Construction of the classical time crystal Lagrangians from Sisyphus dynamics and duality description with the Liénard type equation

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Abstract

We explore the connection between the equations describing sisyphus dynamics and the generic Liénard type or Liénard II equation from the viewpoint of branched Hamiltonians. The former provides the appropriate setting for classical time crystal being derivable from a higher order Lagrangian. However it appears the equations of Sisyphus dynamics have a close relation with the Liénard-II equation when expressed in terms of the 'velocity' variable. Another interesting feature of the equations of Sisyphus dynamics is the appearance of velocity dependent "mass function" in contrast to the more commonly encountered position dependent mass. The consequences of such mass functions seem to have connections to cosmological time crystals .

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1 Introduction

The issue of time independent classical dynamical systems exhibiting motion in their lowest energy states has been instrumental in the introduction of a time analogue of spatial order as in a crystalline substance [1] (the so called *time crystals*) and its spontaneous breaking. Generically time crystals refer to regular periodic behavior not in the spatial dimensions but in the time domain. In the classical domain, time crystals are related to the periodic evolution of a system possessing the lowest energy in which the motion does not reach a standstill. Despite this apparent contradiction Shapere and Wilczek showed that if the kinetic energy of a particle on a ring is a quartic function of its velocity then the minimal energy state corresponds to a particle moving along a ring with a non-zero velocity. In particular, the minimal time crystal Lagrangian for a single degree of freedom is given by

$$L = \frac{1}{12}\dot{y}^4 - \frac{1}{2}\dot{y}^2 - V(y),$$

where the higher temporal derivative terms violates time translation symmetry. Note that Hamiltonian dynamics forbids the existence of classical time crystal, since $H(p, y)$ minimizes at $\frac{\partial H}{\partial p} = 0 = \frac{\partial H}{\partial y}$ with coordinate y and conjugate momentum p. As a consequence in the minimum energy state $\dot{y} = \partial H/\partial p = 0$ means that y is a constant. However, this negative result can be overcome if the structure of H is such that the canonical momentum p leads to a multivalued Hamiltonian as a function of y with cusps at $\frac{\partial p}{\partial \dot{y}} = 0$, in which case the Hamiltonian equations of motion cease to be valid. A common feature shared by all the models considered by Shapere and Wilczek [1, 2] is that the energy function (Hamiltonian) or Lagrangian systems become multivalued in terms of the canonical phase space variables.

The the study of systems with non-standard and/or non-convex Lagrangians especially with regard to spontaneous breaking of time translation symmetry has also been investigated by a number of other authors following the initial work of Shapere and Wilczek [4, 5, 6, 7].

Of late it has become clear that, for special kinds of mechanical systems, there are choices of Hamiltonian structures in which certain fundamental aspects of classical canonical Hamiltonian mechanics are changed. It has been observed in [4, 5, 6], that one can change the phase space variables which makes the Hamiltonian and symplectic structures on the phase space simultaneously well defined at the price of introducing a non-canonical symplectic structure. Curtright and Zachos [7], on the other hand, studied certain simple unified Lagrangian prototype systems which by virtue of non-convexity in their velocity dependence branch into double-valued (but still self-adjoint) Hamiltonians.

In this context it is therefore natural to investigate the issue of time translation breaking from the perspective of second-order differential equations within the general framework of Lagrangian/Hamiltonian mechanics and this is in fact constitutes our primary motivation. There are several interesting applications of time crystal in molecular phenomena, ion trapping problem etc. [8, 9]. The classical time crystal model is a generalization of the Friedmann-Robertson-Walker (FRW) cosmology endowed with noncommutative geometry corrections [10].

1.1 Motivation, result and organization

As already mentioned in the introduction in [1] Shapere and Wilczek showed that a classical system can reveal periodic motion in its lowest energy state. This essentially occurs when the Hamiltonian is a multivalued function of the momentum with cusps precisely corresponding to the energy minima. The multivaluedness of the Hamiltonian is a consequence of the fact that in theories containing higher powers of the velocities there are generally several solutions to the inversion equations between the velocities and the momenta. The latter also prevents the use of the Ostragradsky method. A particularly useful method for dealing with higher derivative theories is the Dirac formalism wherein one enlarges the phase space by including new additional variables while imposing suitable constraints. The advantage of the method is that it allows one to define correct phase space coordinates in which one can solve the constraints and reduce the number of variables to the original number of degrees of freedom and thereby construct a Hamiltonian which is single valued with a canonical symplectic structure on the phase space. In this sense the problem of dealing with higher derivative theories and multivalued Hamiltonians may be dealt with within the canonical formalism. Our chief motivation is to understand the duality between the Liénard equation of the second kind and the equation of motion of Sisyphus dynamics introduced by Shapere and Wilczek by invoking the Dirac formalism as formulated by Avraham and Brustein in [11].

In [12] the author studied the second class of Liénard system $\ddot{x}+f(x)\dot{x}^2+g(x)=0$, with a center at the origin 0 and investigated conditions under which it exhibited isochronicity. The Liénard equation of the second kind admits a Hamiltonian description using the Jacobi Last Multiplier and it has a profound applications in isochronous systems and the Hamiltonization of the Painlevé-Gambier type equations [13, 14, 15]. It should be noted that one finds up to a constant shift, the square of this Hamiltonian describes systems giving rise to spontaneous time translation symmetry breaking provided the potential function is negative [16]. The phenomenon of spontaneous symmetry breaking is essentially connected to Lagrangian theories with higher powers of velocities in the kinetic energy term. In this article by using the equations of Sisyphus dynamics we investigate the dual picture of symmetry breaking in terms of the Liénard equation of the second kind with a Hamiltonian involving a position dependent mass and Lagrangian for a single degree of freedom given by $L=\frac{1}{4}\dot{y}^4-\frac{1}{2}\dot{y}^2-V(y)$.

It is quite easy to construct models of time-independent, conservative dynamical systems with local ground states in which $\phi_x \neq 0$. The potential energy

$$V(\phi)=c_1\phi_x^4-c_2\phi_x^2,$$

exhibits space translation is spontaneously broken in the ground states. Our case is time-dependent one, where the potential energy

$$\tilde{V}(\phi)=b_1\phi_t^4-b_2\phi_t^2$$

shows spontaneous breaking of time translation The condition $\phi_t \neq 0$ at the ground state seems to imply that the system undergoes perpetual motion in its lowest energy.

The main result of the paper is given below.

Proposition 1.1 *Consider the system of equations describing Sisyphus dynamics*

$$\mu\ddot{x} = f'(x)\dot{y} - g'(x)$$

$$\dot{x}f'(x) = -V'(y)$$

define an invertible auxiliary function $h(x)$ such that

$$g(x) = \int f'(x)h(x)dx.$$

1. *In the limit as $\mu \to 0$ the equations of motion expressed in terms of the y coordinate has the Newtonian form*
$$m(\dot{y})\ddot{y} = -V'(y)$$
with the mass function being manifestly velocity dependent and is given by
$$m(\dot{y}) = h^{-1\prime}(\dot{y})f'(h^{-1}(\dot{y})).$$
In particular, for (1) $h(x) = x$, (2) $h(x) = x^n$, (3) $h(x) = x^2 + a$ and (4) $h(x) = \frac{b-xd}{cx-a}$, $ad - bc \neq 0$ we obtain the corresponding Lagrangians (1) $L_1 = \frac{\dot{y}^4}{12} - \frac{\dot{y}^2}{2} - V(y)$, (2) $L_2 = \frac{n}{3(n+3)}\dot{y}^{\frac{n+3}{n}} - \frac{n}{(n+1)}\dot{y}^{\frac{n+1}{n}} - V(y)$, (3) $L_3 = \frac{2}{15}(\dot{y} - a)^{5/2} - \frac{2}{3}(\dot{y} - a)^{3/2} - V(y)$, (4) $L_4 = \frac{1}{c}\left[\left(1 - \frac{a^2}{c^2}\right)\frac{\Delta}{c}\log|c\dot{y} + d| - \frac{a\Delta^2}{c^3(c\dot{y}+d)} + \frac{\Delta^3}{6c^3}\frac{1}{(c\dot{y}+d)^2}\right] - V(y)$ where $\Delta = ad - bc \neq 0$ respectively for the y coordinate equations.

2. *The equation of motion when expressed in terms of the x coordinate (with the potential function taken to be $V(y) = y^2/2$ is of the Lienard-II type and admits a Hamiltonian description.*

3. *The multivaluedness of the Hamiltonian is studied by employing Dirac brackets to construct the appropriate Legendre transformation and the resulting Hamilton's equations after a change of variable reproduces the Lienard -II equation upon elimination of one of the dynamical variables. This serves to illustrate the duality between the two descriptions.*

The computation of the proof is given in Section 4. Case (1), i.e., $h(x) = x$ coincides with the result of Shapere and Wilczek.

The **Organization** of the article is as follows. In section 2 we briefly outline the Lagrangian and Hamiltonian features of the Liénard equation of the second kind. We then consider the equations for Sisyphus dynamics and establish its connection with the Liénard equation of the second kind in Section 3. In the next section we analyse the duality between the Liénard equation and the equations of Sisyphus dynamics when expressed in terms of the x variable. We give the proof of our main result in section 4.1. Finally in Section 5 we outline an alternative procedure for arriving at the notion of spontaneous symmetry breaking.

2 The Liénard-II equation and its Hamiltonian aspects

The generic form of the Liénard equation of the second kind is

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0. \tag{2.1}$$

Using the concept of a Jacobi Last Multiplier (JLM) one can derive a suitable Lagrangian for this equation [15]. The JLM for a second-order ordinary differential equation is defined to be a solution of the equation

$$\frac{d}{dt}\log M + \frac{\partial \mathcal{F}(x,\dot{x})}{\partial \dot{x}} = 0, \quad \text{where } \mathcal{F}(x,\dot{x}) = -f(x)\dot{x}^2 - g(x). \tag{2.2}$$

In the present case the JLM for (2.1) is given by

$$M(x) = e^{2F(x)}, \quad \text{with} \quad F(x) := \int^x f(s)ds. \tag{2.3}$$

There exists a close connection between the JLM and the Lagrangian which is provided by [18]

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}. \tag{2.4}$$

In view of (2.3) it follows from (2.4) that a Lagrangian for the Liénard-II equation (2.1) is

$$L(x,\dot{x}) = \frac{1}{2}e^{2F(x)}\dot{x}^2 - V(x), \tag{2.5}$$

where the potential term

$$V(x) = \int^x e^{2F(s)}g(s)ds. \tag{2.6}$$

Clearly the conjugate momentum

$$p := \frac{\partial L}{\partial \dot{x}} = \dot{x}e^{2F(x)} \ \text{implies} \ \dot{x} = pe^{-2F(x)}, \tag{2.7}$$

and the final expression for the Hamiltonian using the standard Legendre transformation yields:

$$H = \frac{p^2}{2M(x)} + \int^x M(s)g(s)ds, \tag{2.8}$$

where $p = M(x)\dot{x}$ and $M(x) = \exp(2F(x))$ with $F(x) = \int^x f(s)ds$. The canonical variables are x and p and they satisfy the standard Poisson brackets $\{x,p\} = 1$. In terms of the canonical Poisson brackets the equations of motion appear as

$$\dot{x} = \{x,H\} = \frac{p}{M(x)}, \quad \dot{p} = \{p,H\} = \frac{M'(x)}{2M(x)}p^2 - M(x)g(x), \tag{2.9}$$

from which we can recover (2.1) upon elimination of the conjugate momentum p. Here we have purposely written the Hamiltonian H in terms of the last multiplier $M(x)$ to highlight the latter's role as a position dependent mass term.

As for the existence of a minima of H, considered as a function of x and p, it is necessary that

$$\frac{\partial H}{\partial x} = 0 \quad \text{and} \quad \frac{\partial H}{\partial p} = 0 \tag{2.10}$$

whose solutions then define the stationary points. The former yields

$$-p^2 \frac{M'(x)}{2M^2(x)} + M(x)g(x) = 0$$

while the latter implies $p/M(x) = 0$. Therefore the stationary points are characterized by $p = 0$ and the value(s) of x for which $g(x) = 0$. If $x = x^\star$ denotes a root of $g(x) = 0$ then $(x^\star, p = 0)$ is a stationary point ($s.p$). For a $s.p$ to be a minimum one requires that the principal minors of

$$\Delta = \left| \begin{array}{cc} H_{xx} & H_{xp} \\ H_{px} & H_{pp} \end{array} \right|_{s.p}$$

be positive definite, i.e.,

$$g'(x^\star) > 0 \quad \text{and} \quad M(x^\star)g'(x^\star) > 0.$$

Consistency therefore requires $M(x^\star) > 0$. Note that $M(x)$, which may be thought of as some kind of 'effective mass' such as within a spatial crystal, may be negative for $x \neq x^\star$. Clearly the fact that $p = 0$ in the minimum energy state (ground state) of the system precludes the possibility of any motion. However as explained in Section 5 in order to arrive at spontaneous time translation symmetry breaking we need to consider a modification of the above Hamiltonian (2.8).

3 Sisyphus dynamics

The equations of motion for Sisyphus dynamics are given by

$$\mu\ddot{x} = f'(x)\dot{y} - g'(x), \quad \dot{x}f'(x) = -V'(y). \tag{3.1}$$

In [1] the authors consider the limit where the mass $\mu \to 0$. By making specific choices for the functions

$$f = \frac{1}{3}x^3 - x, \qquad g = \frac{1}{4}x^4 - \frac{1}{2}x^2, \tag{3.2}$$

it immediately from the first equation of the Sisyphus dynamics (3.1) that as $\mu \to 0$, $\dot{y} = x$ and the equation of motion for the y coordinate turns out to be

$$(\dot{y}^2 - 1)\ddot{y} = -V'(y), \qquad \text{where} \qquad f' = \dot{y}^2 - 1. \tag{3.3}$$

on using the second equation $\dot{x} = -\frac{V'(y)}{f'}$.

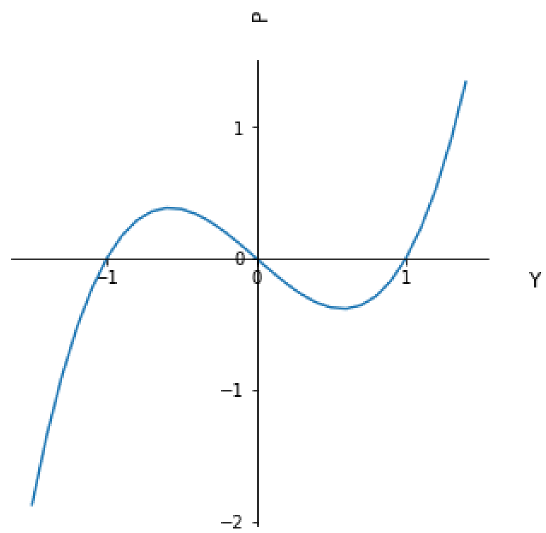

Figure 1: Plot of $p = y^3 - y$

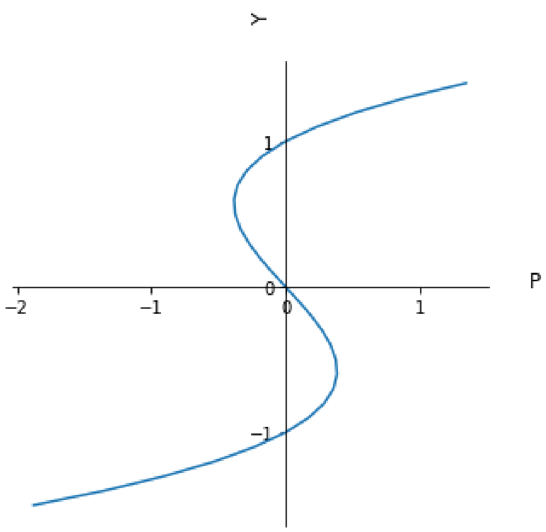

Figure 2: Plot of the inverse function $\left(-\frac{(\sqrt{3}(\sqrt{27x^2-4}-9x)^{1/3}}{2^{1/3}3^{2/3}}\right.$ - conjugate $\left.\right)$ given in fig.1

This equation is clearly of the Newtonian type with 'mass' being velocity dependent. Such equations are less common than the more familiar equations with position dependent mass (PDM) in nonlinear dynamics. Thus (3.3) equation is really to be viewed as a velocity dependent mass (VDM) equation of the Newtonian type.

Let us now focus on the equation for the variable x. To this end we begin by assuming that the function $V(y)$ is single valued with an isolated minimum at $y = 0$. We shall assume for concreteness, $V(y) = y^2/2$, so that the latter equation in (3.1) leads to $y = -\dot{x}f'(x)$. Eliminating now $\dot{y}$ from the first equation we arrive at the following second-order differential equation

$$\ddot{x} + \frac{f'f''}{\mu + f'^2}\dot{x}^2 + \frac{g'(x)}{\mu + f'^2} = 0. \tag{3.4}$$

Clearly this is an equation of the Liénard-II type and in the limit as $\mu \to 0$ we have

$$\ddot{x} + \frac{f''}{f'}\dot{x}^2 + \frac{g'}{f'^2} = 0,$$

which still retains the form of the Liénard-II equation.

Let us for the time being dispense with the explicit forms of the functions f and g and go back to (3.1) and consider the limit $\mu \to 0$; then we have $\dot{y} = g'/f' = h(x)$, (say) and $\dot{x}f'(x) = -V'(y)$. In this scenario we may derive a corresponding second-order equation for the variable y by assuming $h(x)$ is invertible. We have formally $\dot{x} = h^{-1\prime}(\dot{y})\ddot{y}$ so that

$$h^{-1\prime}(\dot{y})f'(h^{-1}(\dot{y}))\ddot{y} = -V'(y), \tag{3.5}$$

which may be regarded as a generalized version of (3.3.) Clearly (3.5) is a Newtonian equation of motion and it is plain that the mass function is given by

$$m(\dot{y}) = h^{-1\prime}(\dot{y})f'(h^{-1}(\dot{y})),$$

which is velocity dependent. As the JLM is related to the Lagrangian of a system by the formula, $M = \partial^2 L/\partial\dot{y}^2$, and in one dimension represents physically the mass function of the system, therefore for the inverse problem we are justified in assuming

$$\frac{\partial^2 L}{\partial \dot{y}^2} = h^{-1\prime}(\dot{y})f'(h^{-1}(\dot{y})), \tag{3.6}$$

which implies that the Lagrangian is of the form

$$L = \int d\dot{y} \int^{\dot{y}} dz h^{-1\prime}(z)f'(h^{-1}(z)) - V(y). \tag{3.7}$$

It will be noticed that if the functional forms of f and g are chosen as in [3] then their substitution into the above formula (3.7) for the Lagrangian gives

$$L = \frac{1}{12}\dot{y}^4 - \frac{1}{2}\dot{y}^2 - V(y), \tag{3.8}$$

as it should. The velocity dependent mass functions in this case is $M(\dot{y}) = (\dot{y}^2 - 1)$. If we assume the potential function $V(y) = y^2/2$ then the conjugate momenta turns out to be $p = \dot{y}^3 - \dot{y}$ and from $H = p\dot{y} - L$ we obtain

$$H = \frac{11}{12}\dot{y}^4 - \frac{1}{2}\dot{y}^2 + \frac{1}{2}y^2.$$

Obviously the Hamiltonian here is not written in the correct variables. To express the Hamiltonian in terms of the canonical variables it is necessary to invert the equation $p = \dot{y}^3 - \dot{y}$. This clearly leads to multi valued solutions of $\dot{y}$ in terms of p as depicted in the figures shown below. Therefore the Hamiltonian with $\dot{y}^4$ like terms cannot be expressed as a single valued function of the phase space variables. The energy has to be regarded as a multivalued function of the momentum p with cusps $\frac{\partial p}{\partial \dot{y}} = 0$. At the cusps the usual condition that the gradient should vanish at a minimum does not apply and hence the conjugate momentum is not a good variable for writing down the corresponding Hamiltonian uniquely.

One can of course propose a more general form of the mass function/JLM namely $M = a_0(y)\dot{y}^2 + a_2(y)$. The resulting Lagrangian is then given by

$$L = \frac{a_0(y)}{12}\dot{y}^4 + \frac{a_2(y)}{2}\dot{y}^2 - V(y)$$

and such a Lagrangian has indeed been considered in [10] in the context of cosmological time crystals in quadratic gravity.

When the above Lagrangian is inserted into the the Euler-Lagrange equation then it yields the following equation,

$$(a_0(y)\dot{y}^2 + a_2(y))\ddot{y} + \frac{1}{4}a_0'(y)\dot{y}^4 + \frac{1}{2}a_2'(y)\dot{y}^4 + B'(y) = 0. \tag{3.9}$$

This is a higher degree version of the usual Liénard-II equation. Clearly when a_0 and a_2 are constants we get

$$(c\dot{y}^2 + a)\ddot{y} + B'(y) = 0,$$

which is nothing but the (3.3).

We have already seen that when, $V(y) = y^2/2$, then upon eliminating $\dot{y}$ from the Sisyphus dynamics we obtain the equation (3.4) which we write as

$$\ddot{x} + \tilde{f}(x)\dot{x}^2 + \tilde{g}(x) = 0, \quad \text{with} \quad \tilde{f}(x) = \frac{f'f''}{\mu + f'^2}, \quad \tilde{g}(x) = \frac{g'(x)}{\mu + f'^2}.$$

The solution of the JLM is given by

$$M_{Sisy} = e^{2\tilde{F(x)}} = (\mu + f'^2), \qquad \tilde{F}(x) := \int^x \tilde{f}(s)ds,$$

and the Lagrangian of the equation is

$$L_{Sisy}(x, \dot{x}) = \frac{1}{2}(\mu + f'^2)\dot{x}^2 - V(x), \tag{3.10}$$

where

$$V(x) = \int^x (\mu + f'^2(s))\tilde{g}(s)\, ds = g(x). \tag{3.11}$$

Thus the momentum is given by $p = \dot{x}(\mu + f'^2(x))$ and the corresponding Hamiltonian is

$$H_{Sisy}(x, p) = \frac{p^2}{2(\mu + f'^2(x))} + g(x). \tag{3.12}$$

In this article we retain the general forms of $f(x)$ and $g(x)$ for the most part and attempt to understand the duality between the velocity dependent mass form of the Sisyphus equation of motion in terms of the y variable and the corresponding x equation of motion which is of the Liënard-II type.

4 Duality from Dirac brackets and Legendre transformation

We consider again the expression for the Lagrangian as given by (3.7), i.e,

$$L = \int d\dot{y} \int^{\dot{y}} dz h^{-1\prime}(z) f'(h^{-1}(z)) - V(y). \tag{4.1}$$

As already illustrated higher powers of $\dot{y}$ in the R.H.S. of (4.1) lead to a complicated algebraic equation connecting the canonical momentum

$$p_y = \frac{\partial L}{\partial \dot{y}} = \int^{\dot{y}} dz h^{-1\prime}(z) f'(h^{-1}(z)) \tag{4.2}$$

and the velocity $\dot{y}$. This usually leads to a multivalued Hamiltonian in terms of p_y.

Following Avraham and Brustein [11] we treat the velocity $\dot{y}$ as a new coordinate Q and after introducing a Lagrange multiplier write a modified Lagrangian

$$\tilde{L} = L(y, Q) + \lambda(\dot{y} - Q). \tag{4.3}$$

A standard Legendre transformation leads to the Hamiltonian

$$\tilde{H} = \dot{y}\frac{\partial \tilde{L}}{\partial \dot{y}} - \tilde{L} = \lambda Q - L(y, Q). \tag{4.4}$$

The primary or *first class* constraints are given by

$$\phi_1 = p_Q = \frac{\partial \tilde{L}}{\partial Q} = 0, \quad \phi_3 = p_3 - \lambda \approx 0, \quad \phi_4 = p_\lambda \approx 0, \tag{4.5}$$

where $p_y = \frac{\partial \tilde{L}}{\partial \dot{y}} = \lambda$. As the constraints ϕ_3 and ϕ_4 are not dynamical we can insert their values in the Hamiltonian and write

$$H_1 = \tilde{H} + \mu_1\phi_1 + \mu_3\phi_3 + \mu_4\phi_4 = \lambda Q - L(y, Q) + \mu_1 p_Q. \tag{4.6}$$

The secondary or *second class* constraint follows from the condition

$$\dot{\phi} = \{\phi_1, H_1\} = 0 \Rightarrow \phi_2 = \frac{\partial L}{\partial Q} - \lambda \approx 0,$$

using $\{Q, p_Q\} = 1$. Hence we have the total Hamiltonian

$$H_T = H_1 + \mu_2\phi_2 = \lambda Q - L(y, Q) + \mu_1 p_Q + \mu_2(\frac{\partial L}{\partial Q} - \lambda). \tag{4.7}$$

To construct Dirac brackets we note that

$$\{\phi_1, \phi_2\} = -\frac{\partial^2 L}{\partial Q^2}, \tag{4.8}$$

and the Dirac matrix C is defined as a skew symmetric matrix having entries $C_{ij} = \{\phi_i, \phi_j\}$ so that in the present case has the appearance

$$C = -\frac{\partial^2 L}{\partial Q^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with its inverse given by

$$C^{-1} = \frac{1}{\frac{\partial^2 L}{\partial Q^2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The expression for the Dirac bracket of A and B is

$$\{A, B\}_D = \{A, B\} - \frac{1}{\frac{\partial^2 L}{\partial Q^2}}\big(\{A, \phi_1\}\{\phi_2, B\} - \{A, \phi_2\}\{\phi_1, B\}\big). \tag{4.9}$$

On shell, after inserting the constraints the Hamiltonian becomes

$$H_F = Q\frac{\partial L}{\partial Q} - L(y, Q). \tag{4.10}$$

However the Dirac bracket of the phase space variables (y, Q) is not unity but

$$\{y, Q\}_D = -\frac{1}{\frac{\partial^2 L}{\partial Q^2}}. \tag{4.11}$$

To arrive at a canonical structure one is therefore led to consider a change of variables

$$(y, Q) \longrightarrow (F(y, Q), G(y, Q), \qquad \text{s.t.} \qquad \{F, G\}_D = 1, \tag{4.12}$$

which translates to the requirement

$$\frac{\partial F}{\partial y}\frac{\partial G}{\partial Q} - \frac{\partial F}{\partial Q}\frac{\partial G}{\partial y} = \frac{\partial^2 L}{\partial Q^2} = {h^{-1}}'(Q) f'(h^{-1}(Q)). \tag{4.13}$$

The simple choice $G = Q$ then implies $F = yh^{-1\prime}(Q)f'(h^{-1}(Q))$ and we may therefore re-express $H_F(y, Q)$ in terms of variables F and G. As a consequence we have

$$\dot{F}(y, Q) = \{(F(y, Q), H(F(y, Q), G(y, Q))\}_D = \{F, F\}_D \frac{\partial H}{\partial F} + \{F, G\}_D \frac{\partial H}{\partial G} = \frac{\partial H}{\partial G}, \quad (4.14)$$

and similarly

$$\dot{G}(y, Q) = \{(G(y, Q), H(F(y, Q), G(y, Q))\}_D = -\frac{\partial H}{\partial F}. \quad (4.15)$$

Thus in summary we have the following change of variables

$$Q = G, \quad y = \frac{F}{h^{-1\prime}(Q)f'(h^{-1}(Q))}, \quad (4.16)$$

together with the Hamiltonian written in terms of the new variables (F, G) which leads to a canonical structure:

$$H(F, G) = G\int^G dzh^{-1\prime}(z)f'(h^{-1}(z)) - \int dG\int^G dzh^{-1\prime}(z)f'(h^{-1}(z)) + V\left(\frac{F}{h^{-1\prime}(G)f'(h^{-1}(G)}\right). \quad (4.17)$$

Using (4.14) and (4.15) the canonical equations of motion are

$$\dot{F} = \int^a dzh^{-1\prime}(z)f'(h^{-1}(z)) - \int^G dzh^{-1\prime}(z)f'(h^{-1}(z)) + V'(r)F\frac{\partial}{\partial G}\Big(\frac{1}{h^{-1\prime}(G)f'(h^{-1}(G))}\Big) \quad (4.18)$$

$$\dot{G} = -\frac{\partial H}{\partial F} = -\frac{V'(r)}{h^{-1\prime}(G)f'(h^{-1}(G))}, \quad \text{with } r = \frac{F}{h^{-1\prime}(G)f'(h^{-1}(G))}. \quad (4.19)$$

4.1 Some cases of invertible functions $h^{-1}(z)$

We consider the following cases.

1. $h^{-1}(z) = z$ or $h(z) = z$.
2. $h^{-1}(z) = z^{1/n}$, i.e. $h(z) = z^n$.
3. $h^{-1}(z) = \sqrt{z - a}$, $a > 0$, $z \geq a$, i.e., $h(z) = z^2 + a$.
4. $h^{-1}(z) = \frac{az+b}{cz+d}$, then $h(z) = \frac{b-zd}{cz-a}$, $\quad ad - bc \neq 0$.

Note that as $h(z) = \frac{g'(z)}{f'(z)}$, once $h(z)$ and $f(z)$ have been chosen then $g(z)$ is obtained from

$$g(z) = \int f'(z)h(z)dz.$$

The Lagrangians corresponding to the above cases with the explicit choice $f(z) = z^3/3 - z$ are given below:

1. $L_1 = \frac{\dot{y}^4}{12} - \frac{\dot{y}^2}{2} - V(y)$.
2. $L_2 = \frac{n}{3(n+3)}\dot{y}^{\frac{n+3}{n}} - \frac{n}{(n+1)}\dot{y}^{\frac{n+1}{n}} - V(y)$.
3. $L_3 = \frac{2}{15}(\dot{y}-a)^{5/2} - \frac{2}{3}(\dot{y}-a)^{3/2} - V(y)$.
4. $L_4 = \frac{1}{c}\left[\left(1-\frac{a^2}{c^2}\right)\frac{\Delta}{c}\log|c\dot{y}+d| - \frac{a\Delta^2}{c^3(c\dot{y}+d)} + \frac{\Delta^3}{6c^3}\frac{1}{(c\dot{y}+d)^2}\right] - V(y)$ where $\Delta = ad - bc \neq 0$

Case 1 : In this case it follows that $Q = G$, $y = \frac{F}{(G^2-1)}$, if $V(z)$ is taken as $\frac{1}{2}z^2$. Hence from the Hamiltonian

$$H = G\int^G dz(z^2-1) - \int dG \int^G dz(z^2-1) + V(\frac{F}{G^2-1}) \tag{4.20}$$

we obtain

$$H = \frac{1}{4}G^4 - \frac{1}{2}G^2 + \frac{1}{2}\frac{F^2}{(G^2-1)^2}.$$

This yields

$$\dot{F} = \frac{\partial H}{\partial G} = G^3 - G^2 - \frac{2F^2G}{(G^2-1)^3}, \qquad \dot{G} = -\frac{\partial H}{\partial F} = -\frac{F}{(G^2-1)^2}. \tag{4.21}$$

If we switch back to the old variable then as, $\dot{y} = Q = G$, the second equation in (4.21) actually coincides with the result of Shapere and Wilczek [3], namely $(\dot{y}^2-1)\ddot{y} = -y$, as already mentioned.

Case 2: In this case it follows that

$$H(F,G) = \frac{1}{n+3}G^{\frac{n+3}{n}} - \frac{1}{n+1}G^{\frac{n+1}{n}} + V(\frac{F}{s})$$

where

$$s = \frac{1}{n}\left(G^{\frac{3}{n}-1} - G^{\frac{1}{n}-1}\right)$$

This leads to the canonical equations of motion, namely

$$\dot{F} = \frac{\partial H}{\partial G} = \frac{1}{n}\left(G^{\frac{3}{n}} - G^{\frac{1}{n}}\right) - V'(\frac{F}{s})\frac{F}{ns^2}\left((\frac{3}{n}-1)G^{\frac{3}{n}-2} - (\frac{1}{n}-1)G^{\frac{1}{n}-2}\right), \tag{4.22}$$

$$\dot{G} = -\frac{\partial H}{\partial F} = -V'(\frac{F}{s})\frac{1}{s}. \tag{4.23}$$

If one assumes $V(z) = z^2/2$ then straightforward calculation gives the following forms of the equations of motion

$$\dot{F} = \frac{1}{n}\left(G^{\frac{3}{n}} - G^{\frac{1}{n}}\right) - \frac{F^2}{ns^3}\left((\frac{3}{n}-1)G^{\frac{3}{n}-2} - (\frac{1}{n}-1)G^{\frac{1}{n}-2}\right), \tag{4.24}$$

$$\dot{G} = -\frac{F}{s^2}. \tag{4.25}$$

Upon eliminating F from the above system we then arrive at the second-order equation

$$\ddot{G} + \frac{\dot{G}^2}{ns}\left((\frac{3}{n}-1)G^{\frac{3}{n}-2} - (\frac{1}{n}-1)G^{\frac{1}{n}-2}\right) + \frac{1}{ns^2}\left(G^{\frac{3}{n}} - G^{\frac{1}{n}}\right) = 0. \tag{4.26}$$

This is an equation of the form of a Liénard equation of the second kind.

Case 3: This is a special case of the situation considered above, involving a translation of the z coordinate after setting the value of the exponent $n = 2$ and therefore we do not consider its details.

Case 4: In this case the Lagrangian is given by

$$L = \frac{1}{c}\left[\left(1 - \frac{a^2}{c^2}\right)\frac{\Delta}{c}\log|c\dot{y} + d| - \frac{a\Delta^2}{c^3(c\dot{y}+d)} + \frac{\Delta^3}{6c^3}\frac{1}{(c\dot{y}+d)^2}\right] - V\left(\frac{F}{s}\right)$$

where

$$s(\dot{y}) = \Delta\left(\frac{(a\dot{y}+b)^2}{(c\dot{y}+d)^4} - \frac{1}{(c\dot{y}+d)^2}\right), \text{ with } \Delta = ad - bc \neq 0$$

In this case the Hamiltonian when written in terms of the transformed variables (F, G) is given by

$$H(F,G) = \frac{\Lambda_1 G + \Lambda_2}{(cG+d)} + \frac{\Lambda_3 G + \Lambda_4}{(cG+d)^2} + \Lambda_5 \log|cG+d| + V\left(\frac{F}{u(G)}\right)$$

where the coefficients have the following values

$$\Lambda_1 = \frac{\Delta}{c}\left(1 - \frac{a^2}{c^2}\right), \quad \Lambda_2 = \frac{a\Delta^2}{c^4}, \quad \Lambda_3 = \frac{a\Delta^2}{c^3}, \quad \Lambda_4 = -\frac{\Delta^3}{6c^4}, \quad \Lambda_5 = -\frac{\Lambda_1}{c}$$

Assuming as before $V(z) = z^2/2$ the equations of motion following from the above Hamiltonian are given by

$$\dot{F} = \frac{\partial H}{\partial G} = \frac{c\Lambda_5}{(cG+d)} + \frac{\Lambda_1 d - \Lambda_2 c + \Lambda_3}{(cG+d)^2} - \frac{2c(\Lambda_3 G + \Lambda_4)}{(cG+d)^3} - \frac{s'(G)}{s^3(G)}F^2,$$

$$\dot{G} = -\frac{F}{s^2(G)}.$$

Upon elimination of F we arrive at the following second-order equation of the Liénard type namely:

$$\ddot{G} + \frac{s'(G)}{s(G)}\dot{G}^2 + K(G) = 0,$$

where

$$K(G) = \frac{1}{s^2(G)}\left[\frac{c\Lambda_5}{(cG+d)} + \frac{\Lambda_1 d - \Lambda_2 c + \Lambda_3}{(cG+d)^2} - \frac{2c(\Lambda_3 G + \Lambda_4)}{(cG+d)^3}\right],$$

and

$$s(G) = \Delta\left[\frac{(aG+d)^2}{(cG+d)^4} - \frac{1}{(cG+d)^2}\right].$$

5 A modified Hamiltonian and spontaneously broken time translation symmetry

Consider a one-dimensional generalized Hamiltonian system $\widetilde{H} = \mathcal{F}(H)$ with Hamiltonian vector field given in terms of the canonical form

$$\mathbb{X}_{\widetilde{H}} = \frac{\partial \widetilde{H}}{\partial p}\frac{\partial}{\partial x} - \frac{\partial \widetilde{H}}{\partial x}\frac{\partial}{\partial p}, \qquad \{G, \widetilde{H}\} = \dot{G}.$$

In the symplectic coordinates (x, p) this is equivalent to canonical Hamiltonian equations

$$\dot{x} = \mathcal{F}(H)'\{x, H\}, \qquad \dot{p} = \mathcal{F}(H)'\{p, H\}, \qquad \text{where} \quad \mathcal{F}(H)' > 0.$$

It may be easily verified that the above set of Hamiltonian equations may be obtained from the modified symplectic form $\omega = \mathcal{F}(H)'dx \wedge dp$. Moreover this change of Hamiltonian structure will not change the partition function, hence all thermodynamic quantities will remain unchanged.

Let us consider a new Hamiltonian [5] defined by the square of H as given by (2.8) and a shift, i.e.,

$$\widetilde{H} = \left(\frac{p^2}{2M(x)} + \int^x M(s)g(s)ds\right)^2 + E_0 = H^2 + E_0, \tag{5.1}$$

where E_0 is an arbitrary constant. As the new Hamiltonian is anticipated to generate a dynamics which is distinct from that of H, let us also introduce the following Poisson structure $\{x, p\} = \xi(x, p)$ so that the equations of motion which follow from

$$\dot{x} = \{x, \widetilde{H}\}, \quad \dot{p} = \{p, \widetilde{H}\} \tag{5.2}$$

give

$$\dot{x} = 2\xi H \frac{p}{M(x)} \tag{5.3}$$

$$\dot{p} = -2\xi H\left(-\frac{M'(x)}{2M^2(x)}p^2 + M(x)g(x)\right). \tag{5.4}$$

At this point we need to make a clear distinction regarding the two Poisson structures we have introduced. It will be noticed that if one assumes $\{x, p\} = \xi(x, p) = \frac{1}{2H(x,p)}$ then we get back the original Liénard-II equation (2.1), if however we persist with $\xi = 1$, i.e., assume x and p are canonical then the equation of motion resulting from the Hamiltonian $\widetilde{H}$ is of the form

$$\ddot{x} + 2H(f(x)\dot{x}^2 + g(x)) = 0. \tag{5.5}$$

Although (5.5) appears to be different from (2.1) it is interesting to note that (5.5) can be mapped to the original set of Hamiltonian equations (2.9) by using a (nonlocal) Sundman transformation [17] through a transformation of the independent temporal variable t to a new independent variable s given by $ds = 2Hdt$, whence we obtain

$$x' = \frac{p}{M(x)}, \qquad p' = -\left(-\frac{M'(x)}{2M^2(x)}p^2 + M(x)g(x)\right), \tag{5.6}$$

where $' = \frac{d}{ds}$. In fact such transformations were used by Sundman while attempting to solve the restricted three body problem.

As for the stationary points of the Hamiltonian $\widetilde{H}$, these follow from the solutions of $\partial\widetilde{H}/\partial x = 0$ and $\partial\widetilde{H}/\partial p = 0$. The latter yields either $p = 0$ or $H = 0$. If $p = 0$ then the former condition gives either $H = 0$ or $g(x) = 0$, i.e $x = x^\star$. The pair $(x^\star, p = 0)$ leads by the previous analysis to the case

$$\widetilde{H}_{min} = \left(\int^{x^\star} M(s)g(s)ds\right)^2 + E_0. \tag{5.7}$$

From the above equation it is clear that the local minimum of $\widetilde{H}$ is in general greater than the constant E_0 because the potential $V(x^\star)$ is not required to vanish at $x = x^\star$. As the stationary point corresponds to $p = 0$ the time translation symmetry is not broken and we have the same situation as previously discussed in section 2.

However one also has now the possibility wherein $H = 0$ which implies that the locus of the stationary points lie on the curve [16]

$$\frac{p^2}{2M(x)} + \int^x M(s)g(s)ds = 0. \tag{5.8}$$

This condition obviously implies that $\widetilde{H}$ has a minima with $\widetilde{H}_{min} = E_0$ which is less than that given by (5.7). Now for real values of p it is then necessary that

$$V(x) = \int^x M(s)g(s)ds < 0.$$

The force dV/dx is clearly not necessarily zero and motion can therefore occur in the ground state. The existence of motion under such circumstances is indicative of the spontaneous breaking of the time-translation symmetry [1]. This then provides an alternative procedure for obtaining motion in the minimal energy state.

6 Conclusion

The equations governing Sisyphus dynamics are expressed in terms of the auxiliary variable x instead of y as done in [3] by Shapere and Wilczek who assumed that the constitute functions f and g are such that $\dot{y}$, is linear in $\dot{y}$. In this paper we have considered a more general form such that $\dot{y} = h(x)$, hence it is a function of the momentum conjugate to y. In this situation we have derived a corresponding second-order equation, for the " momentum" coordinate x, $\ddot{x}+f(x)\dot{x}+g(x)=0$, which belongs to the Liénard-II type.

In our computation we have considered $h(x)=\frac{g'(x)}{f'(x)}$, for fixed $f(x)=x^3/3-x$ and for different invertible functions $h(x)$ using which we have obtained the corresponding classical time crystal Lagrangians. The Lagrangians contain higher powers of the time derivative and display a nonlinear relationship between the velocities and momenta, thus leading to a multivalued canonical description. The multivalued nature of phase space originates due to an inappropriate choice of coordinates. Following Avraham and Brustein [11], the Dirac formalism is introduced to define a generalized Legendre transformation. In this paper we have shown that the phase space coordinates may be so chosen as to enable one to define a single valued Hamiltonian for the generalized classical time crystal Lagrangians.

We have also outlined the the branched Hamiltonian aspects of the Liénard type equation corresponding to the " momentum " coordinate. Although a number of articles have appeared on branched Hamiltonians, there appears to be no uniform consensus on the physical interpretations of the results of these analyses. The Hamiltonians studied have almost invariably time independent. In this context it is interesting to mention that there are examples of time-dependent Hamiltonian systems for which one can define a suitable conjugate set of the canonical Hamilton's equations and they offer a alternative scenario to test for multivaluedness and branching to the Hamiltonian thereby leading to possibly a new dynamics.

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