Octonionic Clifford Algebra for the Internal Space of the Standard Model

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Abstract

We explore the $\mathbb{Z}_2$ graded product $C\ell_{10} = C\ell_4 \hat{\otimes} C\ell_6$ as a finite internal space algebra of the Standard Model of particle physics. The gamma matrices generating $C\ell_{10}$ are expressed in terms of left multiplication by the imaginary octonion units and the Pauli matrices. The subgroup of $Spin(10)$ that fixes an imaginary unit (and thus allows to write $\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3$ expressing the quark-lepton splitting) is the Pati-Salam group $G_{\text{PS}} = Spin(4) \times Spin(6)/\mathbb{Z}_2 \subset Spin(10)$. If we identify the preserved imaginary unit with the $C\ell_6$ pseudoscalar $\omega_6 = \gamma_1 \cdots \gamma_6$, $\omega_6^2 = -1$, then $\mathcal{P} = \frac{1}{2}(1 - i\omega_6)$ will be the projector on the extended particle subspace, including the right-handed (sterile) neutrino. We express the generators of $C\ell_4$ and $C\ell_6$ in terms of fermionic oscillators $a_\alpha, a_\alpha^*, \alpha = 1, 2$ and $b_j, b_j^*, j = 1, 2, 3$ describing flavour and colour, respectively. The internal space observables belong to the Jordan subalgebra of hermitian elements of the complexified Clifford algebra $\mathbb{C} \otimes C\ell_{10}$ which commute with the weak hypercharge $\frac{1}{2}Y = \frac{1}{3}\sum_{j=1}^3 b_j^* b_j - \frac{1}{2}\sum_{\alpha=1}^2 a_\alpha^* a_\alpha$. We only distinguish particles from antiparticles if they have different eigenvalues of Y. Thus the sterile neutrino and antineutrino (both with $Y = 0$) are allowed to mix into Majorana neutrinos. Restricting $C\ell_{10}$ to the particle subspace, which consists of leptons with $Y < 0$ and quarks, allows a natural definition of the Higgs field Φ, the scalar of Quillen's superconnection, as an element of $C\ell_4^1$, the odd part of the first factor in $C\ell_{10}$. As an application we express the ratio $\frac{m_H}{m_W}$ of the Higgs and the W-boson masses in terms of the cosine of the *theoretical* Weinberg angle.

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1 Introduction

The elaboration of the Standard Model (SM) of particle physics was completed in the early 1970's. To quote John Baez [B21] 50 "years trying to go beyond the Standard Model hasn't yet led to any clear success". The present paper belongs to an equally long albeit less fashionable effort to clarify the algebraic (or geometric) roots of the SM, more specifically, to find a natural framework featuring its internal space properties. After discussing some old ideas motivating our approach among others, we review some recent developments, clarifying on the way the role of different projection operators, expressed in terms of Clifford algebra pseudoscalars and their interrelations.

Most ideas on the natural framework of the SM originate in the 1970's, the first decade of its existence. (Two exceptions: the Jordan algebras were introduced and classified in the 1930's [J, JvNW]; the noncommutative geometry approach originated in the late 1980's, [C, DKM, CL] and is still vigorously developed by Connes and collaborators [CC, CCS, CIS, NS].)

First, early in 1973, the ultimate division algebra, the octonions[1] were introduced by Gürsey[2] and his student Günaydin [GG, G] for the description of quarks and their $SU(3)$ colour symmetry. The idea was taken up and extended to incorporate all four division algebras by G. Dixon (see [D10, D14] and earlier work cited there) and is further developed by Furey [F14, F15, F16, F, F18, FH1, FH]. Dubois-Violette (D-V) arrives at the octonions via the quark-lepton symmetry and the unimodularity of the colour group [D16]. Thus, the octonions appear with an additional complex structure,

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3 \,, \tag{1.1}$$

preserved by the subgroup $SU(3)$ of the automorphism group G_2 of $\mathbb{O}$.

1.1 Octonions as a composition algebra. The Cayley-Dickson construction

One can in fact provide a basis free definition of the octonions starting with the splitting (1.1). To this end one uses the skew symmetric vector product and the standard inner product on $\mathbb{C}^3$ to define a noncommutative and non-associative distributive product xy on $\mathbb{O}$ and a real valued nondegenerate symmetric bilinear form $\langle x, y\rangle = \langle y, x\rangle$ such that the quadratic norm $N(x) = \langle x, x\rangle$ is multiplicative:

$$N(xy) = N(x)N(y) \quad \text{for} \quad N(x) = \langle x, x\rangle \tag{1.2}$$

(cf. [D16, TD]). Furthermore, defining the real part of $x \in \mathbb{O}$ by $\mathrm{Re}\, x = \langle x, 1\rangle$ and the octonionic conjugation $x \to x^* = 2\langle x, 1\rangle - x$, we shall have

$$xx^* = N(x)\mathbb{1} \Leftrightarrow x^2 - 2\langle x, 1\rangle x + N(x)\mathbb{1} = 0\,. \tag{1.3}$$

[1] For a pleasant to read review of octonions, their history and applications – see [B02].

[2] See Witten's eloquent characterization of his personality and work in the Wikipedia entry on Feza Gürsey (1921-1991).

A unital algebra with a non-degenerate quadratic norm obeying (1.2) is called a *composition algebra*.

Another basis free definition of the octonions $\mathbb{O}$ and of their split version $\widetilde{\mathbb{O}}$ can be given in terms of quaternions by the Cayley-Dickson construction. We represent the quaternion as scalars plus vectors

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3,\ x = u + U,\ y = v + V,\ u, v \in \mathbb{R},\ U, V \in \mathbb{R}^3,$$
$$xy = uv - \langle U, V \rangle + uV + Uv + U \times V \tag{1.4}$$

with the vector product $U \times V \in \mathbb{R}^3$ satisfying

$$U \times V = -V \times U,\ (U \times V) \times W = \langle U, W \rangle V - \langle V, W \rangle U\,. \tag{1.5}$$

The product (1.4) is clearly noncommutative but one verifies that it is associative. The Cayley-Dickson construction defines the octonions $\mathbb{O}$ and the split octonions $\widetilde{\mathbb{O}}$ in terms of a pair of quaternions and a new "imaginary unit" ℓ as:

$$x = u + U + \ell(v + V),\ \ell(v + V) = (v - V)\ell\,,$$
$$\ell^2 = \begin{cases} -1 & \Rightarrow \quad x \in \mathbb{O} \\ \ 1 & \Rightarrow \quad x \in \widetilde{\mathbb{O}}\,. \end{cases} \tag{1.6}$$

1.2 Jordan algebras; grand unified theories; Clifford algebras

D-V suggests that classical observables (real valued functions) are replaced by an algebra of functions on space-time with values in a *finite dimensional euclidean Jordan algebra*[3]. As a particularly attractive choice, which incorporates the idea of quark-lepton symmetry, D-V proposes [D16] the exceptional Jordan algebra of 3×3 hermitian matrices with octonionic entries,

$$J_3^8 = \mathcal{H}_3(\mathbb{O})\,. \tag{1.7}$$

This approach is further pursued in [TD, TD-V, DT, T, DT20].

A second development, *Grand Unified theory* (GUT), anticipated during the same 1973 by Pati and Salam [PS], became for a time mainstream[4]. Fundamental chiral fermions fit the complex spinor representation of $Spin(10)$, introduced as a GUT group by Fritzsch and Minkowski and by Georgi. A preferred symmetry breaking yields the maximal rank semisimple Pati-Salam subgroup,

$$G_{\mathrm{PS}} = \frac{Spin(4) \times Spin(6)}{\mathbb{Z}_2} \subset Spin(10),$$
$$Spin(4) = SU(2)_L \times SU(2)_R,\ Spin(6) = SU(4)\,. \tag{1.8}$$

We note that G_{PS} is the only GUT group which does not predict a gauge triggered proton decay. It is also encountered in the noncommutative geometry approach to the SM [CCS, BF]. In general, GUTs provide a nice home for

[3]These algebras are defined and classified in [JvNW]; for concise reviews see Sect. 3.2 in [D16] and Sect. 2 of [T].

[4]For an enlightening review of the algebra of GUTs and some 40 references see [BH].

the fundamental fermions, as displayed by the two 16-dimensional complex conjugate "Weyl spinors" of $Spin(10)$. Their other representations, however, like the 45-dimensional adjoint representation of $Spin(10)$ are much too big, involve unobserved beasts like leptoquarks which cause difficulties.

A central role in our approach will be given to the Clifford algebra[5] $C\ell_{10}$, viewed as a $\mathbb{Z}_2$-graded tensor product [F16, F, T21]:

$$C\ell_{10} = C\ell_4 \,\widehat{\otimes}\, C\ell_6 \,. \tag{1.9}$$

The complexified Clifford algebra has a single faithful irreducible representation (IR) of dimension $2^5 = 32$ which fits precisely the fundamental (anti)fermions of one generation. Clifford algebras were also applied to the SM in the 1970's – see [CG] and references therein. There are two new points in our approach.

1) We use the presence of the octonions with a preferred complex structure in $C\ell_{8+\nu}$, $\nu = 0, 1, 2$ to derive the gauge group of the SM (for $C\ell_9$),

$$G_{\rm SM} = S(U(2) \times U(3)) \tag{1.10}$$

and its left right symmetric extension (for $C\ell_{10}$) [B] (see also the talks of J. Baez [B21], K. Krasnov [K21] and L. Boyle at the Perimeter Institute Workshop, as well as [FH1, FH, T21]). One relies, in particular, on the nonassociativity of the octonions (as emphasized in [K]) which implies noncommutativity of left and right multiplication L_x, R_y $(x, y \in \mathbb{O})$.

2) We make essential use of the $\mathbb{Z}_2$ grading of the Clifford algebra. The Higgs field, which intertwines left and right chiral fermions, belongs to the odd part of the factor $C\ell_4$ in (1.9) [DT20, T21]. This fits perfectly the *super-connection* approach to the SM, pioneered by Ne'eman [N] and Fairlie [F79] well before the notion was coined (and named) by mathematicians [Q, MQ].

Octonions by themselves are not fitted to describe observables. Their Jordan subalgebra of hermitian elements consists just of the real numbers. They do enter however the Jordan *spin factors* J_2^ν of degree $\nu \geq 7$ whose associative envelopes are $C\ell_{\nu+1}$ (as well as the exceptional Jordan algebra (1.7)):

$$J_2^\nu \subset C\ell'_{\nu+1}(\nu = 7, 8, 9, \cdots), \ \dim(J_2^\nu) = \nu + 2, \ J_2^8 \subset J_3^8 \,. \tag{1.11}$$

As already noted, for $\nu = 8, 9$ their Clifford envelopes may describe the internal space observables of one generation of fundamental fermions. It will be recalled in Sect. 3 that the gauge group of the SM (1.10) is recovered by considering the restriction of J_3^8 to J_2^8. More precisely, $G_{\rm SM}$ appears as the intersection of two subgroups of the automorphism group F_4 of J_3^8: the centralizer F_4^ω of $\omega \in SU(3)_c \subset F_4$, $\omega^2 + \omega + 1 = 0$ and $Spin(9)$, the stabilizer of J_2^8, the subalgebra of 3×3 matrices in J_3^8 with zero first row and first column:

$$G_{\rm SM} = F_4^\omega \cap Spin(9) \subset F_4 \,, \tag{1.12}$$

[5] Aptly called *geometric algebra* by its inventor – see [DL].

$$F_4^\omega = \frac{SU(3)_c \times SU(3)}{\mathbb{Z}_3}\,,\ \omega(z+Z) = z + exp\left(\frac{2\pi i}{3}\right) Z,\ z \in \mathbb{C},\ Z \in \mathbb{C}^3\,. \quad (1.13)$$

($x = z + Z$ being a realization of the splitting (1.1), [TD-V].) We shall see, however, that the representation of $G_{\rm SM}$, obtained by restriction from $Spin(9)$ only involves $SU(2)_L$-doublets, it has no room for e_R, u_R, d_R. This is, in fact, a manifestation of a general result (see, e.g. [CD], Proposition 15.2 (p. 674)): the only simple compact gauge groups allowing to accomodate chiral fermions are $SU(n)$, $n \geq 3$, $Spin(4n+2)$ and E_6.

2 Triality realization of $Spin(8)$; $C\ell_{-6}$

2.1 The action of octonions on themselves. $Spin(8)$ as a subgroup of $SO(8) \times SO(8) \times SO(8)$

The group $Spin(8)$, the double cover of the orthogonal group $SO(8) = SO(\mathbb{O})$, can be defined (see [Br, Y]) as the set of triples $(g_1, g_2, g_3) \in SO(8) \times SO(8) \times SO(8)$ such that

$$g_2(xy) = g_1(x)\, g_3(y) \text{ for any } x, y \in \mathbb{O}\,. \quad (2.1)$$

If u is a unit octonion, $u^* u = 1$, then the left and right multiplications by u are examples of isometries of $\mathbb{O}$

$$|L_u\, x|^2 = \langle ux, ux\rangle = \langle x, x\rangle,\ |R_u\, x|^2 = \langle xu, xu\rangle = \langle x, x\rangle \text{ for } \langle u, u\rangle = 1\,. \quad (2.2)$$

Using the *Moufang identity*[6]

$$u(xy)u = (ux)(yu) \text{ for any } x, y, u \in \mathbb{O}\,, \quad (2.3)$$

one verifies that the triple $g_1 = L_u,\ g_2 = L_u\, R_u,\ g_3 = R_u$ satisfies (1.1) and hence belongs to $Spin(8)$. It turns out that triples of this type generate $Spin(8)$ (see [Br] or Yokota's book [Y] for more details).

The mappings $x \to L_x$ and $x \to R_x$ are, of course, not algebra homomorphisms as L_x and R_y generate each an associative algebra while the algebra of octonions is non-associative. They do preserve, however, the quadratic relation $xy^* + yx^* = 2\langle x, y\rangle 1\!\!1$:

$$L_x L_{y^*} + L_y L_{x^*} = 2\langle x, y\rangle 1\!\!1 = R_x R_{y^*} + R_y R_{x^*}\,. \quad (2.4)$$

Eq. (2.4), applied to the span of the first six imaginary octonion units e_j, $j = 1, \cdots, 6$, setting $L_{e_j} =: L_j$, $R_{e_j} =: R_j$ becomes the defining relation of the Clifford algebra $C\ell_{-6}$:

$$L_j L_k + L_k L_j = -2\delta_{jk} = R_j R_k + R_k R_j,\ j, k = 1, \cdots, 6\,. \quad (2.5)$$

[6] See [S16] for a reader friendly review of Moufang loops and for a glimps of the personality of Ruth Moufang (1905-1971).

In general, $L_x L_y \neq L_{xy}$ (and similarly for R), but remarkably, as noted in [F16], the relation $(e_1(e_2(e_3(e_4(e_5(e_6\, e_a)))))) = e_7\, e_a$ is satisfied for all $a = 1, \cdots, 8$, so that

$$L_1 L_2 \cdots L_6 = L_{e_7} =: L_7, \;\; R_1 R_2 \cdots R_6 = R_{e_7} =: R_7\,. \tag{2.6}$$

While $L_a R_a = R_a L_a$ (for $a \in \mathbb{O}$) the non-associativity of the algebra of octonions is reflected in the fact that for $x \neq y$, L_x and R_y, in general, do not commute.

2.2 $C\ell_{-6}$ as a generating algebra of $\mathbb{O}$ and of $so(\mathbb{O})$

The Lie algebra $so(8)$ is spanned by the elements of negative square of $C\ell_{-6}$. If we denote the exterior algebra on the span of $L_1, \cdots, L_6$ by

$$\Lambda^* \equiv \Lambda^* C\ell_{-6} = \Lambda^0 + \Lambda^1 + \cdots + \Lambda^6 \; \Big(\Lambda^1 = \operatorname*{Span}_{1\leq j\leq 6} L_j, \;\Lambda^6 = \{\mathbb{R}\, L_7\}\Big)$$

then $so(8) = \Lambda^1 + \Lambda^2 + \Lambda^5 + \Lambda^6$. A basis of the Lie algebra, given by

$$L_{\alpha 8} = \tfrac{1}{2}\, L_\alpha, \;\; L_{\alpha\beta} = -\tfrac{1}{4}\, [L_\alpha, L_\beta], \;\; \alpha, \beta = 1, \cdots, 7 \tag{2.7}$$

obeys the standard commutation relations (CRs)

$$[L_{ab}, L_{cd}] = \delta_{bc}\, L_{ad} - \delta_{bd}\, L_{ac} + \delta_{ad}\, L_{bc} - \delta_{ac}\, L_{bd},$$

$$L_{ab} = \tfrac{1}{4}(L_a L_b^* - L_b L_a^*), \;\; a, b, c, d = 1, 2, \cdots, 8 \tag{2.8}$$

(and similarly for R_{ab}). Each element of $so(8)$ of square -1 defines a *complex structure*. (For a review of this notion in the context of Clifford algebras and spinors – see [D-V].) Following [FH] we shall single out the *Clifford pseudoscalars* L_7 and R_7 (2.6) (called *volume forms* in the highly informative lectures [M] and Coxeter elements in [T11]). We shall use the (mod 7) multiplication rules of [B02] for the imaginary octonion units

$$L_i e_j (= e_i e_j) = -\delta_{ij} + f_{ijk}\, e_k, \;\; f_{ijk} = 1$$

$$\text{for } (i, j, k) = (1, 2, 4)(2, 3, 5)(3, 4, 6)(4, 5, 7)(5, 6, 1)(6, 7, 2)(7, 1, 3) \tag{2.9}$$

and f_{ijk} is fully antisymmetric within each of the above seven triples. The Clifford pseudoscalar is naturally associated with the Cartan subalgebra of $so(6)$ spanned by

$$(L_{13}, L_{26}, L_{45}) \text{ as } L_7(e_1, e_2, e_4) = (e_3, e_6, e_5)\,. \tag{2.10}$$

We can write

$$L_7 = 2^3 L_{13} L_{26} L_{45} \text{ (as } 2L_{13} = L_1 L_3^* = -L_1 L_3 \text{ etc.)} \tag{2.11}$$

The infinitesimal counterpart of (2.1) reads

$$T_\alpha(x, y) = (L_\alpha\, x) y + x (R_\alpha\, y) \text{ for } \alpha, x, y \in \mathbb{O}, \;\; \alpha^* = \alpha\,,$$

$$\text{i.e.} \qquad T_\alpha = L_\alpha + R_\alpha\,. \tag{2.12}$$

There is an involutive outer automorphism π of the Lie algebra $so(8)$ such that

$$\pi(L_\alpha) = T_\alpha,\ \pi(R_\alpha) = -R_\alpha,\ \pi(T_\alpha) = L_\alpha\ (\pi^2 = id)\,. \tag{2.13}$$

As proven in Appendix A

$$\pi(L_{ab}) = E_{ab} \text{ where } E_{ab}\, e_c = \delta_{bc}\, e_a - \delta_{ac}\, e_b\ (a, b, c = 1, 2, \cdots, 8,\ e_8 = 1) \tag{2.14}$$

$(L_{ab}), (E_{ab})$ and (R_{ab}) provide three bases of $so(8)$, each obeying the CRs (2.8). They are expressed by each other in terms of the involution π:

$$L_{ab} = \pi(E_{ab}),\ E_{\alpha 8} = L_{\alpha 8} + R_{\alpha 8},\ \alpha = 1, \cdots, 7\,. \tag{2.15}$$

We find, in particular – see Appendix A:

$$L_7 = 2L_{78} = E_{78} - E_{13} - E_{26} - E_{45}\,,$$

$$R_7 = 2R_{78} = E_{78} + E_{13} + E_{26} + E_{45} = -L_{78} - L_{13} - L_{26} - L_{45}\,. \tag{2.16}$$

While $L_{78} = 4L_{13}L_{26}L_{45}$ (2.11) commutes with the entire Lie algebra $spin(6) = su(4)$ the $u(1)$ generator

$$C_1 = L_{13} + L_{26} + L_{45} \text{ centralizes } u(3) = u(1) \oplus su(3) \subset su(4) \tag{2.17}$$

(that is the unbroken part of the gauge Lie algebra of the SM). The reader may verify the identity $R_7^2 = -1$ for the right hand side of (2.16) using the relations

$$L_{jk}^2 = -\frac{1}{4},\ C_1^2 = -\frac{3}{4} + 2C_2,\ -C_1 L_{07} = C_2 := L_{13}L_{26} + L_{13}L_{45} + L_{26}L_{45}\,. \tag{2.18}$$

The above relations will be useful for the study of higher Clifford and Lie algebras that involve $so(8)$ (expressed in terms of L_{ab} or R_{ab}) as a subalgebra. We shall apply them in the next section to the chain of nested Clifford algebras and their derivation (Lie) algebras

$$(C\ell_{-6} \subset)\, C\ell_8 \subset C\ell_9 \subset C\ell_{10} \leftrightarrow so(8) \subset so(9) \subset so(10)\,. \tag{2.19}$$

In order to accomodate the duality between antihermitian symmetry generators (of a compact gauge group) and the corresponding conserved hermitian observables within the same (internal space counterpart of) Haag's [H] field algebra we need multiplication by an imaginary unit. Thus the algebraic counterpart of Nœther's theorem (cf. [B20]) requires a complexification of the algebras (2.19). In particular, the Cartan subalgebra of $so(8)$ singled out by the complex structure L_7 is spanned by the four commuting hermitian elements

$$2i\, L_{78},\ 2i\, L_{j\,3j\,(\mathrm{mod}\,7)} = 2i(L_{13}, L_{26}, L_{45})\ (j = 1, 2, 4) \tag{2.20}$$

of square one, where the complex imaginary unit i ($i^2 = -1$) commutes with the octonion units e_α. We shall single out the $u(3)$ Lie subalgebra of the derivation algebra $su(4) = so(6)$ that contains the colour $su(3)$ by identifying its centralizer $u(1)$ with the sum of the operator $2i\, L_{j3j}$ (2.20). It is a multiple of the observable

$$B - L = \frac{2i}{3}\,(L_{13} + L_{26} + L_{45})\,, \tag{2.21}$$

the difference between the baryon and the lepton numbers. $B - L$ takes eigenvalues $\pm\frac{1}{3}$ for (anti)quarks and ∓ 1 for (anti)leptons so that

$$[(B-L)^2 - 1][9(B-L)^2 - 1] = 0\,. \tag{2.22}$$

3 $C\ell_{10} = C\ell_4 \,\widehat{\otimes}\, C\ell_6$ as internal space algebra

3.1 Equivalence class of Lorentz like Clifford algebras

Nature appears to select real Clifford algebras $C\ell(s,t)$ of the equivalence class of $C\ell(3,1)$ (with Lorentz signature in four dimensions) in Elie Cartan's classification[7]:

$$C\ell(s,t) = \mathbb{R}[2^n],\ \text{for } s-t = 2(\text{mod}\,8),\ s+t = 2n\,. \tag{3.1}$$

They act on $2n$ dimensional *Majorana spinors* that transform irreducibly under the *real* 2^n dimensional representation of the spin group $Spin(s,t)$. If $\gamma_1, \cdots, \gamma_{2n}$ is an orthonormal basis of the underlying vector space $\mathbb{R}^{s,t}$ then the Clifford pseudoscalar defines a complex structure

$$\omega_{s,t} = \gamma_1 \cdots \gamma_{2n},\ 2n = s+t,\ \omega_{s,t}^2 = -1\,, \tag{3.2}$$

which commutes with the action of $Spin(s,t)$. Upon complexification the resulting *Dirac spinor* splits into two *inequivalent* 2^{n-1}-dimensional complex *Weyl* (or *chiral*) *spinor* representations irreducible *over* $\mathbb{C}$ under $Spin(s,t)$. The corresponding projectors Π_L and Π_R on left and right spinors are given in terms of the chirality χ which involves the imaginary unit i:

$$\Pi_L = \tfrac{1}{2}(1-\chi),\ \Pi_R = \tfrac{1}{2}(1+\chi),\ \chi = i\omega_{s,t}\,,$$

$$\chi^2 = \mathbb{I} \Leftrightarrow \Pi_L^2 = \Pi_L,\ \Pi_R^2 = \Pi_R,\ \Pi_L\Pi_R = 0,\ \Pi_L + \Pi_R = \mathbb{I}\,. \tag{3.3}$$

Another interesting example of the same equivalence class (also with indefinite metric) is the *conformal Clifford algebra* $C\ell(4,2)$ (with isometry group $O(4,2)$). We shall demonstrate that just as $C\ell_{-6}$ was viewed (in Sect. 2) as the *Clifford algebra of the octonions*, $C\ell(4,2)$ plays the role of the *Clifford algebra of the split octonions* (cf. (1.6)):

$$x = v + V + \ell(w + W),\ v, w \in \mathbb{R},\ V = iV_1 + jV_2 + kV_3,\ W = iW_1 + jW_2 + kW_3$$

$$i^2 = j^2 = k^2 = ijk = -1,\ \ell^2 = 1,\ V\ell = -\ell V\,. \tag{3.4}$$

[7]For any associative ring $\mathbb{K}$, in particular, for the division rings $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we denote by $\mathbb{K}[m]$ the algebra of $m \times m$ matrices with entries in $\mathbb{K}$.

Indeed, defining the mapping

$$i \to \gamma_{-1},\ j \to \gamma_0,\ \ell \to \gamma_1,\ j\ell \to \gamma_2,\ \ell k \to \gamma_3,\ \ell i \to \gamma_4$$

$$[\gamma_\mu, \gamma_\nu]_+ = 2\eta_{\mu\nu}\mathbb{I},\ \eta_{11} = \eta_{22} = \eta_{33} = \eta_{44} = 1 = -\eta_{-1,-1} = -\eta_{00} \tag{3.5}$$

we find that the missing split-octonion (originally, quaternion) imaginary unit k $(= ij = -ji)$ can be identified with the $C\ell(4, 2)$ pseudoscalar:

$$\omega_{4,2} = \gamma_{-1}\,\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3\,\gamma_4 \to k,\ \omega_{4,2}^2 = -1,\ [w_{4,2}, \gamma_\nu]_+ = 0\,. \tag{3.6}$$

The conjugate to the split octonion x (3.4) and its norm are

$$x^* = v - V - \ell(w + W),\ N(x) = xx^* = v^2 + V^2 - w^2 - W^2$$

so that the isometry group of $\widetilde{\mathbb{O}}$ is $O(4, 4)$.

As we are interested in the geometry of the internal space of the SM, acted upon by a compact gauge group we shall work with (positive or negative) definite Clifford algebras $C\ell_{2\ell},\ \ell = 1(\mathrm{mod}\,4)$. The algebra $C\ell_{-6}$, considered in Sect. 2, belongs to this family (with $\ell = -3$). For $\ell = 1$ we obtain the Clifford algebra of 2-dimensional conformal field theory; the 1-dimensional Weyl spinors correspond to analytic and antianalytic functions. Here we shall argue that for the next allowed value, $\ell = 5$, the algebra $C\ell_{10} = C\ell_4 \hat{\otimes} C\ell_6$ (1.9), fits beautifully the internal space of the SM, if we associate the two factors to colour and flavour degrees of freedom, respectively. We shall strongly restrict the physical interpretation of the generators $\gamma_{ab}\left(= \frac{1}{2}[\gamma_a, \gamma_b],\ a, b = 1, \cdots, 10\right)$ of the derivations of $C\ell_{10}$ by demanding that the splitting (1.9) of $C\ell_{10}$ into $C\ell_4$ and $C\ell_6$ is preserved. This reflects the demand of preserving the lepton-quark splitting (1.1) and amounts to select a first step of symmetry breakings of the GUT group $Spin(10)$ leading to the semisimple Pati-Salam group $(Spin(4) \times Spin(6))/\mathbb{Z}_2$ (1.8). Furthermore, recalling the discussion of Sect. 2, we identify the first seven γ_α with multiples of the left imaginary units L_α.

3.2 Realization in terms of Fermi oscillators

We start with a basis of γ-matrices adapted to the chain of subalgebras (2.19):

$$\gamma_\alpha = \sigma_0 \otimes \epsilon \otimes L_\alpha,\ \sigma_0 = \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \epsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \alpha = 1, \cdots, 7,$$

$$\gamma_8 = \sigma_0 \otimes \sigma_1 \otimes \mathbb{I}_8,\ \gamma_9 = \sigma_2 \otimes \sigma_3 \otimes \mathbb{I}_8,\ \gamma_{10} = \sigma_1 \otimes \sigma_3 \otimes \mathbb{I}_8, \tag{3.7}$$

σ_k being the 2×2 hermitian Pauli matrices. The internal space algebra $C\ell_4 \hat{\otimes} C\ell_6$ is most suggestively expressed in terms of Fermi oscillators [F] setting (in the notation of [T21]):

$$\tfrac{1}{2}(\gamma_1 + i\gamma_3) = b_1,\ \left(\tfrac{1}{2}(\gamma_1 - i\gamma_3) = b_1^*\right),$$
$$\tfrac{1}{2}(\gamma_2 + i\gamma_6) = b_2,\ \tfrac{1}{2}(\gamma_4 + i\gamma_5) = b_3$$
$$\Longrightarrow i\gamma_{13} = [b_1^*, b_1],\ i\gamma_{26} = [b_2^*, b_2],\ i\gamma_{45} = [b_3^*, b_3]\ \left(\gamma_{jk} = \frac{1}{2}[\gamma_j, \gamma_k]\right); \tag{3.8}$$

$$\gamma_7 = a_2 + a_2^*,\ i\gamma_8 = a_2 - a_2^*;\ \gamma_9 = a_1 + a_1^*,\ i\gamma_{10} = a_1 - a_1^*;$$
$$[a_\alpha,\ a_\beta^*]_+ = \delta_{\alpha\beta},\ [b_j, b_k^*]_+ = \delta_{jk},\ [a_\alpha^{(*)}, b_j^{(*)}]_+ = 0\,. \tag{3.9}$$

We shall use five pairs of commuting orthogonal projections:

$$\pi_\alpha = a_\alpha a_\alpha^*,\ \pi'_\alpha = a_\alpha^* a_\alpha = 1 - \pi_\alpha,\ \alpha = 1,2;\ p_j = b_j b_j^* = 1 - p'_j,\ j = 1,2,3, \tag{3.10}$$

α $(= 1,2)$ and j $(= 1,2,3)$ playing the role (roughly) of flavour and colour indices, respectively. In fact, the weak hypercharge Y involves both:

$$\frac{1}{2}Y = \frac{1}{3}\sum_{j=1}^{3} b_j^* b_j - \frac{1}{2}\sum_{\alpha=1}^{2} a_\alpha^* a_\alpha = \frac{1}{3}(p'_1 + p'_2 + p'_3) - \frac{1}{2}(\pi'_1 + \pi'_2) =$$
$$= \frac{1}{2}(\pi_1 + \pi_2) - \frac{1}{3}(p_1 + p_2 + p_3)\,. \tag{3.11}$$

The left and right chiral (weak) isospin components are expressed entirely in terms of $a_\alpha^{(*)}$:

$$I_+^L = a_1^*\, a_2,\ I_-^L = a_2^*\, a_1,\ [I_+^L, I_-^L] = 2I_3^L = \pi'_1\pi_2 - \pi_1\pi'_2 = \pi'_1 - \pi'_2\,;$$
$$I_+^R = a_2\, a_1,\ I_-^R = a_1^*\, a_2^*,\ [I_+^R, I_-^R] = 2I_3^R = \pi_1\pi_2 - \pi'_1\pi'_2 = \pi_2 - \pi'_1\,. \tag{3.12}$$

We note that the projection on non-zero left and right isospin are mutually orthogonal:

$$P_1 := (2I_3^L)^2 = \pi'_1\pi_2 + \pi_1\pi'_2 (= P_1^2),\ P'_1 := (2I_3^R)^2 = \pi_1\pi_2 + \pi'_1\pi'_2 (= (P'_1)^2)\,,$$
$$P_1 P'_1 = 0,\ P_1 + P'_1 = 1\,. \tag{3.13}$$

The generators of $su(3)_c$, on the other hand, are written in terms of $b_j^{(*)}$:

$$T_a = \tfrac{1}{2} b^* \lambda_a\, b,\ \lambda_a \in \mathcal{H}_3(\mathbb{C}),\ \mathrm{tr}\,\lambda = 0,\ \mathrm{tr}\,\lambda_a\,\lambda_b = 2\delta_{ab},\ a,b = 1,\cdots,8\,. \tag{3.14}$$

The $u(1)$ generator (corresponding to C_1 (2.17)) is a multiple of $B - L$ (2.21)

$$B - L = \frac{i}{3}(\gamma_{13} + \gamma_{26} + \gamma_{45}) = \frac{1}{3}\sum_{j=1}^{3}[b_j^*, b_j] = \frac{1}{3}\sum_j (p'_j - p_j)\,. \tag{3.15}$$

The states of the fundamental (anti)fermions are given by the primitive idempotents of $C\ell_{10}$, represented by the $2^5 = 32$ different products of the five pairs of basic projectors $\pi_\alpha^{(\prime)}, p_j^{(\prime)}$ (3.10). All but two of them are labelled by the eigenvalues of the weak hypercharge $Y = B - L + 2I_3^R$ (3.11) and the electric charge

$$Q = \frac{1}{2}Y + I_3^L = \frac{1}{3}\sum_{j=1}^{3} b_j^* b_j - a_2^*\, a_2 = \frac{1}{3}(p_1' + p_2' + p_3') - \pi_2'\,. \qquad (3.16)$$

Setting $|Q,Y\rangle$ and $\langle Q,Y|$ for the corresponding ket and bra vectors we find:

$$\begin{aligned}
(\nu_L) &= \ell\, \pi_1'\pi_2 = |0,-1\rangle\langle 0,-1| = |\nu_L\rangle\langle\nu_L|\,,\\
(e_L) &= \ell\, \pi_1\pi_2' = |-1,-1\rangle\langle -1,-1| = |e_L\rangle\langle e_L|\,,\ \ \ell := p_1\, p_2\, p_3\,;\\
(e_R) &= \ell\, \pi_1'\pi_2' = |-1,-2\rangle\langle -1,-2| = |e_R\rangle\langle e_R|\,; \qquad (3.17)
\end{aligned}$$

$$\begin{aligned}
(u_L^j) &= q_j\, \pi_1'\pi_2 = |\tfrac{2}{3},\tfrac{1}{3}\rangle\langle\tfrac{2}{3},\tfrac{1}{3}| = |u_L^j\rangle\langle u_L^j|\,,\ \ q_1 = p_1\, p_2'\, p_3' (= p_1\, p_3'\, p_2') \text{ etc.}\\
(d_L^j) &= q_j\, \pi_1\pi_2' = |-\tfrac{1}{3},\tfrac{1}{3}\rangle\langle -\tfrac{1}{3},\tfrac{1}{3}| = |d_L^j\rangle\langle d_L^j|\,; j = 1,2,3,\\
(u_R^j) &= q_j\, \pi_1\pi_2 = |\tfrac{2}{3},\tfrac{4}{3}\rangle\langle\tfrac{2}{3},\tfrac{4}{3}| = |u_R^j\rangle\langle u_R^j|\,,\\
(d_R^j) &= q_j\, \pi_1'\pi_2' = |-\tfrac{1}{3},-\tfrac{2}{3}\rangle\langle -\tfrac{1}{3},-\tfrac{2}{3}| = |d_R^j\rangle\langle d_R^j|\,,\\
q_j &= p_j\, p_k'\, p_\ell' \text{ for } (j,k,\ell) \in \text{Perm}(1,2,3)\,, \qquad (3.18)
\end{aligned}$$

where j stands for the colour label. (As the colour is unobservable we do not bother to assign to it eigenvalues of the diagonal operators $i\gamma_{13}, i\gamma_{26}, i\gamma_{45}$.)

Remark. – The factorisation of the primitive idempotents (3.17) (3.18) into bra and kets include choices. We demand, following [T21], that they are hermitian conjugate elements of $C\ell_{10}$, homogeneous in $a_\alpha^{(*)}$ and $b_j^{(*)}$ such that the kets corresponding to a left(right)chiral *particle* contains an odd (respectively even) number of factors. The result is:

$$\begin{aligned}
|\nu_R\rangle &= \ell\, \pi_1\pi_2 (= \langle\nu_R| = (\nu_R))\,,\ |\nu_L\rangle = a_1^*|\nu_R\rangle = a_1^*\pi_2\, \ell\,,\\
|e_L\rangle &= I_-^L|\nu_L\rangle = \pi_1 a_2^*\, \ell\,,\ |e_R\rangle = a_1^*|e_L\rangle = a_1^* a_2^*\, \ell\,;\\
|d_L^j\rangle &= \pi_1 a_2^*\, q_j\,,\ |u_L^j\rangle = I_+^L|d_L^j\rangle = a_1^*\pi_2\, q_j\,,\\
|d_R^j\rangle &= a_1^*|d_L^j\rangle = a_1^* a_2^*\, q_j\,,\ u_R^j = a_1|u_L^j\rangle = \pi_1\pi_2\, q_j\,, \qquad (3.19)
\end{aligned}$$

$q_j = p_j\, p_k'\, p_\ell'$, $j,k,\ell \in \text{Perm}(1,2,3)$, i.e. $q_1 = p_1\, p_2'\, p_3' = p_1\, p_3'\, p_2'$ etc. We note that all above kets as well as all primitive idempotents (3.18) obey a system of 5 equations (specific for each particle), $a_\alpha|\nu_R\rangle = 0 = b_j|\nu_R\rangle$, $a_1^*|\nu_L\rangle = a_2|\nu_L\rangle = 0 = b_j|\nu_L\rangle$, $\alpha = 1,2$, $j = 1,2,3$, etc. so that they are minimal rigth ideals in accord with the philosophy of Furey [F16].

The exceptional pair consists of the right handed sterile neutrino ν_R and its antiparticle $\overline{\nu}_L$, both with $Q = 0 = Y$. They could be distinguished by introducing a third quantum number, I_3^R or $B - L$,

$$2I_3^R = L - B \ (= 1 \text{ for } \nu_R \text{ and } -1 \text{ for } \overline{\nu}_L).$$

It is argued in [T21] that, if the generator of the centre $\frac{1}{2}Y$ (3.11) of the gauge Lie algebra of the SM is superselected, [WWW], chiral particles and antiparticles are mandatory separated iff $Y \neq 0$. The sterile neutrino and its antiparticle (both with $Y = 0$) can mix (as they do in the popular theory of neutrino oscillations) into a Majorana neutrino. We shall return to the implications of this assumption in Sect. 4 below. Here we shall stay with the majority's convention and include the right handed (sterile) neutrino ν_R, such that

$$(2I_3^R - 1)|\nu_R\rangle = 0 \ (= Y|\nu_R\rangle = Q|\nu_R\rangle)\,, \tag{3.20}$$

in the list of 16 particle states. The corresponding list of antiparticle projectors is obtained by exchanging primed and unprimed π_α and p_j, reversing the signs of Q, Y (and I_3^R) and exchanging left and right. The sum of four flavours (3.17) and (3.20) of leptons and (3.18) of quarks gives the 4-dimensional projector ℓ on leptons and the 12 dimensional projector q on coloured quarks:

$$\ell = (\nu_L) + (e_L) + (\nu_R) + (e_R) = p_1\,p_2\,p_3,\ \ell^2 = \ell,\ \mathrm{tr}\,\ell = 4\,; \tag{3.21}$$

$$q_j = (u_L^j) + (d_L^j) + (u_R^j) + (d_R^j) = p_j\,p_k'\,p_\ell',\ q_i\,q_j = \delta_{ij}q_j,\ \mathrm{tr}\,q_j = 4\,;$$

$$(j,k,\ell) \in \mathrm{Perm}(1,2,3),\ q = q_1 + q_2 + q_3 = q^2,\ \mathrm{tr}\,q = 12\,. \tag{3.22}$$

3.3 Expressing the $C\ell_6$ pseudoscalar in terms of (anti)particle projectors

We now proceed to displaying a remarkable relation between the total particle and antiparticle projectors

$$\mathcal{P} = \ell + q,\ \mathcal{P}' = \ell' + q' \quad \mathcal{P}^{(')2} = \mathcal{P}^{(')}, \mathcal{P}\mathcal{P}' = 0,\ \mathcal{P} + \mathcal{P}' = \mathbb{I}_{32}$$
$$\ell' = p_1'\,p_2'\,p_3'\,,\ q' = p_1'\,p_2\,p_3 + p_1\,p_2'\,p_3 + p_1\,p_2\,p_3'\,, \tag{3.23}$$

and the $C\ell_6$ counterpart of the complex structure L_7 (2.11), proposed as a first step in the sequence of symmetry breakings of the $Spin(10)$ GUT in [FH].

We define the $C\ell_6$ pseudoscalar in the graded tensor product (1.9) by

$$\omega_6 = \gamma_1\gamma_2\cdots\gamma_6 = -\gamma_{13}\,\gamma_{26}\,\gamma_{45} = \sigma_0 \otimes \epsilon^6 \otimes L_7 = -\mathbb{I}_4 \otimes L_7$$
$$\gamma_{jk} = \tfrac{1}{2}[\gamma_j, \gamma_k]\,,\ L_7 = L_1 \cdots L_6\,, \tag{3.24}$$

implying (in view of (3.8))

$$i\omega_6 = (p_1' - p_1)(p_2' - p_2)(p_3' - p_3) = \mathcal{P}' - \mathcal{P}((\mathcal{P}' - \mathcal{P})^2 = \mathcal{P}' + \mathcal{P} = \mathbb{I}_{32})\,. \tag{3.25}$$

We thus find that the $C\ell_6$ pseudoscalar complex structure ω_6 gives rise to the projector

$$\mathcal{P} = \frac{1 - i\omega_6}{2} \quad (\mathcal{P}^2 = \mathcal{P},\ tr\,\mathcal{P} = 16) \tag{3.26}$$

on the particle subspace, invariant under the Pati-Salam group $G_{\rm PS}$ (1.8), which preserves the splitting (1.9).

If we omit the first factor σ_0 (the 2×2 unit matrix) from γ_a for $a = 1, \cdots, 8$, (3.2), we obtain an irreducible representation of $C\ell_8$. We keep the same Fermi oscillator realization (3.8) for the $C\ell_8$ γ-matrices, so that, in particular

$$i\gamma_{13} = [b_1^*, b_1] = p_1' - p_1\,,\; i\gamma_{26} = [b_2^*, b_2] = p_2' - p_2\,,\; \gamma_{45} = [b_3^*, b_3] = p_3' - p_3\,. \quad (3.27)$$

Thus $i\omega_6$ is given by the same expression (3.20) for $C\ell_8$ (but with $tr\,\mathcal{P} = 8$) and for $C\ell_9$ but has a smaller invariance Lie algebra

$$u(4) = su(4) \oplus u(1) \subset so(8) \text{ for } C\ell_8\,; \quad su(4) \oplus su(2) \subset so(9) \text{ for } C\ell_9\,. \quad (3.28)$$

Inspired by [K21, FH] we shall display in both cases the complex structure given by the Clifford pseudoscalar corresponding to the right action of the octonions:

$$\omega_6^R = \gamma_1^R \cdots \gamma_6^R \text{ for } \gamma_\alpha^R = \epsilon \otimes R_\alpha \quad \alpha = 1, \cdots, 7\,. \quad (3.29)$$

We shall view, following [FH], its invariance group, $G_{\rm LR}$, as the second of the nested subgroups of $Spin(10)$: $(Spin(10) \supset)\, G_{\rm PS} \supset G_{\rm LR} \cdots \supset G_{\rm SM} \cdots$ in the sequence of consecutive symmetry breakings. Written in terms of the colour projectors p_j and p_j' the hermitian pseudoscalar $i\omega_6^R$ assumes the form:

$$i\omega_6^R = \tfrac{1}{2}(\mathcal{P}' - \mathcal{P} - 3(B-L)) = \ell + q' - \ell' - q\,, \quad (3.30)$$

since

$$L = \ell - \ell'\,,\; 3B = q - q'\,. \quad (3.31)$$

While the term $\mathcal{P}' - \mathcal{P}$ (3.25) commutes with the entire derivation algebra $spin(6) = su(4)$ of $C\ell_6$ the centralizer of $B-L$ in $su(4)$ is $u(3)$ – see Proposition A2 in Appendix A. It follows that the commutant of ω_6^R in $so(8)$ is $u(3) \oplus u(1)$ while its centralizer in $so(9)$ is the gauge Lie algebra $\mathcal{G}_{\rm SM} = su(3) + su(2) + u(1)$ of the SM; finally, in $so(10)$, ω_6^R is invariant under the left-right symmetric extension of $\mathcal{G}_{\rm SM}$:

$$\mathcal{G}_{\rm LR} = su(3)_c \oplus su(2)_L \oplus su(2)_R \oplus u(1)_{B-L}\,. \quad (3.32)$$

Furthermore, as proven in [K], the subgroup of $Spin(9)$ that leaves ω_6^R invariant is precisely the gauge group[8] $G_{\rm SM} = S(U(2) \times U(3))$ (1.10) of the SM (with the appropriate $\mathbb{Z}_6$ factored out). One is then tempted to assume that $C\ell_9$, the associative envelope of the Jordan algebra $J_2^8 = \mathcal{H}_2(\mathbb{O})$, may play the role of the internal algebra of the SM, corresponding to one generation of fundamental fermions, with $Spin(9)$ as a GUT group [TD, DT]. We shall demonstrate that although $G_{\rm SM}$ appears as a subgroup of $Spin(9)$ its representation, obtained by restricting the (unique) spinor irreducible representation (IR) **16** of $Spin(9)$ to $S(U(2) \times U(3))$ only involves $SU(2)$ doublets, so it has no room

[8] As noted in the introduction the correct $G_{\rm SM}$ was earlier obtained as the stabilizer of the automorphism ω of order 3 (see (1.12), (1.13)).

for $(e_R),(u_R),(d_R)$ (3.17) (3.18). We shall see how this comes about when restricting the realization (3.12) of $\mathbf{I}^L$ and $\mathbf{I}^R$ to $Spin(9) \subset C\ell_9$. It is clear from (3.9) that only the sum $a_1 + a_1^* = \gamma_9$ (not a_1 and a_1^* separately) belongs to $C\ell_9$. So the $su(2)$ subalgebra of $spin(9)$ corresponds to the diagonal embedding $su(2) \hookrightarrow su(2)_L \oplus su(2)_R$:

$$I_+ = I_+^L + I_+^R = (a_1^* + a_1)\, a_2 = \gamma_9\, a_2, \;\; I_- = I_-^L + I_-^R = a_2^* \gamma_9$$

$$2I_3 = 2I_3^L + 2I_3^R = [a_2, a_2^*] = \pi_2 - \pi_2' \,. \tag{3.33}$$

In other words the spinorial IR **16** of $Spin(9)$ is an eigensubspace of the projector $P_1 = (2I_3^L)^2$. It consists of four $SU(2)_L$ particle doublets and of their right chiral antiparticles. More generally, as recalled in the introduction the only simple orthogonal groups with a pair of inequivalent complex conjugate fundamental IRs, are $Spin(4n+2)$. They include $Spin(10)$ but not $Spin(9)$.

There is one more pseudoscalar, ω_4, associated with the first factor, $C\ell_4$, of the tensor product (1.9):

$$\omega_4 = \gamma_7\, \gamma_8\, \gamma_9\, \gamma_{10} = [a_1, a_1^*][a_2^*, a_2] = P_1 - P_1' \,, \tag{3.34}$$

$P_1 = \pi_1'\pi_2 + \pi_1\pi_2'$ is the projector (3.13) on the subspace with $(2I_3^L)^2 = 1$ and $P_1' = \pi_1\pi_2 + \pi_1'\pi_2'$ is its orthogonal complement. (We have $\omega_4^2 = 1$; such a ω_4 is called a pseudo complex structure.)

The $C\ell_{10}$ pseudoscalar $\omega_{10} = \omega_6\, \omega_4$ defines the ($spin(10)$ invariant) *chirality*

$$\chi = i\omega_{10} = i\omega_6\, \omega_4 = (\mathcal{P}' - \mathcal{P})(P_1 - P_1') = \Pi_R - \Pi_L \,. \tag{3.35}$$

It gives rise to the projector

$$\Pi_L = \frac{1-\chi}{2} = \mathcal{P} P_1 + \mathcal{P}' P_1^1 \tag{3.36}$$

on the left chiral particles (four $SU(2)_L$ doublets) and the 8 antiparticles (the conjugates to the eight right chiral $SU(2)_L$-singlets).

A direct description of the IR $\mathbf{16}_L$ of $Spin(10)$ acting on $\mathbb{CH} \otimes \mathbb{CO}$ is given in [FH1]. (Here $\mathbb{CH}$ and $\mathbb{CO}$ are a short hand for the complexified quaternions and octonions: $\mathbb{CH} := \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$.) The right action of $\mathbb{CH}$ on elements of $\mathbb{CH} \otimes \mathbb{CO}$ which commutes with the left acting $spin(10)$, is interpreted in [FH1] as Lorentz ($SL(2,\mathbb{C})$) transformation of (unconstrained) 2-component Weyl spinors.

The left-right symmetric exension $\mathcal{G}_{\rm LR}$ (3.32) of $\mathcal{G}_{\rm SM}$ has a long history, starting with [MP] and vividly (with an admitted bias) told in [S17]. It has been recently invigurated in [HH, DHH]. The group $G_{\rm LR}$ was derived by Boyle [B] starting with the automorphism group E_6 of the complexified exceptional Jordan algebra $\mathbb{C}J_3^8$ and following the procedure of [TD-V].

4 Particle subspace and the Higgs field

4.1 Particle projection and chirality

Theories whose field algebra is a tensor product of a Dirac spinor bundle on a spacetime manifold with a finite dimensional "quantum" internal space usually encounter the problem of fermion doubling [GIS] (still discussed over 20 years later, [BS]). It was proposed in [DT20] as a remedy to consider the algebra $\mathcal{P}C\ell_{10}\mathcal{P}$ where $\mathcal{P}$ is the projector (3.18) on the 16 dimensional particle subspace (including the hypothetical right-handed sterile neutrino). The resulting subspace is $\mathbb{Z}_2$ graded by the chirality operator separating left and right chiral *particles* (with antiparticles projected out):

$$\chi_{\mathcal{P}} = i\omega_{10}\,\mathcal{P} = \mathcal{P}(\Pi_R - \Pi_L)\,, \quad \mathcal{P}\Pi_L = \mathcal{P}P_1\,, \tag{4.1}$$

where P_1 (3.13) projects on $SU(2)_L$ doublets. The Dirac operator $\not{D} = \gamma^\mu D_\mu$ ($D_\mu = \partial_\mu + A_\mu$) anticommutes with space-time chirality $\gamma_5 = i\,\gamma^1\gamma^2\gamma^3\gamma^0$ and hence intertwines – like the Higgs field – left and right chiral spinors. This has motivated Connes and coworkers [C, CL, CC] to introduce an internal space Dirac operator in the framework of noncommutative (almost commutative) geometry that involves the Higgs field. Following the pioneering work of Ne'eman and Fairlie [N, F79], Thierry-Mieg and Ne'eman [T-MN] developed effectively a superconnection approach to the SM, prior to its introduction (and naming) in mathematics [Q]. (For later reviews and more references – see [R, BMV, T-M].) The Clifford algebra approach with the chirality operator $\chi_{\mathcal{P}}$ (4.1), developed in [DT20] appears to be ideally suited for a geometric interpretation of the Higgs field. (An alternative approach to internal space connection involving scalar fields is been pursued by Dubois-Violette and coworkers for over thirty years [DKM, D-V, D21].) It turns out that there is another unanticipated benefit in introducing the projector $\mathcal{P}$: it kills odd polynomials of colour carrying Fermi operators:

$$\mathcal{P}b^{(*)}\mathcal{P} = 0 \ (= \mathcal{P}C\ell_6^1\,\mathcal{P}) \quad \text{for} \quad \omega_6\, C\ell_6^1 = -C\ell_6^1\,\omega^6 \tag{4.2}$$

while projecting a^*_α into non-zero odd elements:

$$\mathcal{P}a_\alpha^{(*)}\mathcal{P} = \mathcal{P}a_\alpha^{(*)} = a_\alpha^{(*)}\mathcal{P}\,,\ [\mathcal{P}a_\alpha, \mathcal{P}a^*_\beta]_+ = \delta_{\alpha\beta}\,\mathcal{P}\,. \tag{4.3}$$

One may thus place the Higgs field in the odd part, $C\ell_4^1$, of the first factor $C\ell_4$ of the product (1.4) and hence mediate the breaking of the electroweak flavour symmetry without affecting the quark colour $SU(3)_c$ symmetry which is known to be exact. While the odd part $C\ell_6^1$ of $C\ell_6$ maps the particle subspace into its orthogonal complement the $u(3)$ generators $\frac{1}{2}[b^*_j, b_k] \in C\ell_6^0$ are projected onto non-zero elements of $C\ell_6^0$ obeying the same CRs; in particular, for (j,k,ℓ) a permutation of $(1,2,3)$ we have

$$\mathcal{P}\,b^*_j\,b_k\,\mathcal{P} = q_k\,b^*_j\,b_k\,q_j = b^*_j\,b_k\,p'_\ell =: B_{jk} \Rightarrow [B_{jk}, B_{k\ell}] = B_{j\ell}\,. \tag{4.4}$$

4.2 The Higgs as a scalar part of a superconnection

Let D be the Yang-Mills connection 1-form of the SM,

$$D = dx^\mu(\partial_\mu + A_\mu(x))\,,$$

$$iA_\mu = W_\mu^+ I_+^L + W_\mu^- I_-^L + W_\mu^3 I_3^L + \frac{N}{2} Y B_\mu + G_\mu^a T_a\,, \tag{4.5}$$

where $Y, \mathbf{I}^L$ and T_a are given by (3.11), (3.12) and (3.14), respectively, G_μ^a is the gluon field, $\mathbf{W}_\mu$ and B_μ provide an orthonormal basis of electroweak gauge bosons. Then one defines a superconnection $\mathbb{D}$ by

$$\mathbb{D} = \chi D + \Phi\,, \quad \Phi = \sum_\alpha (\phi_\alpha\, a_\alpha^* - \overline{\phi}_\alpha\, a_\alpha)\,. \tag{4.6}$$

(We omit, for the time being, the projector $\mathcal{P}$ in A_μ and Φ.) The factor χ (first introduced in this context in [T-M]) insures the anticommutativity of Φ and $\chi\mathcal{D}$ without changing the Yang-Mills curvature $D^2 = (\chi D)^2$.

The projector $\mathcal{P}$ (3.23) on the 16 dimensional particle subspace that includes the hypothetical right chiral neutrino (and is implicit in (4.6)) was adopted in [DT20]. By contrast, particles are only distinguished from antiparticles in [T21] if they have different quantum numbers with respect to the Lie algebra of the SM. In fact, $\mathcal{G}_{\rm SM} = s(u(2)\oplus u(3))$ is precisely the Lie subalgebra of $\mathcal{G}_{\rm LR}$ (3.29) which annihilates the sterile (anti)neutrino:

$$\mathcal{G}_{\rm SM} = \{\alpha \in \mathcal{G}_{\rm LR}; \alpha(\nu_R) = 0 = \alpha(\overline{\nu}_L)(= \alpha(a_1 a_2\, b_1 b_2\, b_3 + b_3^*\, b_2^*\, b_1^* a_2^*\, a_1^*))\,. \tag{4.7}$$

Thus, in [T21] $\mathcal{P}$ is restricted to the 15-dimensional projector $\mathcal{P}_r$ on the *restricted particle space*:

$$\mathcal{P}_r = \mathcal{P} - (\nu_R) = q + \ell_r\,, \quad \ell_r = \ell(1 - \pi_1\pi_2)\,. \tag{4.8}$$

The projected odd operators $a_\alpha^{(*)}$ in the lepton sector,

$$\ell_r\, a_\alpha\, \ell_r = \ell(1-\pi_1\pi_2)\, a_\alpha,\ \ \ell_r\, a_\alpha^*\, \ell_r = \ell\, a_\alpha^*(1-\pi_1\pi_2) \Rightarrow$$

$$\ell_r\, a_1\, \ell_r = \ell\, a_1 \pi_2',\ \ \ell_r\, a_2\, \ell_r = \ell\, a_2\, \pi_1',\ \ \ell_r\, a_1^*\, \ell_r = \ell\, a_1^* \pi_2',\ \ \ell_r\, a_2^*\, \ell_r = \ell\, a_2^*\, \pi_1'\,, \tag{4.9}$$

have modified anticommutation relations. In fact, they provide a realization of the four odd elements of the 8-dimensional simple Lie superalgebra $s\ell(2|1)$ whose even part is the 4-dimensional Lie algebra $u(2)$ of the Weinberg-Salam model of the electroweak interactions (see [T21] for details). It is precisely the Lie superalgebra proposed in 1979 independently by Ne'eman and by Fairlie [N, F79] (and denoted by them $su(2|1)$) in their attempt to unify $su(2)_L$ with $u(1)_Y$ (and explain the spectrum of the weak hypercharge). Let us stress that the representation space of $s\ell(2|1)$ consists of the observed left and right chiral leptons (rather than of bosons and fermions like in the popular speculative theories in which the superpartners are hypothetical). Note in passing that the trace of Y on negative chirality leptons (ν_L, e_L) is equal to its eigenvalue on the unique positive chirality (e_R) (equal to -2) so that only the supertrace of Y vanishes on the lepton (as well as on the quark) space. This observation is useful in the treatment of anomaly cancellation (cf. [T-M20]).

We shall sketch the main steps in the application of the superconnection (4.6) to the bosonic sector of the SM emphasizing specific additional hypotheses used on the way (for a detailed treatment see [T21]).

The canonical curvature form

$$\mathbb{D}^2 = D^2 + \chi[D, \Phi] + \Phi^2,\ [D, \Phi] = dx^\mu(\partial_\mu \Phi^* [A_\mu, \Phi]) \tag{4.10}$$

satisfies the *Bianchi identity*

$$\mathbb{D}\mathbb{D}^2 = \mathbb{D}^2\mathbb{D}\ (\Rightarrow \chi(d\Phi^2 + [A, \phi^2] + [\Phi, D\Phi]_+) = 0)\,, \tag{4.11}$$

equivalent to the (super) Jacobi identity of our Lie superalgebra. It is important that the Bianchi identity, needed for the consistency of the theory still holds if we add to $\mathbb{D}^2$ a constant matrix term with a similar structure. Without such a term the Higgs potential would be a multiple of $\mathrm{Tr}\,\Phi^4$ and would only have a trivial minimum at $\Phi = 0$ yielding no symmetry breaking. The projected form of Φ (4.6) and hence the admissible constant matrix addition to Φ^2 depends on whether we use the projector $\mathcal{P}$ (as in [DT20]) or P_r (as in [T21]). In the first case we just replace $a_\alpha^{(*)}$ with $a_\alpha^{(*)}\mathcal{P}$. In the second, however, the odd generators for leptons and quarks differ and we set:

$$\Phi = \ell[(\phi_1 a_1^* - \overline{\phi}_1 a_1)\pi_1' + (\phi_2\, a_2^* - \overline{\phi}_2\, a_2)\pi_2'] + \rho q \sum_{\alpha=1}^{2} (\phi_\alpha\, a_\alpha^* - \overline{\phi}_\alpha\, a_\alpha)\,, \tag{4.12}$$

where ρ (like N in (4.5)) is a normalization constant that will be fixed later. Recalling that ℓ and q are mutually orthogonal ($\ell q = 0 = q\ell,\ \ell + q = \mathcal{P}$) we find

$$\begin{aligned}\Phi^2 = {} & \ell(\phi_1\overline{\phi}_2\, I_+^L + \overline{\Phi}_1\phi_2\, I_-^L - \phi_1\overline{\phi}_1\pi_2' - \phi_2\,\overline{\phi}_2\pi_1') \\ & -\rho^2 q(\phi_1\overline{\phi}_1 + \phi_2\,\overline{\phi}_2)\ (\phi_\alpha = \phi_\alpha(x))\,.\end{aligned} \tag{4.13}$$

This sugggests defining the SM field strength (the extended curvature form) as

$$\mathbb{F} = i(\mathbb{D}^2 + \widehat{m}^2)\,, \quad \widehat{m}^2 = m^2(\ell(1 - \pi_1\,\pi_2) + \rho^2 q) \tag{4.14}$$

($\widehat{m}^2 = m^2\mathcal{P}$ for the 16 dimensional particle subspace of [DT20]).

4.3 Higgs potential and mass formulas

This yields the bosonic Lagrangian

$$\mathcal{L}(x) = \mathrm{Tr}\left\{\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} - (\partial_\mu\Phi + [A_\mu, \Phi])(\partial^\mu\Phi + [A^\mu, \Phi])\right\} - V(\Phi) \tag{4.15}$$

where the Higgs potential $V(\Phi)$ is given by

$$V(\Phi) = \mathrm{Tr}\,(\widehat{m}^2 + \Phi^2)^2 - \tfrac{1}{4}m^4 = \tfrac{1}{2}(1+6\rho^4)(\phi\overline{\phi} - m^2)^2\,. \qquad (4.16)$$

Minimizing $V(\Phi)$ gives the expectation value of the square of $\phi = (\phi_1, \phi_2)$:

$$\langle \phi\,\overline{\phi}\rangle = \phi_1^m\,\overline{\phi_1^m} + \phi_2^m\,\overline{\phi_2^m} = m^2, \text{ for } \Phi^m = \sum_{\alpha=1}^{2} \phi_a^m a_\alpha^* (\ell\pi'_{3-\alpha} + \rho q) + c\cdot c\,. \quad (4.17)$$

(The superscript m indicates that ϕ_α take constant in x values depending on the mass parameter m.) The mass spectrum of the gauge bosons is determined by the term $-\,\mathrm{Tr}\,[A_\mu, \Phi][A^\mu, \Phi]$ of the Lagrangian (4.15) with A_μ and Φ given by (4.5) and (4.17) for $\phi_\alpha = \phi_\alpha^m$. The gluon field G_μ does not contribute to the mass term as $C\ell_6^0$ commutes with $C\ell_4^1$. The resulting quadratic form is, in general, not degenerate, so it does not yield a massless photon. It does so however if we assume that Φ^m is electrically neutral (i.e. commutes with Q (3.16)):

$$[\Phi^m, Q] = 0 \Rightarrow \phi_2^m = 0\ (= \overline{\phi_2^m})\,. \qquad (4.18)$$

The normalization constant $N (= \mathrm{tg}\,\theta_w)$ is fixed by assuming that $2I_3^L$ and NY are equally normalized:

$$N^2 = \frac{\mathrm{Tr}\,(2I_3^L)^2}{\mathrm{Tr}\,Y^2} = \frac{3}{5}\left(= (\mathrm{tg}\,\theta_w)^2 \Rightarrow \sin^2\theta_w = \frac{3}{8}\right). \qquad (4.19)$$

As $Y(\nu_R) = 0 = I_3^L(\nu_R)$ this result for the "Weinberg angle at unification scale" is independent on whether we use $\mathcal{P}$ or $\mathcal{P}_r$. If one takes the trace over the leptonic subspace the result would have been $(\mathrm{tg}\,\theta_w)^2 = \frac{1}{3}(\Rightarrow \sin\theta_w = \frac{1}{2}$, [F79]) closer to the measured low energy value.

Demanding, similarly, that the leptonic contribution to Φ^2 is the same as that for a coloured quark (which gives $\rho = 1$ for the unrestricted projector $\mathcal{P}$) we find

$$\rho^2 = \frac{\mathrm{Tr}(\ell(1-\pi_1\pi_2)\Phi^2)}{\mathrm{Tr}\,q_j\,\Phi^2} = \frac{Tr(\pi'_1\pi'_2\,\phi\,\overline{\phi} + \pi'_1\pi_2\,\phi_2\,\overline{\phi}_2 + \pi_1\pi'_2\,\phi_1\,\overline{\phi}_1)}{4\,\phi\,\overline{\phi}} = \frac{1}{2}\,. \qquad (4.20)$$

The ratio $\frac{m_H^2}{m_W^2}$, on the other hand is found to be

$$\frac{m_H^2}{m_W^2} = 4\,\frac{1+6\rho^4}{1+6\rho^2} = \begin{cases} 4 \text{ for } \rho^2 = 1 \text{ ([N], [DT20])} \\ \frac{5}{2} \text{ for } \rho^2 = \frac{1}{2} \text{ ([T21])} \end{cases}. \qquad (4.21)$$

The result of [T21], much closer to the observed value, can also be written in the form $m_H^2 = 4\cos^2\theta_W\, m_H^2$, where θ_W is the theoretical Weinberg angle (4.19).

5 Outlook

5.1 Coming to $C\ell_{10}$

The search for an appropriate choice of a finite dimensional algebra suited to represent the internal space $\mathcal{F}$ of the SM is still going on. Our road to the choice of $C\ell_{10}$, adopted in this survey, has been convoluted.

In view of the lepton-quark correspondence which is embodied in the splitting (1.1) of the normed division algebra $\mathbb{O}$ of the octonions, the choice of Dubois-Violette [D16] of the exceptional Jordan algebra $\mathcal{F} = \mathcal{H}_3(\mathbb{O})$ (1.7) looked particularly attractive. We realized [TD, TD-V] that the simpler to work with subalgebra

$$J_2^8 = \mathcal{H}_2(\mathbb{O}) \subset \mathcal{H}_3(\mathbb{O}) = J_3^8 \tag{5.1}$$

corresponds to the observables of one generation of fundamental fermions. The associative envelope of J_2^8 is $C\ell_9 = \mathbb{R}[16] \oplus \mathbb{R}[16]$ with associated symmetry group $Spin(9)$. It was proven in [TD-V] that the SM gauge group $G_{\rm SM}$ (1.10) is the intersection of $Spin(9)$ with the subgroup F_4^ω (1.13) of the automorphism group F_4 of J_3^8 that preserves the splitting (1.1) of $\mathbb{O}$, yielding (1.12).

So we were inclined to identify $Spin(9)$ as a most economic GUT group. As demonstrated in Sect. 3.3, however, the restriction of the spinor IR **16** of $Spin(9)$ to its subgroup $G_{\rm SM}$ gives room to only half of the fundamental fermions: the $SU(2)_L$ doublets; the right chiral singlets, e_R, u_R, d_R, are left out. It was thus recognized that the Clifford algebra $C\ell_{10}$ (which also involves the octonions) does the job.

After a synopsis of the triality realization of $Spin(8)$ on the octonions (Sect. 2) the present survey starts directly with the (complexified) Clifford algebra $C\ell_{10}$ displaying in Sect. 3.1 its salient features which place it in the same equivalence family under the Cartan classification as the Lorentzian Clifford algebra $C\ell(3,1)$. The particle interpretation of $C\ell_{10}$ is dictated by the choice of a (maximal) set of five commuting operators in the derivation algebra $so(10)$ of $C\ell_{10}$. It follows the presentation of $C\ell_{10}$ by the $\mathbb{Z}_2$ graded tensor product (1.9),

$$C\ell_{10} = C\ell_6 \,\hat{\otimes}\, C\ell_4\,, \tag{5.2}$$

which is preserved by the Pati-Salam subgroup $G_{\rm PS}$ (1.8) of $Spin(10)$. This led us to presenting all chiral leptons and quarks of one generation as mutually orthogonal idempotents (3.17) (3.18).

Furay [F] arrived (back in 2018) at the tensor product (5.2) following the $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ road. In fact, Clifford algebras have arisen as an outgrow of Grassmann algebras and the quaternions[9]. The 32 products $e_a\varepsilon_\nu (= \varepsilon_\nu e_a)$, $a = 1, \cdots, 8$ ($e_8 = 1\!\!1$), $\nu = 0, 1, 2, 3$ of octonion and quaternion units may

[9]The Dublin Professor of Astronomy William Rowan Hamilton (1805-1865) and the Stettin Gymnasium teacher Hermann Günter Grassmann (1809-1877) published their papers, on quaternions and on "extensive algebras", respectively, in the same year 1844. William Kingdom Clifford (1845-1879) combined the two in a "geometric algebra" in 1878, a year before his death, aged 33, referring to both of them.

serve as components of a $Spin(10)$ Dirac (bi)spinor, acted upon by $C\ell_{10}$ (with generators (3.7) involving the operators L_α of left multiplication by octonion units) – cf. [FH1].

5.2 Two ways to avoid fermion doubling

There are two inequivalent possibilities to avoid fermion doubling within $C\ell_{10}$. One, adopted in [DT20, T21] and in Sect. 3 of the present survey consists in projecting on the particle subspace, which incorporates four $SU(2)_L$ doublets and eight $SU(2)_L$ (right chiral) singlets, with projector

$$\mathcal{P} = \ell + q = \frac{1 - i\omega_6}{2}\,,\ \ell = p_1\,p_2\,p_3\,,\ q = q_1 + q_2 + q_3 \tag{5.3}$$

(see (3.22), (3.23) and (3.25)). Here ω_6 is the $C\ell_6$ pseudoscalar, the distinguished complex structure, used in [FH] as a first step in the "cascade of symmetry breakings". The particle projector (5.3) is only invariant under the Pati-Salam subgroup (1.8) of $Spin(10)$. The more popular alternative, adopted in [FH1], projects on left chiral fermions (4 particle doublets and 8 antiparticle singlets) with projector (3.36), defined in terms of the $C\ell_{10}$ chirality $\chi = i\omega_{10}$:

$$\Pi_L = \frac{1-\chi}{2} = \mathcal{P}P_1 + \mathcal{P}'P_1' \quad (\mathcal{P} + \mathcal{P}' = 1 = P_1 + P_1')\,, \tag{5.4}$$

where P_1 projects on $SU(2)_L$ doublets, invariant under the entire $Spin(10)$. The components of the resulting $\mathbf{16}_L$ are viewed in [FH1] as Weyl spinors; the right action of (complexified) quaternions (which commutes with the left $spin(10)$ action) is interpreted as an $s\ell(2,\mathbb{C})$ (Lorentz) transformation.

The difference of the two approaches which can be labeled by the projectors $\mathcal{P}$ and Π_L (on left and right particles and on left particles and antiparticles, respectively) has implications in the treatment of generalized connection (including the Higgs) and anomalies. Thus, for the Π_L (anti)leptons (ν_L, e_L), $\overline{e}_L$, $\overline{\nu}_L$ we have vanishing trace of the hypercharge, $\mathrm{tr}\,\Pi_L Y = 0$. For $\mathcal{P}$ leptons, $(\nu_L, e_L), \nu_R, e_R$, the traces of the left and right chiral hypercharge are equal: $\mathrm{tr}(\mathcal{P}\Pi_L Y) = -2 = \mathrm{tr}(\mathcal{P}\Pi_R Y)$, so that, as noted in Sect. 4.2, only the supertrace vanishes in this case. The associated Lie superalgebra fits ideally Quillen's notion of super connection. A real "physical difference" only appears under the assumption that the electroweak hypercharge is superselected and the particle projector is restricted to the projector $\mathcal{P}_r$ on the 15-dimensional particle subspace (with the sterile neutrino ν_R, with vanishing hypercharge, excluded). Then the leptonic (electroweak) part of the SM is governed by the Lie superalgebra $s\ell(2|1)$, whose four odd generators are given by third degree monomials in $a_\alpha^{(*)}$, the $C\ell_4$ Fermi oscillators. The replacement of $\mathcal{P}$ by $\mathcal{P}_r$ breaks the quark-lepton symmetry: while each coloured quark q_j appears in four flavours, the colourless leptons are just three. This yields a relative normalization factor between the quark and leptonic projection of the Higgs field and allows to derive (in [T21]) the relation (see (4.21))

$$m_H^2 = \tfrac{5}{2}\,m_W^2 = 4\cos^2\theta_{\mathrm{th}}\,m_W^2\,, \tag{5.5}$$

where θ_{th} is the *theoretical* Weinberg angle, such that $\text{tg}^2\,\theta_W = \frac{3}{5}$. The relation (5.5) is satisfied within 1% accuracy by the observed Higgs and $W^{\pm}$ masses.

5.3 A challenge

What is missing for completing the "Algebraic Design of Physics" – to quote from the title of the 1994 book by Geoffrey Dixon – is a true understanding of the *three generations* of fundamental fermions. None of the attempts in this direction [F14, D16, T, B] has brought a clear success so far. The exceptional Jordan algebra $J_3^8 = \mathcal{H}_3(\mathbb{O})$ (1.7) with its built in triality was first proposed to this end in [D16] (continued in [DT]); in its most naive form, however, it corresponds to the triple coupling of left and right chiral spinors with a vector in internal space, rather than to three generations of fermions. As recalled in (Sect. 5.2 of) [T] any finite-dimensional unital module over $\mathcal{H}_3(\mathbb{O})$ has the (disappointingly unimaginative) form of a tensor product of $\mathcal{H}_3(\mathbb{O})$ with a finite dimensional real vector space E. It was further suggested there that the dimension of E should be divisible by 3 but the idea was not pursued any further. Boyle [B] proposed to consider the complexified exceptional Jordan algebra whose automorphism group is the compact form of E_6. This led to a promising left-right symmetric extension of the gauge group of the SM but the discussion has not yet shed new light on the 3 generation problem.

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Appendix A

Inter relations between the L, E, and R bases of $so(8)$

The imaginary octonion units $e_1, \cdots, e_7$ obey the anticommutation relations of $C\ell_{-7}$,

$$[e_\alpha, e_\beta]_+ := e_\alpha e_\beta + e_\beta e_\alpha = -2\,\delta_{\alpha\beta}\,,\ \alpha, \beta = 1, \cdots, 7 \qquad \text{(A.1)}$$

and give rise to the seven generators $L_\alpha = L_{e_\alpha}$ of the Lie algebra $so(8)$:

$$L_{\alpha 8} := \tfrac{1}{2}\, L_\alpha =: -L_{8\alpha}\,,\ L_{\alpha\beta} := [L_{\alpha 8}, L_{8\beta}] \in so(7) \subset so(8)\,. \qquad \text{(A.2)}$$

For $\alpha \neq \beta$ there is a unique γ such that

$$L_\alpha\, e_\beta = f_{\alpha\beta\gamma}\, e_\gamma = \pm e_\gamma\,,\ f_{\alpha\beta\gamma} = -f_{\beta\alpha\gamma} = f_{\gamma\alpha\beta}\,. \qquad \text{(A.3)}$$

The *structure constants* $f_{\alpha\beta\gamma}$ (which only take values $0, \pm 1$) obey for different triples (α, β, γ) the relations

$$f_{\alpha\beta\gamma} = f_{\alpha+1\,\beta+1\,\gamma+2} = f_{2\alpha, 2\beta, 2\gamma} \pmod 7\,. \qquad \text{(A.4)}$$

The list (2.9) follows from $f_{124} = 1$ and the first equation (A.4), taking into account relations like $f_{679} \equiv f_{672} \pmod 7$ etc. Note that for $f_{\alpha\beta\gamma} \neq 0$ $f_{\alpha\beta\gamma}$ are the structure constants of a (quaternionic) $su(2)$ Lie algebra. They are *not* structure constants of $so(7) \subset so(8)$.

Define the involutive outer automorphism π of the Lie algebra $so(8)$ by its action (2.13) on left and right multiplication L_α and R_α of octonions by imaginary octonions $\alpha = -\alpha^*$:

$$\pi(L_\alpha) = L_\alpha + R_\alpha =: T_\alpha\,,\ \pi(R_\alpha) = -R_\alpha \Rightarrow \pi(T_\alpha) = L_\alpha\,. \qquad \text{(A.5)}$$

In the basis (A.1) (A.3) of imaginary octonion units e_α ($\alpha = 1, \cdots, 7$), setting $e_8 = \mathbb{1}$ and $L_{\alpha 8} = \frac{1}{2}\, L_\alpha$ (A.2), $R_{\alpha 8} = \frac{1}{2}\, R_\alpha = -R_{8\alpha}$, we define E_{ab} by the second relation (2.14)

$$E_{ab}\, e_c := \delta_{bc}\, e_a - \delta_{ac}\, e_b\,,\ a, b, c = 1, \cdots, 8 \quad (e_8 = 1)\,. \qquad \text{(A.6)}$$

Proposition A.1 – Under the above assumptions/definitions we have

$$\pi(L_{ab}) = E_{ab} \quad (\text{for } L_{\alpha\beta} := [L_{\alpha 8}, L_{8\beta}]\,,\ L_{\alpha 8} = \tfrac{1}{2}\, L_\alpha = -L_{8\alpha})\,. \qquad \text{(A.7)}$$

Proof. – From the first equation (A.5) and from (A.1) (A.2) and (A.6) it follows that

$$E_{\alpha 8} = L_{\alpha 8} + R_{\alpha 8} = \pi(L_{\alpha 8})\,. \qquad \text{(A.8)}$$

The proposition then follows from the relations

$$L_{\alpha\beta} = [L_{\alpha 8}, L_{8\beta}]\,, \quad E_{\alpha\beta} = [E_{\alpha 8}, E_{8\beta}] \qquad \text{(A.9)}$$

and from the assumption that π is a Lie algebra homomorphism.

Corollary. – From (A.7) and the involutive character of π it follows that, conversely,

$$\pi(E_{ab}) = L_{ab}\,. \tag{A.10}$$

To each $\alpha = 1, \cdots, 7$ there correspond 3 pairs $\beta\gamma$ such that $L_{\beta\gamma}$ and $E_{\beta\gamma}$ commute with L_α and among themselves and allow to express $L_\alpha = 2L_{\alpha 8}$ in terms of $E_{\alpha 8}$ and the corresponding $E_{\beta\gamma}$:

$$\begin{aligned}
L_1 &= 2\,L_{18} = E_{18} - E_{24} - E_{37} - E_{56}\,, \\
L_2 &= 2\,L_{28} = E_{28} + E_{14} - E_{35} - E_{67}\,, \\
L_3 &= 2\,L_{38} = E_{38} + E_{17} + E_{25} - E_{46}\,, \\
L_4 &= 2\,L_{48} = E_{48} - E_{12} + E_{36} - E_{57}\,, \\
L_5 &= 2\,L_{58} = E_{58} + E_{16} - E_{23} - E_{47}\,, \\
L_6 &= 2\,L_{68} = E_{68} - E_{15} + E_{27} - E_{34}\,, \\
L_7 &= 2\,L_{78} = E_{78} - E_{13} - E_{26} - E_{45}\,, \text{ or } L_\alpha = E_{\alpha 8} - \sum_{\beta<\gamma} f_{\alpha\beta\gamma}\,E_{\beta\gamma}\,. \quad \text{(A.11)}
\end{aligned}$$

Recalling that $E_{ab} = \pi(L_{ab})$ (A.8) and the fact that π is involutive, so that $\pi(E_{ab}) = L_{ab}$ (A.10) we deduce, in particular,

$$2\,E_{78} = L_{78} - L_{13} - L_{26} - L_{45}\,,$$

$$R_7 = 2\,E_{78} - 2\,L_{78} = -L_{78} - L_{13} - L_{26} - L_{45}\,, \tag{A.12}$$

thus reproducing (2.16).

We now proceed to displaying the commutant of $i\omega_6$ and $i\omega_6^R$ in $so(7+j)$, $j = 1, 2, 3$.

Proposition A.2 – While the Lie algebra $spin(6) = su(4)$ commutes with L_7, the commutant of R_7 (A.12) in $su(4) \subset s\ell(4,\mathbb{C})$ is $u(3) (\subset s\ell(4,\mathbb{C}))$ given by

$$u(3) = \left\{ \sum_{j,k=1}^{3} C_{jk} [b_j^*, b_k]\,;\ C_{jk} \in \mathbb{C}\,,\ C_{kj} = -\overline{C_{jk}} \right\} \tag{A.13}$$

in the fermionic oscillator relalization of $C\ell_6(\mathbb{C})$ (the bar over C_{jk} standing for complex conjugation).

Proof. – The fact that $L_7 = 2\,L_{78}$ commutes with the generators $L_{\alpha\beta}$ ($\alpha, \beta = 1, \cdots, 6$) of $so(6)$ follows from (2.8). To find the commutant of R_7 (A.12) it is convenient to use the fermionic realization of the complexification $s\ell(4,\mathbb{C})$ of $su(4)$ which is spanned by the 9 commutators $[b_j^*, b_k]$ in (A.13) and the 6 products

$$b_j\,b_k = -b_k\,b_j\,,\ b_j^*\,b_k^* = -b_k^*\,b_j^*\,,\ j, k = 1, 2, 3,\ j \neq k\,. \tag{A.14}$$

The sum $L_{13} + L_{26} + L_{45}$ in (A.12) is a multiple of $B - L$ (3.10), the hermitian generator of the centre of $s\ell(3, \mathbb{C})$,

$$B - L \left(= \frac{i}{3} (\gamma_{13} + \gamma_{26} + \gamma_{45}) \right) = \frac{1}{3} \sum_{j=1}^{3} [b_j^*, b_j] \, . \tag{A.15}$$

The relations

$$[B - L, b_j^* \, b_k^*] = \tfrac{2}{3} \, b_j^* \, b_k^* \, , \; [B - L, b_j \, b_k] = -\tfrac{2}{3} \, b_j \, b_k \, ,$$
$$\big[[B - L, [b_j^*, b_k]] \big] = 0 \, , \; j, k = 1, 2, 3, \; j \neq k \, , \tag{A.16}$$

show that the commutant of $B - L$ (and hence of R_7) in $su(4)$ is $u(3)$.

Corollary. – The commutant of ω_6^R in $so(8)$ is $u(3) \oplus u(1)$; the commutant of ω_6^R in $spin(9)$ is the gauge Lie algebra of the SM:

$$\mathcal{G}_{\rm SM} = \{ a \in spin(9) \, ; \; [a, \omega_6^R] = 0 \} = u(3) \oplus su(2) \, . \tag{A.17}$$

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