Universal cocycles and the graph complex action on homogeneous Poisson brackets by diffeomorphisms

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Universal cocycles and the graph complex action on homogeneous Poisson brackets by diffeomorphisms

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Abstract

The graph complex acts on the spaces of Poisson bi-vectors $\mathcal{P}$ by infinitesimal symmetries. We prove that whenever a Poisson structure is homogeneous, i.e. $\mathcal{P} = \mathrm{L}_{\vec{V}}(\mathcal{P})$ w.r.t. the Lie derivative along some vector field $\vec{V}$, but not quadratic (the coefficients of $\mathcal{P}$ are not degree-two homogeneous polynomials), and whenever its velocity bi-vector $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$, also homogeneous w.r.t. $\vec{V}$ by $\mathrm{L}_{\vec{V}}(\mathcal{Q}) = n\mathcal{Q}$ whenever $\mathcal{Q}(\mathcal{P}) = \mathrm{O\vec{r}}(\gamma)(\mathcal{P}^{\otimes^n})$ is obtained using the orientation morphism $\mathrm{O\vec{r}}$ from a graph cocycle γ on n vertices and $2n-2$ edges, then the 1-vector $\vec{\mathcal{X}} = \mathrm{O\vec{r}}(\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes^{n-1}})$ is a Poisson cocycle. Its construction is uniform for all Poisson bi-vectors $\mathcal{P}$ satisfying the above assumptions, on all finite-dimensional affine manifolds M. Still, if the bi-vector $\mathcal{Q} \not\equiv 0$ is exact in the respective Poisson cohomology, so there exists a vector field $\vec{\mathcal{Y}}$ such that $\mathcal{Q}(\mathcal{P}) = [\![\vec{\mathcal{Y}}, \mathcal{P}]\!]$, then the universal cocycle $\vec{\mathcal{X}}$ does not belong to the coset of $\vec{\mathcal{Y}}$ mod $\ker[\![\mathcal{P}, \cdot]\!]$. We illustrate the construction using two examples of cubic-coefficient Poisson brackets associated with the R-matrices for the Lie algebra $\mathfrak{gl}(2)$.

Introduction. Bi-vector cocycles $\mathcal{Q}(\mathcal{P}) = \mathrm{O\vec{r}}(\gamma)(\mathcal{P}^{\otimes^n}) \in \ker[\![\mathcal{P}, \cdot]\!]$ are obtained by Kontsevich's graph orientation morphism $\mathrm{O\vec{r}}$ from graph cocycles γ on n vertices and $2n-2$ edges in a way which is uniform for all finite-dimensional affine Poisson manifolds (M^r, $\mathcal{P}$). The (non)triviality of cocycles $\mathcal{Q}(\mathcal{P})$ in the second Poisson cohomology w.r.t. the differential $\partial_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$ remains an open problem, twenty-five years after the discovery of the graph complex and orientation morphism (see [11]). In all the Poisson geometries probed so far, the known infinitesimal symmetries $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ of the Jacobi identity $\tfrac{1}{2}[\![\mathcal{P}, \mathcal{P}]\!] = 0$ are $\partial_{\mathcal{P}}$-exact: there always exists a vector field $\vec{\mathcal{Y}}$ such that $\mathcal{Q}(\mathcal{P}) = [\![\vec{\mathcal{Y}}, \mathcal{P}]\!]$. The evolution $\mathcal{P}(\varepsilon = 0) \longmapsto \mathcal{P}(\varepsilon > 0)$ of the tensor $\mathcal{P}$ then amounts to its reparametrisations under the diffeomorphisms of Poisson manifold which are induced by the shifts along the integral trajectories of the vector field $\vec{\mathcal{Y}}$. This is why, instead of producing new Poisson brackets from a given one, the Kontsevich graph flows on the spaces of Poisson bi-vectors induce (non)linear

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diffeomorphisms of the base manifold M, although no more than its affine structure was the initial assumption and no possibility of smooth coordinate reparametrizations was presumed.

For a much used class of (scaling-)homogeneous Poisson bi-vectors $\mathcal{P} = \mathrm{L}_{\vec{V}}(\mathcal{P})$, we obtain an explicit formula, $\vec{\mathcal{X}} = \vec{\mathrm{Or}}(\gamma)(\vec{V} \otimes \mathcal{P}^{\otimes^{n-1}})$, of a 1-vector cocycle $\vec{\mathcal{X}}(\gamma, \vec{V}, \mathcal{P}) \in \ker[\![\mathcal{P}, \cdot]\!]$ which is built from the graph cocycles γ uniformly for all homogeneous Poisson bi-vectors $\mathcal{P}$ on affine manifolds $M^{r<\infty}$. The cocycle $\vec{\mathcal{X}}$ is however not necessarily a 1-vector representative of the coset $\vec{\mathcal{Y}} \bmod \{\vec{\mathcal{Z}} \in \ker[\![\mathcal{P}, \cdot]\!]\}$ which would trivialise the value $\mathcal{Q}(\mathcal{P}) = [\![\vec{\mathcal{Y}}, \mathcal{P}]\!]$ of Kontsevich's symmetries at homogeneous Poisson structures. Indeed, the Poisson cocycle $\mathcal{Q}(\mathcal{P})$ can be, we show, a nonzero bi-vector on M^r, whereas the bi-vector $[\![\vec{\mathcal{X}}, \mathcal{P}]\!]$ is identically zero on M^r by construction. We contrast the formulas of universal cocycles $\vec{\mathcal{X}}(\gamma, \vec{V}, \mathcal{P})$ and trivialising vector fields $\vec{\mathcal{Y}}$ for nonzero symmetries $\dot{\mathcal{P}} = \vec{\mathrm{Or}}(\gamma)(\mathcal{P})$ by two examples, namely, using cubic-coefficient Poisson brackets associated with the R-matrices for $\mathfrak{gl}(2)$.

This paper is organized as follows. In §1 we recall elements of Poisson cohomology theory in the context of Kontsevich's universal deformations of bi-vectors by using the unoriented graph cocycles. In §2 we phrase the notion of structures which are homogeneous w.r.t. a 1-vector field, and we prove the main theorem. Finally, we illustrate the result (cf. [10]).

1. Poisson cohomology and the graph complex. A Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P}}$ on a real manifold M is a bi-linear skew-symmetric bi-derivation which takes $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ and satisfies the Jacobi identity $\frac{1}{2}\sum_{\sigma\in S_3}\{\{\sigma(f),\sigma(g)\}_{\mathcal{P}},\sigma(h)\}_{\mathcal{P}} = 0$ for any $f, g, h \in C^\infty(M)$. The fact that both the arguments f, g and their bracket $\{f, g\}_{\mathcal{P}}$ are scalars dictates the tensor transformation law of the components $\mathcal{P}^{ij}$ of a bi-vector $\mathcal{P} = \sum_{i,j} \mathcal{P}^{ij}(\boldsymbol{x})\partial_i \otimes \partial_j = \frac{1}{2}\sum_{i,j}\mathcal{P}^{ij}(\boldsymbol{x})(\partial_i \otimes \partial_j - \partial_j \otimes \partial_i) = \frac{1}{2}\sum_{i,j}\mathcal{P}^{ij}\,\partial_i \wedge \partial_j$ whenever the structure is referred to a system of coordinates $\boldsymbol{x} = (x^1, \ldots, x^r)$ and $\partial_i = \partial/\partial x^i$ is a shorthand notation.

The calculus on the space of multivectors $\Gamma(\bigwedge^\bullet TM) \cong C^\infty(\Pi T^*M)$ is simplified if one uses the parity-odd coordinates ξ_i along the directions dx^i in the fibres of the cotangent bundle T^*M over points $\boldsymbol{a} \in M$ (which are parametrized by x^i). The symbol ξ_i thus corresponds to $\partial/\partial x^i$ dual to dx^i, and bi-vectors are $\mathcal{P} = \frac{1}{2}\sum_{i,j}\mathcal{P}^{ij}\xi_i\xi_j$, so that $\{f, g\}_{\mathcal{P}}(\boldsymbol{a}) = (f)\overleftarrow{\partial}/\partial x^\mu \cdot \vec{\partial}/\partial\xi_\mu\,(\mathcal{P})\,\overleftarrow{\partial}/\partial\xi_\nu \cdot \vec{\partial}/\partial x^\nu(g)$; here, both the coefficients $\mathcal{P}^{ij}$ and derivatives $\partial/\partial x^k$ are evaluated at the point $\boldsymbol{a} \in M$ as in the left-hand side.[1]

The space of multivectors is endowed with the parity-odd Poisson bracket $[\![\cdot,\cdot]\!]$ (the Schouten bracket, or antibracket) of own degree -1. For arbitrary multivectors $\mathcal{P}, \mathcal{Q}$, the formula is $[\![\mathcal{P}, \mathcal{Q}]\!] = (\mathcal{P})\overleftarrow{\partial}/\partial\xi_i \cdot \vec{\partial}/\partial x^i(\mathcal{Q}) - (\mathcal{P})\overleftarrow{\partial}/\partial x^i \cdot \vec{\partial}/\partial\xi_i(\mathcal{Q})$; in particular, $[\![\vec{\mathcal{X}}, \vec{\mathcal{Y}}]\!] = [\vec{\mathcal{X}}, \vec{\mathcal{Y}}]$ is the usual commutator of vector fields $\vec{\mathcal{X}}, \vec{\mathcal{Y}}$ on M. The Schouten bracket $[\![\cdot,\cdot]\!]$ is shifted-graded skew-symmetric: $[\![\mathcal{Q}, \mathcal{P}]\!] = -(-)^{(|\mathcal{P}|-1)\cdot(|\mathcal{Q}|-1)}[\![\mathcal{P}, \mathcal{Q}]\!]$ for $\mathcal{P}$ and $\mathcal{Q}$ grading-homogeneous. This is why, unlike the tautology $[\![\vec{\mathcal{X}}, \vec{\mathcal{X}}]\!] \equiv 0$, the equation $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ is a nontrivial restriction for bi-vectors $\mathcal{P}$, containing the tri-vector in the l.-h.s. of the Jacobi identity $\frac{1}{2}[\![\mathcal{P}, \mathcal{P}]\!](f, g, h) = 0$ for the bracket $\{f, g\}_{\mathcal{P}} = [\![[\![f, \mathcal{P}]\!], g]\!]$. The Schouten bracket itself satisfies the graded Jacobi identity $[\![\mathcal{P}, [\![\mathcal{Q}, \mathcal{R}]\!]]\!] - (-)^{(|\mathcal{P}|-1)\cdot(|\mathcal{Q}|-1)}[\![\mathcal{Q}, [\![\mathcal{P}, \mathcal{R}]\!]]\!] = [\![[\![\mathcal{P}, \mathcal{Q}]\!], \mathcal{R}]\!]$ with $\mathcal{P}$ and $\mathcal{Q}$ grading-homogeneous. This identity implies that for Poisson bi-vectors $\mathcal{P}$, their adjoint action by $\partial_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$ is a differential of degree $+1$ on the space of multivectors on M. The Poisson differential $\partial_{\mathcal{P}}$ gives rise to the Poisson cohomology $H^i_{\mathcal{P}}(M)$ of the manifold M (see [13]).[2]

[1]The dot $\cdot$ denotes the coupling of iterated variations of the objects f, $\mathcal{P}$, and g with respect to the canonically conjugate variables x^i and ξ_i, see [9] and references therein.

[2]The group $H^0_{\mathcal{P}}(M)$ spans the Casimirs, i.e. the functions which Poisson-commute with any $f \in C^\infty(M)$; the group $H^1_{\mathcal{P}}(M)$ consists of vector fields which preserve the Poisson structure but do not amount to the Hamiltonian vector fields $\vec{\mathcal{X}}_h = [\![\mathcal{P}, h]\!]$; the second group $H^2_{\mathcal{P}}(M) \ni \mathcal{Q}$ contains infinitesimal symmetries $\mathcal{P} \mapsto \mathcal{P} + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ of Poisson bi-vectors, whereas the next group $H^3_{\mathcal{P}}(M)$ stores the obstructions to formal

If a bi-vector $\mathcal{Q} = [\![\vec{\mathcal{X}}, \mathcal{P}]\!]$ is a trivial Poisson cocycle, then it certainly is an infinitesimal symmetry of the Jacobi identity $\frac{1}{2}[\![\mathcal{P}, \mathcal{P}]\!] = 0$. But the infinitesimal change $[\![\vec{\mathcal{X}}, \mathcal{P}]\!]$ of the tensor $\mathcal{P}$ then amounts to its reparametrisation under the infinitesimal change of coordinates $\mathbf{x}'(\boldsymbol{x}) \rightleftarrows \boldsymbol{x}(\mathbf{x}')$ along the integral trajectories of the vector field $\vec{\mathcal{X}}$ on the manifold M. The following fact is true for all multivectors (regardless of the concept of Poisson cohomology).

Proposition 1. *Let $\boldsymbol{a} \in M$ be a point of an r-dimensional manifold and $\vec{\mathcal{X}} \in \Gamma(TM)$ be a vector field on it. For every $\varepsilon \in \mathcal{I} \subseteq \mathbb{R}$ such that there is the integral trajectory bringing $\boldsymbol{b}(-\varepsilon) := \exp(-\varepsilon\vec{\mathcal{X}})(\boldsymbol{a})$ to $\boldsymbol{a}$ by the $(+\varepsilon)$-shift, and for any choice of the r-tuple $\boldsymbol{x} = (x^1, \ldots, x^r)$ of local coordinates in a chart U_α around $\boldsymbol{a} \in M$ (and for $|\varepsilon|$ small enough for the points $\boldsymbol{b}(-\varepsilon)$ to not yet run out of the chart U_α), introduce a new parametrization*[3] *for the point $\boldsymbol{a}$ by using the new r-tuple $\mathbf{x}'$. By definition, put $\mathbf{x}'(\boldsymbol{a}) := \boldsymbol{x}(\boldsymbol{b}(-\varepsilon))$. Let $\boldsymbol{\Omega}$ be any multi-vector field near $\boldsymbol{a}$ on M. Under the reparametrization $\mathbf{x}'(\boldsymbol{x})$, the speed at which the components of $\boldsymbol{\Omega}$ at the point $\boldsymbol{a}$ change in ε, as $\varepsilon \to 0$, equals $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\boldsymbol{\Omega}(\boldsymbol{a}) = [\![\vec{\mathcal{X}}, \boldsymbol{\Omega}]\!](\boldsymbol{a})$. In particular, a 1-vector field $\vec{\mathcal{Y}}$ near $\boldsymbol{a}$ would change at $\boldsymbol{a}$ as fast as its commutator with the vector field $\vec{\mathcal{X}}$: $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\vec{\mathcal{Y}}(\boldsymbol{a}) = [\vec{\mathcal{X}}, \vec{\mathcal{Y}}](\boldsymbol{a})$.*

The geography of the set of Poisson structures near a given bracket $\{\cdot,\cdot\}_\mathcal{P}$ on a given manifold M^r is, generally speaking, unknown. All the more it was a priori unclear whether Poisson bi-vectors $\mathcal{P}$, irrespective of the dimension $r \geqslant 3$, topology of M^r, etc., can be infinitesimally shifted by Poisson 2-cocycles $\mathcal{Q}(\mathcal{P})$, the construction of which would be universal for all $\mathcal{P}$. The discovery of the graph complex in 1993–94 allowed Kontsevich to state (in [11]) the affirmative answer to the above question. Namely, the graph orientation morphism $\vec{\text{Or}}(\cdot)(\mathcal{P})\colon \ker \mathrm{d} \ni \gamma \mapsto \mathcal{Q}(\mathcal{P}) \in \ker \partial_\mathcal{P}$ takes graph cocycles on n vertices and $2n-2$ edges in each term (e.g., the tetrahedron, cf. [1, 3, 5, 6]) to Poisson cocycles whenever the bi-vector $\mathcal{P}$ itself is Poisson. Willwacher [15] revealed that the generators of Drinfeld's Grothendieck--Teichmüller Lie algebra $\mathfrak{grt}$ are source of at least countably many such cocycles in the vertex-edge bi-grading $(n, 2n-2)$; these cocycles are marked by the $(2\ell+1)$-wheel graphs (e.g., see [6, 7]). Brown proved in [2] that, under the Willwacher isomorphism $\mathfrak{grt} \cong H^0(\text{GRA})$ these graph cocycles with wheels generate a free Lie subalgebra in $\mathfrak{grt}$, which means effectively that the iterated commutators of already known cocycles – under the bracket in the differential graded Lie algebra GRA of graphs – would never vanish. The commutator of two cocycles is a cocycle by the Jacobi identity. All of them again being of the bi-grading $(n, 2n-2)$, these graph cocycles determine countably many infinitesimal symmetries of a given Poisson bi-vector $\mathcal{P}$; the construction is uniform for all the geometries $(M^r, \mathcal{P})$.

Lemma 2. *For a given Poisson bi-vector $\mathcal{P}$, the graph orientation mapping $\vec{\text{Or}}(\cdot)(\mathcal{P})\colon \ker \mathrm{d} \ni \gamma \mapsto \mathcal{Q}(\mathcal{P}) \in \ker \partial_\mathcal{P}$ is a Lie algebra morphism that takes the bracket of two cocycles in bi-grading $(n, 2n-2)$ to the commutator $[\frac{d}{d\varepsilon_1}, \frac{d}{d\varepsilon_2}](\mathcal{P})$ of two symmetries $\frac{d}{d\varepsilon_i}(\mathcal{P}) = \mathcal{Q}_i(\mathcal{P})$.*[4]

By construction, the components of universal symmetry bi-vectors $\mathcal{Q}(\mathcal{P})$ are differential polynomials w.r.t. the components $\mathcal{P}^{ij}$ of the Poisson bi-vector $\mathcal{P}$ that evolves. It can of course be that a graph flow $\dot{\mathcal{P}} = \vec{\text{Or}}(\gamma)(\mathcal{P})$ vanishes identically over the manifold M^r

integration $\mathcal{P} \mapsto \mathcal{P}(\varepsilon) = \mathcal{P} + \sum_{k\geqslant 1} \varepsilon^k \mathcal{Q}_{(k)}$ of infinitesimal symmetries $\mathcal{Q} = \mathcal{Q}_{(1)}$ to Poisson bi-vector formal power series satisfying $[\![\mathcal{P}(\varepsilon), \mathcal{P}(\varepsilon)]\!] = 0$.

[3] Actually, this is a way to construct new coordinates for *all* points of M near $\boldsymbol{a}$ in U_α, i.e. not only those which lie on a piece of the integral trajectory of $\vec{\mathcal{X}}$ passing through $\boldsymbol{a}$.

[4] By Brown [2], the commutator does in general not vanish for Willwacher's odd-sided wheel cocycles.

whenever $\mathcal{Q}$ is evaluated at a particular class of Poisson structures $\mathcal{P}$.[5] Nevertheless, there is no mechanism which would force a given Kontsevich's graph flow to vanish at all Poisson structures on all manifolds of all dimensions.[6] Independently, it remains an open problem (cf. [10]) whether there is a Poisson manifold $(M^r, \mathcal{P})$ and a graph cocycle γ such that the Poisson cohomology class of $\mathcal{Q}(\mathcal{P}) := \vec{\mathrm{Or}}(\gamma)(\mathcal{P})$ would be *nontrivial* in $H^2_{\mathcal{P}}(M)$. In other words, for all the shifts $\mathcal{Q} = \vec{\mathrm{Or}}(\gamma)$ and all Poisson bi-vectors tried so far, the Poisson coboundary equation $\mathcal{Q}(\mathcal{P}) = [\![\vec{\mathcal{X}}, \mathcal{P}]\!]$ did have vector field solutions $\vec{\mathcal{X}}$ on the manifolds M.

Remark 1. Obtained from the graphs $\gamma \in \ker \mathrm{d}$, the symmetries $\mathcal{Q}(\mathcal{P}) = \vec{\mathrm{Or}}(\gamma)(\mathcal{P}) \in \ker[\![\mathcal{P}, \cdot]\!]$ are independent of a choice of local coordinates x^i (hence ξ_i) on a chart if, the Kontsevich construction requires, the manifold M^r is endowed with an *affine* structure: all the coordinate transformations amount to $\mathbf{x}' = A\boldsymbol{x} + \vec{\boldsymbol{b}}$ with a constant (over the intersection of charts) Jacobian matrix A. The parity-odd fibre variables are transformed using the inverse Jacobian matrix, $\xi_i = A_i^{i'} \xi'_{i'}$, making sense of the couplings $\overleftarrow{\partial}/\partial\xi_i \cdot \vec{\partial}/\partial x^i$ which decorate the oriented edges of Kontsevich's graphs after the morphism $\vec{\mathrm{Or}}$ works (see [3, 11]). The problem of Poisson cohomology class (non)triviality for the Kontsevich infinitesimal symmetries $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) \in \ker[\![\mathcal{P}, \cdot]\!]$ thus acquires two diametrally opposite interpretations:

1 (as in [11])**.** The Poisson manifold $M^{r<\infty}$ is equipped with *both* the smooth and affine structures.[7] By definition, two Poisson bi-vectors are equivalent, $\mathcal{P}_1 \sim \mathcal{P}_2$, if they are related by a diffeomorphism of the manifold M: using its smooth structure, the diffeomorphism identifies points in two copies of M, then relating the Poisson tensors by local coordinate reparametrizations near the respective points. The affine structure on M is now used to run the Kontsevich flows in two initial value problems $\dot{\mathcal{P}}_i(\varepsilon) = \mathcal{Q}(\mathcal{P}_i(\varepsilon))$, $\mathcal{P}_i(\varepsilon = 0) = \mathcal{P}_i$. The Poisson triviality $\mathcal{Q}(\mathcal{P}(\varepsilon)) = [\![\vec{\mathcal{X}}(\varepsilon), \mathcal{P}(\varepsilon)]\!]$ would relate either of bi-vectors $\mathcal{P}_i(\varepsilon)$ back to the Cauchy datum $\mathcal{P}_i$ by diffeomorphisms (as long as $|\varepsilon|$ is small enough). Consequently, the Poisson bi-vectors $\mathcal{P}_1(\varepsilon) \sim \mathcal{P}_2(\varepsilon)$ do not run out of the old equivalence class. In conclusion, the goal is to produce essentially new Poisson brackets by using a nontrivial cocycle $\mathcal{Q}$, two given structures on the manifold M^r, and its diffeomorphism. No examples of nontrivial action, so that $\mathcal{P}_2(\varepsilon) \not\sim \mathcal{P}_i \not\sim \mathcal{P}_1(\varepsilon)$ at $\varepsilon > 0$, have ever been produced since 1996 (see [7, 11]).

2 (as in [10])**.** The Poisson manifold $M^{r<\infty}$ is equipped only with an affine structure. The countably many $\mathfrak{grt}$-related graph cocycles on n vertices and $2n-2$ edges in every term (the tetrahedron, the pentagon-wheel cocycle, etc., see [6, 15]) generate a noncommutative Lie algebra of infinitesimal symmetries $\mathcal{Q}(\mathcal{P}) = \vec{\mathrm{Or}}(\gamma)(\mathcal{P})$ for a given Poisson structure $\mathcal{P}$. Consider the extreme case when *all* the cocycles $\mathcal{Q}(\mathcal{P}) \in \ker[\![\mathcal{P}, \cdot]\!]$ are exact in the coho-

[5]**Example.** So it is for the Kontsevich tetrahedral flow ([11] and [1]) evaluated at the Kirillov–Kostant linear Poisson brackets on the duals $\mathfrak{g}^*$ of Lie algebras because in every term within the cocycle $\mathcal{Q}(\mathcal{P})$ under study, at least one copy is $\mathcal{P}$ is differentiated at least twice with respect to the global coordinates on $\mathfrak{g}^*$.

[6]**Example.** The Poisson bi-vectors $\mathcal{P} = \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_m / \mathrm{dvol}(\mathbb{R}^{m+2})$ of Nambu type with arbitrary Casimirs $a_1, \ldots, a_m \in C^\infty(\mathbb{R}^{m+2})$ and an arbitrary density in the volume element can have polynomial components $\mathcal{P}^{ij} \in \mathbb{R}[x^1, \ldots, x^{m+2}]$ of degrees as high as need be w.r.t. the global Cartesian coordinates x^α on the vector space $\mathbb{R}^{m+2}$. The universal symmetries $\dot{\mathcal{P}} = \vec{\mathrm{Or}}(\gamma)(\mathcal{P})$ obtained from Kontsevich's graph cocycles deform the symplectic foliation (which is given in $\mathbb{R}^{m+2}$ by the intersections of the level sets for the Casimirs $a_1, \ldots, a_m$) in a regular way on an open dense subset of $\mathbb{R}^{m+2}$, so that the symmetries $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ preserve this Nambu class of Poisson brackets: the flows force the evolution of the Casimirs and the volume density. Its integrability is an open problem; by Lemma 2 and [2], the evolutions induced by different graph cocycles do not commute.

[7]On the circle $\mathbb{S}^1$, the affine coordinate 'angle' is obvious whereas the smooth structure is used in the realm of Poincaré topology. A smooth atlas is always available for the spheres $\mathbb{S}^r$, but not for all $r \in \mathbb{N}$ would the r-dimensional sphere admit an affine structure.

mology group $H^2_{\mathcal{P}}(M)$ w.r.t. the Poisson differential $\partial_{\mathcal{P}}$. This assumption gives rise to the countable set of vector fields $\vec{\mathcal{Y}}(\gamma,\mathcal{P})$ on M such that $\mathcal{Q}(\mathcal{P}) = [\![\vec{\mathcal{Y}},\mathcal{P}]\!]$. (Some of these vector fields can be identically zero over M.) But if at least one such vector field is not constant w.r.t. the affine structure on M, then the shifts along its integral trajectories are nonlinear diffeomorphisms of M. The evolution of bi-vector $\mathcal{P}$ is $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P}) = [\![\vec{\mathcal{Y}},\mathcal{P}]\!]$ or similarly, $\dot{\Omega} = [\![\vec{\mathcal{Y}},\Omega]\!]$ for any multi-vector Ω on M (see Proposition 1); this evolution is now seen as mutlivectors' response to the diffeomorphism whose construction refers only to the simple, affine local portrait of M. Summarizing, the store of flows $\mathrm{O\vec{r}}(\gamma)(\mathcal{P})$ from the $\mathfrak{grt}$-related graph cocycles γ could be enough to approximate arbitrary smooth vector fields on M^r, that is, imitate its smooth structure. Whether this theoretical possibility is actually realised in relevant Poisson models is an open problem.

The Kontsevich symmetry construction is, therefore, either a generator of new Poisson brackets or the mechanism that provides diffeomorphisms of the underlying manifold.

2. Homogeneous Poisson structures. By definition, a bi-vector $\mathcal{P}$ on a manifold M is called *homogeneous* (of scale λ) with respect to a vector field $\vec{\mathcal{V}}$ on M if $[\![\vec{\mathcal{V}},\mathcal{P}]\!] = \lambda\cdot\mathcal{P}$.

Example 1. Let $M = \mathbb{R}^r$ be a vector space (only linear reparametrizations $\mathbf{x}' = A\boldsymbol{x}$ are allowed, so that the polynomial degrees of monomials in the ring $\mathbb{R}[x^1,\ldots,x^r]$ is well defined). Introduce the Euler vector field $\vec{\mathcal{E}} = \sum_{i=1}^r x^i\,\partial/\partial x^i$, and let all the components $\mathcal{P}^{ij}$ of a bi-vector $\mathcal{P}$ be homogeneous polynomials of degree d in the variables x^i. Then we have that $[\![\vec{\mathcal{E}},\mathcal{P}]\!] = (d-2)\cdot\mathcal{P}$, which means that $\mathcal{P}$ is homogeneous of scale $d-2$ w.r.t. the Euler vector field $\vec{\mathcal{E}}$. In particular, if $d\neq 2$ (i.e. if the coefficients of bi-vector $\mathcal{P}$ are not quadratic), then we set $\vec{\mathcal{V}} = (d-2)^{-1}\cdot\vec{\mathcal{E}}$ and from the equality $\mathcal{P} = [\![\vec{\mathcal{V}},\mathcal{P}]\!]$ we obtain that the same bi-vector $\mathcal{P}$ has homogeneity scale $\lambda = 1$ w.r.t. the multiple $\vec{\mathcal{V}}$ of the Euler vector field $\vec{\mathcal{E}}$ on $\mathbb{R}^r$.

Example 2. Under the same assumptions, suppose further that $\gamma = \sum_a c_a\gamma_a$ is a graph cocycle with n vertices and $2n-2$ edges in every term γ_a (e.g., take the tetrahedron). Orient the ordered (by First $\prec\cdots\prec$ Last) edges in every γ_a using the edge decoration operators $\vec{\Delta}_{ij} = \sum_{\mu=1}^r(\vec{\partial}/\partial\xi_\mu^{(i)}\otimes\vec{\partial}/\partial x^\mu_{(j)} + \vec{\partial}/\partial\xi_\mu^{(j)}\otimes\vec{\partial}/\partial x^\mu_{(i)})$. By placing a copy of bi-vector $\mathcal{P} = \frac{1}{2}\mathcal{P}^{kl}(\boldsymbol{x})\xi_k\xi_l$ in each vertex $v^{(i)}$ of γ_a and taking the sum (over the graph index a) of products of the content of vertices in γ_a after all the edge operators $\vec{\Delta}_{ij}$ work, we obtain[8] the bi-vector $\mathcal{Q}(\mathcal{P}) := \mathrm{O\vec{r}}(\gamma)(\mathcal{P})$. Then the coefficients of the bi-vector $\mathcal{Q}(\mathcal{P})$ are homogeneous polynomials of degree $n\cdot d-(2n-2)$ with respect to x^1, $\ldots$, x^r, so that $[\![\vec{\mathcal{E}},\mathcal{Q}(\mathcal{P})]\!] = n(d-2)\mathcal{Q}(\mathcal{P})$. In particular, if $d\neq 2$, then $[\![\vec{\mathcal{V}},\mathcal{Q}(\mathcal{P})]\!] = n\cdot\mathcal{Q}(\mathcal{P})$, whereas quadratic-coefficient bi-vectors $\mathcal{P}$ (with $d=2$) are deformed within their subspace by the quadratic bi-vectors $\mathcal{Q}(\mathcal{P})$ which are obtained from the Kontsevich graph cocycles.

Lemma 3. *If a Poisson bi-vector $\mathcal{P} = [\![\vec{\mathcal{V}},\mathcal{Q}(\mathcal{P})]\!]$ is homogeneous and $\mathcal{Q}(\mathcal{P}) = \mathrm{O\vec{r}}(\gamma)(\mathcal{P}^{\otimes^n})$ is built from a graph cocycle γ on n vertices, now containing a copy of $\mathcal{P}$ in each vertex, then the bi-vector $\mathcal{Q}(\mathcal{P})$ is also homogeneous: $[\![\vec{\mathcal{V}},\mathcal{Q}(\mathcal{P})]\!] = n\cdot\mathcal{Q}(\mathcal{P})$, so that its scale is n.*[9]

Remark 2 ([14, Rem. 4.9]). Consider a Nambu-type Poisson bi-vector $\mathcal{P} = \mathrm{d}a/\mathrm{d}x\mathrm{d}y\mathrm{d}z$ on $\mathbb{R}^3$ with Cartesian coordinates x,y,z; here $a\in\mathbb{R}[x,y,z]$ is a weight-homogeneous polynomial

[8] We refer to the original paper [11] and to [3] for illustrations and discussion how the graph orientation morphism works in practice.

[9] The proof amounts to the Leibniz rule: let us inspect how fast the bi-vector $\mathcal{Q}(\mathcal{P})$, which by construction is a homogeneous differential polynomial of degree n in $\mathcal{P}$, evolves along the vector field $\vec{\mathcal{V}}$.

with an isolated singularity at the origin[10], so that $(w_{(x)} \cdot x\partial/\partial x + w_{(y)} \cdot y\partial/\partial y + w_{(z)} \cdot z\partial/\partial z)(a) = w_{(a)} \cdot a$. Then a vector field $\vec{\mathcal{V}}$ with polynomial components satisfying the first-order PDE $\mathcal{P} = [\![\vec{\mathcal{V}}, \mathcal{P}]\!]$ exists if and only if[11] the weight degree $w_{(a)}$ of the polynomial a is not equal to the sum $w_{(x)} + w_{(y)} + w_{(z)}$ of weight degrees for the variables x, y, z.[12]

Summarizing, the homogeneity assumption about bi-vectors $\mathcal{P}$ is restrictive; it is not always satisfied in Poisson models.

Theorem 4. *Let $(M, \mathcal{P})$ be an affine finite-dimensional real Poisson manifold with $\mathcal{P} = [\![\vec{\mathcal{V}}, \mathcal{P}]\!]$ homogeneous. Let $\gamma = \sum_a c_a \cdot \gamma_a$ be a graph cocycle consisting of unoriented graphs γ_a over n vertices and $2n-2$ edges (with a fixed ordering of edges in each γ_a). Then the 1-vector $\vec{X}(\gamma, \vec{\mathcal{V}}, \mathcal{P}) = \mathrm{O}\vec{\mathrm{r}}(\gamma)(\vec{\mathcal{V}} \otimes \mathcal{P}^{\otimes^{n-1}})$, which is obtained by representing each edge i—j with the operator $\vec{\Delta}_{ij}$ and by (graded-)symmetrizing over all the ways $\sigma \in \mathbb{S}_n$ to send the n-tuple $\vec{\mathcal{V}} \otimes \mathcal{P}^{\otimes^{n-1}}$ into the n vertices in each γ_a, is a Poisson cocycle: $\vec{\mathcal{X}} \in \ker [\![\mathcal{P}, \cdot]\!]$.*[13]

The vector field $\vec{\mathcal{X}}$ is defined up to adding arbitrary Poisson 1-cocycles $\vec{\mathcal{Z}} \in \ker [\![\mathcal{P}, \cdot]\!]$.

Proof. The expansion $0 = \mathrm{O}\vec{\mathrm{r}}(\mathrm{d}\gamma)(\vec{\mathcal{V}} \otimes \mathcal{P}^{\otimes^n})$ for $\gamma \in \ker \mathrm{d}$ goes along the lines of [11] and [3, 7, 8], but the $(n+1)$-tuple of multivectors now contains one 1-vector and only n copies of the Poisson bi-vector $\mathcal{P}$. By assumption, $\mathrm{d}\gamma = \mathbf{0} \in \mathrm{G{\scriptsize RA}}$; recall that $\mathrm{O}\vec{\mathrm{r}}(\mathbf{0})(\text{any multivectors}) = 0 \in \Gamma(\bigwedge^\bullet TM)$. This zero l.-h.s. equates $0 = \big(\pi_S \vec{\circ} \mathrm{O}\vec{\mathrm{r}}(\gamma) - (-)^{(-1)\cdot(-N)} \mathrm{O}\vec{\mathrm{r}}(\gamma) \vec{\circ} \pi_S\big)(\vec{\mathcal{V}} \otimes \mathcal{P}^{\otimes^n})$.[14]

The appointment of graded (multi)vectors into the vertices of $\mathrm{d}\gamma$ (hence, into the argument slots of the endomorphism $\mathrm{O}\vec{\mathrm{r}}(\mathrm{d}\gamma)$) is achieved by the graded symmetrization using $((n+1)!)^{-1} \mathrm{O}\vec{\mathrm{r}}(\mathrm{d}\gamma)\big(\pm\sigma(\vec{\mathcal{V}} \otimes \mathcal{P}^{\otimes^n})\big)$. Fortunately, the field $\vec{\mathcal{V}}$ is the only parity-odd object, so its transpositions with the parity-even bi-vectors $\mathcal{P}$ produce no sign factor: these $\pm$ are all $+$. Likewise, the $n!$ permutations of n indistinguishable copies of $\mathcal{P}$ leave only $n+1$ from $(n+1)!$ in the denominator; to get rid of it, let us multiply by $n+1$ both sides of the equality $0 = \mathrm{O}\vec{\mathrm{r}}(\mathrm{d}\gamma)(\vec{\mathcal{V}} \otimes \mathcal{P}^{\otimes^n})$. The symmetrization thus amounts, by the linearity of $\mathrm{O}\vec{\mathrm{r}}(\gamma)$, to its evaluation at the sum of arguments, $\vec{\mathcal{V}} \cdot \mathcal{P}^n + \mathcal{P} \cdot \vec{\mathcal{V}} \cdot \mathcal{P}^{n-1} + \ldots + \mathcal{P}^n \cdot \vec{\mathcal{V}}$, in which the ordering of (multi)vectors now matches an arbitrary fixed enumeration of the vertices.

The rest of the proof is standard.[15] There remains $0 = \mathrm{O}\vec{\mathrm{r}}(\gamma)\big(\pi_S(\vec{\mathcal{V}}, \mathcal{P}) \cdot \mathcal{P}^{n-1}\big) + \mathcal{P} \cdot$

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[10]The Milnor number is the dimension $\dim_{\mathbb{R}} \mathbb{R}[x, y, z]/(\partial a/\partial x, \partial a/\partial y, \partial a/\partial z)$ – here, $< \infty$ by assumption.

[11]This means that not all Nambu-type Poisson bi-vectors $\mathcal{P} = \mathrm{d}a/\mathrm{d}x\mathrm{d}y\mathrm{d}z$ are homogeneous w.r.t. a vector field $\vec{\mathcal{V}}$ with polynomial components; the PDE $\mathcal{P} = [\![\vec{\mathcal{V}}, \mathcal{P}]\!]$ with polynomial coefficients and unknown $\vec{\mathcal{V}}$ can in principle admit non-polynomial solutions.

[12]**Example.** If the weights of (x, y, z) are $(1, 1, 1)$ and $a = \frac{1}{3}(x^3 + y^3 + z^3)$ is cubic-homogeneous, then the components of Poisson bi-vector $\mathcal{P}$ are quadratic and (by the above and also by [12, Exerc. 4.5.7c]) not of the form $\mathcal{P} = [\![\vec{\mathcal{V}}, \mathcal{P}]\!]$ for any polynomial-coefficient vector field $\vec{\mathcal{V}}$. The non-existence of a solution $\vec{\mathcal{V}}$ with smooth non-polynomial coefficients is a separate problem.

[13]**Open problem** (for $\mathcal{P}$ homogeneous and Poisson)**.** Is the universal 1-vector field $\vec{\mathcal{X}}(\gamma, \vec{\mathcal{V}}, \mathcal{P}) \in \ker \partial_{\mathcal{P}}$ Hamiltonian, i.e. $\vec{\mathcal{X}} = [\![\mathcal{P}, h]\!]$ for $h \in C^\infty(M)$ or at least, $\vec{\mathcal{X}} = \mathcal{P} \lrcorner\, \eta$ for a maybe not exact 1-form η on M?

[14]Here, π_S is the graded-symmetric Schouten bracket (so $\pi_S(F, G) = (-)^{|F|-1} [\![F, G]\!]$), the graph insertion $\vec{\circ}$ into vertices is now the endomorphism insertion into argument slots, $|\pi_S| = -1$, and $N = 2n - 2$ is the even number of edges in γ, hence minus the even number of $\partial/\partial\xi_\mu$ in the edge operators $\vec{\Delta}_{ij}$ making $\mathrm{O}\vec{\mathrm{r}}(\gamma)$.

[15]We have $0 = \mathrm{O}\vec{\mathrm{r}}(\gamma)(\pi_S(\vec{\mathcal{V}}, \mathcal{P}), \mathcal{P}^{n-1}) + \mathrm{O}\vec{\mathrm{r}}(\gamma)(\pi_S(\mathcal{P}, \vec{\mathcal{V}}), \mathcal{P}^{n-1}) + \mathrm{O}\vec{\mathrm{r}}(\gamma)(\pi_S(\mathcal{P}, \mathcal{P}), \vec{\mathcal{V}}, \mathcal{P}^{n-2}) + \ldots + \mathrm{O}\vec{\mathrm{r}}(\gamma)(\pi_S(\mathcal{P}, \mathcal{P}), \mathcal{P}^{n-2}, \vec{\mathcal{V}}) + \mathrm{O}\vec{\mathrm{r}}(\gamma)(\vec{\mathcal{V}}, \pi_S(\mathcal{P}, \mathcal{P}), \mathcal{P}^{n-2}) + \mathrm{O}\vec{\mathrm{r}}(\gamma)(\mathcal{P}, \pi_S(\vec{\mathcal{V}}, \mathcal{P}), \mathcal{P}^{n-2}) + \mathrm{O}\vec{\mathrm{r}}(\gamma)(\mathcal{P}, \pi_S(\mathcal{P}, \vec{\mathcal{V}}), \mathcal{P}^{n-2}) + \mathrm{O}\vec{\mathrm{r}}(\gamma)(\mathcal{P}, \pi_S(\mathcal{P}, \mathcal{P}), \vec{\mathcal{V}}, \mathcal{P}^{n-3}) + \ldots + \mathrm{O}\vec{\mathrm{r}}(\gamma)(\mathcal{P}, \pi_S(\mathcal{P}, \mathcal{P}), \mathcal{P}^{n-3}, \vec{\mathcal{V}}) + \ldots$ (the Schouten bracket π_S passes along the slots towards the end) $+ \mathrm{O}\vec{\mathrm{r}}(\gamma)(\vec{\mathcal{V}}, \mathcal{P}^{n-2}, \pi_S(\mathcal{P}, \mathcal{P})) + \ldots + \mathrm{O}\vec{\mathrm{r}}(\gamma)(\mathcal{P}^{n-2}, \vec{\mathcal{V}}, \pi_S(\mathcal{P}, \mathcal{P})) + \mathrm{O}\vec{\mathrm{r}}(\gamma)(\mathcal{P}^{n-1}, \pi_S(\vec{\mathcal{V}}, \mathcal{P})) + \mathrm{O}\vec{\mathrm{r}}(\gamma)(\mathcal{P}^{n-1}, \pi_S(\mathcal{P}, \vec{\mathcal{V}})) - (-)^N \cdot \big[\pi_S\big(\mathrm{O}\vec{\mathrm{r}}(\gamma)(\vec{\mathcal{V}} \cdot \mathcal{P}^{n-1} + \mathcal{P} \cdot \vec{\mathcal{V}} \cdot \mathcal{P}^{n-2} + \ldots + \mathcal{P}^{n-1} \cdot \vec{\mathcal{V}}), \mathcal{P}\big) + \pi_S(\mathrm{O}\vec{\mathrm{r}}(\gamma)(\mathcal{P}^n), \vec{\mathcal{V}}) + \pi_S(\vec{\mathcal{V}}, \mathrm{O}\vec{\mathrm{r}}(\gamma)(\mathcal{P}^n)) + \pi_S\big(\mathcal{P}, \mathrm{O}\vec{\mathrm{r}}(\gamma)(\vec{\mathcal{V}} \cdot \mathcal{P}^{n-1} + \mathcal{P} \cdot \vec{\mathcal{V}} \cdot \mathcal{P}^{n-2} + \ldots + \mathcal{P}^{n-1} \cdot \vec{\mathcal{V}})\big)\big]$. For $\mathcal{P}$ Poisson, $\pi_S(\mathcal{P}, \mathcal{P}) = 0$, so we exclude all such terms ([4]). The remaining graded-symmetric Schouten brackets π_S contain a bi-vector as one of the arguments, hence those can be swapped at no sign factor; all doubles, so let us divide by 2.

$\pi_S(\vec{\mathcal{V}},\mathcal{P})\cdot\mathcal{P}^{n-2})+\ldots+\mathcal{P}^{n-1}\cdot\pi_S(\vec{\mathcal{V}},\mathcal{P})\big)-(-)^N\big[\pi_S\big(\mathrm{O\vec{r}}(\gamma)\big(\vec{\mathcal{V}}\cdot\mathcal{P}^{n-1}+\mathcal{P}\cdot\vec{\mathcal{V}}\cdot\mathcal{P}^{n-2}+\ldots+\mathcal{P}^{n-1}\cdot\vec{\mathcal{V}}\big),\mathcal{P}\big)+\pi_S(\mathrm{O\vec{r}}(\gamma)(\mathcal{P}^n),\vec{\mathcal{V}})\big]$. By the homogeneity assumption, $\pi_S(\vec{\mathcal{V}},\mathcal{P})=(-)^{1-1}[\![\vec{\mathcal{V}},\mathcal{P}]\!]=\mathcal{P}$, and by construction, $\mathrm{O\vec{r}}(\gamma)(\mathcal{P}^n)=\mathcal{Q}(\mathcal{P})$, whence the minuend equals $n\cdot\mathcal{Q}(\mathcal{P})$. By Lemma 3, the graph flow is also homogeneous: $[\![\vec{\mathcal{V}},\mathcal{Q}(\mathcal{P})]\!]=\lambda\cdot\mathcal{Q}(\mathcal{P})$ with the vertex count $\lambda=n$. We obtain the equality

$$(-)^{2n-2}\cdot[\![\mathrm{O\vec{r}}(\gamma)\big(\vec{\mathcal{V}}\cdot\mathcal{P}^{n-1}+\mathcal{P}\cdot\vec{\mathcal{V}}\cdot\mathcal{P}^{n-2}+\ldots+\mathcal{P}^{n-1}\cdot\vec{\mathcal{V}}\big),\mathcal{P}]\!]=$$
$$=n\cdot\mathcal{Q}(\mathcal{P})-(-)^{2n-2}\lambda\cdot\mathcal{Q}(\mathcal{P})=(n-(-)^{\text{even}}\,n)\cdot\mathcal{Q}(\mathcal{P})\equiv 0.$$

We conclude that the 1-vector $\vec{\mathcal{X}}:=\mathrm{O\vec{r}}(\gamma)(\vec{\mathcal{V}}\otimes\mathcal{P}^{\otimes^{n-1}})$ lies in $\ker[\![\mathcal{P},\cdot]\!]$.[16] □

Example 3. Take the Lie algebra $\mathfrak{gl}_2(\mathbb{R})$ with its four-dimensional vector space structure; denote by x, y, z, v the Cartesian coordinates. Consider the R-matrix $\left(\begin{smallmatrix}x&y\\z&v\end{smallmatrix}\right)\mapsto\left(\begin{smallmatrix}0&y\\-z&0\end{smallmatrix}\right)$ known from [12]; the standard construction then yeilds the Poisson bi-vector in the algebra of coordinate functions, $\mathcal{P}=(x^2y+y^2z)\,\partial_x\wedge\partial_y+(x^2z+yz^2)\,\partial_x\wedge\partial_z+(2\,xyz+2\,yzv)\,\partial_x\wedge\partial_v+(y^2z+yv^2)\,\partial_y\wedge\partial_v+(yz^2+zv^2)\,\partial_z\wedge\partial_v$. This bracket has cubic-nonlinear homogeneous polynomial coefficients, hence $d=3$. The vector field $\vec{\mathcal{V}}=(d-2)^{-1}\cdot\vec{\mathcal{E}}$ is the (multiple of the) Euler vector field on $\mathbb{R}^4$. As the graph cocycle γ, we take the tetrahedron (see [1, 11]); then the symmetry flow is $\dot{\mathcal{P}}=\mathcal{Q}(\mathcal{P})=(-48\,x^5y-288\,x^3y^2z-240\,xy^3z^2+192\,y^3z^2v-384\,xy^2zv^2-192\,y^2zv^3)\partial_x\wedge\partial_y+(-48\,x^5z-288\,x^3yz^2-240\,xy^2z^3+192\,y^2z^3v-384\,xyz^2v^2-192\,yz^2v^3)\partial_x\wedge\partial_z+(-336\,x^4yz-480\,x^2y^2z^2-576\,x^3yzv+480\,y^2z^2v^2+576\,xyzv^3+336\,yzv^4)\partial_x\wedge\partial_v+(192\,x^3y^2z-192\,xy^3z^2+288\,y^2zv^3+48\,yv^5+48\,(8\,x^2y^2z+5\,y^3z^2)v)\partial_y\wedge\partial_v+(192\,x^3yz^2-192\,xy^2z^3+288\,yz^2v^3+48\,zv^5+48\,(8\,x^2yz^2+5\,y^2z^3)v)\partial_z\wedge\partial_v$. We detect that this bi-vector is a coboundary, $\mathcal{Q}(\mathcal{P})=[\![\vec{\mathcal{Y}},\mathcal{P}]\!]$ with the vector $\vec{\mathcal{Y}}=(-24\,x^4+120\,y^2z^2-96\,yzv^2)\partial_x+(96\,x^3y-96\,yv^3)\partial_y+(96\,x^3z-96\,zv^3)\partial_z+(96\,x^2yz-120\,y^2z^2+24\,v^4)\partial_v$ mod $\ker[\![\mathcal{P},\cdot]\!]$. The vector field $\vec{\mathcal{Y}}\notin\ker\partial_{\mathcal{P}}$ cannot be Poisson-exact (clearly, $\mathcal{Q}(\mathcal{P})\not\equiv 0$), hence $\vec{\mathcal{Y}}$ does not mark the Poisson cocycle of zero 1-vector.[17] But the universal vector field $\vec{\mathcal{X}}(\gamma,\vec{\mathcal{V}},\mathcal{P})\in\ker\partial_{\mathcal{P}}$ is identically zero on $\mathbb{R}^4$. Indeed, the Euler field $\vec{\mathcal{E}}=\vec{\mathcal{V}}$ is linear, yet it is readily seen from the figures in [1] that in every orgraph from the 1-vector $\mathrm{O\vec{r}}(\gamma)(\vec{\mathcal{V}}\otimes\mathcal{P}^{\otimes^{n-1}})$, the vertex with $\vec{\mathcal{V}}$ is differentiated at least twice (and at most thrice), so $\vec{\mathcal{X}}\equiv 0$.

Proposition 5. *The flow $\dot{\mathcal{P}}=\mathrm{O\vec{r}}($tetrahedron $\boldsymbol{\gamma}_3)(\mathcal{P})$ preserves the Nambu class of Poisson brackets, $\{f,g\}_{\mathcal{P}}=\varrho(x,y,z)\cdot\det\big(\partial(a,f,g)/\partial(x,y,z)\big)$ with arbitrary ϱ and global Casimir a on $\mathbb{R}^3$: the flow forces the nonlinear evolution $\dot{a}$, $\dot{\varrho}$ with differential-polynomial r.-h.s.*

- *This flow $\dot{\mathcal{P}}=\mathcal{Q}(\mathcal{P})$ is* not *Poisson-exact in terms of any vector field $\vec{\mathcal{Y}}$ with differential-polynomial coefficients (cubic in both a and ϱ, of total differential order eight).*

[16]**Exercise.** Extend the proof to the case $n=1$, $\gamma=\bullet$, $\mathrm{d}\gamma=-\bullet\!\!-\!\!\bullet$ (so that the l.-h.s. was nonzero).

[17]Likewise, by using another R-matrix for $\mathfrak{gl}_2(\mathbb{R})$, namely $\left(\begin{smallmatrix}x&y\\z&v\end{smallmatrix}\right)\mapsto\left(\begin{smallmatrix}x&y\\-z&v\end{smallmatrix}\right)$ also from [12], we obtain the Poisson bi-vector $\mathcal{P}=2\,x^2y\partial_x\wedge\partial_y+2\,yz^2\partial_x\wedge\partial_z+(2\,xyz+2\,yzv)\partial_x\wedge\partial_v+(-2\,xyz+2\,yzv)\partial_y\wedge\partial_z+2\,yv^2\partial_y\wedge\partial_v+2\,yz^2\partial_z\wedge\partial_v$ on $\mathbb{R}^4$ with Cartesian coordinates x, y, z, v. The tetrahedral flow then equals $\dot{\mathcal{P}}=\mathcal{Q}(\mathcal{P})=(-384\,x^5y-384\,x^3y^2z-1536\,xy^2zv^2+384\,(x^2y^2z-4\,y^3z^2)v)\partial_x\wedge\partial_y+(-384\,x^3yz^2-2688\,xy^2z^3+1152\,xyz^2v^2+384\,yz^2v^3-384\,(3\,x^2yz^2-7\,y^2z^3)v)\partial_x\wedge\partial_z+(-384\,x^4yz-2688\,x^2y^2z^2-1536\,x^3yzv+2688\,y^2z^2v^2+1536\,xyzv^3+384\,yzv^4)\partial_x\wedge\partial_v+(384\,x^4yz+384\,x^2y^2z^2+1536\,y^3z^3-384\,xyzv^3+384\,yzv^4+384\,(x^2yz+y^2z^2)v^2-384\,(x^3yz-2\,xy^2z^2)v)\partial_y\wedge\partial_z+(1536\,xy^3z^2+1536\,x^2y^2zv-384\,xy^2zv^2+384\,y^2zv^3+384\,yv^5)\partial_y\wedge\partial_v+(-384\,x^3yz^2-2688\,xy^2z^3+1152\,xyz^2v^2+384\,yz^2v^3-384\,(3\,x^2yz^2-7\,y^2z^3)v)\partial_z\wedge\partial_v$. It is Poisson-trivial: $\mathcal{Q}(\mathcal{P})=[\![\vec{\mathcal{Y}},\mathcal{P}]\!]$ with a representative $\vec{\mathcal{Y}}=(-96\,x^4+576\,y^2z^2-384\,yzv^2)\partial_x+(-192\,xy^2z+192\,y^2zv-384\,yv^3)\partial_y+(-96\,x^3z-96\,xzv^2+96\,zv^3+96\,\big(x^2z-4\,yz^2\big)v)\partial_z+(-576\,y^2z^2-384\,xyzv+96\,v^4)\partial_v$. These explicit examples of Poisson-exact bi-vector flows $\dot{\mathcal{P}}=\mathcal{Q}(\mathcal{P})=[\![\vec{\mathcal{Y}},\mathcal{P}]\!]$ will be useful in the future study of the mechanism $\vec{\mathcal{Y}}=\vec{\mathcal{Y}}(\gamma,\vec{\mathcal{V}},\mathcal{P})$ of their observed $\partial_{\mathcal{P}}$-triviality.

The cocycle equation at hand, $\mathcal{E}(\boldsymbol{\gamma}_3, a, \varrho) = \{\dot{\mathcal{P}} = [\![\vec{\mathcal{Y}}, \mathcal{P}]\!]\}$, is a first-order PDE with differential-polynomial coefficients (their skew-symmetry under permutations of x, y, z is inherited from the property of the Jacobian determinant and from the transformation law for the density ϱ in $\mathcal{P}$). Whether this equation $\mathcal{E}$ does not admit any non-polynomial solutions $\vec{\mathcal{Y}}(a_{|\sigma|\leqslant 3}, \varrho_{|\tau|\leqslant 2})$ is an open problem.

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