

Big Picard theorem for moduli spaces of polarized manifolds

Ya DENG



Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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YA DENG

ABSTRACT. Consider a smooth projective family of complex polarized manifolds with semi-ample canonical sheaf over a quasi-projective manifold V . When the associated moduli map $V \rightarrow P_h$ from the base to coarse moduli space is quasi-finite, we prove that the generalized big Picard theorem holds for the base manifold V : for any projective compactification Y of V , any holomorphic map $f : \Delta - \{0\} \rightarrow V$ from the punctured unit disk to V extends to a holomorphic map of the unit disk Δ into Y . This result generalizes our previous work on the Brody hyperbolicity of V (*i.e.* there are no entire curves on V), as well as a more recent work by Lu-Sun-Zuo on the Borel hyperbolicity of V (*i.e.* any holomorphic map from a quasi-projective variety to V is algebraic). We also obtain generalized big Picard theorem for bases of log Calabi-Yau families.

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0. INTRODUCTION

The classical big Picard theorem says that any holomorphic map from the punctured disk Δ^* into \mathbb{P}^1 which omits three points can be extended to a holomorphic map $\Delta \rightarrow \mathbb{P}^1$, where Δ denotes the unit disk. Therefore, we say the (generalized) big Picard theorem holds for a quasi-projective variety V if for some (thus any) projective compactification Y of V , any holomorphic map $f : \Delta^* \rightarrow V$ extends to a holomorphic map $\tilde{f} : \Delta \rightarrow X$. This property is interesting for it implies the *Borel hyperbolicity*¹ of V : any holomorphic map from a quasi-projective variety to V is necessarily *algebraic*. A natural question is to find algebraic varieties satisfying the big Picard theorem. By the fundamental work of Kobayashi (see [Kob98, Theorem 6.3.6]), big Picard theorem holds for the quasi-projective manifold V which admits a projective compactification Y such that V is *hyperbolically embedded into Y* (see [Kob98, Chapter 3. §3] for the definition). This gives

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¹The notion of Borel hyperbolicity was first introduced by Javanpeykar-Kucharczyk in [JK18].

us an important criteria for varieties satisfying the big Picard theorem. By the work of Fujimoto [Fuj72] and Green [Gre77], complements of $2n + 1$ general hyperplanes in \mathbb{P}^n are hyperbolically embedded into \mathbb{P}^n . More recently Brotbek and the author [BD19] proved that the complement of a general hypersurface in \mathbb{P}^n of high degree is also hyperbolically embedded into \mathbb{P}^n . By A. Borel [Bor72] and Kobayashi-Ochiai [KO71], the quotients of bounded symmetric domains by torsion free arithmetic groups are hyperbolically embedded into its Baily-Borel compactification. Hence these results provide examples of quasi-projective manifolds satisfying the big Picard theorem.

However, quasi-projective manifold V being hyperbolically embedded into some projective compactification Y is *minimal* in the sense that, for any birational modification $\mu : \tilde{Y} \rightarrow Y$ by blowing-up the boundary $Y \setminus V$, $\mu^{-1}(V) \simeq V$ is no more hyperbolically embedded into \tilde{Y} , while the big Picard theorem does not depends on its compactification. The first result in this paper is to establish a more flexible criteria for big Picard theorem. For our definition of Finsler metric (which is slightly different from the usual one in the literature), see Definition 1.6.

Theorem A (=Theorem 1.7). *Let X be a projective manifold and let D be a simple normal crossing divisor on X . Let h be a (possibly degenerate) Finsler metric of $T_X(-\log D)$. Assume that $f : \Delta^* \rightarrow X \setminus D$ is a holomorphic map from the punctured unit disk Δ^* to $X \setminus D$ such that $|f'(t)|_h^2 \neq 0$, and*

$$\sqrt{-1}\partial\bar{\partial} \log |f'(t)|_h^2 \geq f^* \omega$$

for some smooth Kähler metric ω on X . Then f extends to a holomorphic map $\tilde{f} : \Delta \rightarrow X$ of the unit disk into X .

Theorem A is inspired by *fundamental vanishing theorem* for jet differentials vanishing on some ample divisor by Siu-Yeung [SY97] and Demailly [Dem97, §4], and its proof is mainly based on a *logarithmic derivative lemma* by Noguchi [Nog81].

The motivation of Theorem A is to study the hyperbolicity of moduli spaces of polarized manifolds with semi-ample canonical bundle. By the fundamental work of Viehweg-Zuo [VZ02, VZ03], and the recent development by Popa et al. [PS17, PTW18] and [Den18a, Den18b, Den19, LSZ19], a special Higgs bundle, the so-called Viehweg-Zuo Higgs bundle in Definition 2.1 below, turns out to be a powerful technique in studying hyperbolicity problems. For a quasi-projective manifold V equipped with a Viehweg-Zuo Higgs bundle, in [Den18a, Den18b] we prove that V can be equipped with a *generically positively definite* Finsler metric whose holomorphic sectional curvature is bounded from above by a negative constant. In particular, we prove that V is always *pseudo Kobayashi hyperbolic*. The second aim of this article is to give a new curvature estimate for that Finsler metric on V .

Theorem B (=Theorem 2.8). *Let X be a projective manifold equipped with a smooth Kähler metric ω and let D be a simple normal crossing divisor on X . Assume that there is a Viehweg-Zuo Higgs bundle over (X, D) . Then there are a positive constant δ and a Finsler metric h on $T_X(-\log D)$ which is positively definite on a dense Zariski open set V° of $V := X \setminus D$, such that for any holomorphic map $\gamma : C \rightarrow V$ from any open subset C of \mathbb{C} to V with $\gamma(C) \cap V^\circ \neq \emptyset$, one has*

$$\sqrt{-1}\partial\bar{\partial} \log |\gamma'|_h^2 \geq \delta \gamma^* \omega.$$

In particular, by Theorem A, for any holomorphic map $f : \Delta^* \rightarrow X \setminus D$, with $f(\Delta^*) \cap V^\circ \neq \emptyset$, it extends to a holomorphic map $\tilde{f} : \Delta \rightarrow X$.

By the work [VZ02, VZ03, PTW18], Viehweg-Zuo Higgs bundles exist on bases of maximally varying, smooth family of projective manifolds with semi-ample canonical bundle. Combining their results and Theorem B, we prove the big Picard theorem for moduli of polarized manifolds with semi-ample canonical bundle.

Theorem C. Consider the moduli functor \mathcal{P}_h of polarized manifolds with semi-ample canonical sheaf introduced by Viehweg [Vie95, §7.6], where h is the Hilbert polynomial associated to the polarization. Assume that for some quasi-projective manifold V over which there exists a smooth polarized family $(f_U : U \rightarrow V, \mathcal{L}) \in \mathcal{P}_h(V)$ such that the induced moduli map $\varphi_U : V \rightarrow P_h$ is quasi-finite. Let Y be an arbitrary projective compactification of V . Then any holomorphic map $\gamma : \Delta^* \rightarrow V$ from the punctured unit disk Δ^* to V extends to a holomorphic map from the unit disk Δ to Y .

Under the same assumption as Theorem B, we have already in [Den18b] proved the Brody hyperbolicity of V : there exists no entire curves $\gamma : \mathbb{C} \rightarrow V$. Based on the infinitesimal Torelli-type theorem proven in [Den18b, Theorem C] (see Theorem 2.3 below), more recently, Lu-Sun-Zuo [LSZ19] proved the Borel hyperbolicity of V : any holomorphic map from a quasi-projective variety to V is algebraic. The use of Nevanlinna theory in this article is inspired by their work, although our methods are different from theirs (see Remark 3.1).

Finally, let us mention that in [Den19], we construct Viehweg-Zuo Higgs bundles over bases of maximally varying, log smooth families of Calabi-Yau families (see Definition 3.2 for the definition of log smooth family). Applying Theorems A and B to this result, we also obtain big Picard theorem for these base manifolds.

Theorem D (=Theorem 3.4). Let $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ be a log smooth family over a quasi-projective manifold Y° . Assume that each fiber $(X_y, D_y) := (f^\circ)^{-1}(y)$ of f° is a klt pair, and $K_{X_y} + D_y \equiv_{\mathbb{Q}} 0$. Assume that the logarithmic Kodaira-Spencer map

$$T_{Y^\circ, y} \rightarrow H^1(X_y, T_{X_y}(-\log D_y))$$

is injective for any $y \in Y^\circ$. Then for any projective compactification Y of the base Y° , any holomorphic map $\gamma : \Delta^* \rightarrow Y^\circ$ extends to a holomorphic map from Δ to Y .

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1. A DIFFERENTIAL GEOMETRIC CRITERIA FOR BIG PICARD THEOREM

In the similar vein as the fundamental vanishing theorem for jet differentials vanishing on some ample divisor by Siu-Yeung [SY97] and Demailly [Dem97], in this section we will establish a differential geometric criteria for big Picard type theorem via the logarithmic derivative lemma by Noguchi [Nog81].

1.1. Preliminary in Nevanlinna theory. Let $\mathbb{D}^* := \{t \in \mathbb{C} \mid |t| > 1\}$, and $\mathbb{D} := \mathbb{D}^* \cup \infty$. Then via the map $z \mapsto \frac{1}{z}$, \mathbb{D}^* is isomorphic to the punctured unit disk Δ^* and \mathbb{D} is isomorphic to the unit disk Δ . Therefore, for any holomorphic map f from the punctured disk Δ^* into a projective variety Y , f extends to the origin if and only if $f(\frac{1}{z}) : \mathbb{D}^* \rightarrow Y$ extends to the infinity.

Let (X, ω) be a compact Kähler manifold, and $\gamma : \mathbb{D}^* \rightarrow X$ be a holomorphic map. Fix any $r_0 > 1$. Write $\mathbb{D}_r := \{z \in \mathbb{C} \mid r_0 < |z| < r\}$. The *order function* is defined by

$$T_{\gamma, \omega}(r) := \int_{r_0}^r \frac{d\tau}{\tau} \int_{\mathbb{D}_\tau} \gamma^* \omega.$$

As is well-known, the asymptotic behavior of $T_{\gamma, \omega}(r)$ as $r \rightarrow \infty$ characterizes whether γ can be extended over the ∞ (see e.g. [Dem97, 2.11. Cas «local »] or [NW14, Remark 4.7.4.(ii)]).

Lemma 1.1. $T_{\gamma, \omega}(r) = O(\log r)$ if and only if γ is extended holomorphically over ∞ . \square

The following lemma is well-known to experts (see e.g. [Dem97, Lemme 1.6]).

Lemma 1.2. *Let X be a projective manifold equipped with a hermitian metric ω and let $u : X \rightarrow \mathbb{P}^1$ be a rational function. Then for any holomorphic map $\gamma : \mathbb{D}^* \rightarrow X$, one has*

$$T_{u \circ \gamma, \omega_{FS}}(r) \leq CT_{\gamma, \omega}(r) + O(1)$$

where ω_{FS} is the Fubini-Study metric for \mathbb{P}^1 . \square

The following logarithmic derivative lemma by Noguchi is crucial in the proof of Theorem A.

Lemma 1.3 ([Nog81, Lemma2.12], [Dem97, 3.4. Cas local]). *Let $u : \mathbb{D}^* \rightarrow \mathbb{P}^1$ be any meromorphic function. Then we have*

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ |(\log u)'(re^{i\theta})| d\theta \leq C(\log^+ T_{u, \omega_{FS}}(r) + \log r) + O(1) \quad \|\|,$$

for some constant $C > 0$ which does not depend on r . Here the symbol $\|\|$ means that the inequality holds outside a Borel subset of $(r_0, +\infty)$ of finite Lebesgue measure. \square

We need the lemma by E. Borel.

Lemma 1.4 ([NW14, Lemma 1.2.1]). *Let $\phi(r) \geq 0$ ($r \geq r_0 \geq 0$) be a monotone increasing function. For every $\delta > 0$,*

$$\frac{d}{dr} \phi(r) \leq \phi(r)^{1+\delta} \quad \|\|.$$

\square

We recall two useful formulas (the second one is the well-known Jensen formula).

Lemma 1.5. *Write $\log^+ x := \max(\log x, 0)$.*

$$(1.1.1) \quad \log^+ \left(\sum_{i=1}^N x_i \right) \leq \sum_{i=1}^N \log^+ x_i + \log N \quad \text{for } x_i \geq 0.$$

$$(1.1.2) \quad \frac{1}{\pi} \int_{r_0}^r \frac{d\tau}{\tau} \int_{\mathbb{D}_\tau} \sqrt{-1} \partial \bar{\partial} v = \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta}) d\theta - \frac{1}{2\pi} \int_0^{2\pi} v(r_0 e^{i\theta}) d\theta$$

for all functions v so that $\sqrt{-1} \partial \bar{\partial} v$ exists as measures (e.g. v is the difference of two subharmonic functions). \square

1.2. A criteria for big Picard theorem.

Definition 1.6 (Finsler metric). *Let E be a holomorphic vector bundle on a complex manifold X . A Finsler metric on E is a real non-negative continuous function $h : E \rightarrow [0, +\infty[$ such that*

$$h(av) = |a|h(v)$$

for any $a \in \mathbb{C}$ and $v \in E$. The metric h is degenerate at a point $x \in X$ if $h(v) = 0$ for some nonzero $v \in E_x$, and the set of such degenerate points is denoted by Δ_h .

We shall mention that our definition is a bit different from that in [Kob98, Chapter 2, §3], which requires convexity, and the Finsler metric therein can be upper-semi continuous.

Let us now state and prove the main result in this section.

Theorem 1.7 (Criteria for big Picard theorem). *Let X be a projective manifold and let D be a simple normal crossing divisor on X . Let h be a (possibly degenerate) Finsler metric of $T_X(-\log D)$. Assume that $f : \mathbb{D}^* \rightarrow X \setminus D$ is a holomorphic map such that $|f'(t)|_h^2 \not\equiv 0$, and*

$$(1.2.1) \quad \frac{1}{\pi} \sqrt{-1} \partial \bar{\partial} \log |f'(t)|_h^2 \geq f^* \omega$$

for some smooth Kähler metric ω on X . Then f extends to a holomorphic map $\bar{f} : \mathbb{D} \rightarrow X$.

The proof is an application of logarithmic derivative lemma, which is inspired by [Dem97, §4] and [NW14, Lemma 4.7.1].

Proof of Theorem 1.7. We take a finite affine covering $\{U_\alpha\}_{\alpha \in I}$ of X and rational functions $(x_{\alpha 1}, \dots, x_{\alpha n})$ on X which are holomorphic on U_α so that

$$\begin{aligned} dx_{\alpha 1} \wedge \cdots \wedge dx_{\alpha n} &\neq 0 \text{ on } U_\alpha \\ D \cap U_\alpha &= (x_{\alpha, s(\alpha)+1} \cdots x_{\alpha n} = 0) \end{aligned}$$

Hence

$$(1.2.2) \quad (e_{\alpha 1}, \dots, e_{\alpha n}) := \left(\frac{\partial}{\partial x_{\alpha 1}}, \dots, \frac{\partial}{\partial x_{\alpha s(\alpha)}}, x_{\alpha, s(\alpha)+1} \frac{\partial}{\partial x_{\alpha, s(\alpha)+1}}, \dots, x_{\alpha n} \frac{\partial}{\partial x_{\alpha n}} \right)$$

is a basis for $T_X(-\log D)|_{U_\alpha}$. Write

$$(f_{\alpha 1}(t), \dots, f_{\alpha n}(t)) := (x_{\alpha 1} \circ f, \dots, x_{\alpha n} \circ f).$$

With respect to the trivialization of $T_X(-\log D)$ induced by the basis (1.2.2), $f'(t)$ can be written as

$$f'(t) = f'_{\alpha 1}(t)e_{\alpha 1} + \cdots + f'_{\alpha s(\alpha)}(t)e_{\alpha s(\alpha)} + (\log f_{\alpha, s(\alpha)+1})'(t)e_{\alpha, s(\alpha)+1} + \cdots + (\log f_{\alpha n})'(t)e_{\alpha n}.$$

Let $\{\rho_\alpha\}_{\alpha \in I}$ be a partition of unity subordinated to $\{U_\alpha\}_{\alpha \in I}$. Since h is Finsler metric for $T_X(-\log D)$ which is continuous and locally bounded from above by Definition 1.6, and I is a finite set, there is a constant $C > 0$ so that

$$(1.2.3) \quad \rho_\alpha \circ f \cdot |f'(t)|_h^2 \leq C \left(\sum_{j=1}^{s(\alpha)} \rho_\alpha \circ f \cdot |f'_{\alpha j}(t)|^2 + \sum_{i=s(\alpha)+1}^n |(\log f_{\alpha i})'(t)|^2 \right) \quad \forall t \in \mathbb{D}^*$$

for any α . Hence

$$\begin{aligned} T_{f, \omega}(r) &:= \int_{r_0}^r \frac{d\tau}{\tau} \int_{\mathbb{D}_\tau} f^* \omega \stackrel{(1.2.1)}{\leq} \int_{r_0}^r \frac{d\tau}{\tau} \int_{\mathbb{D}_\tau} \frac{1}{\pi} \sqrt{-1} \partial \bar{\partial} \log |f'|_h^2 \\ &\stackrel{(1.1.2)}{\leq} \frac{1}{2\pi} \int_0^{2\pi} \log |f'(re^{i\theta})|_h d\theta - \frac{1}{2\pi} \int_0^{2\pi} \log |f'(r_0 e^{i\theta})|_h d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f'(re^{i\theta})|_h d\theta + O(1) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \sum_\alpha |\rho_\alpha \circ f \cdot f'(re^{i\theta})|_h d\theta + O(1) \\ &\stackrel{(1.1.1)}{\leq} \sum_\alpha \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'(re^{i\theta})|_h d\theta + O(1) \\ &\stackrel{(1.2.3)}{\leq} \sum_\alpha \sum_{i=s(\alpha)+1}^n \frac{1}{2\pi} \int_0^{2\pi} \log^+ |(\log f_{\alpha i})'(re^{i\theta})| d\theta \\ &\quad + \sum_\alpha \sum_{j=1}^{s(\alpha)} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha j}(re^{i\theta})| d\theta + O(1) \\ &\stackrel{\text{Lemma 1.3}}{\leq} C_1 \sum_\alpha \sum_{i=s(\alpha)+1}^n (\log^+ T_{f_{\alpha i}, \omega_{FS}}(r) + \log r) \\ &\quad + \sum_\alpha \sum_{j=1}^{s(\alpha)} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha j}(re^{i\theta})| d\theta + O(1) \quad \parallel \end{aligned}$$

(1.2.4)

$$\stackrel{\text{Lemma 1.2}}{\leq} C_2 (\log^+ T_{f, \omega}(r) + \log r) + \sum_\alpha \sum_{j=1}^{s(\alpha)} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha j}(re^{i\theta})| d\theta + O(1) \quad \parallel.$$

where C_1 and C_2 are two positive constants which do not depend on r .

Claim 1.8. *For any $\alpha \in I$ and any $j \in \{1, \dots, s(\alpha)\}$, one has*

$$(1.2.5) \quad \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha_j}(re^{i\theta})| d\theta \leq C_3(\log^+ T_{f,\omega}(r) + \log r) + O(1) \quad \parallel$$

for positive constant C_3 which does not depend on r .

Proof of Claim 1.8. The proof of the claim is borrowed from [NW14, eq.(4.7.2)]. Pick $C > 0$ so that $\rho_\alpha \sqrt{-1} dx_{\alpha_j} \wedge d\bar{x}_{\alpha_j} \leq C\omega$. Write $f^*\omega := \sqrt{-1}B(t)dt \wedge d\bar{t}$. Then

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \log^+ |\rho_\alpha \circ f \cdot f'_{\alpha_j}(re^{i\theta})| d\theta = \frac{1}{4\pi} \int_0^{2\pi} \log^+ (|\rho_\alpha \circ f| \cdot |f'_{\alpha_j}(re^{i\theta})|^2) d\theta \\ & \leq \frac{1}{4\pi} \int_0^{2\pi} \log^+ B(re^{i\theta}) d\theta + O(1) \leq \frac{1}{4\pi} \int_0^{2\pi} \log(1 + B(re^{i\theta})) d\theta + O(1) \\ & \leq \frac{1}{2} \log\left(1 + \frac{1}{2\pi} \int_0^{2\pi} B(re^{i\theta}) d\theta\right) + O(1) = \frac{1}{2} \log\left(1 + \frac{1}{2\pi r} \frac{d}{dr} \int_{\mathbb{D}_r} rBdrd\theta\right) + O(1) \\ & = \frac{1}{2} \log\left(1 + \frac{1}{2\pi r} \frac{d}{dr} \int_{\mathbb{D}_r} f^*\omega\right) + O(1) \\ & \stackrel{\text{Lemma 1.4}}{\leq} \frac{1}{2} \log\left(1 + \frac{1}{2\pi r} \left(\int_{\mathbb{D}_r} f^*\omega\right)^{1+\delta}\right) + O(1) \quad \parallel \\ & = \frac{1}{2} \log\left(1 + \frac{r^\delta}{2\pi} \left(\frac{d}{dr} T_{f,\omega}(r)\right)^{1+\delta}\right) + O(1) \quad \parallel \\ & \stackrel{\text{Lemma 1.4}}{\leq} \frac{1}{2} \log\left(1 + \frac{r^\delta}{2\pi} (T_{f,\omega}(r))^{(1+\delta)^2}\right) + O(1) \quad \parallel \\ & \leq 4 \log^+ T_{f,\omega}(r) + \delta \log r + O(1) \quad \parallel. \end{aligned}$$

Here we pick $0 < \delta < 1$ to apply Lemma 1.4. The claim is proved. \square

Putting (1.2.5) to (1.2.4), one obtains

$$T_{f,\omega}(r) \leq C(\log^+ T_{f,\omega}(r) + \log r) + O(1) \quad \parallel$$

for some positive constant C . Hence $T_{f,\omega}(r) = O(\log r)$. We apply Lemma 1.1 to conclude that f extends to the ∞ . \square

2. THE NEGATIVELY CURVED METRIC VIA VIEHWEG-ZUO HIGGS BUNDLES

In [VZ02, VZ03], Viehweg-Zuo introduced a special type of Higgs bundles over the bases of smooth families of polarized manifolds with semi-ample canonical sheaves to study the hyperbolicity of such bases. It was later developed in [PTW18]. For any quasi-projective manifold V endowed with a Viehweg-Zuo Higgs bundle, we construct in [Den18a, Den18b] a generically positively definite Finsler metric over V whose holomorphic sectional curvature is bounded from above by a negative constant. In this section we will refine curvature estimate in [Den18a] to prove big Picard theorem for such quasi-projective manifold V .

2.1. Viehweg-Zuo Higgs bundles and their proper metrics. The definition for Viehweg-Zuo Higgs bundles we present below follows from the formulation in [VZ02, VZ03] and [PTW18].

Definition 2.1 (Abstract Viehweg-Zuo Higgs bundles). Let V be a quasi-projective manifold, and let $Y \supset V$ be a projective compactification of V with the boundary $D := Y \setminus V$ simple normal crossing. A *Viehweg-Zuo Higgs bundle* over (Y, D) (or say over V abusively) is a logarithmic Higgs bundle $(\tilde{\mathcal{E}}, \tilde{\theta})$ over Y consisting of the following data:

- (i) a divisor S on Y so that $D + S$ is simple normal crossing,
- (ii) a big and nef line bundle L over Y with $\mathbf{B}_+(L) \subset D \cup S$,
- (iii) a Higgs bundle $(\mathcal{E}, \theta) := (\bigoplus_{q=0}^n E^{n-q,q}, \bigoplus_{q=0}^n \theta_{n-q,q})$ induced by the lower canonical extension of a polarized VHS defined over $Y \setminus (D \cup S)$,
- (iv) a sub-Higgs sheaf $(\mathcal{F}, \eta) \subset (\tilde{\mathcal{E}}, \tilde{\theta})$,

which satisfy the following properties.

- (1) The Higgs bundle $(\tilde{\mathcal{E}}, \tilde{\theta}) := (L^{-1} \otimes \mathcal{E}, \mathbb{1} \otimes \theta)$. In particular, $\tilde{\theta} : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}} \otimes \Omega_Y(\log(D+S))$, and $\tilde{\theta} \wedge \tilde{\theta} = 0$.
- (2) The sub-Higgs sheaf (\mathcal{F}, η) has log poles only on the boundary D , that is, $\eta : \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_Y(\log D)$.
- (3) Write $\tilde{\mathcal{E}}_k := L^{-1} \otimes E^{n-k,k}$, and denote by $\mathcal{F}_k := \tilde{\mathcal{E}}_k \cap \mathcal{F}$. Then the first stage \mathcal{F}_0 of \mathcal{F} is an *effective line bundle*. In other words, there exists a non-trivial morphism $\mathcal{O}_Y \rightarrow \mathcal{F}_0$.

As shown in [VZ02], by iterating η for k -times, we obtain

$$\mathcal{F}_0 \xrightarrow{\overbrace{\eta \circ \cdots \circ \eta}^{k \text{ times}}} \mathcal{F}_k \otimes (\Omega_Y(\log D))^{\otimes k}.$$

Since $\eta \wedge \eta = 0$, the above morphism factors through $\mathcal{F}_k \otimes \text{Sym}^k \Omega_Y(\log D)$, and by (3) one thus obtains

$$\mathcal{O}_Y \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_k \otimes \text{Sym}^k \Omega_Y(\log D) \rightarrow L^{-1} \otimes E^{n-k,k} \otimes \text{Sym}^k \Omega_Y(\log D).$$

Equivalently, we have a morphism

$$(2.1.1) \quad \tau_k : \text{Sym}^k T_Y(-\log D) \rightarrow L^{-1} \otimes E^{n-k,k}.$$

It was proven in [VZ02, Corollary 4.5] that τ_1 is always non-trivial. In [Den18b] we prove that $\tau_1 : T_Y(-\log D) \rightarrow L^{-1} \otimes E^{n-1,1}$ in (2.1.1) is generically injective.

We will follow [PTW18] to give some ‘‘proper’’ metric on $\tilde{\mathcal{E}} = \bigoplus_{k=0}^n L^{-1} \otimes E^{n-k,k}$. Write the simple normal crossing divisor $D = D_1 + \cdots + D_k$ and $S = S_1 + \cdots + S_\ell$. Let $f_{D_i} \in H^0(Y, \mathcal{O}_Y(D_i))$ and $f_{S_i} \in H^0(Y, \mathcal{O}_Y(S_i))$ be the canonical section defining D_i and S_i . We fix smooth hermitian metrics g_{D_i} and g_{S_i} on $\mathcal{O}_Y(D_i)$ and $\mathcal{O}_Y(S_i)$. After rescaling g_{D_i} and g_{S_i} , we assume that $|f_{D_i}|_{g_{D_i}} < 1$ and $|f_{S_j}|_{g_{S_j}} < 1$ for $i = 1, \dots, k$ and $j = 1, \dots, \ell$. Set

$$r_D := \prod_{i=1}^k (-\log |f_{D_i}|_{g_{D_i}}^2), \quad r_S := \prod_{j=1}^{\ell} (-\log |f_{S_j}|_{g_{S_j}}^2).$$

Let g be a singular hermitian metric with analytic singularities of the big and nef line bundle L such that g is smooth on $Y \setminus \mathbf{B}_+(L) \supset Y \setminus D \cup S$, and the curvature current $\sqrt{-1}\Theta_g(L) \geq \omega$ for some smooth Kähler form ω on Y . For $\alpha \in \mathbb{N}$, define

$$h_L := g \cdot (r_D \cdot r_S)^\alpha$$

The following proposition is a slight variant of [PTW18, Lemma 3.1, Corollary 3.4].

Proposition 2.2 ([PTW18]). *When $\alpha \gg 0$, after rescaling f_{D_i} and f_{S_i} , there exists a continuous, positively definite hermitian form ω_α on $T_Y(-\log D)$ such that*

- (i) over $V_0 := Y \setminus D \cup S$, the curvature form

$$\sqrt{-1}\Theta_{h_L}|_{V_0} \geq r_D^{-2} \cdot \omega_\alpha|_{V_0}, \quad \sqrt{-1}\Theta_{h_L}(L) \geq \omega$$

where ω is a smooth Kähler metric on Y .

- (ii) The singular hermitian metric $h := h_L^{-1} \otimes h_{\text{hod}}$ on $\tilde{\mathcal{E}} = L^{-1} \otimes \mathcal{E}$ is locally bounded on Y , and smooth outside $(D + S)$, where h_{hod} is the Hodge metric for the Hodge bundle \mathcal{E} . Moreover, h is degenerate on $D + S$.

(iii) *The singular hermitian metric $r_D^2 h$ on $L^{-1} \otimes \mathcal{E}$ is also locally bounded on Y and is degenerate on $D + S$.* \square

Hence by Definition 1.6, h and $r_D^2 h$ are both Finsler metrics on $\tilde{\mathcal{E}}$.

2.2. Construction of negatively curved Finsler metric. We adopt the same notations as § 2.1 throughout this subsection. Assume that the log manifold (Y, D) is endowed with a Viehweg-Zuo Higgs bundle. In [Den18a, §3.4] we construct Finsler metrics F_1, \dots, F_n on $T_Y(-\log D)$ as follows. By (2.1.1), for each $k = 1, \dots, n$, there exists

$$(2.2.1) \quad \tau_k : \text{Sym}^k T_Y(-\log D) \rightarrow L^{-1} \otimes E^{n-k, k}.$$

Then it follows from Proposition 2.2.(ii) that the Finsler metric h on $L^{-1} \otimes E^{n-k, k}$ induces a Finsler metric F_k on $T_Y(-\log D)$ defined as follows: for any $e \in T_Y(-\log D)_y$,

$$(2.2.2) \quad F_k(e) := h(\tau_k(e^{\otimes k}))^{\frac{1}{k}}$$

Let $C \subset \mathbb{C}$ be any open set of \mathbb{C} . For any $\gamma : C \rightarrow V$, one has

$$(2.2.3) \quad d\gamma : T_C \rightarrow \gamma^* T_V \hookrightarrow \gamma^* T_Y(-\log D).$$

We denote by $\partial_t := \frac{\partial}{\partial t}$ the canonical vector fields in $C \subset \mathbb{C}$, $\bar{\partial}_t := \frac{\partial}{\partial \bar{t}}$ its conjugate. The Finsler metric F_k induces a continuous Hermitian pseudo-metric on C , defined by

$$(2.2.4) \quad \gamma^* F_k^2 = \sqrt{-1} G_k(t) dt \wedge d\bar{t}.$$

Hence $G_k(t) = |\tau_k(d\gamma(\partial_t)^{\otimes k})|_h^{\frac{2}{k}}$, where τ_k is defined in (2.1.1). The reader might worried that all $G_k(t)$ will be identically equal to zero. In [Den18b, Theorem C], we prove that ‘generically’ this cannot happen.

Theorem 2.3 ([Den18b]). *There is a dense Zariski open set $V^\circ \subset V_0 = Y \setminus (D + S)$ of V° so that $\tau_1 : T_Y(-\log D)|_{V^\circ} \rightarrow L^{-1} \otimes E^{n-1, 1}|_{V^\circ}$ is injective.* \square

We now fix any $\gamma : C \rightarrow V$ with $\gamma(C) \cap V^\circ \neq \emptyset$. By Proposition 2.2.(ii), the metric h for $L^{-1} \otimes \mathcal{E}$ is smooth and positively definite over V_0 . It then follows from Theorem 2.3 that $G_1(t) \neq 0$. Let $C^\circ \subset C$ be an (non-empty) open set whose complement $C \setminus C^\circ$ is a discrete set so that

- The image $\gamma(C^\circ) \subset V^\circ$.
- For every $k = 1, \dots, n$, either $G_k(t) \equiv 0$ on C° or $G_k(t) > 0$ for any $t \in C^\circ$.
- $\gamma'(t) \neq 0$ for any $t \in C^\circ$.

By the definition of $G_k(t)$, if $G_k(t) \equiv 0$ for some $k > 1$, then $\tau_k(\partial_t^{\otimes k}) \equiv 0$ where τ_k is defined in (2.1.1). Note that one has $\tau_{k+1}(\partial_t^{\otimes(k+1)}) = \tilde{\theta}(\tau_k(\partial_t^{\otimes k}))(\partial_t)$, where $\tilde{\theta} : L^{-1} \otimes \mathcal{E} \rightarrow L^{-1} \otimes \mathcal{E} \otimes \Omega_Y(\log(D + S))$ is defined in Definition 2.1. We thus conclude that $G_{k+1}(t) \equiv 0$. Hence it exists $1 \leq m \leq n$ so that the set $\{k \mid G_k(t) > 0 \text{ over } C^\circ\} = \{1, \dots, m\}$, and $G_\ell(t) \equiv 0$ for all $\ell = m + 1, \dots, n$. From now on, *all the computations* are made over C° .

In [Den18a] we proved the following curvature formula.

Theorem 2.4 ([Den18a, Proposition 3.12]). *For $k = 1, \dots, m$, over C° one has*

$$(2.2.5) \quad \frac{\partial^2 \log G_1}{\partial t \partial \bar{t}} \geq \Theta_{L, h_L}(\partial_t, \bar{\partial}_t) - \frac{G_2^2}{G_1} \quad \text{if } k = 1,$$

$$(2.2.6) \quad \frac{\partial^2 \log G_k}{\partial t \partial \bar{t}} \geq \frac{1}{k} \left(\Theta_{L, h_L}(\partial_t, \bar{\partial}_t) + \frac{G_k^k}{G_{k-1}^{k-1}} - \frac{G_{k+1}^{k+1}}{G_k^k} \right) \quad \text{if } k > 1.$$

Here we make the convention that $G_{n+1} \equiv 0$ and $\frac{0}{0} = 0$. We also write ∂_t (resp. $\bar{\partial}_t$) for $d\gamma(\partial_t)$ (resp. $d\gamma(\bar{\partial}_t)$) abusively, where $d\gamma$ is defined in (2.2.3). \square

Let us mention that in [Den18a, eq. (3.3.58)] we drop the term $\Theta_{L, h_L}(\partial_t, \bar{\partial}_t)$ in (2.2.6), though it can be easily seen from the proof of [Den18a, Lemma 3.9]. As we will see below, such a term is crucial in deriving the new curvature estimate.

By Theorem 2.4 we see that the curvature of F_k is not the desired type (1.2.1) for applying the criteria for big Picard theorem in Theorem 1.7. In [Den18a, §3.4], following ideas by [TY15, Sch17] we introduce a new Finsler metric F on $T_Y(-\log D)$ by taking convex sum in the following form

$$(2.2.7) \quad F := \sqrt{\sum_{k=1}^n k \alpha_k F_k^2}.$$

where $\alpha_1, \dots, \alpha_n \in \mathbb{R}^+$ are some constants which will be fixed later. By Theorem 2.3, the set of degenerate points of F defined in Definition 1.6, denoted by Δ_F , is contained in a proper Zariski closed subset $Y \setminus V^\circ$. In [Den18a, Theorem 3.8] we prove that the holomorphic sectional curvature of F is bounded from above by a negative constant. Let us now prove a new curvature formula for F in this section.

For the above $\gamma : C \rightarrow V$ with $\gamma(C) \cap V^\circ \neq \emptyset$, we write

$$\gamma^* F^2 = \sqrt{-1} H(t) dt \wedge d\bar{t}.$$

Then

$$(2.2.8) \quad H(t) = \sum_{k=1}^n k \alpha_k G_k(t),$$

where G_k is defined in (2.2.4). Recall that for $k = 1, \dots, m$, $G_k(t) > 0$ for $t \in C^\circ$.

We first recall a computational lemma by Schumacher.

Lemma 2.5 ([Sch17, Lemma 17]). *Let $\alpha_j > 0$ and G_j be positive real numbers for $j = 1, \dots, n$. Then*

$$(2.2.9) \quad \begin{aligned} & \sum_{j=2}^n \left(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1} \frac{G_j^j}{G_{j-1}^{j-2}} \right) \\ & \geq \frac{1}{2} \left(-\frac{\alpha_1^3}{\alpha_2^2} G_1^2 + \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} G_n^2 + \sum_{j=2}^{n-1} \left(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} \right) G_j^2 \right) \end{aligned}$$

□

Now we are ready to compute the curvature of the Finsler metric F based on Theorem 2.4.

Theorem 2.6. *Fix a smooth Kähler metric ω on Y . There exist universal constants $0 < \alpha_1 < \dots < \alpha_n$ and $\delta > 0$, such that for any $\gamma : C \rightarrow V$ with C an open set of \mathbb{C} and $\gamma(C) \cap V^\circ \neq \emptyset$, one has*

$$(2.2.10) \quad \sqrt{-1} \partial \bar{\partial} \log |\gamma'(t)|_F^2 \geq \delta \gamma^* \omega$$

Proof. By Theorem 2.3 and the assumption that $\gamma(C) \cap V^\circ \neq \emptyset$, $G_1(t) \neq 0$. We first recall a result in [Den18a, Lemma 3.11], and we write its proof here for it is crucial in what follows.

Claim 2.7. *There is a universal constant $c_0 > 0$ (i.e. it does not depend on γ) so that $\Theta_{L, h_L}(\partial_t, \bar{\partial}_t) \geq c_0 G_1(t)$ for all t .*

Proof of Claim 2.7. Indeed, by Proposition 2.2.(i), it suffice to prove that

$$(2.2.11) \quad \frac{|\partial_t|_{\gamma^*(r_D^{-2} \cdot \omega_\alpha)}^2}{|\tau_1(d\gamma(\partial_t))|_h^2} \geq c_0$$

for some $c_0 > 0$, where ω_α is a positively definite Hermitian metric on $T_Y(-\log D)$. Note that

$$\frac{|\partial_t|_{\gamma^*(r_D^{-2} \cdot \omega_\alpha)}^2}{|\tau_1(d\gamma(\partial_t))|_h^2} = \frac{|\partial_t|_{\gamma^*(r_D^{-2} \cdot \omega_\alpha)}^2}{|\partial_t|_{\gamma^* \tau_1^* h}^2} = \frac{|\partial_t|_{\gamma^*(\omega_\alpha)}^2}{|\partial_t|_{\gamma^* \tau_1^*(r_D^2 \cdot h)}^2},$$

where $\tau_1^*(r_D^2 \cdot h)$ is a Finsler metric (indeed continuous pseudo hermitian metric) on $T_Y(-\log D)$ by Proposition 2.2.(iii). Since Y is compact, there exists a constant $c_0 > 0$ such that

$$\omega_\alpha \geq c_0 \tau_1^*(r_D^2 \cdot h).$$

Hence (2.2.11) holds for any $\gamma : C \rightarrow V$ with $\gamma(C) \cap V^\circ \neq \emptyset$. The claim is proved. \square

By [Sch12, Lemma 8],

$$(2.2.12) \quad \sqrt{-1} \partial \bar{\partial} \log \left(\sum_{j=1}^n j \alpha_j G_j \right) \geq \frac{\sum_{j=1}^n j \alpha_j G_j \sqrt{-1} \partial \bar{\partial} \log G_j}{\sum_{i=1}^n j \alpha_j G_i}$$

Putting (2.2.5) and (2.2.6) to (2.2.12), and making the convention that $\frac{0}{0} = 0$, we obtain

$$\begin{aligned} \frac{\partial^2 \log H(t)}{\partial t \partial \bar{t}} &\geq \frac{1}{H} \left(-\alpha_1 G_2^2 + \sum_{k=2}^n \alpha_k \left(\frac{G_k^{k+1}}{G_{k-1}^{k-1}} - \frac{G_{k+1}^{k+1}}{G_k^{k-1}} \right) \right) + \frac{\sum_{k=1}^n \alpha_k G_k}{H} \Theta_{L, h_L}(\partial_t, \bar{\partial}_t) \\ &= \frac{1}{H} \left(\sum_{j=2}^n \left(\alpha_j \frac{G_j^{j+1}}{G_{j-1}^{j-1}} - \alpha_{j-1} \frac{G_j^j}{G_{j-1}^{j-2}} \right) \right) + \frac{\sum_{k=1}^n \alpha_k G_k}{H} \Theta_{L, h_L}(\partial_t, \bar{\partial}_t) \\ &\stackrel{(2.2.9)}{\geq} \frac{1}{H} \left(-\frac{1}{2} \frac{\alpha_1^3}{\alpha_2^2} G_1^2 + \frac{1}{2} \sum_{j=2}^{n-1} \left(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} \right) G_j^2 + \frac{1}{2} \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} G_n^2 \right) \\ &\quad + \frac{\sum_{k=1}^n \alpha_k G_k}{H} \Theta_{L, h_L}(\partial_t, \bar{\partial}_t) \\ &\stackrel{\text{Claim 2.7}}{\geq} \frac{1}{H} \left(\alpha_1 \left(c_0 - \frac{\alpha_1^2}{\alpha_2^2} \right) G_1^2 + \frac{1}{2} \sum_{j=2}^{n-1} \left(\frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} \right) G_j^2 + \frac{1}{2} \frac{\alpha_{n-1}^{n-1}}{\alpha_n^{n-2}} G_n^2 \right) \\ &\quad + \frac{1}{H} \left(\frac{1}{2} \alpha_1 G_1 + \sum_{k=2}^n \alpha_k G_k \right) \Theta_{L, h_L}(\partial_t, \bar{\partial}_t) \end{aligned}$$

One can take $\alpha_1 = 1$, and choose the further $\alpha_j > \alpha_{j-1}$ inductively so that

$$(2.2.13) \quad c_0 - \frac{\alpha_1^2}{\alpha_2^2} > 0, \quad \frac{\alpha_{j-1}^{j-1}}{\alpha_j^{j-2}} - \frac{\alpha_j^{j+2}}{\alpha_{j+1}^{j+1}} > 0 \quad \forall j = 2, \dots, n-1.$$

Hence

$$\frac{\partial^2 \log H(t)}{\partial t \partial \bar{t}} \geq \frac{1}{H} \left(\frac{1}{2} \alpha_1 G_1 + \sum_{k=2}^n \alpha_k G_k \right) \Theta_{L, h_L}(\partial_t, \bar{\partial}_t) \stackrel{(2.2.8)}{\geq} \frac{1}{n} \Theta_{L, h_L}(\partial_t, \bar{\partial}_t)$$

over C° . By Proposition 2.2.(i), this implies that

$$(2.2.14) \quad \sqrt{-1} \partial \bar{\partial} \log |\gamma'|_F^2 = \sqrt{-1} \partial \bar{\partial} \log H(t) \geq \frac{1}{n} \gamma^* \sqrt{-1} \Theta_{L, h_L} \geq \delta \gamma^* \omega$$

over C° for some positive constant δ , which does not depend on γ . Since $|\gamma'(t)|_F^2$ is continuous and locally bounded from above over C , by the extension theorem of subharmonic function, (2.2.14) holds over the whole C . Since $c_0 > 0$ is a constant which does not depend on γ , so are $\alpha_1, \dots, \alpha_n$ by (2.2.13). The theorem is thus proved. \square

In summary of results in this subsection, we obtain the following theorem.

Theorem 2.8. *Let X be a projective manifold and let D be a simple normal crossing divisor on X . Assume that there is a Viehweg-Zuo Higgs bundle over (X, D) . Then there are a Finsler metric h on $T_X(-\log D)$ which is positively definite on a dense Zariski open set V° of $V := X \setminus D$, and a smooth Kähler form on X such that for any holomorphic map $\gamma : C \rightarrow V$ from any open subset C of \mathbb{C} with $\gamma(C) \cap V^\circ \neq \emptyset$, one has*

$$\sqrt{-1} \partial \bar{\partial} \log |\gamma'|_h^2 \geq \gamma^* \omega.$$

In particular, by Theorem A, for any holomorphic map $f : \Delta^ \rightarrow X \setminus D$, with $f(\Delta^*) \cap V^\circ \neq \emptyset$, it extends to a holomorphic map $\tilde{f} : \Delta \rightarrow X$. \square*

3. GENERALIZED BIG PICARD THEOREMS

We will apply Theorems 1.7 and 2.8 to prove the big Picard theorem for moduli of polarized manifolds with semi-ample canonical sheaf, and for bases of log smooth families of Calabi-Yau pairs.

3.1. Big Picard theorem for moduli of polarized manifolds.

Proof of Theorem C. Let Z be the Zariski closure of $\gamma(\Delta^*)$ in Y . Take an embedded desingularization of singularities $\mu : \tilde{Y} \rightarrow Y$ so that the strict transform of Z , denoted by \tilde{Z} , is smooth. Write $\tilde{Z}^\circ := \tilde{Z} \cap \mu^{-1}(V)$, which is a dense Zariski open set of \tilde{Z} . We take the base change

$$\begin{array}{ccc} X^\circ = U \times_V \tilde{Z}^\circ & \longrightarrow & U \\ f_{X^\circ} \downarrow & & \downarrow f_U \\ \tilde{Z}^\circ & \xrightarrow{\iota} & V \end{array}$$

Then polarized family $(f_{X^\circ} : X^\circ \rightarrow \tilde{Z}^\circ, \iota^* \mathcal{L}) \in \mathcal{P}_h(\tilde{Z}^\circ)$. We denote by $\varphi_{X^\circ} : \tilde{Z}^\circ \rightarrow P_h$ the moduli map associated to f_{X° . Then $\varphi_{X^\circ} = \varphi_U \circ \iota$, which is generically finite. Hence f_{X° is of maximal variation. By [VZ02,PTW18], after passing to a birational modification $\nu : W \rightarrow \tilde{Z}$, there exists a Viehweg-Zuo Higgs bundle on $W^\circ := \nu^{-1}(\tilde{Z}^\circ)$. By Theorem 2.8, there is a dense Zariski open set $W' \subset W^\circ$ so that any holomorphic map $\Delta^* \rightarrow W^\circ$ extends to $\Delta \rightarrow W$ provided that its image is not contained in $W \setminus W'$. Since $\gamma : \Delta^* \rightarrow Z$ is Zariski dense, it thus does not lie on the discriminant locus of the birational morphism $\mu|_{\tilde{Z}} \circ \nu : W \rightarrow Z$, and thus $\tilde{\gamma} = (\mu|_{\tilde{Z}} \circ \nu)^{-1} \circ \gamma : \Delta^* \rightarrow W$ exists with its image contained in W° . Moreover, $\tilde{\gamma} : \Delta^* \rightarrow W$ is also Zariski dense, and thus $\tilde{\gamma}(\Delta^*) \cap W' \neq \emptyset$. By Theorem 1.7, $\tilde{\gamma} : \Delta^* \rightarrow W$ extends to a holomorphic map $\tilde{\tilde{\gamma}} : \Delta \rightarrow W$. The holomorphic map $\mu \circ \nu \circ \tilde{\tilde{\gamma}} : \Delta \rightarrow Y$ is the desired extension of $\gamma : \Delta^* \rightarrow V$. The theorem is proved. \square

Remark 3.1. Based on the fundamental work [VZ02,VZ03,PTW18], in [Den18b] we prove that the base V in Theorem C is both Brody hyperbolic and pseudo Kobayashi hyperbolic. In [LSZ19], Lu-Sun-Zuo combine the original approach in [VZ03] with our Torelli type result Theorem 2.3 to construct negatively curved pseudo hermitian metric on any algebraic curve in V , so that they can apply the celebrated work of Griffiths-King [GK73] to prove the Borel hyperbolicity of V .

3.2. Big Picard theorem for bases of log Calabi-Yau families. In [Den19], we prove that for maximally varying, log smooth family of Calabi-Yau pairs, its base can be equipped with a Viehweg-Zuo Higgs bundle. Let us start with the following definition.

Definition 3.2 (log smooth family). Let X° and Y° be quasi-projective manifolds, and let $D^\circ = \sum_{i=1}^m a_i D_i^\circ$ be a Kawamata log terminal (klt for short) \mathbb{Q} -divisor on X° with simple normal crossing support. The morphism $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ is a *log smooth family* if $f^\circ : X^\circ \rightarrow Y^\circ$ is a smooth projective morphism with connected fibers, and D° is *relatively normal crossing over Y°* , namely each stratum $D_{i_1}^\circ \cap \cdots \cap D_{i_k}^\circ$ of D° is dominant onto and smooth over Y° under f° .

Let us recall the main result in [Den19].

Theorem 3.3 ([Den19, Theorem A]). *Let $f^\circ : (X^\circ, D^\circ) \rightarrow Y^\circ$ be a log smooth family over a quasi-projective manifold Y° . Assume that each fiber $(X_y, D_y) := (f^\circ)^{-1}(y)$ is a klt pair, and $K_{X_y} + D_y \equiv_{\mathbb{Q}} 0$. Assume that the logarithmic Kodaira-Spencer map $\rho : T_{Y^\circ} \rightarrow R^1 f_* (T_{X^\circ/Y^\circ}(-\log D^\circ))$ is generically injective. Then after replacing Y° by a birational model, there exists a Viehweg-Zuo Higgs bundle over Y° . \square*

By Theorem 3.3, one can perform the same proof as that of Theorem C to conclude the following result.

Theorem 3.4. *In the setting of Theorem 3.3, assume additionally that the logarithmic Kodaira-Spencer map*

$$T_{Y^\circ, y} \rightarrow H^1(X_y, T_{X_y}(-\log D_y))$$

is injective for any $y \in Y^\circ$. Then for any projective compactification Y of the base Y° , any holomorphic map $\gamma : \Delta^ \rightarrow Y^\circ$ extends into the origin. \square*

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INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, UNIVERSITÉ PARIS-SACLAY, 35 ROUTE DE CHARTRES,
91440, BURES-SUR-YVETTE, FRANCE

Email address: deng@ihes.fr

URL: <https://www.ihes.fr/~deng>