Solutions of loop equations are random matrices

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Abstract:
For a given polynomial $V(x) \in \mathbb{C}[x]$, a random matrix eigenvalues measure is a measure $\prod_{1\le i<j\le N}(x_i - x_j)^2 \prod_{i=1}^N e^{-V(x_i)} dx_i$ on γ^N. Hermitian matrices have real eigenvalues $\gamma = \mathbb{R}$, which generalize to γ a complex Jordan arc, or actually a linear combination of homotopy classes of Jordan arcs, chosen such that integrals are absolutely convergent. Polynomial moments of such measure satisfy a set of linear equations called "loop equations". We prove that every solution of loop equations are necessarily polynomial moments of some random matrix measure for some choice of arcs. There is an isomorphism between the homology space of integrable arcs and the set of solutions of loop equations. We also generalize this to a 2-matrix model and to the chain of matrices, and to cases where V is not a polynomial but $V'(x) \in \mathbb{C}(x)$.

1 Introduction

Let us recall a few basic facts, from Mehta's book [9] for instance.

1.1 Hermitian random matrices

Let $V \in \mathbb{R}[x]$ a real polynomial bounded from below on $\mathbb{R}$ (i.e. of even degree with positive leading coefficient). Let $\mathcal{H}_N$ the set of $N \times N$ Hermitian matrices, and recall that every Hermitian matrix $M \in \mathcal{H}_N$ can be diagonalized by a unitary conjugation

$$M = UXU^\dagger \tag{1-1}$$

where $U \in U(N)$ and

$$X = \mathrm{diag}(x_1, \dots, x_N) \tag{1-2}$$

is the set of its eigenvalues. To make the decomposition unique, notice that U can be right multiplied by any diagonal unitary matrix, and thus we shall consider U in the quotient group $U(N)/U(1)^N$, and eigenvalues can be permuted by multiplying U with a permutation matrix, eventually we roughly have

$$\mathcal{H}_N \sim (U(N)/U(1)^N \ \times \mathbb{R}^N)/\mathfrak{S}_N. \tag{1-3}$$

(remark: we abusively oversimplified the discussion, in fact when some eigenvalues are not distinct, the non-uniqueness group = the stabilizer is larger, and we should quotient $U(N)$ by the stabilizer of X rather than $U(1)^N \times \mathfrak{S}_N$. This can be written as an orbifold, however, degenerate spectra will be of measure 0 in what follows and can be ignored).

It is well known [9] that the Lebesgue measure $\mathcal{D}M$ on $\mathcal{H}_N$ can be rewritten as a measure on $U(N)/U(1)^N \ \times \mathbb{R}^N$ as

$$\mathcal{D}M = \prod_{i,j} dM_{i,j} = \Delta(X)^2 \ \mathcal{D}U \ \mathcal{D}X \tag{1-4}$$

where $\mathcal{D}X = \prod_{i=1}^N dX_i$ is the Lebesgue measure on $\mathbb{R}^N$ and $\mathcal{D}U$ is the Haar measure on the Lie group $U(N)/U(1)^N$, and

$$\Delta(X) = \prod_{i<j}(x_i - x_j) \tag{1-5}$$

is called the Vandermonde determinant.

A Boltzmann weight probability measure on $\mathcal{H}_N$ of the form

$$\frac{1}{\hat{Z}} e^{-\operatorname{Tr} V(M)} \mathcal{D}M \tag{1-6}$$

yields a marginal probability measure for eigenvalues

$$\frac{1}{Z} \ \Delta(X)^2 \mathrm{e}^{-\operatorname{Tr} V(X)} \mathcal{D}X \tag{1-7}$$

where Z and $\hat{Z}$ are normalization factors, however, we shall from now on not normalize the measures.

Loop equations are a set of relationships (proved by integration by parts) among expectation values of symmetric polynomials of the eigenvalues, for example:

$$\begin{aligned} & \mathbb{E}(\operatorname{Tr} V'(X)) = 0 \\ \forall\ k \geq 1 \qquad & \mathbb{E}(\operatorname{Tr} X^k V'(X)) = \sum_{j=0}^{k-1} \mathbb{E}(\operatorname{Tr} X^j \operatorname{Tr} X^{k-j-1}), \end{aligned} \tag{1-8}$$

and many other such relations between expectation values of product of traces of powers, that we shall detail further below.

1.2 Generalization to normal matrices

Let $\gamma : \mathbb{R} \to \mathbb{C}$ a piecewise C^1 Jordan arc in the complex plane. We generalize Hermitian matrices to normal matrices (= diagonalizable by a unitary conjugation) with eigenvalues on γ:

$$\mathcal{H}_N(\gamma) = \{M = UXU^\dagger \mid U \in U(N),\ X = \mathrm{diag}(x_1, \dots, x_N),\ x_i \in \gamma\}. \tag{1-9}$$

We equip it with measure:

$$\mathcal{D}M = \Delta(X)^2 \mathcal{D}U \mathcal{D}X \tag{1-10}$$

where $\mathcal{D}U$ is the Haar measure on $U(N)/U(1)^N$ and $\mathcal{D}X = \prod_{i=1}^N dx_i$ where dx_i is the curvilinear measure on γ defined as

$$x_i = \gamma(s_i) \quad \to \quad dx_i = \gamma'(s_i) ds_i \qquad , \ s_i \in \mathbb{R} \tag{1-11}$$

which is in fact independent of the chosen parametrization of the Jordan arc.

For examples:

- $\gamma = \mathbb{R}$ gives $\mathcal{H}_N(\mathbb{R}) = \mathcal{H}_N$ and $\mathcal{D}M$ is the usual Lebesgue measure on $\mathcal{H}_N$.
- $\gamma = S^1$ the unit circle, gives $\mathcal{H}_N(S^1) = U(N)$ and $\mathcal{D}M$ is related to the Haar measure on $U(N)$ as

$$\mathcal{D}M = \mathrm{i}^{\,N^2} \ \det M^N \ \mathcal{D}_{\mathrm{Haar}(U(N))} M. \tag{1-12}$$

  (indeed $\mathrm{i}^{\,-N^2} \mathcal{D}M \det M^{-N}$ is a real measure, right invariant). This formalism of normal matrices unifies Hermitian ensembles with circular ensembles (as well as many others). See [7] for examples and applications.

A Boltzmann weight measure (possibly complex) $e^{-\operatorname{Tr} V(M)} \mathcal{D}M$ on $\mathcal{H}_N(\gamma)$ yields a marginal measure for the eigenvalues on γ^N:

$$\Delta(X)^2 e^{-\operatorname{Tr} V(X)} \mathcal{D}X. \tag{1-13}$$

Integrals of symmetric polynomials of the eigenvalues will satisfy the same loop equations (1-8) as in the Hermitian case.

Notice that the measure (1-13) can be integrated on γ^N only for some choices of γ, namely we need the integral be absolutely convergent and thus if γ goes to ∞, then $|e^{-V(x)}|$ must tend to zero. In order to define integrals of all symmetric polynomials of eigenvalues we shall require that $|x^k e^{-V(x)}| \to 0$ at ∞ on γ, for all $k \in \mathbb{Z}_+$.

In order to have the same loop equations as for the Hermitian case, we need to do integration by parts, and we need that there is no boundary term, therefore we shall require that γ has no boundary except at ∞ (the case where γ has finite boundaries at which $e^{-V(x)} \neq 0$ is called "hard edges", loop equations for hard edges can be found in [5]).

Let us now study the set of acceptable Jordan arcs for a given polynomial potential $V(x)$. We shall study in section 5 the generalization to $V'(x) \in \mathbb{C}(x)$, i.e. rational case.

2 Loop equations and measures

2.1 Arcs and homology

Let $V \in \mathbb{C}[x]$ a polynomial of degree ≥ 2 written

$$V(x) = \sum_{k=1}^{d+1} \frac{t_k}{k} x^k \qquad , \; t_{d+1} \neq 0. \tag{2-1}$$

Consider the set of Jordan arcs $\gamma : \mathbb{R} \to \mathbb{C}$, piecewise C^1, such that

$$\begin{aligned} \gamma(-\infty) &= \infty \\ \gamma(+\infty) &= \infty \\ \forall k \in \mathbb{Z}_+, \quad |x^k e^{-V(x)}| & \text{ bounded on } \gamma \end{aligned} \tag{2-2}$$

Consider the group of homotopy classes of those Jordan arcs, with addition by concatenation, and the homology space of K-linear combinations with K a ring or field, typically $K = \mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$.

We define

Definition 2.1 *the homology space of admissible integration classes for the measure*

$$\begin{aligned} H_1(e^{-V(x)}dx, K) \quad = \quad & \Big\{ K - \textit{linear combinations of Jordan arcs } \gamma, \\ & \textit{going from } \infty \textit{ to } \infty, \\ & \textit{and } \; \forall \; k \geq 0, \; |x^k e^{-V(x)}| \textit{ bounded on } \gamma \Big\}. \end{aligned} \tag{2-3}$$

It is a vector space if K is a field (or a module if K is a ring, let us focus on fields from now on).

The notion of integral of a holomorphic 1-form ω is well defined on a homology class $\gamma \in H_1(e^{-V(x)}dx, K)$. Indeed since the form is holomorphic, the integral is invariant under homotopic deformations, and for a linear combination of homotopy classes $\gamma = \sum_i c_i \gamma_i$, we define by linearity

$$\int_\gamma \omega \stackrel{\text{def}}{=} \sum_i c_i \int_{\gamma_i} \omega. \tag{2-4}$$

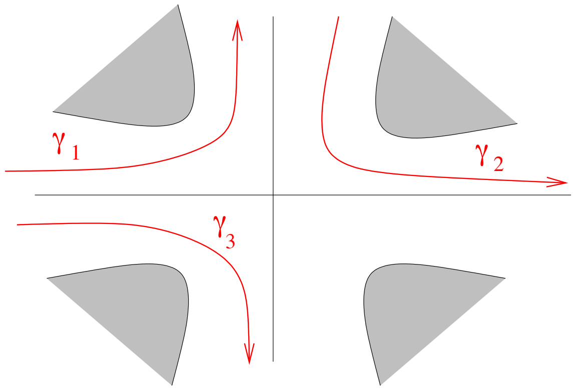

Figure 1: Example with $\deg V = 4$. There are 4 sectors $\Re V > 0$ and 4 sectors $\Re V < 0$ near ∞ (shaded). Paths in H_1 can go from positive sector to positive sector, they must not go to ∞ in shaded sectors. 3 consecutive make a basis of H_1. For example $\mathbb{R} = \gamma_1 + \gamma_2$ and $\mathrm{i}\,\mathbb{R} = \gamma_1 - \gamma_3$ are both in H_1.

It is clear that if $\gamma \in H_1(e^{-V(x)}dx, K)$, the integral

$$\int_\gamma e^{-V(x)}dx \tag{2-5}$$

is absolutely convergent.

Proposition 2.1 *If* $\deg V = d + 1 \geq 2$*,* $H_1(e^{-V(x)}dx, K)$ *has dimension*

$$\dim H_1(e^{-V(x)}dx, K) = d. \tag{2-6}$$

proof: See [2, 4]. Any Jordan arc going from ∞ to ∞ such that $|x^k e^{-V(x)}|$ is bounded must start and end in sectors near ∞, in which $\Re V(x) \to +\infty$. There are $d+1$ angular sectors near ∞ in which $\Re V(x) > 0$ separated by $d+1$ sectors where $\Im V(x) < 0$. A generating family of arcs is constructed by arcs going from a sector to the next, there are $d+1$ such, and only d are independent. This is illustrated on fig.1. □

2.2 Eigenvalues measure

We now consider the N dimensional generalization, the homology space of admissible N dimensional integration domains $\subset \mathbb{C}^N$, on which an N–dimensional spectral–matrix–model–measure is absolutely integrable, it is the symmetric N tensor product:

$$H_N\left(\Delta(X)^2 \prod_{i=1}^N e^{-V(x_i)}dx_i, K\right) = \mathrm{Sym}\left(H_1\left(e^{-V(x)}dx, K\right)^{\otimes N}\right). \tag{2-7}$$

Let $\gamma_1,\dots,\gamma_d$ an arbitrary basis of $H_1(e^{-V(x)}dx,K)$. For every d-uple $n=(n_1,n_2,\dots,n_d)$ of non-negative integers $n_i\in\mathbb{Z}_+$ such that $\sum_{i=1}^d n_i=N$, we define

$$\gamma^n=\mathrm{sym}(\gamma_1^{n_1}\times\gamma_2^{n_2}\times\cdots\times\gamma_d^{n_d})=\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_N}\sigma_*\gamma_1^{n_1}\times\gamma_2^{n_2}\times\cdots\times\gamma_d^{n_d} \tag{2-8}$$

We may thus write

$$H_N\left(\Delta(X)^2\prod_{i=1}^N e^{-V(x_i)}dx_i,K\right)=\{\sum_{n_1+\cdots+n_d=N}c_{n_1,\dots,n_d}\mathrm{sym}(\gamma_1^{n_1}\times\gamma_2^{n_2}\times\cdots\times\gamma_d^{n_d})\} \tag{2-9}$$

For short we shall call it H_N.

It is clear that if $\Gamma\in H_N$, the following integral

$$Z(\Gamma)=\int_\Gamma\Delta(X)^2\prod_{i=1}^N e^{-V(x_i)}dx_i \tag{2-10}$$

is absolutely convergent, as well as all its polynomial moments.

Proposition 2.2

$$\dim H_N=\binom{N+d-1}{N}=\frac{(N+d-1)!}{N!(d-1)!}. \tag{2-11}$$

proof: This dimension is the number of d-uples $n=(n_1,\dots,n_d)$ such that $n_i\geq 0$ and $\sum_{i=1}^d n_i=N$. $\square$

2.3 Polynomial moments

The integral $Z(\Gamma)$ is called a matrix integral, it is in fact the integral of the marginal eigenvalue distribution induced by the measure $e^{-\mathrm{Tr}\,V(M)}\mathcal{D}M$ on $\mathcal{H}_N(\Gamma)$.

Let $\mathcal{P}_N=\mathbb{C}[x_1,\dots,x_N]^{\mathrm{Sym}}$ the vector space of all symmetric polynomials of N variables.

Definition 2.2 *For $\Gamma\in H_N$, the measure $\Delta(X)^2\prod_i e^{-V(x_i)}dx_i$ defines the following map:*

$$\begin{aligned}\mathbb{E}_\Gamma:\ \mathcal{P}_N &\to\mathbb{C}\\ p &\mapsto\int_\Gamma p(x_1,\dots,x_N)\ \Delta(X)^2\prod_{i=1}^N e^{-V(x_i)}dx_i\end{aligned} \tag{2-12}$$

which is a linear form on $\mathcal{P}_N$:

$$\mathbb{E}_\Gamma\in\mathcal{P}_N^*. \tag{2-13}$$

Since H_N is a vector space, and the map $\mathbb{E} : \Gamma \mapsto \mathbb{E}_\Gamma$ is clearly linear, we have a homeomorphism of vector spaces. A key result is that this homeomorphism is injective:

Theorem 2.1 (Injectivity) $\mathbb{E}$ *is an injective homeomorphism of vector spaces*

$$\begin{aligned} \mathbb{E} : \quad H_N & \to \mathcal{P}_N^* \\ \Gamma & \mapsto \mathbb{E}_\Gamma \end{aligned} \tag{2-14}$$

proof: We sketch the proof here, the full proof is detailed in appendix B. We need to prove that $\operatorname{Ker}\mathbb{E} = 0$. Let us assume that $0 \neq \Gamma \in \operatorname{Ker}\mathbb{E}$. Writing

$$\Gamma = \sum_{n=(n_1,\dots,n_d),\ n_1+\dots+n_d=N} c_n \gamma^n, \tag{2-15}$$

if $\Gamma \neq 0$, there must exist some n such that $c_n \neq 0$. The idea is to construct a family of symmetric polynomials $p_{r,m} \in \mathcal{P}_N$ for any d-uple $m = (m_1, \dots, m_d)$ with $\sum_{i=1}^d m_i = N$, such that we have

$$\lim_{r\to\infty} \int_{\gamma^n} p_{r,m}(x_1,\dots,x_N)\ \Delta(X)^2 \prod_{i=1}^N e^{-V(x_i)} dx_i = \delta_{n,m}. \tag{2-16}$$

This will imply that

$$\lim_{r\to\infty} \mathbb{E}_\Gamma(p_{r,n}) = c_n \neq 0, \tag{2-17}$$

which is a contradiction since we assumed that $\Gamma \in \operatorname{Ker}\mathbb{E}$. The construction of $p_{r,m}$ is done in appendix B. See also exercise in [7]. $\square$

Symmetric polynomials

Let the power sums be defined as the following symmetric polynomials:

$$p_k(x_1,\dots,x_N) = \sum_{i=1}^N x_i^k = \operatorname{Tr} X^k. \tag{2-18}$$

For $\mu = (\mu_1 \geq \mu_2 \geq \mu_3 \geq \dots \geq \mu_\ell)$ a partition, we denote

$$p_\mu(x_1,\dots,x_N) = \prod_{j=1}^\ell p_{\mu_j}(x_1,\dots,x_N). \tag{2-19}$$

We shall also use the same notation when $\mu = (\mu_1, \dots, \mu_\ell)$ is a ℓ–uple (no ordering assumed). We recall the notations:

- weight of a partition (resp. a upple)

$$|\mu| = \sum_i \mu_i, \tag{2-20}$$

- length of a partition (resp. a upple)

$$\ell(\mu) = \#\{i \mid \mu_i \neq 0\}. \tag{2-21}$$

We recall the classical lemma:

Lemma 2.1 (Basis of $\mathcal{P}_N$) *A basis of $\mathcal{P}_N$ is given by*

$$\{p_\mu \mid \ell(\mu) \leq N\}. \tag{2-22}$$

proof: Easy by recursion on N. See appendix A.

2.4 Loop equations

Define

Definition 2.3 *For a n-uple $\mu = (\mu_1, \dots, \mu_n)$ (not necessarily ordered), let the following symmetric polynomial*

$$\begin{aligned} Q_\mu \;=\; & \sum_{j=0}^{d} t_{j+1} p_{\mu_1+j} p_{\mu_2} \dots p_{\mu_n} \\ & - \sum_{j=0}^{\mu_1-1} p_j p_{\mu_1-1-j} p_{\mu_2} \dots p_{\mu_n} \\ & - \sum_{i=2}^{n} \mu_i \; p_{\mu_1+\mu_i-1} \prod_{k\neq j} p_{\mu_k} \end{aligned} \tag{2-23}$$

They generate

$$\mathcal{L} = Span \left< Q_\mu \right>_{n,\ \mu=(\mu_1,\dots,\mu_n)} \subset \mathcal{P}_N. \tag{2-24}$$

Definition 2.4 (Loop equations) *A linear form $E \in \mathcal{P}_N^*$ is called a solution of loop equations iff*

$$E(\mathcal{L}) = 0. \tag{2-25}$$

The set of solutions of loop equations is denoted

$$\mathcal{L}^\perp = \{E \in \mathcal{P}_N^* \mid \forall p \in \mathcal{L},\ E(p) = 0\}. \tag{2-26}$$

Theorem 2.2 (Matrix integrals satisfy loop equations)

$$\forall \Gamma \in H_N \qquad , \quad \mathbb{E}_\Gamma \in \mathcal{L}^\perp. \tag{2-27}$$

The map $\mathbb{E} : H_N \to \mathcal{L}^\perp$ is an injective homeomorphism, and

$$\dim \mathcal{L}^\perp \geq \frac{(N+d-1)!}{N!(d-1)!}. \tag{2-28}$$

proof: This is a well known theorem in random matrix theory, it is a special case of Schwinger-Dyson equations (Schwinger-Dyson equations are more generally defined for quantum field theories (QFT)). When the QFT is a matrix integral, these were called "loop equations" by Migdal [10]. Schwinger-Dyson equations merely reflect the fact that an integral is invariant under change of variable. They can also be rewritten as just integration by parts [3, 7]. Indeed notice that

$$Q_\mu(X)\Delta(X)^2 e^{-N\,\mathrm{Tr}\,V(X)} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(x_i^{\mu_1}\, p_{\mu_2}(X)\dots p_{\mu_n}(X)\Delta(X)^2 e^{-N\,\mathrm{Tr}\,V(X)}\right), \quad (2\text{-}29)$$

immediatly implying that

$$\mathbb{E}_\Gamma(Q_\mu) = 0. \quad (2\text{-}30)$$

This relies on the fact that the integrand vanishes at the boundaries of Γ, i.e. at ∞. See [5] in case there would be boundary terms. $\square$

2.5 Every solution of loop equations is a matrix integral

The morphism $\mathbb{E}$ is in fact an isomorphism. This means that to every solution E of loop equations corresponds a $\Gamma \in H_N$ such that $E = \mathbb{E}_\Gamma$. The following is the main theorem of this article

Theorem 2.3 (Solutions of loop equations = matrix eigenvalues integrals) *The map $\mathbb{E} : H_N \to \mathcal{L}^\perp$ is an isomorphism:*

$$\dim \mathcal{L}^\perp = \dim H_N = \frac{(N+d-1)!}{N!(d-1)!}. \quad (2\text{-}31)$$

proof: We need to prove surjectivity. Let $E \in \mathcal{L}^\perp$.

Let

$$A_{N,d} = \{\mu = \text{partitions}\ ,\ \ell(\mu) \leq N \text{ and } \forall i,\ \mu_i \leq d-1\}. \quad (2\text{-}32)$$

We have (see fig.2)

$$\# A_{N,d} = \frac{(N+d-1)!}{N!(d-1)!}. \quad (2\text{-}33)$$

We shall first prove the following lemma:

Lemma 2.2 *There exist some coefficients $c_{\mu,\nu}$ such that for every partition μ*

$$E(p_\mu) = \sum_{\nu \in A_{N,d}} c_{\mu,\nu} E(p_\nu). \quad (2\text{-}34)$$

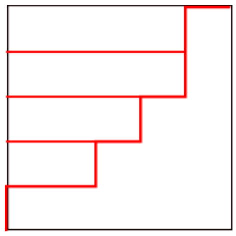

Figure 2: An element of $A_{N,d}$ can be seen as a Ferrer diagram that can fit in a box $(d-1)\times N$. It is thus a line of length $N+d-1$ with N vertical steps and $d-1$ horizontal steps. The number of such lines is the number of ways to choose N vertical steps among $N+d-1$.

Proof of the lemma: We prove it by recursion on $k=|\mu|$. It clearly holds for $k=0$ since the empty partition is already in $A_{N,d}$. Assume that it holds up to $k-1$. Let μ a partition of weight $|\mu|=k$ and length $\ell(\mu)\leq N$. If all $\mu_i\leq d-1$ then $\mu\in A_{N,d}$ so the lemma holds. If there exists $\mu_i\geq d$, up to relabeling assume that it is $\mu_1\geq d$. The loop equation $E(Q_{\mu_1-d,\mu_2,\dots\mu_n})=0$ implies

$$\begin{aligned} t_{d+1}E(p_\mu) &= -\sum_{j=0}^{d-1} t_{j+1}E(p_{\mu_1-d+j}p_{\mu_2}\dots p_{\mu_n}) \\ &\quad -\sum_{j=0}^{\mu_1-1} E(p_j p_{\mu_1-d-1-j}p_{\mu_2}\dots p_{\mu_n}) \\ &\quad -\sum_{i=2}^{n}\mu_i\ E(p_{\mu_1-d+\mu_i-1}\prod_{k\neq j}p_{\mu_k}). \end{aligned} \tag{2-35}$$

We recall that $t_{d+1}\neq 0$ and we notice that all polynomials in the right hand side have weights $<k$. By the recursion hypothesis, this implies that all terms in the right hand side are linear combinations of $E(p_\nu)$ with $\nu\in A_{N,d}$. If μ is such that $\ell(\mu)>N$, according to lemma 2.1, we can rewrite p_μ as a linear combination of p_νs of the same weight $|\nu|=|\mu|$ with $\ell(\nu)\leq N$. This ends the proof of the lemma. □

This implies that the map $E:\mathcal{P}_N\to\mathbb{C}$ is entirely determined by its value on the subspace

$$\text{span}\,\langle p_\mu\rangle_{\mu\in A_{N,d}} \tag{2-36}$$

and therefore

$$\dim\mathcal{L}^\perp\leq \#A_{N,d}=\frac{(N+d-1)!}{N!(d-1)!}. \tag{2-37}$$

Since we already knew the opposite inequality this implies equality:

$$\dim\mathcal{L}^\perp=\frac{(N+d-1)!}{N!(d-1)!}, \tag{2-38}$$

which thus implies that $\mathbb{E}$ is an ismorphism. □

3 2-Matrix model

3.1 Setting, arcs and homology

Consider 2 random normal matrices $M, \tilde{M}$ of size $N \times N$, with eigenvalues on some arcs γ and $\tilde{\gamma}$, i.e. in $\mathcal{H}_N(\gamma) \times \mathcal{H}_N(\tilde{\gamma})$, with a measure

$$e^{-\operatorname{Tr}(V(M)+\tilde{V}(\tilde{M})-M\tilde{M})}\mathcal{D}M\ \mathcal{D}\tilde{M}, \tag{3-1}$$

where V and $\tilde{V}$ are polynomials of respective degrees $d+1$ and $\tilde{d}+1$, written

$$V(x) = \sum_{k=1}^{d+1} \frac{t_k}{k} x^k \quad , \quad \tilde{V}(y) = \sum_{k=1}^{\tilde{d}+1} \frac{\tilde{t}_k}{k} y^k. \tag{3-2}$$

Diagonalizing $M = UXU^\dagger$ and $\tilde{M} = \tilde{U}Y\tilde{U}^\dagger$ we get (we used Harish-Chandra Itzykson-Zuber integral over the group $U(N)$, see [9, 7]) the marginal law of eigenvalues

$$\Delta(X)\Delta(Y)\det(e^{x_i y_j}) \prod_{i=1}^N e^{-V(x_i)}dx_i \prod_{i=1}^N e^{-\tilde{V}(y_i)}dy_i. \tag{3-3}$$

The integration domains for x_i (resp. y_i) must be such that integrals of polynomial moments are absolutely convergent, which leads us to the space of admissible homology classes. The following lemma is obvious:

Lemma 3.1 *If $d\tilde{d} > 1$, we have*

$$H_1(e^{xy-V(x)-\tilde{V}(y)}dxdy) = H_1(e^{-V(x)}dx) \otimes H_1(e^{-\tilde{V}(y)}dy) \tag{3-4}$$

thus

$$\dim H_1 = d\tilde{d}, \tag{3-5}$$

and a basis of H_1 is made of products $\gamma_{i,j} := \gamma_i \times \tilde{\gamma}_j$.

$$H_N(\Delta(X)\Delta(Y)\det(e^{x_i y_j}) \prod_{i=1}^N e^{-V(x_i)}dx_i \prod_{i=1}^N e^{-\tilde{V}(y_i)}dy_i) \tag{3-6}$$

is of dimension

$$\dim H_N = \frac{(N+d\tilde{d}-1)!}{N!(d\tilde{d}-1)!} \tag{3-7}$$

A basis is given by

$$\{\prod_{i,j} \gamma_{i,j}^{n_{i,j}} \mid \sum_{i,j} n_{i,j} = N\}. \tag{3-8}$$

Let $\Gamma \in H_N$, we define the linear map

$$\begin{aligned} \mathbb{E}_\Gamma \; : \; \mathcal{P}_N & \to \mathbb{C} \\ p & \mapsto \int_\Gamma p(X)\Delta(X)\Delta(Y)\det(e^{x_i y_j}) \prod_{i=1}^N e^{-V(x_i)} dx_i \prod_{i=1}^N e^{-\tilde{V}(y_i)} dy_i \end{aligned} \tag{3-9}$$

The integration defines a morphism

$$\begin{aligned} \mathbb{E} \; : \; H_N & \to \mathcal{P}_N^* \\ \Gamma & \mapsto \mathbb{E}_\Gamma \end{aligned} \tag{3-10}$$

Theorem 3.1 (Injectivity) *The morphism $\mathbb{E} : H_N \to \mathcal{P}_N^*$ is injective.*

proof: Similar to the 1-matrix case. □

3.2 Loop equations

The loop equations of the 2-matrix model are slightly more subtle.

Let us define the $N \times N$ matrix $U(X,Y)$ by

$$U(X,Y)_{i,j} = e^{x_i y_j} \tag{3-11}$$

then define

$$R_{i,j}(X,Y) = e^{x_i y_j} (U(X,Y)^{-1})_{j,i} \quad , \quad R_i^{(l)}(X,Y) = \sum_j R_{i,j}(X,Y) y_j^l. \tag{3-12}$$

Notice that

$$\sum_i R_{i,j}(X,Y) = 1 \quad , \quad \sum_j R_{i,j}(X,Y) = 1 \quad , \quad R_i^{(0)}(X,Y) = 1. \tag{3-13}$$

Let us define

$$p_k^{(l)}(X,Y) = \sum_i x_i^k \; R_i^{(l)}(X,Y). \tag{3-14}$$

Theorem 3.2 (Loop equations) *For each n–uple $(\mu_1, \mu_2, \dots, \mu_n)$, there is a symmetric polynomial $Q_{\mu_1,\mu_2,\dots,\mu_n}(X) \in \mathcal{P}_N$, of highest weight term*

$$Q_{\mu_1,\mu_2,\dots,\mu_n}(X) = \tilde{t}_{\tilde{d}+1} (t_{d+1})^{\tilde{d}} p_{\mu_1 + d\tilde{d}, \mu_2, \dots, \mu_n}(X) + \sum_{\nu, \; |\nu| < |\mu| + d\tilde{d}} c_{\mu,\nu} p_\nu(X), \tag{3-15}$$

such that

$$\mathbb{E}_\Gamma (Q_{\mu_1,\mu_2,\dots,\mu_n}(X)) = 0, \tag{3-16}$$

proof: See for instance a proof in [4]. Let us recall it here. For each $k, l, \mu_2, \dots, \mu_n$, we have

$$\begin{aligned}
& \sum_i \frac{\partial}{\partial x_i}\Big(x_i^k R_i^{(l)}(X,Y) p_{\mu_2}(X)\dots p_{\mu_n}(X)\Delta(X)\Delta(Y)\det e^{x_a y_b}\prod_a e^{-V(x_a)}\prod_b e^{-\tilde{V}(y_b)}\Big) \\
= & \Big(p_k^{(l+1)}(X,Y)p_{\mu_2}(X)\dots p_{\mu_n}(X) \\
& - \sum_j t_{j+1} p_{k+j}^{(l)}(X,Y)p_{\mu_2}(X)\dots p_{\mu_n}(X) \\
& + \sum_{j=0}^{k-1} p_j^{(l)}(X,Y)p_{k-1-j}(X)p_{\mu_2}(X)\dots p_{\mu_n}(X) \\
& + \sum_{i=2}^{n} \mu_i p_{k+\mu_i-1}^{(l)}(X,Y)\dots \widehat{p_{\mu_i}(X)}\dots p_{\mu_n}(X)\Big) \\
& \Delta(X)\Delta(Y)\det e^{x_a y_b}\prod_a e^{-V(x_a)}\prod_b e^{-\tilde{V}(y_b)}.
\end{aligned} \tag{3-17}$$

$$\begin{aligned}
& \sum_{i,j} \frac{\partial}{\partial y_j}\Big(x_i^k p_{\mu_2}(X)\dots p_{\mu_n}(X)\Delta(X)\Delta(Y)\det e^{x_a y_b}\prod_a e^{-V(x_a)}\prod_b e^{-\tilde{V}(y_b)}\Big) \\
= & \Big(p_{k+1}(X)p_{\mu_2}(X)\dots p_{\mu_n}(X) - \sum_l \tilde{t}_{l+1} p_k^{(l)}(X,Y)p_{\mu_2}(X)\dots p_{\mu_n}(X)\Big) \\
& \Delta(X)\Delta(Y)\det e^{x_a y_b}\prod_a e^{-V(x_a)}\prod_b e^{-\tilde{V}(y_b)}.
\end{aligned} \tag{3-18}$$

Integration by parts thus implies

$$\begin{aligned}
\mathbb{E}_\Gamma(p_k^{(l+1)}(X,Y)p_{\mu_2}(X)\dots p_{\mu_n}(X)) &= \sum_j t_{j+1}\mathbb{E}_\Gamma(p_{k+j}^{(l)}(X,Y)p_{\mu_2}(X)\dots p_{\mu_n}(X)) \\
& - \sum_{j=0}^{k-1} \mathbb{E}_\Gamma(p_j^{(l)}(X,Y)p_{k-1-j}(X)p_{\mu_2}(X)\dots p_{\mu_n}(X)) \\
& - \sum_{i=2}^{n} \mu_i \mathbb{E}_\Gamma(p_{k+\mu_i-1}^{(l)}(X,Y)\dots \widehat{p_{\mu_i}(X)}\dots p_{\mu_n}(X)) \\
\mathbb{E}_\Gamma(p_{k+1}(X)p_{\mu_2}(X)\dots p_{\mu_n}(X)) &= \sum_l \tilde{t}_{l+1}\mathbb{E}_\Gamma(p_k^{(l)}(X,Y)p_{\mu_2}(X)\dots p_{\mu_n}(X))
\end{aligned} \tag{3-19}$$

The first equation is a recursion on l, with initial condition

$$\mathbb{E}_\Gamma(p_k^{(0)}(X,Y)p_{\mu_2}(X)\dots p_{\mu_n}(X)) = \mathbb{E}_\Gamma(p_k(X)p_{\mu_2}(X)\dots p_{\mu_n}(X)), \tag{3-20}$$

which allows to express for every l, $\mathbb{E}_\Gamma(p_k^{(l)}(X,Y)p_{\mu_2}(X)\dots p_{\mu_n}(X))$ as a linear combination of $\mathbb{E}_\Gamma$ of some symmetric polynomials of x only. The last equation can then be written as an equation relating the $\mathbb{E}_\Gamma$ of some symmetric polynomials of x only, let us write it:

$$\mathbb{E}_\Gamma(Q_{k,\mu_2,\dots,\mu_n}(X)) = 0, \tag{3-21}$$

where $Q_{k,\mu_2,\dots,\mu_n}(X)$ is a symmetric polynomial of x, thus a linear combination of power sums symmetric polynomials. Its highest weigth term is

$$Q_{k,\mu_2,\dots,\mu_n}(X) = \tilde{t}_{\tilde{d}+1}(t_{d+1})^{\tilde{d}} p_{k+d\tilde{d},\mu_2,\dots,\mu_n}(X) + \sum_{\nu,\ |\nu|<|k+d\tilde{d}+\mu_2+\dots+\mu_n} c_{(k,\mu_2,\dots,\mu_n),\nu} p_\nu(X). \tag{3-22}$$

□

Definition 3.1 (Loop equations) *We define the loop equations sub-space*

$$\mathcal{L} = Span\,\langle Q_{\mu_1,\mu_2,\dots,\mu_n}(X)\rangle \subset \mathcal{P}_N, \tag{3-23}$$

and the set of solutions of loop equations

$$\mathcal{L}^\perp = \{E \in \mathcal{P}_N^* \mid E(\mathcal{L}) = 0\} \subset \mathcal{P}_N^*. \tag{3-24}$$

We have proved that

Proposition 3.1 *The map $\mathbb{E} : H_N \to \mathcal{L}^\perp$, $\Gamma \mapsto \mathbb{E}_\Gamma$, is an injective homeomorphism. We thus have*

$$\dim \mathcal{L}^\perp \geq \frac{(N + d\tilde{d} - 1)!}{N!(d\tilde{d} - 1)!}. \tag{3-25}$$

3.3 Solutions of loop equations are matrix integrals

In fact the map is an isomoprhism

Theorem 3.3 (Solutions of loop equations = matrix integrals) *The map $\mathbb{E}$: $H_N \to \mathcal{L}^\perp$, $\Gamma \mapsto \mathbb{E}_\Gamma$, is an isomorphism.*

$$\dim \mathcal{L}^\perp = \frac{(N + d\tilde{d} - 1)!}{N!(d\tilde{d} - 1)!}. \tag{3-26}$$

proof: The proof is very similar to the one matrix model. We prove that $E \in \mathcal{L}^\perp$ is determined by its value on the subspace

$$\text{Span}\,\langle p_\mu\rangle_{\mu\in A_{N,d\tilde{d}}}. \tag{3-27}$$

In other words we show that for any partition μ:

$$E(p_\mu) = \sum_{\nu\in A_{N,d\tilde{d}}} c_{\mu,\nu} E(p_\nu). \tag{3-28}$$

This is proved by recursion on $|\mu|$. This is obviously true when $|\mu| = 0$, assume it is true up to $|\mu| - 1$. If $\mu \notin A_{N,d\tilde{d}}$, this means that one row, let us say $\mu_1 \geq d\tilde{d}$, the loop equation

$$E(Q_{\mu_1-d\tilde{d},\mu_2,\dots,\mu_n}) = 0 \tag{3-29}$$

implies (3-28).

This implies that

$$\dim \mathcal{L}^\perp \leq \# A_{N,d\tilde{d}} = \frac{(N+d\tilde{d}-1)!}{N!(d\tilde{d}-1)!}, \tag{3-30}$$

and since we already have the opposite inequality from injectivity, we conclude that there is equality and $\mathbb{E}_\Gamma$ is an isomorphism. $\square$

4 Chain of normal matrices

The chain of matrices is for example defined in [9, 6, 4].

Consider some complex polynomials $V_1, V_2, \dots, V_L$, of respective degrees

$$\deg V_i' = d_i. \tag{4-1}$$

Consider the measure

$$\mathcal{D}P(X_1,\dots,X_L) = \Delta(X_1)\Delta(X_L) \prod_{l=1}^{L-1} \det_{a,b}(e^{(X_l)_a (X_{l+1})_b}) \prod_{l=1}^{L} e^{-\operatorname{Tr} V_l(X_l)} \mathcal{D}X_l \tag{4-2}$$

that we shall put on $\Gamma \in H_N$: Let

$$H_N = \otimes_{l=1}^L H_N(\Delta(X_l)^2 e^{-\operatorname{Tr} V(X_l)} \mathcal{D}X_l). \tag{4-3}$$

We have

$$\dim H_N = D = \prod_{l=1}^L d_l. \tag{4-4}$$

For $\Gamma \in H_N$ we define

$$\begin{aligned} \mathbb{E}_\Gamma : \quad \mathcal{P}_N &\to \mathbb{C} \\ p &\mapsto \int_\Gamma p(X_1) \mathcal{D}P(X_1,\dots,X_L). \end{aligned} \tag{4-5}$$

and the map

$$\begin{aligned} \mathbb{E} : \quad H_N &\to \mathcal{P}_N^* \\ \Gamma &\mapsto \mathbb{E}_\Gamma. \end{aligned} \tag{4-6}$$

This map is injective, the proof is more or less the same as the 1 matrix model.

Loop equations

Define

$$p_k^{(l_2,\dots,l_L)}(X_1,X_2,\dots,X_L) = \sum_{i_1,\dots,i_L} ((X_1)_{i_1})^k R(X_1,X_2)_{i_1,i_2}((X_2)_{i_2})^{l_2} \\ R(X_2,X_3)_{i_2,i_3}\dots R(X_{L-1},X_L)_{i_{L-1},i_L}((X_L)_{i_L})^{l_L}. \quad (4\text{-}7)$$

Theorem 4.1 (Loop equations) *For each n–uple $(\mu_1,\mu_2,\dots,\mu_n)$, there is a symmetric polynomial $Q_{\mu_1,\mu_2,\dots,\mu_n}(X)\in\mathcal{P}_N$, of highest weight term*

$$Q_{\mu_1,\mu_2,\dots,\mu_n}(X) = C\ p_{\mu_1+D,\mu_2,\dots,\mu_n}(X) + \sum_{\nu,\ |\nu|<|\mu|+D} c_{\mu,\nu}p_\nu(X), \quad (4\text{-}8)$$

with $C\neq 0$, such that

$$\mathbb{E}_\Gamma(Q_{\mu_1,\mu_2,\dots,\mu_n}(X)) = 0, \quad (4\text{-}9)$$

We denote $\mathcal{L} = Span < Q_\mu >$, and $\mathcal{L}^\perp = \{E\in\mathcal{P}_N^* \mid E(\mathcal{L})=0\}$.

This theorem was proved in [4]. The coefficient C is the leading doefficient of $V'_L\circ V'_{L-1}\circ\cdots\circ V'_1$.

The rest is the same as for 1 and 2 matrix models:

Theorem 4.2 $\mathbb{E}$ *is an isomorphism*

$$\begin{aligned} \mathbb{E}\ :\quad & H_N \to \mathcal{L}^\perp \\ & \Gamma \mapsto \mathbb{E}_\Gamma \end{aligned} \quad (4\text{-}10)$$

is an isomorphism

$$\dim\mathcal{L}^\perp = \dim H_N = \frac{(N+D-1)!}{N!(D-1)!} \quad (4\text{-}11)$$

where $D=\prod_{l=1}^L \deg V'_l$.

The proof is exactly the same as 1 and 2 matrix models.

5 Rational potentials

Now we will consider

$$V'(x)\in\mathbb{C}(x) \quad (5\text{-}1)$$

which means that $V(x)$ can also have logarithms. The degree of $V'(x)$ is defined to be the sum of degrees of all poles, including the pole at ∞.

$$\deg V' = \sum_{p=\text{poles}} \deg_p V'. \quad (5\text{-}2)$$

Notice that $e^{-V(x)}$ has essential singularities at pole of $V'(x)$, and if V' has a simple pole p with a non-vanishing residue $r=\operatorname{Res}_p V'$, 3 situations can occur:

- $r \in \mathbb{Z}_-$: then $e^{-V(x)}$ has a zero at p.

- $r \in \mathbb{Z}_+$: then $e^{-V(x)}$ has a pole at p.

- $r \notin \mathbb{Z}$: then $e^{-V(x)}$ is not analytic at p, we need to introduce a cut ending at p.

Let us consider the complex plane from which we remove all poles, and possibly cuts ending at poles, so that e^{-V} is analytic in the considered domain.

The admissible Jordan arcs, are now arcs going from a pole to another (or the same pole), and not crossing cuts. Arcs can arrive at a pole only in a direction in which $\Re V(x) \to +\infty$.

• If $e^{-V(x)}$ has a zero, an arc can end on it from any direction.

• If $e^{-V(x)}$ has a pole, no arc can end on it, but can go around it, for instance a small closed circle around a pole is an admissible arc.

• If $e^{-V(x)}$ has a cut, arcs must go around the cut without crossing it.

These arcs are described in [2, 4], where it is shown thatthe total number of homologically independent arcs is $\deg V'$:

Proposition 5.1 (Homology space [2]) *The dimension of the homology space*

$$\dim H_1(e^{-V(x)}dx, K) = \deg V' = \sum_{p=poles} \deg_p V'. \tag{5-3}$$

5.1 One matrix

Again consider

$$H_N = \text{Sym}\left(H_1^{\otimes N}\right), \tag{5-4}$$

its dimension is

$$\dim H_N = \frac{(N+d-1)!}{N!(d-1)!} \quad , \quad d = \deg V'. \tag{5-5}$$

For any $\Gamma \in H_N$, for any symmetric polynomial $p \in \mathcal{P}_N$, the following integral is absolutely convergent

$$\mathbb{E}_\Gamma(p) = \int_\Gamma p(X)\ \Delta(X)^2 \prod_{i=1}^N e^{-V(x_i)} dx_i. \tag{5-6}$$

The map $\mathbb{E}_\Gamma : \mathcal{P}_N \to \mathbb{C}$ is a linear form on $\mathcal{P}_N$:

$$\mathbb{E}_\Gamma \in \mathcal{P}_N^*, \tag{5-7}$$

and the map $\mathbb{E} : H_N \to \mathcal{P}_N^*$ is a homeomorphism.

Theorem 5.1 *the map $\mathbb{E} : H_N \to \mathcal{P}_N^*$ is an injective homeomorphism.*

proof: The proof is the same as the polynomial case, and is done in appendix B. □

Loop equations

Let us write $V'(x)$ as an irreducible rational fraction of 2 polynomials

$$V'(x) = \frac{R(x)}{D(x)} \tag{5-8}$$

where $D(x)$ is a monic polynomial. Let us assume that $\deg R > \deg D$, and we have

$$\deg R = d = \deg V'. \tag{5-9}$$

Write

$$D(x) = \sum_{k=0}^{\deg D} D_k x^k. \tag{5-10}$$

Define the symmetric polynomials

$$p_k^{(D)}(x_1, \dots, x_N) = \sum_{i=1}^{N} D(x_i) x_i^k \tag{5-11}$$

and for a n–uple $\mu_1, \dots, \mu_n$, define

$$p_\mu^{(D)}(x_1, \dots, x_N) = p_{\mu_1}^{(D)}(x_1, \dots, x_N) p_{\mu_2}(x_1, \dots, x_N) \dots p_{\mu_n}(x_1, \dots, x_N). \tag{5-12}$$

Then define

$$Q_\mu = p_\mu^{(R)} - p_\mu^{(D')} - \sum_{k=0}^{\deg D} D_k \sum_{j=0}^{k+\mu_1-1} p_{j,k+\mu_1-1,\mu_2,\dots,\mu_n} - \sum_{i=2}^{n} \mu_i p^{(D)}_{\mu_1+\mu_i-1,\mu_2,\dots,\widehat{\mu_i}\dots\mu_n}. \tag{5-13}$$

Let

$$\mathcal{L} = \text{Span} \left\langle Q_\mu \right\rangle_\mu . \tag{5-14}$$

Theorem 5.2 (Loop equations) *For any $\Gamma \in H_N$ we have*

$$\mathbb{E}_\Gamma(\mathcal{L}) = 0. \tag{5-15}$$

The map $\mathbb{E} : H_N \to \mathcal{L}^\perp$ is an injective homeomorphism,

$$\dim \mathcal{L}^\perp \geq \frac{(N+d-1)!}{N!(d-1)!}. \tag{5-16}$$

proof:

$$Q_\mu(X)\Delta(X)^2 \, e^{-N \operatorname{Tr} V(X)} = \sum_i \frac{\partial}{\partial x_i}\Big(D(x_i) x_i^{\mu_1} p_{\mu_2}(X) \dots p_{\mu_n}(X) \; \Delta(X)^2 \, e^{-\operatorname{Tr} V(X)} \Big) \tag{5-17}$$

and by integration by parts $\mathbb{E}_\Gamma(Q_\mu) = 0$. □

Theorem 5.3 *The map* $\mathbb{E}: H_N \to \mathcal{L}^\perp$ *is an isomorphism,*

$$\dim \mathcal{L}^\perp = \frac{(N+d-1)!}{N!(d-1)!}. \tag{5-18}$$

proof: same as for polynomial potentials. We just need to notice that

$$Q_\mu = C p_{\mu_1+d,\mu_2,\dots\mu_n} + \sum_{\nu,\ |\nu|<d+|\mu|} c_{\mu,\nu} p_\nu, \tag{5-19}$$

so that if a partition has a row $\mu_i \leq d$ we can shorten it by using $Q_{\mu_i-d,\mu_2,\dots,\widehat{\mu_i},\dots,\mu_n}$, so eventually $E \in \mathcal{L}^\perp$ is entirely determined by its restriction to

$$\mathrm{Span}\,\langle p_\mu \rangle_{\mu \in A_{N,d}} \tag{5-20}$$

and thus

$$\dim \mathcal{L}^\perp \leq \# A_{N,d} = \frac{(N+d-1)!}{N!(d-1)!}. \tag{5-21}$$

$\square$

5.2 Chain of matrices

The same proof generalizes immediately to chain of matrices with rational $V'_l \in \mathbb{C}(x_l)$, we get that

$$\dim \mathcal{L}^\perp = \frac{(N+D-1)!}{N!(D-1)!} \tag{5-22}$$

where $D = \prod_{j=1}^L \deg V'_j$ where $\deg V'_j$ is the sum of degrees of all the poles of V'_j.

6 Examples of applications

6.1 Application: Haar measure on $U(N)$

We already mentioned that if $\gamma = S^1$ the unit circle, we have

$$\mathcal{H}_N(S^1) = U(N) \tag{6-1}$$

and the measure $\mathcal{D}M$ is closely related to the Haar measure (see [7], it is easy to see that $\mathrm{i}^{-N^2} \det M^{-N} \mathcal{D}M$ is a real positive measure and is invariant under right or left multiplication by an element of $U(N)$ so is the Haar measure)

$$\mathrm{i}^{N^2}\, \mathcal{D}_{\mathrm{Haar}} M = \frac{1}{(\det M)^N}\, \mathcal{D}M = e^{-N\,\mathrm{Tr}\log M}\, \mathcal{D}M, \tag{6-2}$$

so that the eigenvalues statistics of a random unitary matrix with Haar measure on $U(N)$, can be rewritten as a normal matrix whose potential is

$$V(x) = N \log(x) \tag{6-3}$$

i.e. V' a rational fraction

$$V'(x) = \frac{N}{x} \quad , \quad d = \deg V' = 1. \tag{6-4}$$

There is thus a unique homology class in H_N which has dimension

$$\dim H_N = \frac{(N+d-1)!}{N!(d-1)!} = 1. \tag{6-5}$$

This unique homology class is $(S^1)^N$, i.e. all eigenvalues are on the circle.

We could also consider a Haar measure with polynomial potential of some degree $k+1$, typically

$$\left|e^{-\operatorname{Tr} V(M)}\right| \mathcal{D}_{\text{Haar}} M = e^{-\operatorname{Tr}(\frac{1}{2}V(M)+\frac{1}{2}V(M^{-1})+N\log M)}\ \mathcal{D}M \tag{6-6}$$

which is a normal matrix model with rational

$$\frac{1}{2}V'(x) - \frac{1}{2x^2}V'(1/x) + \frac{N}{x} \tag{6-7}$$

of total degree

$$d = 2k+2. \tag{6-8}$$

6.2 Application: normal matrices in the complex plane

It is well known that one random complex matrix $M \in M_N(\mathbb{C})$, is closely related to one random normal complex matrix $M \in \mathcal{H}_N(\mathbb{C})$ and equivalent to a 2-matrix model. Let us recall how.

Consider a normal random matrix M in $\mathcal{H}_N(\mathbb{C})$, with a measure

$$e^{-\operatorname{Tr} MM^\dagger} e^{-\operatorname{Tr} V(M)+\tilde{V}(M^\dagger)} \mathcal{D}M\mathcal{D}M^\dagger \tag{6-9}$$

where $\mathcal{D}M\mathcal{D}M^\dagger$ denotes the measure on $\mathcal{H}_N(\mathbb{C})$ defined below. To define it, notice that both M and $M^\dagger$ are normal matrices and can be diagonalized by the same unitary conjugation:

$$M = UXU^\dagger \quad , \quad M^\dagger = UYU^\dagger \quad , \quad Y = \bar{X} \tag{6-10}$$

where $U \in U(N)/U(1)^N$, and X is a diagonal complex matrix.

The measure on $\mathcal{H}_N(\mathbb{C})$ is defined as

$$\mathcal{D}M\mathcal{D}M^\dagger = |\Delta(X)|^2 \mathcal{D}U\mathcal{D}X\mathcal{D}\bar{X}, \tag{6-11}$$

where $\mathcal{D}U$ is as usual the Haar measure on $U(N)/U(1)^N$ and $\mathcal{D}X\mathcal{D}\bar{X} = \prod_{i=1}^N dx_i d\bar{x}_i$ and each $dx_i d\bar{x}_i$ is the Lebesgue measure of $x_i \in \mathbb{C} \sim \mathbb{R}^2$.

The induced marginal measure for eigenvalues is

$$|\Delta(X)|^2 \left|\det\left(e^{-x_a\bar{x}_b}\right)\right| \prod_{i=1}^{N} e^{-(V(x_i)+\tilde{V}(\bar{x}_i))} dx_i d\bar{x}_i. \tag{6-12}$$

It is a real measure when $\tilde{V}$ is the complex conjugate of V.

Considering X and $Y = \bar{X}$ as independent variables we see that it is a 2-matrix model. More precisely, it is a 2-matrix model where X, Y are integrated on a N-dimensional submanifold of $\mathbb{C}^{2N}$ satisfying $Y = \bar{X}$. If the integral is convergent, this manifold must be in H_N. In other words, the normal complex matrix model, is identical to a 2-matrix model on a homology class $\Gamma \in H_N(\Delta(X)\Delta(Y)\det(e^{-x_a y_b})\prod_{i=1}^N e^{-(V(x_i)+\tilde{V}(y_i))}dx_i dy_i)$, such that $\bar{\Gamma} = \sigma^*\Gamma$ with σ the involution $(X,Y) \mapsto (Y,X)$. If γ_i (resp. $\tilde{\gamma}_i$) form a basis of $H_1(e^{-V(x)}dx)$ (resp. $H_1(e^{-\tilde{V}(y)}dy)$), then there must exist some bilinear combination

$$\Gamma = \sum_{i=1}^{\deg V'} \sum_{j=1}^{\deg \tilde{V}'} c_{i,j}\gamma_i \times \tilde{\gamma}_j. \tag{6-13}$$

If $\tilde{V} = \bar{V}$, we may choose $\tilde{\gamma}_i = \bar{\gamma}_i$, and the condition $\bar{\Gamma} = \sigma^*\Gamma$ implies that the matrix $c_{i,j}$ must be Hermitian, and we can choose a basis in which it is diagonal and real, i.e. we can choose

$$\Gamma = \sum_{i=1}^{\deg V'} c_i\gamma_i \times \bar{\gamma}_i \qquad , \ c_i \in \mathbb{R}. \tag{6-14}$$

Let us choose

$$\gamma_i = \zeta_i\mathbb{R}_+ - \zeta_{i-1}\mathbb{R}_+ \tag{6-15}$$

where $\zeta_j = t_{d+1}^{\frac{1}{d+1}}\, e^{2\pi \mathrm{i}\,\frac{j}{d+1}}$ are roots of t_{d+1} the leading coefficient of V'. $\gamma_1, \dots, \gamma_d$ form a basis of H_1 and we have

$$\gamma_{d+1} = -\sum_{i=1}^{d} \gamma_i. \tag{6-16}$$

The following class

$$\Gamma = \sum_{i=1}^{d+1} \gamma_i \times \bar{\gamma}_i \tag{6-17}$$

is (up to a real proportionality constant) a homology class invariant under complex conjugation and under rotations by angles $2\pi/(d+1)$. It is the natural candidate to replace $\mathbb{C}$.

6.3 Application: Combinatorics of maps

See [1, 8, 7] for an introduction to maps and random matrices (Readers not familiar with combinatorics of maps may skip this part.)

Let $t_3, t_4, \dots t_{d+1}$ be complex numbers with $t_{d+1} \neq 0$, and $N \in \mathbb{Z}_+$. Let us denote the formal series $\hat{T}_{k_1,\dots,k_n} \in \mathbb{Q}[t_3, \dots, t_{d+1}, N, N^{-1}][[t]]$

$$\hat{T}_{k_1,\dots,k_n} = t\, N\delta_{n,1} + \sum_{e=2}^{\infty} t^e \sum_{m \in \mathbb{M}(e,k_1,\dots,k_n)} \frac{N^{\chi(m)-n}}{\#\mathrm{Aut}(m)} t_3^{n_3(m)} t_4^{n_4(m)} \dots t_{d+1}^{n_{d+1}(m)} \tag{6-18}$$

where $\mathbb{M}(e, k_1, \dots, k_n)$ is the (finite) set of connected orientable maps with e edges, and made of n_3 triangles, n_4 quadrangles, ..., n_{d+1} $(d+1)$–angles, and with also n marked labeled faces (a marked face is a face with a marked oriented edge on its boundary, so that the marked face is on the right of the marked edge) of respective size $k_1, k_2, \dots, k_n$. We require $k_i \geq 1$, whereas unmarked faces have at least size 3 (triangles up to $(d+1)$–angles). $\#\mathrm{Aut}(m)$ is the automorphism factor of the map, $\#\mathrm{Aut}(m) = 1$ for maps with marked faces, and can be ≥ 1 for $n = 0$ (no marked faces). $\chi(m) = \#\mathrm{faces}(m) - \#\mathrm{edges}(m) + \#\mathrm{vertices}(m)$ is the Euler characteristic of the map. Let us define (again as formal power series of t)

$$T_{\emptyset} = e^{\hat{T}_{\emptyset}} \tag{6-19}$$

and for $n \geq 1$

$$T_{k_1,\dots,k_n} = e^{\hat{T}_{\emptyset}} \sum_{\mu=\text{partitions of}\{k_1,\dots,k_n\}} \prod_{K=\text{parts of } \mu} \hat{T}_K. \tag{6-20}$$

For example:

$$T_{k_1} = T_{\emptyset}\hat{T}_{k_1} \ , \quad T_{k_1,k_2} = T_{\emptyset}(\hat{T}_{k_1,k_2} + \hat{T}_{k_1}\hat{T}_{k_2}), \ \dots \tag{6-21}$$

It is well known that $T_{k_1,\dots,k_n}$ are generating functions for counting non–connected maps.

In the 1960's, W. Tutte [11, 12] found some equations relating these generating functions, by recursion on the number of edges. Tutte's equations can be rewritten as loop equations, let us explain how.

Let

$$V(x) = N\left(\frac{1}{2t}x^2 - \sum_{k=3}^{d+1} \frac{t_k}{k} x^k\right). \tag{6-22}$$

Let $E \in \mathcal{P}_N^*$ defined on the basis of power sum polynomials as

$$E(p_\mu) = T_{\mu_1,\dots,\mu_\ell}. \tag{6-23}$$

Tutte's equations are then exactly the loop equations [8, 7]:

$$\forall\ \mu\ , \quad E(Q_\mu) = 0. \tag{6-24}$$

Theorem 2.3 impies that $\exists\, \Gamma \in H_N(\Delta(X)^2 e^{-\operatorname{Tr} V(X)}\mathcal{D}x, \mathbb{Q})$, such that

$$T_{k_1,\dots,k_n} = \int_\Gamma \operatorname{Tr} x^{k_1} \dots \operatorname{Tr} x^{k_n} \;\; \Delta(X)^2 e^{-\operatorname{Tr} V(X)} \prod_{i=1}^{N} dx_i. \tag{6-25}$$

7 Conclusion

The theorems presented here are some "representation theorems", saying that linear forms on the space of symmetric polynomials, satisfying loop equations can always be represented as matrix-model-like measures (Vandermonde–square times exponential for the case of 1-matrix). It also shows how normal matrices can be extremely useful. We expect to prove similar theorems for the matrix model with external fields, or matrix models with hard edges.

Also we may guess some applications to free probabilities, to be explored further.

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Appendices

A Lemma 2.1

Lemma 2.1: *A basis of $\mathcal{P}_N$ is given by*

$$\{p_\mu \mid \ell(\mu) \leq N\}. \tag{1-1}$$

Extension: A basis of $\{p \in \mathcal{P}_N \mid p$ homogeneous of degree $d\}$ is given by

$$\{p_\mu \mid \ell(\mu) \leq N \text{ and } |\mu| = d\}. \tag{1-2}$$

proof: By recursion on N. It is clearly true for $N = 1$.

Assume it holds for $N-1$, let $P \in \mathcal{P}_N$ a symmetric polynomial of N variables. $P(x_1, x_2, \dots, x_N)$ can be expanded in powers of x_1

$$P(x_1, x_2, \dots, x_N) = \sum_k x_1^k Q_k(x_1, \dots, x_N) \tag{1-3}$$

where each $Q_k \in \mathcal{P}_{N-1}$. By recursion hypothesis there exists somecoeffisients $Q_{k,\nu}$

$$P(x_1, x_2, \dots, x_N) = \sum_k x_1^k \sum_{\nu,\ \ell(\nu)\leq N-1} Q_{k,\nu} p_{\nu_1}(x_2, \dots, x_N) \dots p_{\nu_N}(x_2, \dots, x_N) \tag{1-4}$$

Observe that

$$p_{\nu_i}(x_2, \dots, x_N) = p_{\nu_i}(x_1, x_2, \dots, x_N) - x_1^{\nu_i}, \tag{1-5}$$

therefore we can reexpand in powers of x_1

$$P(x_1, x_2, \dots, x_N) = \sum_k x_1^k \sum_{\nu,\ \ell(\nu)\leq N-1} \tilde{Q}_{k,\nu} p_{\nu_1}(x_1, \dots, x_N) \dots p_{\nu_N}(x_1, \dots, x_N) \tag{1-6}$$

By symmetry we also have $\forall\ i = 1, \dots, N$

$$P(x_1, x_2, \dots, x_N) = \sum_k x_i^k \sum_{\nu,\ \ell(\nu)\leq N-1} \tilde{Q}_{k,\nu} p_{\nu_1}(x_1, \dots, x_N) \dots p_{\nu_N}(x_1, \dots, x_N) \tag{1-7}$$

and by summing over i

$$\begin{aligned} P(x_1, x_2, \dots, x_N) &= \frac{1}{N} \sum_{i=1}^N \sum_k x_i^k \sum_{\nu,\ \ell(\nu)\leq N-1} \tilde{Q}_{k,\nu} p_{\nu_1}(x_1, \dots, x_N) \dots p_{\nu_N}(x_1, \dots, x_N) \\ &= \frac{1}{N} \sum_k p_k(x_1, \dots, x_N) \sum_{\nu,\ \ell(\nu)\leq N-1} \tilde{Q}_{k,\nu} p_{\nu_1}(x_1, \dots, x_N) \dots p_{\nu_N}(x_1, \dots, x_N) \end{aligned} \tag{1-8}$$

which is clearly a linear combination of $p_{\nu'}$ where $\nu' = \nu + (k)$ is a partition obtained by adding one part of length k to ν, and it has thus at most N parts. This concludes the proof.

Notice that if P is homogeneous of some degree d, all steps we have followed conserve the homogeneity and its degree, so the extension also holds. $\square$

B Proof of injectivity theorem 2.1

Theorem 2.1 $\mathbb{E}$ *is an injective homeomorphism of vector spaces*

$$\begin{aligned} \mathbb{E}: \quad H_N &\to \mathcal{P}_N^* \\ \Gamma &\mapsto \mathbb{E}_\Gamma \end{aligned} \tag{2-1}$$

proof: We need to prove that $\operatorname{Ker} \mathbb{E} = 0$.

The proof is the same for polynomial $V(x) \in \mathbb{C}[x]$ or rational potentials $V'(x) \in \mathbb{C}(x)$. Without loss of generality. we shall assume that in the rational case V' has no pole at $x = 0$ (otherwise we should replace $\log x$ in what follows by $\log(x - x_0)$ with x_0 a point which is not a pole of V'. Choosing $x_0 = 0$ makes the proof easier to read.)

Let $d = \deg V'$ (= sum of degrees of all poles in the rational case).

We shall proceed in several steps.

- For r a positive integer, we define

$$V_r(x) = V(x) - r\log x \qquad \Longrightarrow \qquad e^{-V_r(x)} = x^r e^{-V(x)}. \tag{2-2}$$

- The Homology space of admissible arcs for V_r

$$\hat{H}_N^{(r)} = H_N\left(\Delta(X)^2 \prod_{i=1}^N e^{-V_r(x_i)} dx_i\right) \tag{2-3}$$

has dimension

$$\dim \hat{H}_N^{(r)} = \frac{(N+d)!}{N!d!}. \tag{2-4}$$

We have

$$H_N \subset \hat{H}_N^{(r)}, \tag{2-5}$$

and we recover H_N as a subset of $\hat{H}_N^{(r)}$ by restricting to homology classes of arcs that have vanishing boundary at $x = 0$.

- Consider the critical points $\xi_1, \dots, \xi_{d+1}$ of V_r, i.e. the solutions of $V_r'(x) = 0$, i.e. the solutions of $xV'(x) = r$. For r large enough they are all distinct. Asymptotically at large r, they approach the poles of V'.

  $*$ If V is a polynomial, or V behaves as $V(x) \sim t_\infty \frac{x^{d_\infty+1}}{d+1} + \tilde{t}_\infty \frac{x^{d_\infty}}{d}$ at large x, we have $d_\infty + 1$ critical points that are large

$$\xi_{\infty,k} \sim \zeta_{d_\infty+1}^k\ (r/t_\infty)^{\frac{1}{d_\infty+1}} \left(1 - \frac{\tilde{t}_\infty t_\infty^{-\frac{d_\infty}{d_\infty+1}} \zeta_{d_\infty+1}^{-k}}{d_\infty+1} r^{\frac{-1}{d+1}} + O(r^{\frac{-2}{d_\infty+1}})\right) \tag{2-6}$$

where we denote roots of unity as

$$\zeta_d = e^{2\pi \mathrm{i}\, \frac{1}{d}}. \tag{2-7}$$

We also have

$$V_r(\xi_{\infty,k}) \sim \frac{r}{d_\infty+1}(1 - \log r + \log t_\infty) + O(r^{1-\frac{1}{d_\infty+1}}). \tag{2-8}$$

$*$ At a finite pole p (recall we assumed $p \neq 0$), if V' behaves as $V'(x) \sim t_p(x-p)^{-d_p}$, we have d_p critical points that are close to p:

$$\xi_{p,k} \sim p + \zeta_{d_p}^k\ (r/pt_p)^{\frac{-1}{d_p}}(1 + O(r^{\frac{-1}{d_p}})). \tag{2-9}$$

We also have, if $d_p > 1$

$$V(\xi_{p,k}) \sim -\frac{r}{(d_p-1)p}\zeta_{d_p}^k\ (r/pt_p)^{\frac{-1}{d_p}}(1 + O(r^{\frac{-1}{d_p}})). \tag{2-10}$$

and if $d_p = 1$

$$V(\xi_{p,k}) \sim -t_p \log r + O(1). \tag{2-11}$$

Define

$$Q(x) = \prod_j (x - \xi_j) = t_\infty^{-1}\, V_r'(x) \prod_p (x-p)^{d_p}. \tag{2-12}$$

- For each $j = 1, \dots, d$, define

$$\gamma_j \subset \{x \in \mathbb{C}^* \mid V_r(x) - V_r(\xi_j) \in \mathbb{R}_+\} \tag{2-13}$$

  a piecewise connected C^1 Jordan arc from pole to pole, going through ξ_j, on which $V_r(x) - V_r(\xi_j) \in \mathbb{R}_+$ such that $\Re V_r(x)$ increases monotonically when going away from ξ_j in both direction.

  The paths γ_j are called steepest-descent contours. It is clear that asymptotically for r large enough they follow rays emanating from the poles and are linearly independent in $\hat{H}_1^{(r)}$, they form a basis of $\hat{H}_1^{(r)}$ (in fact this is true also for r not large, but we don't need it).

- Let $n = (n_1, \dots, n_{d+1})$ such that $\sum_i n_i = N$. Let S_n the set of maps

$$S_n = \{s : [1, \dots, N] \to [1, \dots, d+1] \mid \forall\, j = 1, \dots, d+1,\ \#\{i \mid s(i) = j\} = n_j\}. \tag{2-14}$$

  Notice that $s \in S_n \implies s \circ \sigma \in S_n$.

- Define the polynomials of one variable

$$f_j(x) = \prod_{j' \neq j} \frac{x - \xi_{j'}}{\xi_j - \xi_{j'}} \ . \tag{2-15}$$

  From these polynomials, let us build symmetric polynomials of N variables $p_{r,m}$, for any $(d+1)$-uple $m = (m_1, \dots, m_{d+1})$ with $\sum_i m_i = N$:

$$p_{r,m}(x_1, \dots, x_N) = \frac{\prod_{i=1}^N x_i^r}{\# S_m} \sum_{s \in S_m} \prod_{i=1}^N f_{s(i)}(x_i)\ , \tag{2-16}$$

  Notice that $s \in S_m \implies s \circ \sigma \in S_m$ for all permutation $\sigma \in \mathfrak{S}_N$ and $p_{r,m}$ is a symmetric polynomial.

- Let $\gamma^n = \mathrm{Sym}(\gamma_1^{n_1} \times \dots \gamma_{d+1}^{n_{d+1}}) \in \hat{H}_N^{(r)}$, and $\tilde{s} \in S_n$.

  For large r, rewrite

  - if $\xi_{\tilde{s}(i)}$ is close to a finite pole p:

$$x_i - p = (\xi_{\tilde{s}(i)} - p)(1 + r^{-\frac{1}{2}} u_i). \tag{2-17}$$

- or if $\xi_{\tilde{s}(i)}$ is large $\sim O(r^{-\frac{1}{d+1}})$, use the same writing with $p=0$:

$$x_i = \xi_{\tilde{s}(i)}(1 + r^{-\frac{1}{2}} u_i). \tag{2-18}$$

In all cases we have

$$e^{-V_r(x_i)} \sim e^{-V_r(\xi_{\tilde{s}(i)})} e^{-\frac{1}{2r} V_r''(\xi_{\tilde{s}(i)})(\xi_{\tilde{s}(i)}-p)^2 u_i^2} (1 + O(r^{-1/2})) \tag{2-19}$$

and

$$f_{s(i)}(x_i) \sim \prod_{j\neq s(i)} \left(\frac{\xi_{\tilde{s}(i)} - \xi_j}{\xi_{s(i)} - \xi_j} + r^{-1/2} u_i \frac{\xi_{\tilde{s}(i)} - p}{\xi_{s(i)} - \xi_j} \right) \tag{2-20}$$

If $s(i) = \tilde{s}(i)$ we have

$$f_{s(i)}(x_i) \sim 1 + O(r^{-1/2}), \tag{2-21}$$

and if $s(i) \neq \tilde{s}(i)$ we have

$$f_{s(i)}(x_i) \sim r^{-1/2} u_i \, \frac{\xi_{\tilde{s}(i)} - p}{\xi_{\tilde{s}(i)} - \xi_{s_i}} \, \frac{Q'(\xi_{\tilde{s}(i)})}{Q'(\xi_{s(i)})} (1 + O(r^{-1/2})), \tag{2-22}$$

Remark that in all cases

$$\frac{\xi_{\tilde{s}(i)} - p}{\xi_{\tilde{s}(i)} - \xi_{s_i}} = O(1). \tag{2-23}$$

This implies that

$$\begin{aligned} \prod_{i=1}^{N} f_{s(i)}(x_i) \quad &\sim \quad \prod_i \left(\delta_{s(i),\tilde{s}(i)} + O(r^{-1/2})\right) \prod_i \frac{Q'(\xi_{\tilde{s}(i)})}{Q'(\xi_{s(i)})} \\ &\sim \quad \prod_i \left(\delta_{s(i),\tilde{s}(i)} + O(r^{-1/2})\right) \prod_a Q'(\xi_a)^{n_a - m_a} \end{aligned}$$

(2-24)

- Asymptotic of the Vandermonde

$$\begin{aligned} \Delta(X)^2 \quad \sim \quad & \prod_{a<b} (\xi_a - \xi_b)^{2 n_a n_b} \prod_{a=1}^{d+1} r^{-\frac{1}{2} n_a (n_a - 1)} (\xi_a - p_a)^{n_a (n_a - 1)} \\ & \prod_{a=1}^{d+1} \prod_{i<j,\ \tilde{s}(i)=\tilde{s}(j)=a} (u_i - u_j)^2. \end{aligned} \tag{2-25}$$

- For large r, and $\gamma^n = \mathrm{Sym}(\gamma_1^{n_1} \times \dots \gamma_{d+1}^{n_{d+1}}) \in \hat{H}_N^{(r)}$, by the Laplace steepest descent method we have

$$\mathbb{E}_{\gamma^n}(p_{r,m}) \underset{r\to\infty}{\sim} \prod_{1\leq i<j\leq d+1} (\xi_i - \xi_j)^{2 n_i n_j} \prod_{i=1}^{d+1} r^{-\frac{1}{2} n_i (n_i - 1)} (\xi_i - p_i)^{n_i (n_i - 1)}$$

$$\prod_{j=1}^{d+1} e^{-n_j V_r(\xi_j)} (V_r''(\xi_j))^{-\frac{1}{2}n_j^2} C_{n_j}\ Q'(\xi_j)^{n_j-m_j}$$
$$\left(\delta_{n,m} + O(r^{-1/2})\right)\ . \tag{2-26}$$

where

$$C_n = \int_{\mathbb{R}^n} \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^{n} e^{-\frac{1}{2}x_i^2} dx_i. \tag{2-27}$$

For us, what matters is that $C_n \neq 0$ and is independent of r. The exact value of C_n is known and worth

$$C_n = (2\pi)^{n/2} \prod_{k=0}^{n-1} k!. \tag{2-28}$$

- Let $\Gamma = \sum_n c_n \gamma^n$ be a nonzero element of $\hat{H}_N^{(r)}$.

  Let J be the set of $(d+1)$-uples n such that $c_n \neq 0$.

  The idea will be to choose $n_{\max} \in J$ that maximizes the asymptotic behavior. Generically, $n_{\max}$ is a unique maximum, and we conclude that

$$\mathbb{E}_\Gamma(p_{r,n_{\max}}) \neq 0 \tag{2-29}$$

  which implies $\operatorname{Ker} \mathbb{E} = 0$.

  To be more precise, let us define an order relation in J:

  $n \leq \tilde{n}$ iff as $r \to +\infty$

$$\frac{A(n)}{A(\tilde{n})} = O(1) \tag{2-30}$$

  where

$$A(n) = \prod_{1\leq i<j\leq d+1} (\xi_i - \xi_j)^{2n_i n_j} \prod_{i=1}^{d+1} r^{-\frac{1}{2}n_i(n_i-1)} (\xi_i - p_i)^{n_i(n_i-1)}$$
$$\prod_{j=1}^{d+1} e^{-n_j V_r(\xi_j)} (V_r''(\xi_j))^{-\frac{1}{2}n_j^2} C_{n_j}\ Q'(\xi_j)^{n_j} \tag{2-31}$$

  Let $J_{\max} \subset J$ the set of maximal elements. Let $m \in J_{\max}$. We then have

$$\lim_{r\to\infty} \frac{\mathbb{E}_\Gamma(p_{r,m}) \prod_j Q'(\xi_j)^{m_j}}{A(m)} = c_m \neq 0. \tag{2-32}$$

  Indeed all n's that belong to $J \setminus J_{\max}$ get damped because they are not maximal, and all $n \in J_{\max}$ get a factor $(\delta_{n,m} + O(r^{-1/2}))$, so that only c_m remains in the limit.

  This shows that $\Gamma \neq 0 \implies \mathbb{E}_\Gamma \neq 0$, in other words $\mathbb{E}$ is injective.

□

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