \rightarrowtail A_1 \cdots A_n$ of the signature of $\mathbb{T}$ the subset $MR \rightarrowtail MA_1 \times \ldots \times MA_n$ is G-closed and for each function symbol $f : A_1 \cdots A_n \to A$ over the signature of $\mathbb{T}$ the map $Mf : MA_1 \times \ldots \times MA_n \to MA$ is G-equivariant. Actions $\alpha_A : G \times MA \to MA$ define, by transposition, group homomorphisms from G to the set of bijections of MA which give all together a group homomorphism $G \to \mathrm{Aut}(M)$; by definition of topology of pointwise convergence on $\mathrm{Aut}(M)$, this group homomorphism if continuous since all the actions α_A are. Conversely, any continuous group homorphism $G \to \mathrm{Aut}(M)$ induces continuous actions α_A of G on the MA such that for each relation symbol $R \rightarrowtail A_1 \cdots A_n$ of the signature of $\mathbb{T}$ the subset $MR \rightarrowtail MA_1 \times \ldots \times MA_n$ is G-closed and for each function symbol $f : A_1 \cdots A_n \to A$ over the signature of $\mathbb{T}$ the map $Mf : MA_1 \times \ldots \times MA_n \to MA$ is G-equivariant. $\square$

Theorem 7.11. *Let $\mathbb{T}'$ be a geometric theory and G a topological group. Then there is a bijective correspondence between the geometric morphisms* $\mathbf{Cont}(G) \to \mathbf{Set}[\mathbb{T}']$ *(where* $\mathbf{Set}[\mathbb{T}']$ *is the classifying topos of $\mathbb{T}'$) and the pairs (N, h), where N is a set-based model of $\mathbb{T}'$ and $h : G \to \mathrm{Aut}(N)$ is a continuous group homomorphism. In particular, if $\mathbb{T}$ is an atomic and complete theory with a special model M then there is a bijective correspondence between the (isomorphism classes of) geometric endomorphisms* $f : \mathbf{Set}[\mathbb{T}] \to \mathbf{Set}[\mathbb{T}]$ *of the classifying topos of $\mathbb{T}$ and the pairs (N, h), where N is a set-based model of $\mathbb{T}$ and $h : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ is a continuous group homomorphism. Under this bijection, the automorphisms of* $\mathbf{Set}[\mathbb{T}]$ *correspond to the pairs (N, h) where N is a special model of $\mathbb{T}$ and $h : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ is a continuous group homomorphism such that the geometric morphism* $\mathbf{Cont}(h) : \mathbf{Cont}(\mathrm{Aut}(M)) \to \mathbf{Cont}(\mathrm{Aut}(N))$ *is an equivalence.*

Proof The first statement of the theorem follows from Lemma 7.10 by the universal property of classifying toposes, while the second follows from the first by taking $\mathbb{T}' = \mathbb{T}$ and $G = \mathrm{Aut}(M)$ (recall that $\mathbf{Set}[\mathbb{T}] \simeq \mathbf{Cont}(\mathrm{Aut}(M))$ by Theorem 7.1). It thus remains to show that the bijection thus defined between the (isomorphism classes of) geometric endomorphisms $f : \mathbf{Set}[\mathbb{T}] \to$

Set[$\mathbb{T}$] of the classifying topos of $\mathbb{T}$ and the pairs (N, h), where N is a set-based model of $\mathbb{T}$ and $h : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$ is a continuous group homomorphism restricts to a bijection between the (isomorphism classes of) equivalences f of **Set**[$\mathbb{T}$] and the pairs (N, h) such that N is special and **Cont**(h) is an equivalence.

If N is special then $\widetilde{N}$ is a universal model of $\mathbb{T}$ in the topos **Cont**(Aut(N)) (by Theorem 7.3) and hence if **Cont**(h) is an equivalence **Cont**(h)$^*(\widetilde{N})$ is a universal model of $\mathbb{T}$ in **Cont**(Aut(M)); so the corresponding geometric morphism f is an equivalence (by the universal property of classifying toposes). Conversely, suppose that f : **Cont**(Aut(M)) $\to$ **Cont**(Aut(M)) is an equivalence. The model N appearing in the pair (N, h) corresponding to it is the set-based model underlying the model $f^*(\widetilde{M})$. Since f is an equivalence, its inverse image f^* sends atoms to atoms and hence for any $\mathbb{T}$-complete formula $\phi(\vec{x})$, the set $[[\vec{x} \, . \, \phi]]_N$, equipped with the action $\alpha : \mathrm{Aut}(M) \times [[\vec{x} \, . \, \phi]]_N \to [[\vec{x} \, . \, \phi]]_N$, is an atom of the topos **Cont**(Aut(M)) (i.e. it is a non-empty transitive action) as it is the image of $[[\vec{x} \, . \, \phi]]_{\widetilde{M}}$ under f^*. But this is the action induced by the canonical one of $\mathrm{Aut}(N)$ on N via the homomorphism $h : \mathrm{Aut}(M) \to \mathrm{Aut}(N)$, that is the canonical action of $\mathrm{Aut}(N)$ on $[[\vec{x} \, . \, \phi]]_N$. So N is special. The fact that **Cont**(h) : **Cont**(Aut(M)) $\to$ **Cont**(Aut(N)) is an equivalence follows from the fact that, N being special, $\widetilde{N}$ is a universal model of $\mathbb{T}$ in **Cont**(Aut(N)) and **Cont**(h)$^*(\widetilde{N})$ is a universal model of $\mathbb{T}$ in **Cont**(Aut(M)), it being the image of the universal model $\widetilde{M}$ in **Cont**(Aut(M)) under the equivalence f. □

Theorem 7.11 provides us with a geometric perspective on the relationships between the automorphism groups of models of an atomic and complete theory, which makes it possible to investigate them by analyzing the endomorphisms of its classifying topos, a task of entirely categorical/geometric nature.

Acknowledgements: We gratefully acknowledge IHÉS and MIUR (the latter in the form of a research grant related to the "Rita Levi Montalcini" position of the first author) for supporting visits respectively of the first author to IHÉS and of the second author to the University of Insubria during which parts of this work were written.

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