Equivariant connective K-theory

Nikita KARPENKO and Alexander MERKURJEV

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Novembre 2019

IHES/M/19/15

EQUIVARIANT CONNECTIVE K-THEORY

NIKITA A. KARPENKO AND ALEXANDER S. MERKURJEV

Abstract. For separated schemes of finite type over a field with an action of an affine group scheme of finite type, we construct the bi-graded equivariant connective K-theory mapping to the equivariant K-homology of Guillot and the equivariant algebraic K-theory of Thomason. It has all the standard basic properties as the homotopy invariance and localization. We also get the equivariant version of the Brown-Gersten-Quillen spectral sequence and study its convergence.

Contents

1. Introduction

Let G be an algebraic group acting on an algebraic variety X of dimension n over a field F. Following Totaro's paper [17], Edidin and Graham defined in [3] the G-equivariant Chow groups $\mathrm{CH}_i(G,X)$ of X as follows. Choose a finite dimensional vector space V with a linear G-action having the property that there is a G-invariant open subset $U \subset V$ such that $\mathrm{codim}_V(V \setminus U) > n-i$ and there exists a G-torsor $U \to U/G$ with U/G a variety. Then $\mathrm{CH}_i(G,X)$ is defined as an appropriate Chow group of the algebraic space $(X \times U)/G$. This is independent of the choice of V and U.

The situation changes if we replace Chow groups by another homology theory H, for example, cobordism theory of [9]: the homology groups $H((X \times U)/G)$ do depend on the choice of V and U. To remedy the situation one can take the (inverse) limit of homology groups $H((X \times U)/G)$ over all U. This approach was adopted in [6]. It has two disadvantages. First, the homology groups are large. For example, the G-equivariant group K_0 of the point $\mathrm{pt} = \operatorname{Spec} F$ defined this way coincides with the completion of

Date: October 30, 2019.

A part of the work by the first author has been done during his visit of the Institut des Hautes Etudes Scientifiques in September-October 2019. The second author has been supported by the NSF grant DMS #1801530.

the representation ring $R(G)$ with respect to the fundamental ideal. Second, the proofs of some standard properties are quite involved. (Note that the proof of the localization property for the equivariant cobordism theory given in [6] is incorrect.)

In the case of algebraic K-theory there is an alternative way to define the equivariant K-groups of X as the K-groups of the category $\mathcal{M}(G,X)$ of coherent G-modules on X. These K-groups are smaller: the group $K_0(G,\mathrm{pt})$ coincides with $R(G)$, not the completion.

The theory of Chow groups is a particular case of a bi-graded theory in two ways. First of all, Chow groups are the K-homology groups $A_{p,q}$ when $p+q=0$ (see Section 2). On the other hand, Chow groups are a special case of the motivic homology groups.

In the motivic world the smallest bi-graded theory that maps both to the motivic homology and algebraic K-theory is the connective K-theory. In [2], Cai has constructed a variant of the connective K-theory that maps to the K-homology and algebraic K-theory. The connective K-groups of a variety X arise as the terms $D^2_{p,q}$ in the Brown-Gersten-Quillen (BGQ) spectral sequence that starts from K-homology groups of X and converges to the K-theory of X (with the topological filtration).

In the present paper we define an equivariant analog of Cai's connective K-theory of a G-variety X. The straightforward generalization of Cai's approach does not work by the same reason as one cannot define the equivariant Chow groups using just G-invariant cycles on X – there are not enough of them (look at $X=\mathrm{pt}$). Instead, we apply the BGQ spectral sequence to $X\times V$ for a friendly G-space V. Note that the G-equivariant K-homology and, in particular, the G-equivariant Chow groups can also be obtained this way (see Corollary 3.3).

The equivariant connective homology K-theory we construct is bi-graded. Even though "zero diagonal" homology groups are of primary interest, the higher homology groups are very useful in various inductive proofs. We prove standard properties of the theory such as localization and homotopy invariance (see Sections 3, 5 and 6).

We construct an equivariant analog of the BGQ spectral sequence and study its convergence properties (see Section 4). We prove that there is a natural homomorphism from the subsequent factor groups of the topological filtration on the G-equivariant K-groups of X to the E^∞-terms of the spectral sequence and show that this homomorphism is always injective but may not be surjective (Example 7.2). However it is always an isomorphism at the zero diagonal. Some examples are provided in Section 7.

We use the following terminology in the paper. We don't impose any restrictions on the base field F. A variety over F in the paper is a separated scheme of finite type over F. An algebraic group over F is an affine group scheme of finite type over F.

2. K-homology

Our references for algebraic spaces are [8], [10], [14].

Consider an algebraic space X which is of finite type and quasi-separated over a field F. Write $|X|$ for the set of points of X and $F(x)$ for the residue field of $x\in|X|$. The dimension of x is the transcendence degree of $F(x)$ over F. Denote $X_{(p)}$ the set of all points $x\in|X|$ of dimension p.

An elementary étale neighborhood of a point x is an étale morphism $(U,u)\to(X,x)$, where U is a scheme, $u\in U$ is a point mapping to x, and $F(x)\to F(u)$ is an isomorphism (see [14, 0EMV]). The *henselian local ring* $\mathcal{O}^h_{X,x}$ of X at x is the henselization $\mathcal{O}^h_{U,u}$ of the local ring $\mathcal{O}_{U,u}$ of U at u. This is independent of the choice of an elementary étale neighborhood (U,u) of x. There is a canonical morphism

$$t_x : \operatorname{Spec} \mathcal{O}^h_{X,x} \to X.$$

Denote $\mathcal{M}_p(X)$ the category of coherent $\mathcal{O}_X$-modules on X with support of dimension at most p and set

$$\mathcal{M}_{i/j}(X) = \mathcal{M}_i(X)/\mathcal{M}_j(X)$$

for $i > j$.

For every $x \in |X|$ write $\mathcal{C}_x$ for the abelian category of finitely generated $\mathcal{O}^h_{X,x}$-modules killed by a power of the maximal ideal. For every quasi-coherent module A on X and $x \in |X|$ denote A_x the $\mathcal{O}^h_{X,x}$-module $t^*_x(A)$. If A is in $\mathcal{M}_p(X)$ and $x \in X_{(p)}$, then A_x lies in $\mathcal{C}_x$. Thus, we have an exact functor

$$\mathcal{M}_p(X) \to \coprod_{x \in X_{(p)}} \mathcal{C}_x, \quad A \mapsto (A_x).$$

This functor takes the subcategory $\mathcal{M}_{p-1}(X)$ to zero, hence it yields a functor

$$\alpha : \mathcal{M}_{p/(p-1)}(X) \to \coprod_{x \in X_{(p)}} \mathcal{C}_x.$$

The authors are grateful to Johan de Jong who provided a proof of the following proposition.

Proposition 2.1. *The functor α is an equivalence of categories.*

Proof. Let M be a module in $\mathcal{C}_x$ for some $x \in X_{(p)}$. Then $N := (t_x)_*(M)$ is a quasi-coherent module on X such that $N_{x'} \simeq M$ if $x' = x$ and $N_{x'} = 0$ if $\dim(x') \geq p$ and $x' \neq x$. By [14, Tag 07UV], N is the filtered colimit of its coherent submodules. It follows that there is a coherent submodule $A \subset N$ such that $A_x = N_x \simeq M$ and $A_x = 0$ for all $x' \in |X|$ with $\dim(x') \geq p$ and $x' \neq x$. Then $A \in \mathcal{M}_p(X)$ and $\alpha(A) \simeq M$, i.e., the functor α is essentially surjective.

Let A and B be coherent modules in $\mathcal{M}_p(X)$. We need to prove that the natural map

$$\varphi : \operatorname{Hom}_{\mathcal{M}_{p/(p-1)}(X)}(A, B) \to \coprod_{x \in X_{(p)}} \operatorname{Hom}_{\mathcal{C}_x}(A_x, B_x)$$

is an isomorphism. To prove injectivity consider a morphism $f : A \to B$ such that $f_x : A_x \to B_x$ is trivial for all $x \in X_{(p)}$. Since $\operatorname{Im}(f)_x = 0$ for all $x \in X_{(p)}$, the module $\operatorname{Im}(f)$ is in $\mathcal{M}_{p-1}(X)$, i.e., it represents a zero object in the factor category $\mathcal{M}_{p/(p-1)}(X)$. As a result, $f = 0$ in $\mathcal{M}_{p/(p-1)}(X)$.

Now we turn to the proof of the surjectivity. For any coherent module C in $\mathcal{M}_p(X)$ denote $\widetilde{C}$ the direct sum of the modules $(t_x)_*(C_x)$ over all $x \in X_{(p)}$. Note that there is a natural homomorphism $\lambda_C : C \to \widetilde{C}$ of quasi-coherent modules on X such that the induced homomorphism $C_x \to \widetilde{C}_x$ is an isomorphism for all $x \in X_{(p)}$. Denote $\overline{C} \subset \widetilde{C}$ the image of λ_C. Then $C_x = \overline{C}_x$ for all $x \in X_{(p)}$ and hence $C \simeq \overline{C}$ in $\mathcal{M}_{p/(p-1)}(X)$.

Let $s_x : A_x \to B_x$ be morphisms in $\mathcal{C}_x$ for all $x \in X_{(p)}$. The collection (s_x) yields a morphism $s : \widetilde{A} \to \widetilde{B}$. Write A' for the submodule $s^{-1}(\overline{B})$ in $\overline{A}$ and set $A'' := \overline{A} \cap A'$.

We have $A_x = A'_x = A''_x$ for all $x \in X_{(p)}$, hence the natural homomorphism $A'' \to A$ is an isomorphism in $\mathcal{M}_{p/(p-1)}(X)$. If g is the composition

$$A \xrightarrow{\sim} A'' \xrightarrow{s} \overline{B} \xrightarrow{\sim} B$$

in $\mathcal{M}_{p/(p-1)}(X)$, we have $\varphi(g) = (s_x)$, i.e., φ is surjective. $\square$

Corollary 2.2. *For every n, there is a natural isomorphism*

$$K_n(\mathcal{M}_{p/(p-1)}(X)) \simeq \coprod_{x\in X_{(p)}} K_n(F(x)). \qquad \square$$

By Corollary 2.2, the connecting homomorphisms for the exact sequences of categories

$$0 \to \mathcal{M}_{p/(p-1)}(X) \to \mathcal{M}_{(p+1)/(p-1)}(X) \to \mathcal{M}_{(p+1)/p}(X) \to 0$$

yield a complex

$$C_{p,q}(X) := \Big[\coprod_{x\in X_{(p+1)}} K_{p+q+1}(F(x)) \to \coprod_{x\in X_{(p)}} K_{p+q}(F(x)) \to \coprod_{x\in X_{(p-1)}} K_{p+q-1}(F(x))\Big].$$

Denote $A_{p,q}(X)$ the homology group of this complex and call it a *K-homology group of X*.

Let X be an algebraic space which is of finite type and quasi-separated over a field F, $U \subset X$ an open subspace and $Z \subset X$ a closed subspace such that $|X|$ is the disjoint union of $|Z|$ and $|U|$. We have a short exact sequence of complexes

$$0 \to C_{p,q}(Z) \to C_{p,q}(X) \to C_{p,q}(U) \to 0$$

and therefore a long localization exact sequence

$$\dots \to A_{p+1,q}(U) \to A_{p,q}(Z) \to A_{p,q}(X) \to A_{p,q}(U) \to A_{p-1,q}(Z) \to \dots$$

Let $f : X' \to X$ be a flat morphism of algebraic spaces (of finite type and quasi-separated over F) of relative dimension d. If A is a module in $\mathcal{M}_p(X)$, then $f^*(A)$ is in $\mathcal{M}_{p+d}(X')$. The induced functor

$$\mathcal{M}_{p/(p-1)}(X) \to \mathcal{M}_{(p+d)/(p+d-1)}(X')$$

yields a morphism of complexes $C_{p,q}(X) \to C_{p+d,q-d}(X')$ and finally, the pull-back homomorphism

$$f^* : A_{p,q}(X) \to A_{p+d,q-d}(X').$$

Proposition 2.3. *If $f : E \to X$ is an affine G-bundle (a torsor under a vector G-bundle) of constant rank r over an algebraic space X which is of finite type and quasi-separated over a field F, then the pull-back homomorphism*

$$f^* : A_{p,q}(X) \to A_{p+r,q-r}(E)$$

is an isomorphism.

Proof. We induct on the dimension of X. By [10, Theorem 6.4.1], there is a dense open subscheme $U \subset X$ and let $Z \subset X$ be a closed subspace such that $|X|$ is the disjoint union of $|Z|$ and $|U|$ (see [14, Tag 03IQ]). Let E_U and E_Z be the restrictions of E to U and Z respectively. There is a morphism of the localization long exact sequence for the triple (X, U, Z) and for (E, E_U, E_Z). The pull-back maps $f_U^* : A_{p,q}(U) \to A_{p+r,q-r}(E_U)$ are isomorphisms by the homotopy invariance of K-homology for varieties ([13, Theorem

3.3] or [12, Proposition 8.6]) and the pull-back maps f_Z^* are isomorphisms by induction. The statement of the proposition follows by the 5-lemma. □

Example 2.4. Let an algebraic group G act freely on a variety X over F, i.e., the group $G(R)$ acts freely on $X(R)$ for every commutative F-algebra R. Then the stack X/G is an algebraic space which is of finite type and quasi-separated over F by [14, Tag 06PH]. In some cases X/G is a scheme, for example, when X is quasi-projective with a linearized G-action, or G is connected and X is equivariantly embedded as a closed subscheme of a normal variety (see [3, Proposition 23]).

Let V be a vector G-space of finite dimension with the property that there is a G-invariant open subset $U \subset V$ such that $\operatorname{codim}_V(V \setminus U) \geq n$ and there exists a G-torsor $U \to U/G$ with U/G a variety. We call V an *n-friendly space* and U an *n-friendly subset.* It was shown in [17, Remark 1.4] that n-friendly spaces exist for every n.

Let X be a G-variety. For a G-variety S write X_S for the product $X \times S$. Let U be an n-friendly subset. Then G acts freely on X_U, so that X_U/G is an algebraic space which is of finite type and quasi-separated over F and the *equivariant K-homology group*

$$A_{p,q}(G,X) := A_{p+v-g,q-v+g}(X_U/G),$$

where $v = \dim(V)$ and $g = \dim(G)$, is independent of the choice of U and V if $n > \dim(X) - p + 1$ (see [5]).

3. CK-homology

Let G be an algebraic group over F and X be a G-variety over F. Denote $\mathcal{M}(G,X)$ the abelian category of all coherent G-modules on X (see [15]). For every integer p write $\mathcal{M}_p(G,X)$ for the abelian category of coherent G-modules M on X such that $\dim(\operatorname{Supp}(M)) \leq p$ and $\mathcal{M}_{i/j}(G,X)$ for the factor category $\mathcal{M}_i(G,X)/\mathcal{M}_j(G,X)$ for $i > j$. In particular, $\mathcal{M}_p(G,X)$ coincides with the category $\mathcal{M}(G,X)$ for $p \geq \dim(X)$.

Set

$$D^1_{p,q}(X) = K_{p+q}(\mathcal{M}_p(G,X)), \qquad E^1_{p,q}(X) = K_{p+q}(\mathcal{M}_{p/(p-1)}(G,X)).$$

We have the exact sequences

$$\dots \to E^1_{p,q+1}(X) \to D^1_{p-1,q+1}(X) \to D^1_{p,q}(X) \to E^1_{p,q}(X) \to \dots$$

which form an exact couple

$$\begin{tikzcd} D^1_{*,*}(X) \arrow[rr] & & D^1_{*,*}(X) \arrow[dl] \\ & E^1_{*,*}(X) \arrow[ul] & \end{tikzcd}$$

Note that $D^1_{p,q}(X) = K'_{p+q}(G,X) := K_{p+q}(\mathcal{M}(G,X))$ if $p \geq \dim(X)$ and $E^1_{p,q}(X) = 0$ if $p > \dim(X)$.

The pull-back functor for a flat G-equivariant morphism $f : X' \to X$ of relative dimension d yields functors $f^* : \mathcal{M}_p(G,X) \to \mathcal{M}_{p+d}(G,X')$ for every p. In particular, there is a morphism of the exact couple for X to the one for X' shifting bi-indices by $(d,-d)$.

Lemma 3.1. *Let X be a G-variety, $Y \subset X$ a closed G-invariant subvariety and $X' = X \setminus Y$. Then*

(1) *The natural homomorphism*
$$E^1_{p,q}(X) \to E^1_{p,q}(X')$$
is an isomorphism for all $p > \dim(Y)$ and q.

(2) *The natural homomorphism*
$$E^2_{p,q}(X) \to E^2_{p,q}(X')$$
is an isomorphism for all $p > \dim(Y) + 1$ and q.

Proof. Let $\mathcal{C}$ be the full subcategory of $\mathcal{M}(G, X)$ of coherent modules with support in Y. Then $\mathcal{C} \subset \mathcal{M}_i(G, X)$ for all $i \geq \dim(Y)$ and the factor category $\mathcal{M}_i(G, X)/\mathcal{C}$ is equivalent to $\mathcal{M}_i(G, X')$. It follows that the category $\mathcal{M}_{p/(p-1)}(G, X)$ is equivalent to $\mathcal{M}_{p/(p-1)}(G, X')$ for $p > \dim(Y)$. This proves (1). For the proof of (2) notice that the differential $E^1_{p,q}(X) \to E^1_{p-1,q}(X)$ is the connecting homomorphism for the exact sequence
$$0 \to \mathcal{M}_{(p-1)/(p-2)}(G, X) \to \mathcal{M}_{p/(p-2)}(G, X) \to \mathcal{M}_{p/(p-1)}(G, X) \to 0$$
of abelian categories. □

Recall that $g = \dim(G)$.

Lemma 3.2. *If G acts freely on a variety X, then*
$$E^1_{p,q}(X) = \coprod_{y \in (X/G)_{(p-g)}} K_{p+q}(F(y)), \quad E^2_{p,q}(X) = A_{p-g,q+g}(X/G).$$

Proof. The morphism $f : X \to X/G$ is a G-torsor. To give a coherent module A on X/G is the same as to give a coherent G-module B on X (see [18, §4.4]). Moreover, $B = f^*(A)$ and $\dim \operatorname{Supp}(f^*(A)) = \dim \operatorname{Supp}(A) + g$. Therefore, there are equivalences of categories
$$\mathcal{M}_p(G, X) \xrightarrow{\sim} \mathcal{M}_{p-g}(X/G), \qquad \mathcal{M}_{p/(p-1)}(G, X) \xrightarrow{\sim} \mathcal{M}_{(p-g)/(p-g-1)}(X/G).$$
By Corollary 2.2,
$$E^1_{p,q}(X) = K_{p+q}(\mathcal{M}_{(p-g)/(p-g-1)}(X/G)) = \coprod_{y \in (X/G)_{(p-g)}} K_{p+q}(F(y))$$
and hence
$$E^2_{p,q}(X) = A_{p-g,q+g}(X/G).$$
□

Corollary 3.3. *Let V be an n-friendly G-space and $v = \dim(V)$. Then*
$$E^2_{p,q}(X_V) \simeq A_{p-v,q+v}(G, X)$$
for $p > \dim(X) + v - n + 1$.

Proof. Let U be an n-friendly subset of V, $Z = V \setminus U$ and $p > \dim(X) + v - n + 1$. By assumption, $\dim(Z) \leq v - n$. Applying Lemma 3.1 to the closed subset X_Z in X_V, we have
$$E^2_{p,q}(X_V) \simeq E^2_{p,q}(X_U)$$
since $p > \dim(X) + v - n + 1 \geq \dim(X_Z) + 1$.

Note that G acts freely on X_U. By Lemma 3.2, applied to the G-variety X_U,

$$E^2_{p,q}(X_U) = A_{p-g,q+g}(X_U/G) = A_{p-v,q+v}(G,X). \qquad \square$$

Set

$$\begin{aligned} \widetilde{\mathrm{CK}}_{p,q}(G,X) := D^2_{p+1,q-1}(X) &= \mathrm{Im}(D^1_{p,q}(X) \to D^1_{p+1,q-1}(X)) \\ &= \mathrm{Coker}(E^1_{p+1,q}(X) \to D^1_{p,q}(X)) \\ &= \mathrm{Ker}(D^1_{p+1,q-1}(X) \to E^1_{p+1,q-1}(X)). \end{aligned} \tag{3.4}$$

Note that

$$\widetilde{\mathrm{CK}}_{p,q}(G,X) = K'_{p+q}(G,X)$$

if $p \geq \dim(X)$.

We have an exact sequence

$$\ldots \to E^2_{p+1,q}(X) \to \widetilde{\mathrm{CK}}_{p-1,q+1}(G,X) \to \widetilde{\mathrm{CK}}_{p,q}(G,X) \to E^2_{p,q}(X) \to \ldots \tag{3.5}$$

A G-equivariant flat morphism $f : X' \to X$ of relative dimension d yields a homomorphism

$$\widetilde{\mathrm{CK}}_{p,q}(G,X) \to \widetilde{\mathrm{CK}}_{p+d,q-d}(G,X')$$

for all p and q.

Lemma 3.6. *Let V be an n-friendly G-space, $v = \dim(V)$ and W any vector G-space of finite dimension w. Then the natural homomorphism*

$$\widetilde{\mathrm{CK}}_{p+v,q-v}(G,X_V) \to \widetilde{\mathrm{CK}}_{p+v+w,q-v-w}(G,X_{V\times W})$$

is an isomorphism for every $p > \dim(X) - n + 1$ and q.

Proof. Let $\mathcal{E}(X)$ be the exact sequence (3.5). Consider the natural morphism

$$\theta : \mathcal{E}(X_V) \to \mathcal{E}(X_{V\times W})$$

of exact sequences of bi-degree $(w,-w)$ (i.e., the bi-index is shifted by $(w,-w)$).

Let U be an n-friendly subset of V. By Corollary 3.3 applied twice for the n-friendly spaces V and $V \times W$, we have

$$E^2_{p+v,q-v}(X_V) \simeq A_{p,q}(G,X) \simeq E^2_{p+v+w,q-v-w}(X_{V\times W})$$

if $p > \dim(X) - n + 1$, i.e., θ is an isomorphism on the terms $E^2_{p+v,q-v}(X_V)$ in this range.

If $p \geq \dim(X)$ then

$$\widetilde{\mathrm{CK}}_{p+v,q-v}(G,X_V) = K'_{p+q}(G,X_V) \simeq K'_{p+q}(G,X_{V\times W}) = \widetilde{\mathrm{CK}}_{p+v+w,q-v-w}(G,X_{V\times W})$$

by homotopy invariance in the equivariant K-theory (see [15, Theorem 4.1]), i.e., θ is an isomorphism on the terms $\widetilde{\mathrm{CK}}_{p+v,q-v}(G,X_V)$ in this range.

Descending induction on p and the 5-lemma applied to the homomorphism θ yield the statement in general. $\square$

It follows from Lemma 3.6 that the group $\widetilde{\mathrm{CK}}_{p+v,q-v}(G, X_V)$ is canonically independent of the choice of the n-friendly G-space V for $n > \dim(X) - p + 1$. We define the *G-equivariant connective K-groups*

$$\mathrm{CK}_{p,q}(G, X) := \widetilde{\mathrm{CK}}_{p+v,q-v}(G, X_V). \tag{3.7}$$

In the case of trivial G, these groups coincide with the connective K-groups of [2].

Remark 3.8. It is crucial that friendly G-spaces V instead of friendly subsets U (used in the definition of G-equivariant K-homology) are employed in the definition of G-equivariant connective K-groups: the stabilization we have for V by Lemma 3.6 fails for U. For instance, for $G = \mathbb{G}_m$, n-friendly subset $U = \mathbb{A}^n_F \setminus \{0\} \subset \mathbb{A}^n_F = V$, and any $p \geq n$, the group

$$\widetilde{\mathrm{CK}}_{p,-p}(G, U) = K'_0(G, U) = K'_0(U/G) = K'_0(\mathbb{P}^{n-1}_F)$$

is free of finite rank n.

Let X be a G-variety and p an integer. Write $K'_n(G, X)_{(p)}$ for the subgroup in $K'_n(G, X)$ generated by the images of the homomorphisms

$$K_n(\mathcal{M}_{p+w}(G, X_W)) \to K_n(\mathcal{M}(G, X_W)) = K'_n(G, X_W) \simeq K'_n(G, X) \tag{3.9}$$

for all vector G-spaces W of finite dimension w. We obtain a filtration

$$\cdots \subset K'_n(G, X)_{(p-1)} \subset K'_n(G, X)_{(p)} \subset \cdots$$

on $K'_n(G, X)$, generalizing the Chow filtration on $R(G)$ defined in [7]. Note that

$$K'_n(G, X)_{(p)} = K'_n(G, X)$$

if $p \geq \dim(X)$.

Theorem 3.10. *Let G be an algebraic group over F. The G-equivariant connective K-groups have the following properties.*

(1) *The assignment $X \mapsto \mathrm{CK}_{*,*}(G, X)$ is a functor from the category of G-varieties and proper morphisms to the category of bi-graded abelian groups.*
(2) *If $f : X' \to X$ is a flat G-equivariant morphism of relative dimension d, there is a functorial pull-back homomorphism*

$$f^* : \mathrm{CK}_{p,q}(G, X) \to \mathrm{CK}_{p+d,q-d}(G, X').$$

(3) *There is an exact sequence*

$$\ldots \to A_{p+1,q}(G, X) \to \mathrm{CK}_{p-1,q+1}(G, X) \xrightarrow{\beta_X} \mathrm{CK}_{p,q}(G, X) \to A_{p,q}(G, X) \to \ldots$$

We call β_X the Bott map.
(4) *There is a natural surjective homomorphism $\mathrm{CK}_{p,q}(G, X) \to K'_{p+q}(G, X)_{(p)}$. This is an isomorphism if $p \geq \dim(X)$.*
(5) *Let $i : Z \to X$ be a G-equivariant closed embedding, $X' = X \setminus Z$, and $j : X' \hookrightarrow X$ the open embedding. Then there is a* localization exact sequence

$$\ldots \to \mathrm{CK}_{p,q}(G, Z) \xrightarrow{i_*} \mathrm{CK}_{p,q}(G, X) \xrightarrow{j^*} \mathrm{CK}_{p,q}(G, X') \to \mathrm{CK}_{p-1,q}(G, Z) \xrightarrow{i_*} \ldots$$

(6) *If $f : E \to X$ is an affine G-bundle (a torsor under a vector G-bundle) of constant rank r, then the pull-back homomorphism*

$$f^* : \mathrm{CK}_{p,q}(G,X) \to \mathrm{CK}_{p+r,q-r}(G,E)$$

is an isomorphism.

(7) *Viewing the graded group $\mathrm{CK}_{*,*}(G,X)$ as a $\mathbb{Z}[s]$-module via $sm = \beta_X(m)$, we have an isomorphism*

$$\left(\coprod_{p+q=n} \mathrm{CK}_{p,q}(G,X)\right)[s^{-1}] \xrightarrow{\sim} K'_n(G,X)[s,s^{-1}]$$

for every n.

Proof. (1) Let $f : X \to Y$ be a proper G-equivariant morphism. Write $\mathcal{C}$ for the full subcategory of $\mathcal{M}_p(G,X)$ of all coherent modules M such that $R^i f_*(M) = 0$ for all $i > 0$. By dévissage, $K_{p+q}(\mathcal{C}) \simeq D^1_{p,q}(X)$. The exact functor $f_* : \mathcal{C} \to \mathcal{M}_p(G,Y)$ then gives a homomorphism $D^1_{p,q}(X) \to D^1_{p,q}(Y)$. The latter yields a push-forward map

$$f_* : \mathrm{CK}_{p,q}(G,X) \to \mathrm{CK}_{p,q}(G,Y).$$

(2) The pull-back homomorphism f^* is induced by the pull-back functor $\mathcal{M}_p(G,X) \to \mathcal{M}_{p+d}(G,X')$.

(3) Apply the exact sequence (3.5) to X_V in place of X for an n-friendly space V with n sufficiently large and use Corollary 3.3.

(4) Let V be an n-friendly space with n large, $v = \dim(V)$. The homomorphism

$$D^1_{p+v,q-v}(X_V) = K_{p+q}(\mathcal{M}_p(G,X_V)) \to K'_{p+q}(G,X)$$

factors through $D^1_{p+v+1,q-v-1}(X_V) = K_{p+q}(\mathcal{M}_{p+1}(G,X_V))$. Therefore, there is a well defined homomorphism

$$h_{p,q} : \mathrm{CK}_{p,q}(G,X) \to K'_{p+q}(G,X)_{(p)}.$$

To show that this map is surjective, let W be a vector G-spaces of dimension w. The homomorphism (3.9) with $n = p+q$ factors through $K_{p+q}(G,\mathcal{M}_{p+v+w}(G,X_{V\times W}))$, hence replacing W by $V \times W$ we may assume that W is an m-friendly space for m sufficiently large. It follows that the image of (3.9) is contained in the image of $h_{p,q}$.

(5) Let V be an n-friendly G-space for sufficiently large n, $v = \dim(V)$. Set $\bar{p} = p + v$ and $\bar{q} = q - v$. Let $\mathcal{C}$ be the subcategory of $\mathcal{M}_{\bar{p}}(G,X_V)$ of all modules with support in Z_V. Then the factor category $\mathcal{M}_{\bar{p}}(G,X_V)/\mathcal{C}$ is equivalent to $\mathcal{M}_{\bar{p}}(G,X'_V)$ and $K_n(\mathcal{C}) = K_n(\mathcal{M}_{\bar{p}}(G,X'_V))$ (see [15, Theorem 2.7]). It follows that the middle row of the diagram

$$\begin{tikzcd}
E^1_{\bar p+1,\bar q}(Z_V) \arrow[r] \arrow[d] & E^1_{\bar p+1,\bar q}(X_V) \arrow[r, two heads, "\alpha"] \arrow[d] & E^1_{\bar p+1,\bar q}(X'_V) \arrow[d] & \mathrm{CK}_{p-1,q}(G,Z) \arrow[r, "i_*"] \arrow[d, hook] & \mathrm{CK}_{p-1,q}(G,X) \arrow[r, "j^*"] \arrow[d, hook] & \mathrm{CK}_{p-1,q}(G,X') \arrow[d, hook] \\
D^1_{\bar p,\bar q}(Z_V) \arrow[r] \arrow[d, two heads] & D^1_{\bar p,\bar q}(X_V) \arrow[r] \arrow[d, two heads] & D^1_{\bar p,\bar q}(X'_V) \arrow[r] \arrow[d, two heads] & D^1_{\bar p,\bar q-1}(Z_V) \arrow[r] \arrow[d] & D^1_{\bar p,\bar q-1}(X_V) \arrow[r] \arrow[d] & D^1_{\bar p,\bar q-1}(X'_V) \arrow[d] \\
\mathrm{CK}_{p,q}(G,Z) \arrow[r, "i_*"] & \mathrm{CK}_{p,q}(G,X) \arrow[r, "j^*"] & \mathrm{CK}_{p,q}(G,X') & E^1_{\bar p,\bar q-1}(Z_V) \arrow[r, hook, "\beta"] & E^1_{\bar p,\bar q-1}(X_V) \arrow[r] & E^1_{\bar p,\bar q-1}(X'_V)
\end{tikzcd}$$

is exact. By Lemma 3.2, α is surjective and β is injective. The columns of the diagram are exact by (3.4). The exact sequence in the statement of (5) can be obtained by diagram chase.

(6) Follows by descending induction on p from (3), (4), homotopy invariance of equivariant K-theory [15, Theorem 4.1], 5-lemma and homotopy invariance of equivariant K-homology (Proposition 2.3).

(7) Follows from (3) and (4). □

Remark 3.11 (Euler classes, Projective Bundle Theorem (PBT), Chern classes)**.** Given a G-equivariant vector bundle E over a G-variety X, its Euler class is, as usual, the composition

$$(\pi^*)^{-1} \circ s_*\colon \operatorname{CK}_{*,*}(G,X) \to \operatorname{CK}_{*-1,*+1}(G,X),$$

where $\pi\colon E \to X$ is the projection and $s\colon X \to E$ is the zero section. With the help of Theorem 3.10(3,4), PBT for equivariant K-theory [15, Theorem 3.1], 5-lemma, and PBT for (equivariant) K-cohomology, one sees by descending induction as in the proof of Theorem 3.10(6) that PBT holds for the equivariant connective K-groups. Therefore (operator) Chern classes

$$c_i(E)\colon \operatorname{CK}_{*,*}(G,X) \to \operatorname{CK}_{*-i,*+i}(G,X)$$

(for $i \geq 0$) of E are defined.

Remark 3.12 (Functoriality in G)**.** Given a homomorphism of algebraic groups $G' \to G$, the restriction of action yields an exact functor $\mathcal{M}_p(G,X) \to \mathcal{M}_p(G',X)$ so that the graded group $\widetilde{\operatorname{CK}}_{*,*}(G,X)$ is cofunctorial in G. Taking a sufficiently friendly G-space V and a sufficiently friendly G'-space V', we get a homomorphism

$$\operatorname{CK}_{p,q}(G,X) = \widetilde{\operatorname{CK}}_{p+v,q-v}(G,X_V) \to \widetilde{\operatorname{CK}}_{p+v,q-v}(G',X_V) \to \\ \widetilde{\operatorname{CK}}_{p+v+v',q-v-v'}(G',X_{V\times V'}) = \operatorname{CK}_{p,q}(G',X)$$

independent of the choice of V and V'. It follows that the graded group $\operatorname{CK}_{*,*}(G,X)$ is cofunctorial in G.

4. Spectral sequence

Let

$$\begin{tikzcd} D^2_{*,*} \arrow[rr, "i"] & & D^2_{*,*} \arrow[dl, "j"] \\ & E^2_{*,*} \arrow[ul, "k"] & \end{tikzcd}$$

be an exact couple (starting from the second page). The maps i, j and k have bi-degrees $(1,-1)$, $(-1,1)$ and $(-1,0)$ respectively. For every $r \geq 2$, the $(r-2)$nd derivative of the exact couple is the diagram

$$\begin{tikzcd} D^r_{*,*} \arrow[rr, "i_r"] & & D^r_{*,*} \arrow[dl, "j_r"] \\ & E^r_{*,*} \arrow[ul, "k_r"] & \end{tikzcd}$$

where

$$D^r_{p,q} = \operatorname{Im}(D^2_{p-r+2,q+r-2} \xrightarrow{i^{r-2}} D^2_{p,q}) \subset D^2_{p,q},$$
$$E^r_{p,q} = k^{-1}(D^r_{p-1,q})/j(\operatorname{Ker}(i^{r-2})) = \operatorname{Ker}(j_{r-1}k_{r-1})/\operatorname{Im}(j_{r-1}k_{r-1}).$$

(In the last formula i^{r-2} is the map $D^2_{p,q} \to D^2_{p+r-2,q-r+2}$.) Thus we have a (homological) spectral sequence $\{E^r_{p,q}\}$.

Define the following two subgroups of $D^2_{p,q}$:

$$D^\infty_{p,q} = \bigcap_r D^r_{p,q}, \qquad D^+_{p,q} = \bigcup_r \operatorname{Ker}(i^r)$$

and the "infinity" terms

$$E^\infty_{p,q} = k^{-1}(D^\infty_{p-1,q})/j(D^+_{p+1,q-1}).$$

For every n let H_n be the colimit of the sequence

$$\dots \xrightarrow{i} D^2_{p,n-p} \xrightarrow{i} D^2_{p+1,n-p-1} \xrightarrow{i} D^2_{p+2,n-p-2} \xrightarrow{i} \dots$$

Write $(H_n)_{(p)}$ for the image of the natural homomorphism $D^2_{p+1,n-p-1} \to H_n$, so we have a filtration

$$\dots \subset (H_n)_{(p-1)} \subset (H_n)_{(p)} \subset (H_n)_{(p+1)} \subset \dots$$

of H_n. We would like to relate the subsequent factor $(H_n)_{(p/p-1)}$ and the infinity term $E^\infty_{p,n-p}$.

Proposition 4.1. *There is an exact sequence*

$$0 \to (H_{p+q})_{(p-1)} \to (H_{p+q})_{(p)} \xrightarrow{\bar{j}} E^\infty_{p,q} \xrightarrow{\bar{k}} D^\infty_{p-1,q} \xrightarrow{\bar{i}} D^\infty_{p,q-1}.$$

In particular, $(H_{p+q})_{(p/p-1)}$ is canonically isomorphic to a subgroup of $E^\infty_{p,q}$.

Proof. An element $\bar{x} \in (H_{p+q})_{(p)}$ is represented by an element $x \in D^2_{p+1,q-1}$ and we set

$$\bar{j}(\bar{x}) = j(x) + j(D^+_{p+1,q-1}) \in E^\infty_{p,q}.$$

An element $e \in E^\infty_{p,q}$ is represented by an element $d \in k^{-1}(D^\infty_{p-1,q})$ and we set $\bar{k}(e) = k(d)$. The map $\bar{i}$ is induced by i. The exactness of the sequence readily follows from the definitions. □

We say that the spectral sequence $\{E^r_{p,q}\}$ *converges at the nth diagonal* if the homomorphism $(H_n)_{(p/p-1)} \to E^\infty_{p,q}$ is an isomorphism for all p and q such that $p + q = n$.

Let X be a G-variety. By Theorem 3.10(3), there is an exact couple with

$$D^2_{p,q} = \mathrm{CK}_{p-1,q+1}(G, X) \quad \text{and} \quad E^2_{p,q} = A_{p,q}(G, X).$$

The map i in the exact couple is the Bott map

$$\beta_X : \mathrm{CK}_{p-1,q+1}(G, X) \to \mathrm{CK}_{p,q}(G, X).$$

In view of Theorem 3.10(4),

$$H_n = K'_n(G,X) \quad \text{and} \quad (H_n)_{(p)} = K'_n(G,X)_{(p)}.$$

Write $\overline{\mathrm{CK}}_{p,q}(G,X)$ for the intersection of $\mathrm{Im}(\mathrm{CK}_{p-r,q+r}(G,X) \xrightarrow{\beta_X^r} \mathrm{CK}_{p,q}(G,X))$ over all $r \geq 0$. Thus,

$$D^{\infty}_{p,q} = \overline{\mathrm{CK}}_{p-1,q+1}(G,X).$$

Proposition 4.1 then gives:

Proposition 4.2. *Let X be a G-variety. Then there is an exact sequence*

$$0 \to K'_{p+q}(G,X)_{(p/p-1)} \to E^{\infty}_{p,q} \to \overline{\mathrm{CK}}_{p-2,q+1}(G,X) \to \overline{\mathrm{CK}}_{p-1,q}(G,X).$$

In particular, $K'_{p+q}(G,X)_{(p/p-1)}$ is canonically isomorphic to a subgroup of $E^{\infty}_{p,q}$.

Note that if $p+q=0$, we have $\overline{\mathrm{CK}}_{p-2,q+1}(G,X) = 0$, hence the spectral sequence converges at the zero diagonal.

5. Gysin homomorphism and external product

Let X be a G-variety and $\mathcal{P}(G,X)$ exact category of locally free coherent G-modules on X. Then $K_*(G,X) := K_*(\mathcal{P}(G,X))$ is a ring and $K'_*(G,X)$ is a module over $K_*(G,X)$.

The tensor product pairing of categories

$$\mathcal{P}(G,X) \times \mathcal{M}_p(G,X) \to \mathcal{M}_p(G,X)$$

for every p yields a pairing

$$K_n(G,X) \otimes D^1_{p,q}(X) \to D^1_{p,q+n}(X)$$

which is consistent with the maps $D^1_{p,*}(X) \to D^1_{p+1,*-1}(X)$. It follows that we have a pairing

$$K_n(G,X) \otimes \widetilde{\mathrm{CK}}_{p,q}(G,X) \to \widetilde{\mathrm{CK}}_{p,q+n}(G,X).$$

Let V be an m-friendly G-space and $p > \dim(X) - m + 1$. Then by (3.7) we have a pairing

$$K_n(G,X) \otimes \mathrm{CK}_{p,q}(G,X) \to K_n(G,X_V) \otimes \mathrm{CK}_{p,q}(G,X) \to \mathrm{CK}_{p,q+n}(G,X).$$

Thus for every p, the graded group $\mathrm{CK}_{p,*}(G,X)$ is a left graded module over $K_*(G,X)$.

Let $f: Y \to X$ be a closed G-equivariant embedding. Write D_f for the deformation variety of f, $C_f \subset D_f$ the normal cone of f. The open complement $D_f \setminus C_f$ is canonically isomorphic to $X \times \mathbb{G}_m$. We view $\mathbb{G}_m$ as a G-variety with trivial G-action.

The coordinate function t on $\mathbb{G}_m$ can be viewed as an element in $K_1(G, X \times \mathbb{G}_m)$. The composition

$$\sigma_f : \mathrm{CK}_{p,q}(G,X) \to \mathrm{CK}_{p+1,q-1}(G, X \times \mathbb{G}_m) \xrightarrow{t} \mathrm{CK}_{p+1,q}(G, X \times \mathbb{G}_m) \xrightarrow{\partial} \mathrm{CK}_{p,q}(G, C_f),$$

where the second homomorphism is multiplication by $t \in K_1(G, X \times \mathbb{G}_m)$ and the third one is the connecting map in the localization exact sequence for the closed subvariety C_f in D_f, is called the *deformation homomorphism*.

Let $f : Y \to X$ be a regular closed G-equivariant embedding of codimension d. The normal cone C_f is a vector bundle over Y of rank d. The composition

$$f^\star : \mathrm{CK}_{p,q}(G, X) \xrightarrow{\sigma_f} \mathrm{CK}_{p,q}(G, C_f) \xrightarrow{\sim} \mathrm{CK}_{p-d,q+d}(G, Y),$$

where the second isomorphism is the inverse to the homotopy invariance isomorphism, is called *Gysin homomorphism*.

Let X and X' be two G-varieties. The obvious pairing of categories

$$\mathcal{M}_p(G, X) \times \mathcal{M}_{p'}(G, X') \to \mathcal{M}_{p+p'}(G, X \times X')$$

yields a pairing

$$D^1_{p,q}(X) \otimes D^1_{p',q'}(X') \to D^1_{p+p',q+q'}(X \times X')$$

which is consistent with the maps $D^1_{p,q}(X) \to D^1_{p+1,q-1}(X)$ and $D^1_{p',q'}(X') \to D^1_{p'+1,q'-1}(X')$. It follows that we have a pairing

$$\times : \mathrm{CK}_{p,q}(G, X) \otimes \mathrm{CK}_{p',q'}(G, X') \to \mathrm{CK}_{p+p',q+q'}(G, X \times X'),$$

called the *external product*.

6. Smooth varieties

Let X be a smooth G-variety over F. If X is equidimensional, we define the *K-cohomology groups of X*

$$A^{p,q}(G, X) = A_{d-p,-d-q}(G, X)$$

and the CK-*cohomology groups*

$$\mathrm{CK}^{p,q}(G, X) := \mathrm{CK}_{d-p,-d-q}(G, X),$$

where $d = \dim(X)$. In general, X is a disjoint union of G-invariant equidimensional varieties $X_1, X_2, \ldots, X_n$, and we set

$$A^{p,q}(G, X) := \coprod_i A^{p,q}(G, X_i) \quad \text{and} \quad \mathrm{CK}^{p,q}(G, X) := \coprod_i \mathrm{CK}^{p,q}(G, X_i).$$

The exact sequence in Theorem 3.10 reads

$$\ldots \to A^{p-1,q}(G, X) \to \mathrm{CK}^{p+1,q-1}(G, X) \xrightarrow{\beta_X} \mathrm{CK}^{p,q}(G, X) \to A^{p,q}(G, X) \to \ldots$$

Let $f : X \to Y$ be a morphism of smooth G-varieties. We would like to define a pull-back homomorphism

$$f^* : \mathrm{CK}^{p,q}(G, Y) \to \mathrm{CK}^{p,q}(G, X).$$

We may assume that X and Y are equidimensional varieties. The morphism

$$h = (1_X, f) : X \to X \times Y$$

is a regular closed embedding of codimension $d_Y = \dim(Y)$. The projection $s : X \times Y \to Y$ is flat of relative dimension $d_X = \dim(X)$. We define f^* as the composition

$$\mathrm{CK}^{p,q}(G, Y) = \mathrm{CK}_{d_Y-p,-d_Y-q}(G, Y) \xrightarrow{s^*} \mathrm{CK}_{d_X+d_Y-p,-d_X-d_Y-q}(G, X \times Y) \xrightarrow{h^\star}$$
$$\mathrm{CK}_{d_X-p,-d_X-q}(G, X) = \mathrm{CK}^{p,q}(G, X).$$

If X is an equidimensional smooth G-variety, the diagonal morphism

$$\Delta : X \to X \times X$$

is a closed regular embedding of codimension $\dim(X)$. The composition

$$\mathrm{CK}^{p,q}(G,X) \otimes \mathrm{CK}^{p',q'}(G,X) \xrightarrow{\times} \mathrm{CK}^{p+p',q+q'}(G,X \times X) \xrightarrow{\Delta^\star} \mathrm{CK}^{p+p',q+q'}(G,X)$$

defines the structure of a bi-graded ring on $\mathrm{CK}^{*,*}(G,X)$.

Thus, the assignment $X \mapsto \mathrm{CK}^{*,*}(G,X)$ yields a contravariant functor from the category of smooth G-varieties to the category of bi-graded associative commutative unital rings.

Remark 6.1. For smooth X, the canonical map $\mathrm{CK}^{*,*}(G,X) \to A^{*,*}(G,X)$ is a ring homomorphism. The group $K'_*(G,X)$ is naturally identified with $K_*(G,X)$ ([16, Remark 1.9(a)]) which is a graded ring and the filtration in Section 3 is a ring filtration. The maps $\mathrm{CK}^{p,q}(G,X) \to K_{-p-q}(G,X)^{(p)}$ in Theorem 3.10(4) yield a ring homomorphism $\mathrm{CK}^{*,*}(G,X) \to K_*(G,X)^{(*)}$. For a homomorphism of algebraic groups $G' \to G$, the map $\mathrm{CK}^{*,*}(G,X) \to \mathrm{CK}^{*,*}(G',X)$ in Remark 3.12 is a ring homomorphism.

Remark 6.2 (Projection Formula)**.** For a proper morphism $f\colon X \to Y$ of smooth varieties, Projection Formula holds:

$$f_*(\alpha \cdot f^*(\gamma)) = f_*(\alpha) \cdot \gamma$$

for every $\alpha \in \mathrm{CK}^{*,*}(G,X)$ and $\gamma \in \mathrm{CK}^{*,*}(G,Y)$. The proof is formally the same as [4, Proof of Proposition 56.9].

Example 6.3. The class $[\mathcal{O}_X] \in \mathrm{CK}^{0,0}(G,X)$ is the unit of the ring $\mathrm{CK}^{*,*}(G,X)$. The image β_X of $[\mathcal{O}_X]$ under the Bott homomorphism $\beta_X : \mathrm{CK}^{0,0}(G,X) \to \mathrm{CK}^{-1,1}(G,X)$ is called the *Bott element* of X. The Bott homomorphism $\beta_X : \mathrm{CK}^{p+1,q-1}(G,X) \to \mathrm{CK}^{p,q}(G,X)$ is given by the product with β_X.

Example 6.4. For a vector G-bundle E on a smooth G-variety X, the ith (operator) Chern class $c_i(E)$ in Remark 3.11 is given by the product with the element $c_i(E)([\mathcal{O}_X]) \in \mathrm{CK}^{i,-i}(G,X)$ also called the *ith Chern class of E*.

An algebraic group G over F acts (trivially) on the point $\mathrm{pt} = \mathrm{Spec}(F)$. The stack $\mathrm{B}G$ coincides with pt/G. We will write $K_*(\mathrm{B}G)$ for the ring $K_*(G,\mathrm{pt})$ and $\mathrm{CK}^{*,*}(\mathrm{B}G)$ for $\mathrm{CK}^{*,*}(G,\mathrm{pt})$. Since $\mathcal{M}(G,\mathrm{pt})$ is the category of finite dimensional representations of G, every object in $\mathcal{M}(G,\mathrm{pt})$ has finite length. By [11, §5, Corollary 1],

$$K_*(\mathrm{B}G) = \coprod K_*(\mathrm{End}_G(V)^{op}),$$

where the direct sum is taken over all isomorphism classes of irreducible G-spaces V. In particular, $K_0(\mathrm{B}G)$ is the representation ring $R(G)$.

If G is a diagonalizable group, every irreducible G-space V is 1-dimensional (a character), hence $\mathrm{End}_G(V) = F$ and therefore,

$$K_*(\mathrm{B}G) = \coprod K_*(F)x = K_*(F) \otimes \mathbb{Z}[\widehat{G}], \tag{6.5}$$

where the direct sum is taken over all characters $x \in \widehat{G} = \mathrm{Hom}(G, \mathbb{G}_m)$.

7. Examples

Example 7.1. Let $G = \mathbb{G}_m$ over F. We have $\widehat{G} = \mathbb{Z}$ and

$$A_* := K_*(\mathrm{B}\mathbb{G}_m) = K_*(F)[t, t^{-1}],$$

where t is the class of the tautological (identity) character of $\mathbb{G}_m$. For any $p \geq 0$, the codimension p term of the topological filtration $K_*(\mathrm{B}\mathbb{G}_m)^{(p)}$ is equal to $s^p A_*$, where $s = 1 - t$.

The classifying space $\mathrm{B}\mathbb{G}_m$ is approximated by projective spaces $\mathbb{P}^r = (\mathbb{A}^{r+1} \setminus \{0\})/\mathbb{G}_m$. By [2], the Bott map for projective spaces is injective, hence

$$\mathrm{CK}^{p,q}(\mathrm{B}\mathbb{G}_m) = \begin{cases} A_{-p-q}, & \text{if } p \leq 0; \\ A_{-p-q}s^p, & \text{if } p \geq 0. \end{cases}$$

In other words, $\mathrm{CK}^{*,*}(\mathrm{B}\mathbb{G}_m)$ is the subring of $A_*[\beta, \beta^{-1}]$, generated by A_*, β and $s\beta^{-1}$:

$$\mathrm{CK}^{*,*}(\mathrm{B}\mathbb{G}_m) = \ldots \oplus A_*\beta^2 \oplus A_*\beta \oplus A_* \oplus A_*s\beta^{-1} \oplus A_*s^2\beta^{-2} \oplus \ldots$$

The spectral sequence for $\mathrm{B}\mathbb{G}_m$ converges at all diagonals.

Example 7.2. Now let $G = \mu_2$. The character group $\widehat{G}$ is cyclic of order 2 generated by the restriction $\bar{t}$ to μ_2 of the tautological character t of $\mathbb{G}_m$ in Example 7.1, hence

$$K_*(\mathrm{B}\mu_2) = K_*(F) \oplus K_*(F)\bar{t} = A_*/(1 - t^2)A_*,$$

and $K_*(\mathrm{B}\mu_2)^{(1)} = K_*(F)\bar{s}$, where $\bar{s} = 1 - \bar{t}$.

The classifying space $\mathrm{B}\mu_2$ is approximated by the lens spaces $L_r \setminus Z_r$, where Z_r is the image of the zero section of the line bundle L_r over $\mathbb{P}^r_F$ with $\mathcal{O}(2)$ the sheaf of sections. By homotopy invariance, $\mathrm{CK}^{*,*}(L_r) \simeq \mathrm{CK}^{*,*}(\mathbb{P}^r_F)$. The composition

$$\mathrm{CK}^{*-1,*+1}(\mathbb{P}^r_F) \xrightarrow{z_*} \mathrm{CK}^{*,*}(L_r) \simeq \mathrm{CK}^{*,*}(\mathbb{P}^r_F),$$

where z is the zero section of L_r is multiplication by the first Chern class of L_r. Computed in $K_0(\mathbb{P}^r_F)$ this Chern class is equal to $1 - [\mathcal{O}(-2)]$. It follows that in the notation of Example 7.1 and by localization, there is an exact sequence

$$\mathrm{CK}^{*-1,*+1}(\mathrm{B}\mathbb{G}_m) \xrightarrow{(1-t^{-2})\beta^{-1}} \mathrm{CK}^{*,*}(\mathrm{B}\mathbb{G}_m) \to \mathrm{CK}^{*,*}(\mathrm{B}\mu_2) \to$$
$$\mathrm{CK}^{*,*+1}(\mathrm{B}\mathbb{G}_m) \xrightarrow{(1-t^{-2})\beta^{-1}} \mathrm{CK}^{*+1,*}(\mathrm{B}\mathbb{G}_m).$$

By Example 7.1, multiplication by $(1 - t^{-2})\beta^{-1}$ is injective, hence

$$\mathrm{CK}^{*,*}(\mathrm{B}\mu_2) \simeq \mathrm{CK}^{*,*}(\mathrm{B}\mathbb{G}_m)/(1 - t^2)\beta^{-1}\,\mathrm{CK}^{*-1,*+1}(\mathrm{B}\mathbb{G}_m).$$

It follows that

$$\mathrm{CK}^{p,q}(\mathrm{B}\mu_2) = A_{-p-q}/(1 - t^2)A_{-p-q} \simeq K_{-p-q}(\mathrm{B}\mu_2) = K_{-p-q}(F) \oplus K_{-p-q}(F)\bar{t}, \quad \text{if } p \leq 0,$$

$$\mathrm{CK}^{p,q}(\mathrm{B}\mu_2) = A_{-p-q}s^p/A_{-p-q}s^{p-1}(1 - t^2) \simeq A_{-p-q}/A_{-p-q}(1 + t) \simeq K_{-p-q}(F), \quad \text{if } p > 0.$$

The Bott homomorphism $\beta : \mathrm{CK}^{p,q}(\mathrm{B}\mu_2) \to \mathrm{CK}^{p-1,q+1}(\mathrm{B}\mu_2)$ is the identity if $p \leq 0$, multiplication by $1 - \bar{t}$ if $p = 1$ and multiplication by 2 if $p > 1$ (we use the equality $(1 - \bar{t})^2 = 2(1 - \bar{t})$). Thus, the Bott map is not injective in general.

Recall that the Bott element β is equal to 1 in $K_0(\mathrm{B}\mu_2) = \mathrm{CK}^{-1,1}(\mathrm{B}\mu_2)$. Write α for $1 \in K_0(F) = \mathrm{CK}^{1,-1}(\mathrm{B}\mu_2)$. We have $\alpha^2\beta = 2\alpha$ and

$$\mathrm{CK}^{*,*}(\mathrm{B}\mu_2) = K_*(F)[\alpha,\beta]/(\alpha^2\beta - 2\alpha).$$

The kernel of $\overline{\mathrm{CK}}_{p-2,q+1}(G,\mathrm{pt}) \to \overline{\mathrm{CK}}_{p-1,q}(G,\mathrm{pt})$ in Proposition 4.2 is equal to

$${}_2K_{p+q-1}(F) \cap \big(\bigcap_r 2^r K_{p+q-1}(F)\big)$$

if $p \leq 0$. It is not trivial in general, for example, if $p+q=2$ and F is quadratically closed of characteristic not 2. For such F, the spectral sequence for $\mathrm{B}\mu_2$ does not converge at the second diagonal.

Example 7.3. Let $G = (\mu_2)^n$. By (6.5), $K_0(\mathrm{B}G) = A/Q$, where $A = \mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm n}]$ and Q is the ideal generated by all $t_i^2 - 1$. In what follows we will be considering factor rings of the polynomial ring $\mathbb{Z}[t_1, \dots, t_n]$ by various ideals containing polynomials $f(t_i)$ with constant term 1 for every i. The classes of t_i are invertible in such factor rings. Therefore, we can skip $t_i{}^{-1}$ as follows.

Let $B = \mathbb{Z}[s_1, s_2, \dots, s_n]$, where $s_i = 1 - t_i$, $I \subset B$ the ideal generated by s_i's, $J \subset B$ the ideal generated by $t_i^2 - 1 = s_i^2 - 2s_i$ for all i. Then as in Example 7.2,

$$K_0(\mathrm{B}G) = B/J,$$
$$K_0(\mathrm{B}G)^{(p)} = (I^p + J)/J \simeq I^p/(I^p \cap J)$$

for $p > 0$.

The classifying space $\mathrm{B}G$ has the same cohomology group as the product of n copies of $\mathrm{B}\mathbb{G}_m$ minus the union of n divisors, inverse images of the zero section of $\mathrm{B}\mathbb{G}_m$ under all n projections $(\mathrm{B}\mathbb{G}_m)^n \to \mathrm{B}\mathbb{G}_m$. By localization,

$$\mathrm{CK}^{p,-p}(\mathrm{B}G) = I^p/I^{p-1}J.$$

It follows that

$$\begin{aligned} K_0(\mathrm{B}G)^{(p/p+1)} &= (I^p + J)/(I^{p+1} + J) \simeq I^p/(I^{p+1} + (I^p \cap J)), \\ \mathrm{CH}^p(\mathrm{B}G) &= \mathrm{Coker}\big(\mathrm{CK}^{p+1,-p-1}(\mathrm{B}G) \to \mathrm{CK}^{p,-p}(\mathrm{B}G)\big) \\ &= I^p/(I^{p+1} + I^{p-1}J). \end{aligned}$$

Let $P \subset B$ be the ideal generated by $s_i^2 s_j - s_i s_j^2$ for all i and j.

Lemma 7.4. *$P \subset IJ$ and $I^p \cap J = I^{p-1}J + I^{p-3}P$.*

Proof. Since $s_i^2 s_j - s_i s_j^2 = s_j(s_i^2 - 2s_i) - s_i(s_j^2 - 2s_j) \in IJ$, we have $P \subset IJ \subset J$. It follows that $I^{p-1}J + I^{p-3}P$ is contained in $I^p \cap J$. Let $f \in I^p \cap J$. The *support* of a nonzero monomial $as^k = as_1^{k_1} \cdots s_n^{k_n}$ with $a \in \mathbb{Z}$ is the set of all i such that $k_i > 0$. Let as^k be a nonzero monomial of f with the smallest cardinality of support. Write f as the sum of two polynomials f' and f'' such that f' is the sum of monomials with the same support as s^k and f'' the sum of the remaining monomials of f. Plugging in $s_i = 0$ for all i that are not in the support of s^k, we see that $f' \in I^p \cap J$. It suffices to show that

$f' \in I^{p-1}J + I^{p-3}P$. By induction on n we may assume that the support of all monomials in f coincides with $\{1, \dots, n\}$.

Write $f = sg$, where $s = s_1 s_2 \cdots s_n$ and $g(2, \dots, 2) = 0$. Then $g \in I^{p-n} \cap M$, where M is the ideal in B generated by all $s_i - 2$. It follows that $g \in I^{p-n}M + I^{p-n-1}N$, where N is the ideal in B generated by all $s_i - s_j$. Therefore,

$$f = sg \in I^{p-n}sM + I^{p-n-1}sN \subset I^{p-1}J + I^{p-3}P$$

since $sM \subset I^{n-1}J$ and $sN \subset I^{n-2}P$. □

Corollary 7.5. *$I^p \cap I^{p-2}J = I^p \cap J$ and $I^{p+1} \cap J \subset I^{p-1}J$.* □

It follows from the corollary that

$$\begin{aligned} \operatorname{Ker}\bigl(\mathrm{CH}^p(\mathrm{B}G) \xrightarrow{\varphi} K_0(\mathrm{B}G)^{(p/p+1)}\bigr) &= (I^{p+1} + (I^p \cap J))/(I^{p+1} + I^{p-1}J) \\ &= (I^p \cap J)/\bigl((I^{p+1} \cap J) + I^{p-1}J\bigr) \\ &= (I^p \cap J)/I^{p-1}J, \end{aligned}$$

since $I^{p+1} \cap J \subset I^{p-1}J$. On the other hand,

$$\begin{aligned} \operatorname{Ker}\bigl(\mathrm{CK}^{p,-p}(\mathrm{B}G) \xrightarrow{\beta} \mathrm{CK}^{p-1,-p+1}(\mathrm{B}G)\bigr) &= (I^p \cap I^{p-2}J)/I^{p-1}J \\ &= (I^p \cap J)/I^{p-1}J \end{aligned}$$

since $I^p \cap I^{p-2}J = I^p \cap J$ by Corollary 7.5. We proved that the kernel of φ lifts isomorphically to the kernel of β. In particular, the image of the differential

$$A^{p-2}(\mathrm{B}G, K_{p-1}) \to \mathrm{CH}^p(\mathrm{B}G)$$

on the second page of the spectral sequence for $\mathrm{B}G$ coincides with $\operatorname{Ker}(\varphi)$. It follows that all differentials coming to zero diagonal on pages $E_r^{*,*}$ with $r \geq 3$ are trivial. Note that by Lemma 7.4, the kernel of φ is generated by the classes of $s_i^2 s_j - s_i s_j^2$ and it is nontrivial if $n \geq 2$.

The group

$$\operatorname{Ker}\bigl(\mathrm{CK}^{p,-p}(\mathrm{B}G) \to K_0(\mathrm{B}G)^{(p)}\bigr) = (I^p \cap J)/I^{p-1}J$$

coincides with the torsion part $\mathrm{CK}^{p,-p}(\mathrm{B}G)_{\mathrm{tors}}$ of $\mathrm{CK}^{p,-p}(\mathrm{B}G)$. Hence $\mathrm{CK}^{p,-p}(\mathrm{B}G)_{\mathrm{tors}}$ is contained in the kernel of β. As the intersection of all $K_0(\mathrm{B}G)^{(p)}$ is zero, the groups $\overline{\mathrm{CK}}_{i,-i}(\mathrm{B}G)$ in Proposition 4.2 are trivial. It follows that the spectral sequence converges at the first diagonal.

Example 7.6. The intersection of the terms $R(G)^{(p)} = K_0(\mathrm{B}G)^{(p)}$ over all p is nonzero in the case of $G = \mu_6$ (cf. [1]). Indeed, $R(G) = \mathbb{Z}[t]/(1 - t^6)$ and the ideal $R(G)^{(p)}$ is generated by $(1 - t)^p$. The polynomial $f := (1 - t)(1 + t)(1 + t + t^2) \in \mathbb{Z}[t]$ yields a nonzero element of $R(G)$. To see that this element is in $R(G)^{(p)}$ for any $p > 0$, note that the polynomials $(1 - t)^p$ and $1 - t + t^2$ generate the unit ideal. Writing 1 as their linear combination and multiplying by f, we get the statement because $(1 - t + t^2)f = 1 - t^6$.

References

[1] Atiyah, M. F. Characters and cohomology of finite groups. *Inst. Hautes Études Sci. Publ. Math.*, 9 (1961), 23–64.

[2] Cai, S. Algebraic connective K-theory and the niveau filtration. *J. Pure Appl. Algebra 212*, 7 (2008), 1695–1715.

[3] Edidin, D., and Graham, W. Equivariant intersection theory. *Invent. Math. 131*, 3 (1998), 595–634.

[4] Elman, R., Karpenko, N., and Merkurjev, A. *The Algebraic and Geometric Theory of Quadratic Forms.* American Mathematical Society, Providence, RI, 2008.

[5] Guillot, P. Geometric methods for cohomological invariants. *Doc. Math. 12* (2007), 521–545.

[6] Heller, J., and Malagón-López, J. Equivariant algebraic cobordism. *J. Reine Angew. Math. 684* (2013), 87–112.

[7] Karpenko, N. A., and Merkurjev, A. S. Chow filtration on representation rings of algebraic groups. *International Mathematics Research Notices* (2019). doi: 10.1093/imrn/rnz049.

[8] Knutson, D. *Algebraic spaces.* Lecture Notes in Mathematics, Vol. 203. Springer-Verlag, Berlin-New York, 1971.

[9] Levine, M., and Morel, F. *Algebraic cobordism.* Springer Monographs in Mathematics. Springer, Berlin, 2007.

[10] Olsson, M. *Algebraic spaces and stacks*, vol. 62 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2016.

[11] Quillen, D. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)* (1973), pp. 85–147. Lecture Notes in Math., Vol. 341.

[12] Rost, M. Chow groups with coefficients. *Doc. Math. 1* (1996), No. 16, 319–393 (electronic).

[13] Sherman, C. Some theorems on the K-theory of coherent sheaves. *Comm. Algebra 7*, 14 (1979), 1489–1508.

[14] Stacks Project Authors. *Stacks Project.* https://stacks.math.columbia.edu, 2019.

[15] Thomason, R. W. Algebraic K-theory of group scheme actions. In *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*, vol. 113 of *Ann. of Math. Stud.* Princeton Univ. Press, Princeton, NJ, 1987, pp. 539–563.

[16] Thomason, R. W. Equivariant algebraic vs. topological K-homology Atiyah-Segal-style. *Duke Math. J. 56*, 3 (1988), 589–636.

[17] Totaro, B. The Chow ring of a classifying space. In *Algebraic K-theory (Seattle, WA, 1997)*, vol. 67 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 1999, pp. 249–281.

[18] Vistoli, A. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, vol. 123 of *Math. Surveys Monogr.* Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.

Nikita A. Karpenko, Department of Mathematical & Statistical Sciences, University of Alberta, Edmonton, AB, CANADA

Email address: `karpenko@ualberta.ca`, *web page*: `www.ualberta.ca/~karpenko`

Alexander S. Merkurjev, Department of Mathematics, University of California, Los Angeles, CA, USA

Email address: `merkurev@math.ucla.edu`, *web page*: `www.math.ucla.edu/~merkurev`