On quantum states over time

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Abstract

In 2017, D. Horsman, C. Heunen, M. Pusey, J. Barrett, and R. Spekkens proved that there is no physically reasonable assignment that takes a quantum channel and an initial state and produces a joint state on the tensor product of the input and output spaces. The interpretation was that there is a clear distinction between space and time in the quantum setting that is not visible classically, where in the latter, one can freely use Bayes' theorem to go between joint states and marginals with noisy channels. In this paper, we prove that there actually is such a physically reasonable assignment, bypassing the no-go result of Horsman et al., and we illustrate that this is achievable by restricting the domain of their assignment to a domain which represents the given data more faithfully. This answers an open question at the end of their work, thus indicating the possibility that such a symmetry between space and time may exist in the quantum setting.

Contents

1 Introduction

Given a joint probability measure p_{XY} on the direct product $X \times Y$ of two finite sets, one can obtain the associated marginals p_X on X and p_Y on Y by pushing these measures forward along the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, respectively. In addition, one also

Key words: Markov category; Bayes; quantum state over time; Choi–Jamiołkowski isomorphism

obtains stochastic maps (i.e., Markov kernels) $p_{Y|X} : X \to Y$ and $p_{X|Y} : Y \to X$, called *conditionals*, such that

$$p_{Y|X}(y|x)p_X(x) = p_{XY}(x, y) = p_{X|Y}(x|y)p_Y(y)$$

for all $(x, y) \in X \times Y$. This allows one to convert a joint state, which is a state at a single time, to an initial state together with a stochastic evolution in two distinct ways based on which marginal is used as the initial state.

Conversely, given a probability measure p_X on X and a stochastic map $p_{Y|X} : X \to Y$, one obtains a joint probability measure p_{XY} on $X \times Y$ by the formula

$$p_{XY}(x, y) := p_{Y|X}(y|x)p_X(x).$$

In fact, one can formalize this duality by stating a bijection between these data (modulo some minor subtleties related to measure zero subsets) [2].

As such, one may view a joint state p_{XY} in classical probability either as a state at a single time whose subsystems are arbitrarily separated in space, or as a state *over* time associated with stochastic evolution. Does such a symmetric treatment of space and time hold for quantum systems, namely quantum states and quantum channels?

The question of when it is possible to go from joint states to marginals and channels was the subject of [11], though that work only established the conditions needed when the marginals were full rank density matrices and for a particular construction that was motivated by categorical probability theory [1,5]. The question of when it is possible to go from initial states and channels to joint states was the subject of the work of Horsman et. al. [8], where they argued that such a construction satisfying a collection of axioms they put forward is not possible. Together, these arguments suggest that the symmetry between time and space that is available (and often taken for granted) in the classical setting might no longer hold for quantum systems and their evolution.

In this paper, we show that the no-go results of [8] can be bypassed, answering an open question posed at the end of [8]. In particular, we show that *there is* a consistent assignment from quantum channels endowed with initial states to joint states over time that satisfies the axioms proposed in [8], where the states are represented by self-adjoint, as opposed to positive, density matrices. The way that the no-go result of [8] is bypassed is by restricting the assignment to a domain that reflects the given data more faithfully, rather than demanding a full binary operation as in [8]. Furthermore, we formulate the definitions, axioms, and theorems for arbitrary hybrid classical/quantum systems (i.e., finite-dimensional C^*-algebras) and show how these specialize to the setting of purely quantum systems when restricted to matrix algebras. As such, we work in the Heisenberg picture for the formulations of our results, but we translate to the Schrödinger picture when specializing to the matrix algebra setting.

The main definition of a family of states over time function is given in Definition 2.8. It contrasts with the definition of [8] in that it assumes exactly the data given in the domain rather than assuming that such an assignment extends to a larger domain (for more on this comment, see Remark 2.15). The main theorem in this paper is Theorem 2.13, which says that a family of states over time function exists, which we prove via an explicit construction.

2 From quantum channels and quantum states to joint states

Our results are formulated in the language of finite-dimensional C^*-algebras to illustrate the similarities between classical and quantum systems and to include all hybrid classical/quantum systems. Furthermore, we use string diagrams on occasion to provide visualizations of some concepts and proofs, which are sometimes more illuminating than the algebraic manipulations of coordinate expressions. However, such string diagrams are *not* essential to follow the main definitions and statements of results. The string diagrams we use are those of quantum Markov categories [10] (in fact, quantum CD categories), and the reader is referred to that work for a thorough introduction. A shorter summary of quantum Markov categories is provided in [11]. Classical versions of Markov categories originated in the works [1,5], which also provide adequate introductions.

Notation 2.1. If m is a natural number, then $\mathbb{M}_m(\mathbb{C})$ denotes the C^*-algebra of $m \times m$ matrices with complex entries. The standard matrix units are denoted by E_{ij} (or $E_{ij}^{(m)}$ for additional clarity), while the identity matrix is denoted by $\mathbb{1}_m$. All C^*-algebras in this work will be finite-dimensional and unital, with the involution always written as $\dagger$. As such, all C^*-algebras $\mathcal{A}$ will be *multi-matrix algebras*, i.e., $\mathcal{A} = \bigoplus_{x \in X} \mathbb{M}_{m_x}(\mathbb{C})$, where X is a finite set and the m_x are natural numbers. If $\mathcal{A}$ is a C^*-algebra, let $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denote the linear product map uniquely determined by sending $A_1 \otimes A_2$ to $A_1 A_2$. The unit in $\mathcal{A}$ is written as $1_{\mathcal{A}}$ and the unique unital map from $\mathbb{C}$ to $\mathcal{A}$ will be denoted by $!_{\mathcal{A}}$. Meanwhile, $i_{\mathcal{A}}$ will be used to denote an inclusion of $\mathcal{A}$ into another algebra with $\mathcal{A}$ as a tensor factor, such as $\mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$. If $F : \mathcal{B} \to \mathcal{A}$ is a linear map, with $\mathcal{A} = \bigoplus_{x \in X} \mathbb{M}_{m_x}(\mathbb{C})$ and $\mathcal{B} = \bigoplus_{y \in Y} \mathbb{M}_{n_y}(\mathbb{C})$, let F_{xy} denote the xy component of F, i.e., the composite $\mathbb{M}_{n_y}(\mathbb{C}) \hookrightarrow \mathcal{B} \xrightarrow{F} \mathcal{A} \xrightarrow{\pi_x} \mathbb{M}_{m_x}(\mathbb{C})$, where π_x is the projection. Also, let $F^* : \mathcal{A} \to \mathcal{B}$ denote the *Hilbert–Schmidt adjoint*, which is the map whose yx component is given by $(F^*)_{yx} = (F_{xy})^* \equiv F^*_{xy}$, where F^*_{xy} is the usual Hilbert–Schmidt adjoint for linear maps between matrix algebras, namely, it is the unique linear map satisfying

$$\operatorname{tr}\big(A_x^\dagger F_{xy}(B_y)\big) = \operatorname{tr}\big(F^*_{xy}(A_x)^\dagger B_y\big)$$

for all $A_x \in \mathbb{M}_{m_x}(\mathbb{C})$ and $B_y \in \mathbb{M}_{n_y}(\mathbb{C})$. We will freely use the fact that a linear map is unital if and only if its Hilbert–Schmidt adjoint is trace-preserving. Furthermore, a linear map $F : \mathcal{B} \to \mathcal{A}$ is $\dagger$*-preserving*, aka *self-adjoint* (meaning $F(B)^\dagger = F(B^\dagger)$ for all $B \in \mathcal{B}$), if and only if its Hilbert–Schmidt adjoint is $\dagger$-preserving. The vector space of all linear maps from $\mathcal{B}$ to $\mathcal{A}$ is denoted by $\mathbf{Hom}(\mathcal{B}, \mathcal{A})$, while the affine subspace of $\dagger$-preserving maps is denoted by $\mathbf{Hom}^{sa}(\mathcal{B}, \mathcal{A})$. In what follows, $\gamma : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ will be used to denote the swap isomorphism. Every finite-dimensional C^*-algebra $\mathcal{A} = \bigoplus_{x \in X} \mathbb{M}_{m_x}(\mathbb{C})$ has a unique positive functional $\operatorname{tr} : \mathcal{A} \to \mathbb{C}$, called the *trace*, such that $\operatorname{tr} \circ \gamma = \operatorname{tr}$ and $\operatorname{tr}(\mathbb{1}_{m_x}) = m_x$ for all $x \in X$. Its evaluation on an element of the form $\bigoplus_{x \in X} A_x$ is given by $\sum_{x \in X} \operatorname{tr}(A_x)$ in terms of the usual trace on matrices. Every functional $\omega : \mathcal{A} \to \mathbb{C}$ is given by $\operatorname{tr}(\rho^\dagger\ \cdot\ )$, where $\rho := \omega^*(1) \in \mathcal{A}$ is the *density* associated to ω. The functional ω is $\dagger$-preserving if and only if ρ is self-adjoint.

Definition 2.2. Let $F : \mathcal{B} \to \mathcal{A}$ be a linear unital map. The ***bloom*** of F is the unital map $i_F : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}$ given by $i_F := \mu_{\mathcal{A}} \circ (\mathrm{id}_{\mathcal{A}} \otimes F)$. The ***swapped bloom*** of F is the unital map ${}_F i : \mathcal{A} \otimes \mathcal{B} \to \mathcal{A}$

given by ${}_{\mathsf{F}}\mathsf{i} := \mu_{\mathcal{A}} \circ (\mathsf{F} \otimes \mathrm{id}_{\mathcal{A}}) \circ \gamma$. The ***channel state*** associated with F is the unital functional on $\mathcal{A} \otimes \mathcal{B}$ given by

$$\mathscr{S}[\mathsf{F}] := \mathrm{tr} \circ \mathsf{i}_{\mathsf{F}} \equiv \mathrm{tr} \circ \mu_{\mathcal{A}} \circ (\mathrm{id} \otimes \mathsf{F}).$$

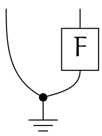
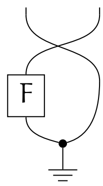
The ***channel density*** associated with F is the element of $\mathcal{A} \otimes \mathcal{B}$ given by

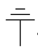
$$\mathscr{D}[\mathsf{F}] := \mathsf{i}_{\mathsf{F}}^{*}(1_{\mathcal{A}}) \equiv (\mathrm{id} \otimes \mathsf{F}^{*})\big(\mu_{\mathcal{A}}^{*}(1_{\mathcal{A}})\big)$$

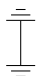
and is the unique element of $\mathcal{A} \otimes \mathcal{B}$ that satisfies $\mathscr{S}[\mathsf{F}] = \mathrm{tr}(\mathscr{D}[\mathsf{F}]^{\dagger} \cdot\ )$. The channel state provides a linear isomorphism $\mathscr{S} : \mathbf{Hom}(\mathcal{B}, \mathcal{A}) \to \mathbf{Hom}(\mathcal{A} \otimes \mathcal{B}, \mathbb{C})$ sending F to $\mathscr{S}[\mathsf{F}]$ and the channel density provides a linear isomorphism $\mathscr{D} : \mathbf{Hom}(\mathcal{B}, \mathcal{A}) \to \mathcal{A} \otimes \mathcal{B}$ sending F to $\mathscr{D}[\mathsf{F}]$. These isomorphisms will both be referred to as the ***Choi–Jamiołkowski isomorphism*** [9]. For additional clarity, these may also be written as $\mathscr{S}_{\mathcal{A},\mathcal{B}}$ and $\mathscr{D}_{\mathcal{A},\mathcal{B}}$.

Remark 2.3. In [8], the element $\mathscr{D}[\mathsf{F}]$ is called the 'channel state' associated with F. We have chosen to call this the channel density to allow for simpler generalizations to the C^{*}-algebraic setting of hybrid classical/quantum systems. We should also point out that the state/density terminology is abusive for two reasons. First, the trace of $\mathscr{D}[\mathsf{F}]$ is not unity, i.e., the channel state $\mathscr{S}[\mathsf{F}]$ is not unital. Second, and more importantly, $\mathscr{D}[\mathsf{F}]$ (and likewise $\mathscr{S}[\mathsf{F}]$) need not be positive. Nevertheless, the terminology is chosen to be somewhat consistent with that of [8] (though what we call 'density' is what [8] calls a 'state').

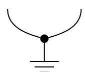
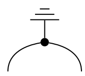
Remark 2.4. The channel state could have equivalently been defined in terms of the swapped bloom as $\mathscr{S}[\mathsf{F}] = \mathrm{tr} \circ {}_{\mathsf{F}}\mathsf{i}$. The fact that these two are equal is a consequence of the properties of the trace. Furthermore, it will often be useful to depict this via string diagrams as

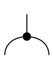
$\mathrm{tr} \circ \mathsf{i}_{\mathsf{F}} =$  and $\mathrm{tr} \circ {}_{\mathsf{F}}\mathsf{i} =$  ,

where  denotes the trace, which is the Hilbert–Schmidt adjoint of $!_{\mathcal{A}} \equiv$ . Note that if $\mathcal{A} = \bigoplus_{x \in X} \mathbb{M}_{m_x}(\mathbb{C})$, then

 $= \sum_{x \in X} m_x$

gives the dimension of the underlying Hilbert space for $\mathcal{A}$. Furthermore, note that if we set

 :=  and  :=  ,

where  is the Hilbert–Schmidt adjoint of , then the Choi–Jamiołkowski isomorphism is easily seen to be an instance of the zig-zag identities from categorical quantum mechanics [3,7].

Example 2.5. When $\mathcal{A} = \mathbb{M}_m(\mathbb{C})$ and $\mathcal{B} = \mathbb{M}_n(\mathbb{C})$, the channel density from Definition 2.2 is the matrix given by

$$\mathscr{D}[\mathsf{F}] = (\mathrm{id} \otimes \mathsf{F}^{*})\big(\mu_{\mathcal{A}}^{*}(\mathbb{1}_m)\big) \equiv \sum_{i,j}^{m} E_{ij}^{(m)} \otimes \mathsf{F}^{*}(E_{ji}^{(m)}),$$

which reproduces the definition from [8, Section 2(a)].[1] In particular,

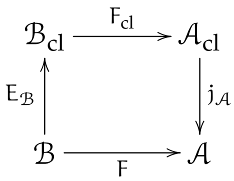
$$\mathscr{D}[\mathrm{id}_{\mathcal{A}}] = \mu^*_{\mathcal{A}}(\mathbb{1}_m) \equiv \sum_{i,j}^{m} E^{(m)}_{ij} \otimes E^{(m)}_{ji}.$$

Definition 2.6. A pair (F, ω), where $F : \mathcal{B} \to \mathcal{A}$ and $\omega : \mathcal{A} \to \mathbb{C}$ are linear, is ***effectively classical***, or ***has a classical model***, iff there exist commutative C^*-subalgebras $\mathcal{A}_{\mathrm{cl}} \subseteq \mathcal{A}$ and $\mathcal{B}_{\mathrm{cl}} \subseteq \mathcal{B}$, a linear map $F_{\mathrm{cl}} : \mathcal{B}_{\mathrm{cl}} \to \mathcal{A}_{\mathrm{cl}}$, and conditional expectations[2] $E_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}_{\mathrm{cl}}$ and $E_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}_{\mathrm{cl}}$ such that

$$\omega_{\restriction} \circ E_{\mathcal{A}} = \omega, \qquad (\omega \circ F)_{\restriction} \circ E_{\mathcal{B}} = \omega \circ F, \qquad \text{and} \qquad F = j_{\mathcal{A}} \circ F_{\mathrm{cl}} \circ E_{\mathcal{B}},$$

where $j_{\mathcal{A}} : \mathcal{A}_{\mathrm{cl}} \to \mathcal{A}$ is the inclusion, and where $\restriction$ is used to denote the restriction, i.e., $\omega_{\restriction} := \omega \circ j_{\mathcal{A}}$ and $(\omega \circ F)_{\restriction} := \omega \circ F \circ j_{\mathcal{B}}$.

The first two equations in Definition 2.6 say that the conditional expectations are state-preserving, i.e., they are particular disintegrations in the terminology of [6]. The last condition is more easily visualized as the commutative diagram



and encapsulates the fact that the quantum dynamics actually factors through a classical system. The conditional expectation condition also guarantees that the composite of effectively classical channels factors through the composite of the underlying classical channels. This is not relevant for the main statement of our theorem and is therefore addressed later in Proposition 3.2. The motivation for Definition 2.6 comes from the argument often provided in the physics literature that density matrices and channels that can be 'simultaneously diagonalized' are classical. The precise statement is given in the next proposition, the proof of which is given in the next section.

Proposition 2.7. *In the notation of Definition 2.6, let $\mathcal{A} = \mathbb{M}_m(\mathbb{C})$ and $\mathcal{B} = \mathbb{M}_n(\mathbb{C})$, and suppose ω and F are $\dagger$-preserving. Write $\omega = \mathrm{tr}(\rho\ \cdot\ )$ and $\omega \circ F = \mathrm{tr}(\theta\ \cdot\ )$. Then, a classical model for (F, ω) exists if and only if there exist orthonormal bases $\{e_i\}$ for $\mathbb{C}^m$ and $\{\epsilon_k\}$ for $\mathbb{C}^n$ such that ρ and θ are diagonal in these bases and the channel density $\mathscr{D}[F]$ associated with F is diagonal[3] in the basis $\{e_i \otimes \epsilon_k\}$. Furthermore, if these equivalent conditions hold, then $[\mathscr{D}[F], \rho \otimes 1_{\mathcal{B}}] = 0$.*

Definition 2.8. Let $\star$ be a family of functions that assign to any pair of finite-dimensional C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ a function $\star : \mathbf{Hom}(\mathcal{B}, \mathcal{A}) \times \mathbf{Hom}(\mathcal{A}, \mathbb{C}) \to \mathbf{Hom}(\mathcal{A} \otimes \mathcal{B}, \mathbb{C})$ taking any linear

[1] There is a small typo in the formula from [8, Section 2(a)] since the matrix $E_{B|A}$ should be an element of $\mathcal{A} \otimes \mathcal{B}$ and not $\mathcal{B} \otimes \mathcal{A}$.

[2] This means $E_{\mathcal{A}}$ and $E_{\mathcal{B}}$ are positive unital and satisfy $E_{\mathcal{A}} \circ j_{\mathcal{A}} = \mathrm{id}_{\mathcal{A}_{\mathrm{cl}}}$ and $E_{\mathcal{B}} \circ j_{\mathcal{B}} = \mathrm{id}_{\mathcal{B}_{\mathrm{cl}}}$. See [6] for further properties.

[3] It suffices to assume that ρ and $\mathscr{D}[F]$ are diagonal in these bases since $\theta = F^*(\rho)$ will be diagonal as a consequence.

map $F : \mathcal{B} \to \mathcal{A}$ and linear functional ω on $\mathcal{A}$ to a functional $F \star \omega$ on $\mathcal{A} \otimes \mathcal{B}$ and satisfying the following conditions.

(a) (Hermiticity and unitality) If (F, ω) is in $\mathbf{Hom}^{sa}(\mathcal{B}, \mathcal{A}) \times \mathbf{Hom}^{sa}(\mathcal{A}, \mathbb{C})$, then $F \star \omega$ is in $\mathbf{Hom}^{sa}(\mathcal{A} \otimes \mathcal{B}, \mathbb{C})$. If F and ω are both unital, then $F \star \omega$ is unital.

(b) (Preservation of probabilistic mixtures/convex bi-linearity) Given any $\lambda \in [0, 1]$ together with maps $F, G \in \mathbf{Hom}(\mathcal{B}, \mathcal{A})$ and $\omega, \xi \in \mathbf{Hom}(\mathcal{A}, \mathbb{C})$, the equalities

$$\Big(\lambda F + (1-\lambda)G\Big) \star \omega = \lambda(F \star \omega) + (1-\lambda)(G \star \omega)$$

and

$$F \star \Big(\lambda \omega + (1-\lambda)\xi\Big) = \lambda(F \star \omega) + (1-\lambda)(F \star \xi)$$

hold.

(c) (Preservation of classical limit) If the pair (F, ω) is effectively classical, then $F \star \omega = \omega \circ i_F$.

(d) (Preservation of marginal states) The initial and final functionals are recovered from the joint functional in the sense that

$$(F \star \omega) \circ i_{\mathcal{A}} = \omega \qquad \text{and} \qquad (F \star \omega) \circ i_{\mathcal{B}} = \omega \circ F.$$

(e) (Compositionality/associativity) Given a composable pair of unital maps $\mathcal{C} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{A}$ and a unital functional ω on $\mathcal{A}$,

$$(!_{\mathcal{A}} \otimes G) \star (F \star \omega) = \Big(\mathscr{S}^{-1}_{\mathcal{A}, \mathcal{B} \otimes \mathcal{C}}\Big[(!_{\mathcal{A}} \otimes G) \star \big(\mathscr{S}_{\mathcal{A}, \mathcal{B}}[F]\big)\Big]\Big) \star \omega.$$

Such a family is called a ***states over time*** function.

The explanation for why the compositionality/associativity formula looks so complicated, but is in fact rather straightforward, will be given in Remark 2.11. In short, it follows from the two natural ways of pairing the construction of states over two successive times and only looks complicated due to the natural isomorphisms coming from the Choi–Jamiołkowski isomorphism. Secondly, although our preservation of the classical limit axiom is expressed differently than in [8], it is equivalent to it by Proposition 2.7 on the domains for which a family of states over time function is defined.

Remark 2.9. The terminology 'a family of *states* over time' is a bit abusive because we are only requiring that the functionals are †-preserving and unital, rather than positive. Note that this is the same restriction imposed in [8]. In other words, there are situations where one might begin with a positive (even completely positive) unital map together with a state and end up with a joint functional that is *not* a state, i.e., it is not necessarily positive. It is an open question whether one can obtain a genuine family of states over time where all maps remain positive under some operation satisfying similar, perhaps slightly weakened, axioms (see Section 5 for more details).

Remark 2.10. Rather than requiring a family of states over time function to be defined as a family of functions of the form $\mathbf{Hom}(\mathcal{B},\mathcal{A}) \times \mathbf{Hom}(\mathcal{A},\mathbb{C}) \to \mathbf{Hom}(\mathcal{A}\otimes\mathcal{B},\mathbb{C})$, we could have required it to be a family of functions of the form $\mathbf{Hom}^{\mathrm{sa}}(\mathcal{B},\mathcal{A}) \times \mathbf{Hom}^{\mathrm{sa}}(\mathcal{A},\mathbb{C}) \to \mathbf{Hom}^{\mathrm{sa}}(\mathcal{A}\otimes\mathcal{B},\mathbb{C})$ so that Hermiticity is in the very definition of the family. However, one can see that by arguments completely analogous to those in the proof of [8, Lemma 4.3], any such function uniquely extends to a function $\mathbf{Hom}(\mathcal{B},\mathcal{A}) \times \mathbf{Hom}(\mathcal{A},\mathbb{C}) \to \mathbf{Hom}(\mathcal{A}\otimes\mathcal{B},\mathbb{C})$ satisfying the same properties, in fact complex bi-linearity. This is achieved by splitting an arbitrary morphism $\mathsf{F} : \mathcal{B} \to \mathcal{A}$ into its Hermitian and anti-Hermitian parts via $\mathsf{F} = \frac{1}{2}(\mathsf{F} + \dagger\circ\mathsf{F}\circ\dagger) + \frac{1}{2}(\mathsf{F} - \dagger\circ\mathsf{F}\circ\dagger)$. Thus, we lose no generality in defining a family of states over time function on all linear maps as opposed to the subspace of $\dagger$-preserving ones.

Remark 2.11. The formula for associativity looks rather complicated because of the way in which we have formulated our definition by avoiding the usage of a binary operation and is one of the two reasons why we are able to bypass the no-go result of [8] (see Remark 2.15 for more details). The formula comes from trying to pair the three different factors in the two possible ways, i.e., the diagram

$$
\begin{array}{ccc}
\mathbf{Hom}(\mathcal{C},\mathcal{B}) \times \mathbf{Hom}(\mathcal{B},\mathcal{A}) \times \mathbf{Hom}(\mathcal{A},\mathbb{C}) & \xrightarrow{(!_{\mathcal{A}}\otimes\,\cdot\,)\times\mathscr{S}_{\mathcal{A},\mathcal{B}}\times\mathrm{id}} & \mathbf{Hom}(\mathcal{C},\mathcal{A}\otimes\mathcal{B}) \times \mathbf{Hom}(\mathcal{A}\otimes\mathcal{B},\mathbb{C}) \times \mathbf{Hom}(\mathcal{A},\mathbb{C}) \\
\downarrow{\scriptstyle \mathrm{id}\times\star} & & \downarrow{\scriptstyle \star\times\mathrm{id}} \\
\mathbf{Hom}(\mathcal{C},\mathcal{B}) \times \mathbf{Hom}(\mathcal{A}\otimes\mathcal{B},\mathbb{C}) & & \mathbf{Hom}(\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C},\mathbb{C}) \times \mathbf{Hom}(\mathcal{A},\mathbb{C}) \\
\downarrow{\scriptstyle (!_{\mathcal{A}}\otimes\,\cdot\,)\times\mathrm{id}} & & \downarrow{\scriptstyle \mathscr{S}^{-1}_{\mathcal{A},\mathcal{B}\otimes\mathcal{C}}\times\mathrm{id}} \\
\mathbf{Hom}(\mathcal{C},\mathcal{A}\otimes\mathcal{B}) \times \mathbf{Hom}(\mathcal{A}\otimes\mathcal{B},\mathbb{C}) & & \mathbf{Hom}(\mathcal{B}\otimes\mathcal{C},\mathcal{A}) \times \mathbf{Hom}(\mathcal{A},\mathbb{C}) \\
& \searrow^{\star} \quad {}^{\star}\swarrow & \\
& \mathbf{Hom}(\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C},\mathbb{C}) &
\end{array}
$$

must commute. Note that the Choi–Jamiołkowski isomorphisms $\mathscr{S}$ are used to transform channels into joint states, while the inverse transforms joint states back into channels. Furthermore, the inclusion $!_{\mathcal{A}}$ is used to guarantee that the domains and codomains match so that the $\star$ operation can be applied. In other words, ignoring these canonical maps, one sees this as associativity of $\star$. Interestingly, when we extend such a $\star$ function to include channels in its second argument in Section 4, we will find that the associated formulation of associativity takes a much simpler form, as it will no longer be necessary to use the Choi–Jamiołkowski isomorphism in its description.

Remark 2.12. We can express the associativity of $\star$ in a manner that is even more closely related to associativity in the usual sense as follows. First, note that by the Choi-Jamiołkowski isomorphism, $\mathbf{Hom}(\mathcal{B},\mathcal{A}) \cong \mathcal{A}\otimes\mathcal{B}$. Furthermore, since $\mathbf{Hom}(\mathcal{A},\mathbb{C}) \cong \mathbb{C}\otimes\mathcal{A} \cong \mathcal{A}$, we can include this into $\mathcal{A}\otimes\mathcal{B}$. In this way, one can imagine an extension of $\star$ to a binary operation $\circledast : (\mathcal{A}\otimes\mathcal{B}) \times (\mathcal{A}\otimes\mathcal{B}) \to \mathcal{A}\otimes\mathcal{B}$ that agrees with $\star$ on the subspace $(\mathcal{A}\otimes\mathcal{B}) \times (\mathcal{A}\otimes\mathbb{C})$, and likewise for other C^*-algebras and their tensor products. Once all these identifications are made, commutativity of the diagram in Remark 2.11 reads

$$\mathscr{D}[!_{A}\otimes\mathsf{G}] \circledast \Big(\big(\mathscr{D}[\mathsf{F}] \circledast (\rho\otimes 1_{\mathcal{B}})\big)\otimes 1_{\mathcal{C}}\Big) = \Big(\mathscr{D}[!_{A}\otimes\mathsf{G}] \circledast \big(\mathscr{D}[\mathsf{F}]\otimes 1_{\mathcal{C}}\big)\Big) \circledast (\rho\otimes 1_{\mathcal{B}}\otimes 1_{\mathcal{C}}).$$

Without writing any of the identity matrices and without writing the inclusions out explicitly, the above formula reduces to

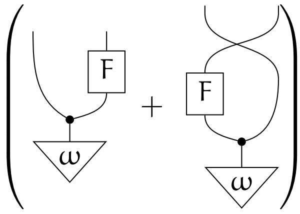
$$\mathscr{D}[G] \circledast \left(\mathscr{D}[F] \circledast \rho\right) = \left(\mathscr{D}[G] \circledast \mathscr{D}[F]\right) \circledast \rho,$$

which looks more like associativity in the usual sense, and in fact coincides with the expression that can be found in [8]. Though perhaps more cumbersome, the benefit of our approach of specifying the domains and codomains more precisely, as well as specifying exactly when the Choi–Jamiołkowski isomorphism is being used, is that we will bypass the no-go result of [8], as will be discussed in Theorem 2.13 and afterwards.

Theorem 2.13. *A family of states over time function exists.*

Proof. Given $F : \mathcal{B} \to \mathcal{A}$ and $\omega : \mathcal{A} \to \mathbb{C}$, the assignment

$$F \star \omega := \frac{1}{2}\left(\omega \circ i_F + \omega \circ {}_F i\right) \equiv \frac{1}{2}$$



satisfies all the required conditions. A detailed proof that this indeed satisfies all the axioms is deferred to Section 3. ■

Remark 2.14. The assignment in the proof of Theorem 2.13 is given by the Jordan product when expressed in terms of the corresponding densities, as described in Lemma 3.6 below.

Remark 2.15. Theorem 2.13 seems to contradict the results of [8]. However, as mentioned in Remark 2.12, there are two reasons for this. First, notice that the *domain* $(\mathcal{A} \otimes \mathcal{B}) \times (\mathcal{A} \otimes \mathcal{B})$ of the family of states over time function in [8] (what we called $\circledast$ in Remark 2.12) is much larger than is necessary for the definition to be made. The authors of [8] then demanded the five axioms to be valid on this larger domain, rather than the domain $(\mathcal{A} \otimes \mathcal{B}) \times \mathcal{A}$, as we have. The second reason, which will be explained in more detail in Section 4, is that the extension of the classical limit axiom as formulated in [8] to the larger domain is more stringent than our extension based on our definition of a classical model. By using only the data given (a channel together with a state), and by using less structure that is nevertheless sufficient to make sense of such a family of states over time function, we have been able to bypass the no-go theorem of [8]. It should be pointed out that the authors of [8] were well aware of the fact that the Jordan product on qubits is associative on the required matrices involved when they discussed the Fitzsimons–Jones–Vedral (FJV) construction [4]. However, they nevertheless demanded a binary operation, so that the Jordan product is no longer associative, and this is what forced their no-go result.

3 Proofs and relevant results

This section contains proofs of all statements made, including the proof of the main theorem. Some lemmas are included as separate statements and a proposition regarding the compositionality of classical models is given.

Proof of Proposition 2.7.

(⇐) Suppose there exist orthonormal bases $\{e_i\}$ and $\{\epsilon_j\}$ as in the assumptions of the reverse claim. Let $P_i \equiv |e_i\rangle\langle e_i|$ and $Q_k \equiv |\epsilon_k\rangle\langle\epsilon_k|$ be the corresponding one-dimensional projection operators. By assumption, there exist numbers $p_i, q_k, f_{ki} \in \mathbb{R}$ such that

$$\rho = \sum_{i=1}^{m} p_i P_i, \qquad \theta = \sum_{k=1}^{n} q_k Q_k, \quad \text{and} \quad \mathscr{D}[F] = \sum_{i=1}^{m}\sum_{k=1}^{n} f_{ki} P_i \otimes Q_k.$$

By Example 2.5, the last equality for the channel density entails

$$F^*\big(|e_j\rangle\langle e_i|\big) = \delta_{ij} F^*(P_i) = \sum_{k=1}^{n} \delta_{ij} f_{ki} Q_k$$

for all $i, j \in \{1, \dots, m\}$. Therefore,

$$F\big(|\epsilon_k\rangle\langle\epsilon_l|\big) = \delta_{kl} \sum_{i=1}^{m} \overline{f_{ki}} P_i$$

for all $k, l \in \{1, \dots, n\}$ by the definition of the Hilbert–Schmidt adjoint (since the f_{ki} are real, $\overline{f_{ki}} = f_{ki}$), since this definition satisfies

$$\operatorname{tr}\left(E_{ij}^{(m)} F\big(E_{kl}^{(n)}\big)\right) \equiv \left\langle E_{ji}^{(m)}, F\big(E_{kl}^{(n)}\big)\right\rangle = \left\langle F^*\big(E_{ji}^{(m)}\big), E_{kl}^{(n)}\right\rangle \equiv \operatorname{tr}\left(F^*\big(E_{ji}^{(m)}\big)^{\dagger} E_{kl}^{(n)}\right)$$

for all $i, j \in \{1, \dots, m\}$ and $k, l \in \{1, \dots, n\}$. From this, we can define the required conditional expectations and classical maps. First, we set $\mathcal{A}_{\text{cl}} := \operatorname{span}_i\{P_i\}$ and $\mathcal{B}_{\text{cl}} := \operatorname{span}_k\{Q_k\}$. These are commutative unital C^*-subalgebras of $\mathcal{A}$ and $\mathcal{B}$, respectively, due to the orthonormality and spanning assumptions on $\{e_i\}$ and $\{\epsilon_k\}$. Next, define $F_{\text{cl}} : \mathcal{B}_{\text{cl}} \to \mathcal{A}_{\text{cl}}$ by specifying

$$F_{\text{cl}}(Q_k) := \sum_{i=1}^{m} \overline{f_{ki}} P_i$$

for all k and then extending linearly. The conditional expectations $E_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}_{\text{cl}}$ and $E_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}_{\text{cl}}$ are defined by

$$E_{\mathcal{A}}(A) := \sum_{i=1}^{m} P_i A P_i \qquad \text{and} \qquad E_{\mathcal{B}}(B) := \sum_{k=1}^{n} Q_k B Q_k.$$

From this, the required conditions of a classical model are all readily checked. Indeed, one has

$$(j_{\mathcal{A}} \circ F_{\text{cl}} \circ E_{\mathcal{B}})\big(|\epsilon_k\rangle\langle\epsilon_l|\big) = (j_{\mathcal{A}} \circ F_{\text{cl}})(\delta_{kl} Q_k) = \delta_{kl} \sum_{i=1}^{m} \overline{f_{ki}} P_i = F\big(|\epsilon_k\rangle\langle\epsilon_l|\big),$$

as needed. The state-preserving condition for the conditional expectations is immediate.

(⇒) Suppose a classical model exists. Then by all the assumptions, there exist orthogonal projections (not necessarily rank 1) $\{P_i\}$ in $\mathcal{A}$ and $\{Q_k\}$ in $\mathcal{B}$ together with coefficients $\{p_i\}$ and $\{q_k\}$ such that

$$\mathcal{A}_{\text{cl}} = \operatorname{span}_i\{P_i\}, \quad \mathcal{B}_{\text{cl}} = \operatorname{span}_k\{Q_k\}, \quad \omega(P_i) = p_i, \quad \text{and} \quad (\omega \circ F)(Q_k) = q_k.$$

Since $E_{\mathcal{A}}$ and $E_{\mathcal{B}}$ are conditional expectations into commutative C^*-algebras, there exist positive functionals $\phi_i : \mathcal{A} \to \mathbb{C}$ and $\psi_k : \mathcal{B} \to \mathbb{C}$ supported on $P_i\mathcal{A}P_i$ and $Q_k\mathcal{B}Q_k$, respectively, such that

$$E_{\mathcal{A}}(A) = \sum_i \phi_i(P_iAP_i)P_i \quad \text{and} \quad E_{\mathcal{B}}(B) = \sum_k \psi_k(Q_kBQ_k)Q_k.$$

Such functionals are necessarily represented by positive matrices $\sigma_i \in P_i\mathcal{A}P_i$ and $\tau_k \in Q_k\mathcal{B}Q_k$ satisfying

$$\phi_i = \mathrm{tr}(\sigma_i \,\cdot\,) \quad \text{and} \quad \psi_k = \mathrm{tr}(\tau_k \,\cdot\,).$$

By the assumption that the conditional expectations $E_{\mathcal{A}}$ and $E_{\mathcal{B}}$ are state-preserving, we conclude that ρ and θ are (orthogonal) linear combinations of these matrices, namely

$$\rho = \sum_i p_i\sigma_i \quad \text{and} \quad \theta = \sum_k q_k\tau_k.$$

Since ρ and θ are self-adjoint, σ_i and τ_k are self-adjoint as well. As such, let $\{|e_{i\alpha_i}\rangle\}$ and $\{|\epsilon_{k\beta_k}\rangle\}$ be orthonormal bases diagonalizing the σ_i and τ_k, respectively. Thus,

$$\rho = \sum_i p_i \sum_{\alpha_i} s_{i\alpha_i}|e_{i\alpha_i}\rangle\langle e_{i\alpha_i}| \quad \text{and} \quad \theta = \sum_k q_k \sum_{\beta_k} t_{k\beta_k}|\epsilon_{k\beta_k}\rangle\langle\epsilon_{k\beta_k}|$$

have been diagonalized in terms of the real coefficients $\{s_{i\alpha_i}\}$ and $\{t_{k\beta_k}\}$. In particular, note that

$$P_i = \sum_{\alpha_i} |e_{i\alpha_i}\rangle\langle e_{i\alpha_i}| \quad \text{and} \quad Q_k = \sum_{\beta_k} |\epsilon_{k\beta_k}\rangle\langle\epsilon_{k\beta_k}|$$

provide orthogonal rank 1 decompositions of the projection operators describing the commutative subalgebras. Now, since $F_{cl} : \mathcal{B}_{cl} \to \mathcal{A}_{cl}$ is linear and †-preserving, there exist numbers $f_{ki} \in \mathbb{R}$ such that

$$F_{cl}(Q_k) = \sum_i f_{ki}P_i$$

for all k. By the assumption that $F = j_{\mathcal{A}} \circ F_{cl} \circ E_{\mathcal{B}}$ together with all the consequences derived thus far, we find

$$\begin{aligned} F\left(|\epsilon_{k\beta_k}\rangle\langle\epsilon_{l\gamma_l}|\right) &= (j_{\mathcal{A}} \circ F_{cl})\Big(\delta_{kl}\mathrm{tr}\big(\tau_k|\epsilon_{k\beta_k}\rangle\langle\epsilon_{l\gamma_l}|\big)Q_k\Big) = \delta_{kl}\delta_{\beta_k\gamma_k}t_{k\beta_k}(j_{\mathcal{A}} \circ F_{cl})(Q_k) \\ &= \delta_{kl}\delta_{\beta_k\gamma_k}t_{k\beta_k}\sum_i f_{ki}P_i = \delta_{kl}\delta_{\beta_k\gamma_k}t_{k\beta_k}\sum_i f_{ki}\sum_{\alpha_i}|e_{i\alpha_i}\rangle\langle e_{i\alpha_i}|. \end{aligned}$$

By a similar Hilbert–Schmidt adjoint calculation as in the proof of the ($\Leftarrow$) direction, this shows that $\mathscr{D}[F]$ is diagonal in these bases.

Finally, the claim that $[\mathscr{D}[F], \rho \otimes 1_{\mathcal{B}}] = 0$ follows from these equivalent conditions by simultaneously diagonalizing $\mathscr{D}[F]$ and $\rho \otimes 1_{\mathcal{B}}$. ■

Remark 3.1. The proof of Proposition 2.7 shows $\sum_k f_{ki} = 1$ if F is unital and $f_{ki} \geqslant 0$ if F is positive. Hence, if F is positive and unital, then the collection f_{ki} determine a stochastic matrix. Analogous statements hold for ρ and θ if ω and $\omega \circ F$ have analogous properties.

After Definition 2.6, it was claimed that the definition of a classical model was made so as to preserve compositionality. This is stated more precisely in the following.

Proposition 3.2. *Suppose* $(\mathcal{B} \xrightarrow{F} \mathcal{A}, \mathcal{A} \xrightarrow{\omega} \mathbb{C})$ *and* $(\mathcal{C} \xrightarrow{G} \mathcal{B}, \mathcal{B} \xrightarrow{\omega \circ F} \mathbb{C})$ *have classical models* $(\mathcal{B}_{\text{cl}} \xrightarrow{F_{\text{cl}}} \mathcal{A}_{\text{cl}}, E_{\mathcal{A}}, E_{\mathcal{B}})$ *and* $(\mathcal{C}_{\text{cl}} \xrightarrow{G_{\text{cl}}} \mathcal{B}_{\text{cl}}, E'_{\mathcal{B}}, E_{\mathcal{C}})$. *Then* $(\mathcal{C} \xrightarrow{G \circ F} \mathcal{A}, \mathcal{A} \xrightarrow{\omega} \mathbb{C})$ *has* $(\mathcal{C}_{\text{cl}} \xrightarrow{F_{\text{cl}} \circ G_{\text{cl}}} \mathcal{A}_{\text{cl}}, E_{\mathcal{A}}, E_{\mathcal{C}})$ *as a classical model.*

Remark 3.3. Note that the two conditional expectations $E'_{\mathcal{B}}$ need not equal $E_{\mathcal{B}}$ in Proposition 3.2, but they are almost everywhere equivalent. None of these subtleties arise if all densities have full rank. See [6] for details.

Proof of Proposition 3.2. The state-preservation conditions hold by assumption, while the factorization follows from

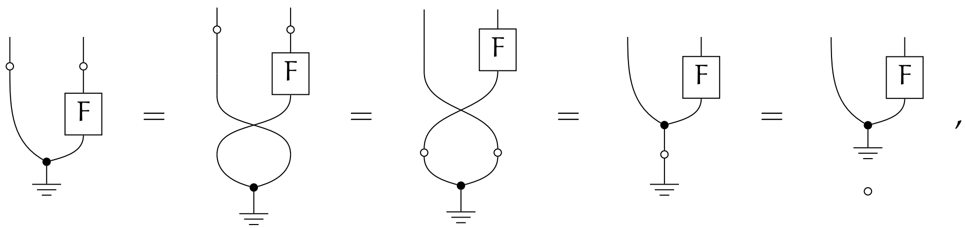
$$j_{\mathcal{A}} \circ (F_{\text{cl}} \circ G_{\text{cl}}) \circ E_{\mathcal{C}} = j_{\mathcal{A}} \circ F_{\text{cl}} \circ E_{\mathcal{B}} \circ j_{\mathcal{B}} \circ G_{\text{cl}} \circ E_{\mathcal{C}} = F \circ G. \qquad \blacksquare$$

Lemma 3.4. *In terms of the notation from Definition 2.2, if* F *is* $\dagger$*-preserving, then the associated channel density is self-adjoint (equivalently, the channel state is* $\dagger$*-preserving).*

Proof. This follows from the fact that F is $\dagger$-preserving and $\mu_{\mathcal{A}}$ is $\dagger$-reversing, namely $\mu_{\mathcal{A}} \circ \dagger = \dagger \circ \mu_{\mathcal{A}} \circ \gamma$, where γ is the swap map. In more detail,

$$\Big((\text{id} \otimes F^*)\big(\mu^*_{\mathcal{A}}(1_{\mathcal{A}})\big)\Big)^\dagger = (\text{id} \otimes F^*)\big(\mu^*_{\mathcal{A}}(1_{\mathcal{A}})\big)^\dagger = (\text{id} \otimes F^*)\Big(\gamma\big(\mu^*_{\mathcal{A}}(1^\dagger_{\mathcal{A}})\big)\Big) = (\text{id} \otimes F^*)\big(\mu^*_{\mathcal{A}}(1_{\mathcal{A}})\big).$$

Equivalently,



where the first identity follows from the properties of the trace, and where $\circ$ denotes the involution (cf. [10]). $\blacksquare$

Lemma 3.5. *Given linear maps* $F : \mathcal{B} \to \mathcal{A}$ *and* $\omega : \mathcal{A} \to \mathbb{C}$, *the densities associated with* $\omega \circ i_F$ *and* $\omega \circ {}_F i$ *are* $(\rho \otimes 1_{\mathcal{B}})\mathscr{D}[F]$ *and* $\mathscr{D}[F](\rho \otimes 1_{\mathcal{B}})$, *respectively, where* $\rho := \omega^*(1)$ *is the density associated with* ω *via* $\omega = \text{tr}(\rho^\dagger \cdot\ )$.

Proof. We prove one of these claims as the other is completely analogous. Indeed, by using the cyclicity property of the trace and the definition of the Hilbert–Schmidt adjoint,

$$\begin{aligned} \text{tr}\Big(\big((\rho \otimes 1_{\mathcal{B}})\mathscr{D}[F]\big)^\dagger (A \otimes B)\Big) &= \text{tr}\Big(\big((\text{id}_{\mathcal{A}} \otimes F^*)\mu^*_{\mathcal{A}}(1_{\mathcal{A}})\big)^\dagger (\rho^\dagger A \otimes B)\Big) \\ &= \text{tr}\Big(\mu^*_{\mathcal{A}}(1_{\mathcal{A}})^\dagger\big(\rho^\dagger A \otimes F(B)\big)\Big) \\ &= \text{tr}\Big(1^\dagger_{\mathcal{A}}\mu_{\mathcal{A}}\big(\rho^\dagger A \otimes F(B)\big)\Big) \\ &= \text{tr}\big(\rho^\dagger A F(B)\big) \\ &= (\omega \circ i_F)(A \otimes B). \end{aligned}$$

Since $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are arbitrary and since the trace is non-degenerate, the density associated with $\omega \circ i_F$ is $(\rho \otimes 1_{\mathcal{B}})\, \mathscr{D}[F]$. ■

Lemma 3.6. *Given unital †-preserving maps $F : \mathcal{B} \to \mathcal{A}$ and $\omega : \mathcal{A} \to \mathbb{C}$, one has*

$$\omega \circ i_F \circ \dagger = \dagger \circ \omega \circ {}_F i \quad \text{and} \quad \omega \circ {}_F i \circ \dagger = \dagger \circ \omega \circ i_F\,.$$

In particular, the functional $F \star \omega := \frac{1}{2}(\omega \circ i_F + \omega \circ {}_F i)$ is unital and †-preserving and has a density given by the Jordan product of the channel density together with the density ρ representing ω, i.e.,

$$(F \star \omega)^*(1) = \frac{1}{2}\Big((\rho \otimes 1_{\mathcal{B}})\, \mathscr{D}[F] + \mathscr{D}[F] (\rho \otimes 1_{\mathcal{B}}) \Big).$$

Proof. The two identities immediately follow from Lemma 3.4, Lemma 3.5, and

$$\big((\rho \otimes 1_{\mathcal{B}})\, \mathscr{D}[F] \big)^\dagger = \mathscr{D}[F]^\dagger (\rho \otimes 1_{\mathcal{B}})^\dagger = \mathscr{D}[F] (\rho \otimes 1_{\mathcal{B}}).$$

The †-preserving property follows from this. The unitality follows from the fact that the composite of unital maps is unital. The formula for the density in terms of the Jordan product follows from the first two identities and Lemma 3.5. ■

Lemma 3.7. *Let $F : \mathcal{B} \to \mathcal{A}$ and $\omega : \mathcal{A} \to \mathbb{C}$ be unital linear maps. Then*

$$\omega \circ i_F \circ i_{\mathcal{A}} = \omega = \omega \circ {}_F i \circ i_{\mathcal{A}} \qquad \text{and} \qquad \omega \circ i_F \circ i_{\mathcal{B}} = \omega \circ F = \omega \circ {}_F i \circ i_{\mathcal{B}}.$$

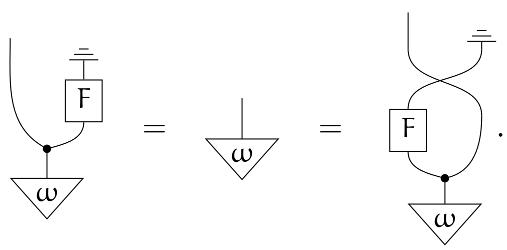
Proof. The proofs are straightforward calculations. For example,

$$(\omega \circ i_F \circ i_{\mathcal{A}})(A) = (\omega \circ \mu_{\mathcal{A}} \circ (\mathrm{id}_{\mathcal{A}} \otimes F) \circ i_{\mathcal{A}})(A) = \omega\big(AF(1_{\mathcal{B}})\big) = \omega(A 1_{\mathcal{A}}) = \omega(A)$$

and

$$(\omega \circ {}_F i \circ i_{\mathcal{A}})(A) = \big(\omega \circ \mu_{\mathcal{A}} \circ (F \otimes \mathrm{id}_{\mathcal{A}}) \circ \gamma \circ i_{\mathcal{A}}\big)(A) = \omega\Big(\mu_{\mathcal{A}}\big(F(1_{\mathcal{B}})A\big)\Big) = \omega\big(F(1_{\mathcal{B}})A\big) = \omega(A)$$

for all $A \in \mathcal{A}$. A similar calculation holds for the second set of claims. The calculations are easily visualized using string diagrams. For example, the two we just showed are given by



Note that unitality of F was used here. ■

Proof of Theorem 2.13.

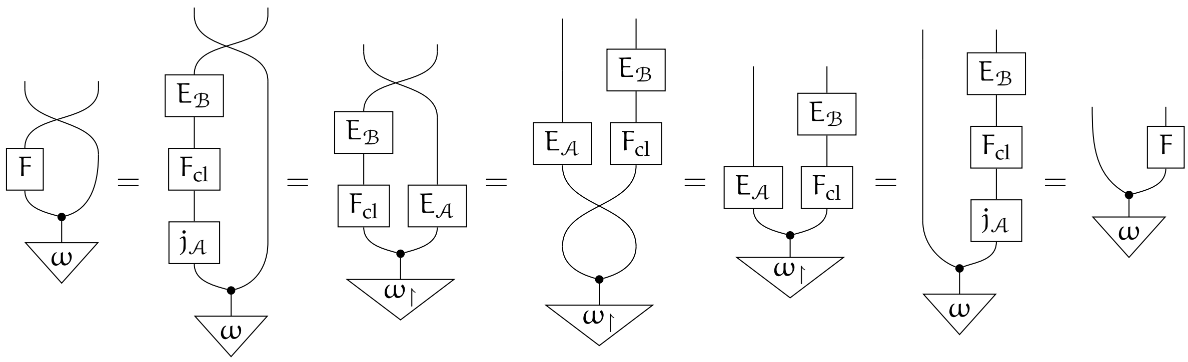
(a) This follows from Lemma 3.6.

(b) This follows from the linearity of i_F and ${}_F i$ in the argument F as well as linearity of ω. Indeed, writing $F \star \omega$ out more explicitly as

$$F \star \omega = \frac{1}{2}\Big(\omega \circ \mu_{\mathcal{A}} \circ (\mathrm{id} \otimes F) + \omega \circ \mu_{\mathcal{A}} \circ (F \otimes \mathrm{id}) \circ \gamma\Big)$$

shows that in fact each term on the right-hand-side depends linearly on both F and ω.

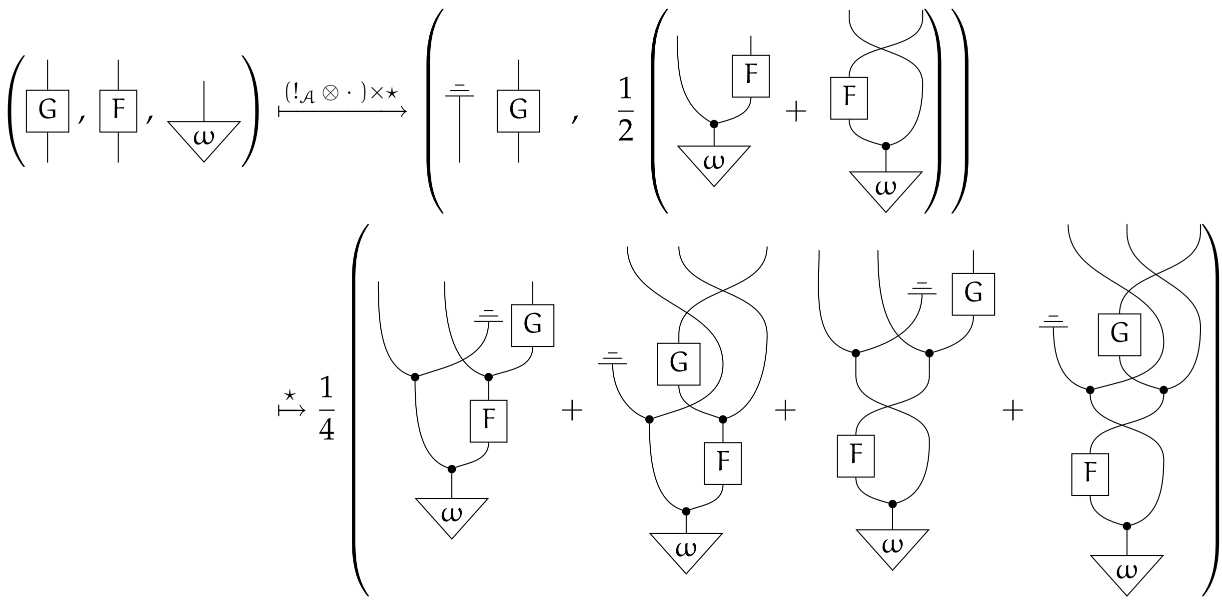
(c) In what follows, we will prove $\omega \circ {}_F i = \omega \circ i_F$. We will use string diagrams for an elegant proof. It will use the fact that state-preserving conditional expectations are Bayesian inverses [6], and it will also crucially use the fact that $\mu_{\mathcal{A}_{\mathrm{cl}}} \circ \gamma = \mu_{\mathcal{A}_{\mathrm{cl}}}$, which only holds for commutative C*-algebras. We implement the notation of Definition 2.6. The calculation

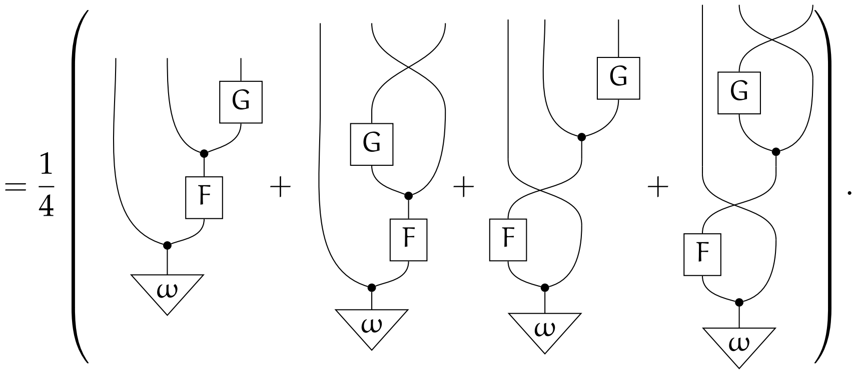


implies the claim.

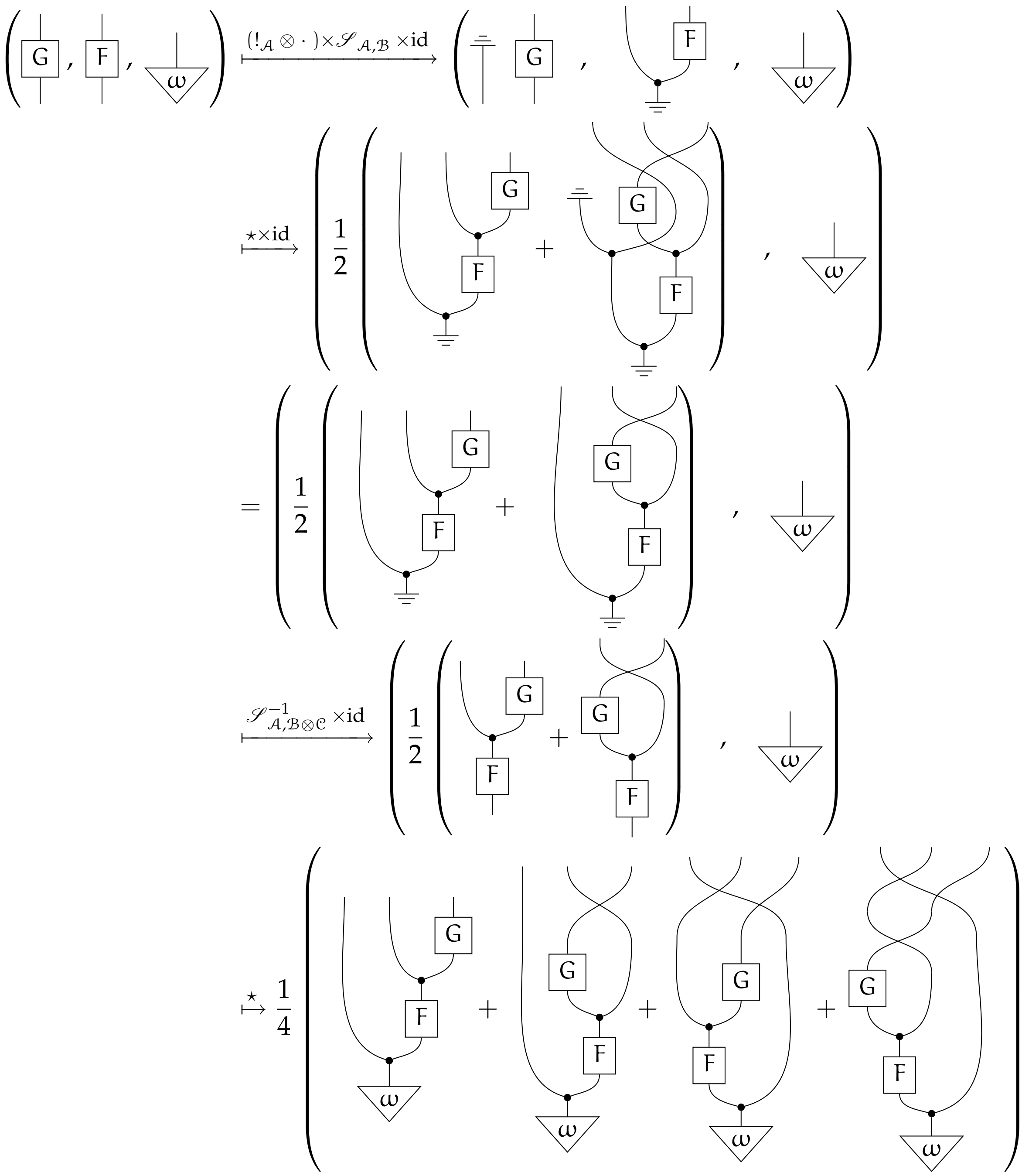
(d) This follows from Lemma 3.7.

(e) The proof of associativity will be achieved by showing commutativity of the diagram in Remark 2.11. Following along the left-hand-side of that diagram results in





Meanwhile, following along the top and right side of the diagram in Remark 2.11 gives



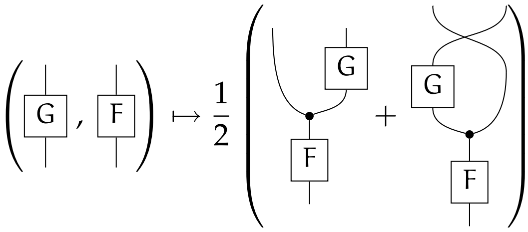
By comparing these two results and using string-diagrammatic manipulations, we immediately see that they are equal. ■

Remark 3.8. When expressed in terms of densities, the preservation of classical limit axiom follows from the fact that the densities $\mathscr{D}[\mathsf{F}]$ and $\mathscr{D}[\omega] \otimes 1_{\mathcal{B}}$ commute, while the associativity axiom follows essentially from the fact that the densities $1_{\mathcal{A}} \otimes \mathscr{D}[\mathsf{G}]$ and $\mathscr{D}[\omega] \otimes 1_{\mathcal{B}} \otimes 1_{\mathcal{C}}$ commute.

4 Extension to channels

The graphical proof of Theorem 2.13 is easily seen to be independent of whether ω is a state or a channel. More precisely, if one defines

$$\begin{aligned} \mathbf{Hom}(\mathcal{C},\mathcal{B}) \times \mathbf{Hom}(\mathcal{B},\mathcal{A}) &\xrightarrow{\star} \mathbf{Hom}(\mathcal{C},\mathcal{A}\otimes\mathcal{B}) \\ (\mathsf{G},\mathsf{F}) &\mapsto \frac{1}{2}\left(\mathsf{F}\circ \mathsf{i}_{\mathsf{G}} + \mathsf{F}\circ {}_{\mathsf{G}}\mathsf{i}\right) \end{aligned}$$



(4.1)

(bypassing the Choi–Jamiołkowski isomorphism altogether), then the proof of Theorem 2.13 goes through without any changes. This seems to go against the no-go theorems of [8]. The resolution to this seeming paradox comes from the ‘preservation of classical limit’ axiom. We have shown that our formulation of this axiom is equivalent to the one of [8] when dealing with turning a channel plus state into a joint state (cf. Proposition 2.7). However, when extending our definition of a classical model (Definition 2.6) to channels, as opposed to just states, we find that our definition is inequivalent to the axiom of commutativity enforced in the no-go theorems of [8]. Thus, by using categorical reasoning and the framework of quantum Markov categories [10], we are able to bypass the no-go theorems of [8]. The present section will illustrate how this works.

Notation 4.2. In all definitions made before, wherever a linear functional $\omega : \mathcal{A} \to \mathbb{C}$ appears, the same exact definition is now made for a linear map $\omega : \mathcal{A} \to \mathcal{Z}$, where $\mathcal{Z}$ is some finite-dimensional C*-algebra. For example, a ***classical model*** for a pair $(\mathcal{B} \xrightarrow{\mathsf{F}} \mathcal{A}, \mathcal{A} \xrightarrow{\omega} \mathcal{Z})$ consists of commutative C*-algebras $\mathsf{j}_{\mathcal{A}} : \mathcal{A}_{\mathrm{cl}} \hookrightarrow \mathcal{A}$, $\mathsf{j}_{\mathcal{B}} : \mathcal{B}_{\mathrm{cl}} \hookrightarrow \mathcal{B}$, conditional expectations $\mathsf{E}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}_{\mathrm{cl}}$, $\mathsf{E}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}_{\mathrm{cl}}$, and a linear map $\mathsf{F}_{\mathrm{cl}} : \mathcal{B}_{\mathrm{cl}} \to \mathcal{A}_{\mathrm{cl}}$ such that[4]

$$\mathsf{F} = \mathsf{j}_{\mathcal{A}} \circ \mathsf{F}_{\mathrm{cl}} \circ \mathsf{E}_{\mathcal{B}}, \qquad \omega = \omega_{\restriction} \circ \mathsf{E}_{\mathcal{A}}, \qquad \text{and} \qquad \omega \circ \mathsf{F} = (\omega \circ \mathsf{F})_{\restriction} \circ \mathsf{E}_{\mathcal{B}},$$

where the $\restriction$ subscript means restriction to the commutative subalgebras. The only difference in notation/terminology between this section and the previous sections is that a ‘family of states

[4]As in the case of states, $\omega \circ \mathsf{F} = (\omega \circ \mathsf{F})_{\restriction} \circ \mathsf{E}_{\mathcal{B}}$ is a consequence of the other two conditions.

over time function' is replaced with a 'family of channels over time function.' Again, this is slightly abusive terminology since channels here need not be positive. Note, however, that the compositionality/associativity axiom can now be formulated much more simply without ever even using the Choi–Jamiołkowski isomorphism. Namely, it says that the diagram

$$\begin{array}{ccc} \mathbf{Hom}(\mathcal{C},\mathcal{B}) \times \mathbf{Hom}(\mathcal{B},\mathcal{A}) \times \mathbf{Hom}(\mathcal{A},\mathcal{Z}) & \xrightarrow{\star\times\mathrm{id}} & \mathbf{Hom}(\mathcal{B}\otimes\mathcal{C},\mathcal{A}) \times \mathbf{Hom}(\mathcal{A},\mathcal{Z}) \\ \Big\downarrow{\scriptstyle (!_{\mathcal{A}}\otimes\,\cdot\,)\times\star} & & \Big\downarrow{\scriptstyle\star} \\ \mathbf{Hom}(\mathcal{C},\mathcal{A}\otimes\mathcal{B}) \times \mathbf{Hom}(\mathcal{A}\otimes\mathcal{B},\mathcal{Z}) & \xrightarrow[\star]{} & \mathbf{Hom}(\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C},\mathcal{Z}) \end{array}$$

commutes, i.e.,

$$(G \star F) \star \omega = (!_A \otimes G) \star (F \star \omega).$$

Proposition 4.3. *Given †-preserving linear maps $(\mathcal{B} \xrightarrow{F} \mathcal{A}, \mathcal{A} \xrightarrow{\omega} \mathcal{Z})$, the following are equivalent.*

i. The identity $\omega \circ i_F = \omega \circ {}_F i$ holds.

ii. The densities $1_{\mathcal{Z}} \otimes \mathscr{D}[F]$ and $\mathscr{D}[\omega] \otimes 1_{\mathcal{B}}$ commute, i.e., $\left[1_{\mathcal{Z}} \otimes \mathscr{D}[F], \mathscr{D}[\omega] \otimes 1_{\mathcal{B}}\right] = 0$.

Furthermore, if the pair (F, ω) admits a classical model, then items i and ii hold.

Remark 4.4. Note that when $\mathcal{Z} = \mathbb{C}$, Proposition 4.3 provides a strengthening of Proposition 2.7. In addition, Proposition 4.3 illustrates in what sense a certain commutativity condition holds when a system is effectively classical. This commutativity condition is what replaces the commutativity condition in [8] and allows us to guarantee that a family of channels over time function exists (see Theorem 4.7 below).

Lemma 4.5. *Let $\mathcal{A} = \bigoplus_{x\in X} \mathbb{M}_{m_x}(\mathbb{C})$. Then*

$$\bigoplus_{x\in X} \mathbb{M}_{m_x}(\mathbb{C}) \xrightarrow{\mu_{\mathcal{A}}^*} \bigoplus_{x',x''\in X} \left(\mathbb{M}_{m_{x'}}(\mathbb{C}) \otimes \mathbb{M}_{m_{x''}}(\mathbb{C})\right),$$

the Hilbert–Schmidt adjoint of the multiplication map, is given explicitly by the formula

$$(\mu_{\mathcal{A}}^*)_{(x',x'')x}(A) = \delta_{xx'}\delta_{xx''} \sum_{i,j,k} A_{ij} E_{ik}^{(m_x)} \otimes E_{kj}^{(m_x)}$$

for all $A \in \mathbb{M}_{m_x}(\mathbb{C})$ and for all $x, x', x'' \in X$. Here, A_{ij} denotes the ij-th entry of A with respect to the standard basis.

Proof of Lemma 4.5. This follows from the fact that $(\mu_{\mathcal{A}})_{x(x',x'')}$ vanishes unless $x = x' = x''$, the definition of the Hilbert–Schmidt inner product, and the fact that multiplication is computed component-wise. ■

The following lemma generalizes Lemma 3.5.

Lemma 4.6. *Given linear maps* $(\mathcal{B} \xrightarrow{F} \mathcal{A}, \mathcal{A} \xrightarrow{\omega} \mathcal{Z})$,

$$\mathscr{D}[\omega \circ i_F] = \big(\mathscr{D}[\omega] \otimes 1_{\mathcal{B}}\big)\big(1_{\mathcal{Z}} \otimes \mathscr{D}[F]\big) \quad \textit{and} \quad \mathscr{D}[\omega \circ {}_F i] = \big(1_{\mathcal{Z}} \otimes \mathscr{D}[F]\big)\big(\mathscr{D}[\omega] \otimes 1_{\mathcal{B}}\big).$$

Proof of Lemma 4.6. By the distributive property of $\otimes$ and $\oplus$ along with Lemma 4.5, it suffices to assume the algebras are matrix algebras. Set $\omega^{ij}_{\beta\alpha} \in \mathbb{C}$ to be the unique numbers satisfying $\omega^*(E_{\beta\alpha}) = \sum_{i,j} \omega^{ij}_{\beta\alpha} E_{ij}$ for all α, β. Then

$$\begin{aligned}
\mathscr{D}[\omega \circ i_F] &= \Big(\mathrm{id}_{\mathcal{Z}} \otimes \big(\omega \circ \mu_{\mathcal{A}} \circ (\mathrm{id}_{\mathcal{A}} \otimes F)\big)^*\Big) \left(\sum_{\alpha,\beta} E_{\alpha\beta} \otimes E_{\beta\alpha}\right) \\
&= (\mathrm{id}_{\mathcal{Z}} \otimes \mathrm{id}_{\mathcal{A}} \otimes F^*) \left(\sum_{\alpha,\beta} E_{\alpha\beta} \otimes \mu^*_{\mathcal{A}}\big(\omega^*(E_{\beta\alpha})\big)\right) \\
&= (\mathrm{id}_{\mathcal{Z}} \otimes \mathrm{id}_{\mathcal{A}} \otimes F^*) \left(\sum_{\alpha,\beta} \sum_{i,j} \omega^{ij}_{\beta\alpha} E_{\alpha\beta} \otimes \mu^*_{\mathcal{A}}(E_{ij})\right) \\
&= \sum_{\alpha,\beta} \sum_{i,j,k} \omega^{ij}_{\beta\alpha} E_{\alpha\beta} \otimes E_{ik} \otimes F^*(E_{kj}).
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
\big(\mathscr{D}[\omega] \otimes 1_{\mathcal{B}}\big)\big(1_{\mathcal{Z}} \otimes \mathscr{D}[F]\big) &= \sum_{\alpha,\beta} \sum_{i,j} E_{\alpha\beta} \otimes \omega^*(E_{\beta\alpha}) E_{ij} \otimes F^*(E_{ji}) \\
&= \sum_{\alpha,\beta} \sum_{i,j,k,l} \omega^{kl}_{\beta\alpha} E_{\alpha\beta} \otimes \underbrace{(E_{kl} E_{ij})}_{\delta_{li} E_{kj}} \otimes F^*(E_{ji}) \\
&= \sum_{\alpha,\beta} \sum_{i,j,k} \omega^{ki}_{\beta\alpha} E_{\alpha\beta} \otimes E_{kj} \otimes F^*(E_{ji}).
\end{aligned}$$

By relabelling the dummy indices, the two expressions are seen to be the same. The other identity follows from similar calculations. ∎

Proof of Proposition 4.3. The proof of the last claim implies item i follows exactly the same argument as in the proof of the 'preservation of classical limit' in Theorem 2.13. The equivalence between item i and item ii follows from Lemma 4.6. ∎

Theorem 4.7. *A family of channels over time function exists and (4.1) provides an explicit construction.*

Proof. The proof is completely analogous to the proof of Theorem 2.13. ∎

5 Discussion

In this paper, we constructed a consistent way of associating a joint 'state' on $\mathcal{A} \otimes \mathcal{B}$ with every state on $\mathcal{A}$ and a quantum channel[5] $\mathcal{A} \to \mathcal{B}$, in such a way that by-passes the no-go result of [8]. The reason 'state' is in quotes is because the associated joint matrix is only self-adjoint in general, but is not necessarily positive. Therefore, we have not answered the more physical question of whether there exists a consistent manner of associating a genuinely positive joint state to an initial (positive) state and a positive (perhaps even completely positive) map. In particular, we do not know whether there is such an assignment satisfying the axioms we have outlined that also includes such a positivity constraint. Furthermore, although we have provided a construction of a family of states over time function, we have made no claim as to the uniqueness of such an assignment. In particular, we do not know if the Jordan product provides the unique function that satisfies these axioms.

An interesting aspect of our work is that the proof of the main theorem was provided in the setting of (enriched) quantum Markov categories [10]. The proof itself also illustrated a natural generalization to channels, where the 'preservation of the classical limit' axiom of [8] was replaced by an alternative one that allowed us to bypass the no-go result of [8]. It seems reasonable to suspect that extensions to certain von Neumann algebras are possible, though this is only a speculation. We leave this question to the interested reader.

Yet another question that arises as a result of our theorem is related to quantum conditionals, which can be viewed as the opposite procedure to the one described in this work. In particular, if one is given a joint state, can one find a process for which the joint state can be expressed in terms of this process and its marginal? In [11], it was shown that one can express a joint state ζ on $\mathcal{A} \otimes \mathcal{B}$ as $\zeta = \omega \circ i_F$ for some *positive* F, and where $\omega = \zeta \circ i_{\mathcal{A}}$, if and only if some non-trivial condition holds. The results of this paper suggest that perhaps one should change the question to the existence of a positive F such that $\zeta = \frac{1}{2}(\omega \circ i_F + \omega \circ {}_F i)$. It is not presently known if such a symmetrization procedure allows more conditionals to exist.

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References

[1] Kenta Cho and Bart Jacobs, *Disintegration and Bayesian inversion via string diagrams*, Math. Struct. Comp. Sci. (2019), 1–34, available at 1709.00322. ↑2, 3

[2] Florence Clerc, Vincent Danos, Fredrik Dahlqvist, and Ilias Garnier, *Pointless learning*, International conference on foundations of software science and computation structures, 2017, pp. 355–369. ↑2

[3] Bob Coecke and Aleks Kissinger, *Picturing quantum processes: A first course in quantum theory and diagrammatic reasoning*, Cambridge University Press, 2017. ↑4

[5] In the Heisenberg picture, the directionality of the arrows is $\mathcal{B} \to \mathcal{A}$ and is the convention followed in the present paper.

[4] Joseph F. Fitzsimons, Jonathan A. Jones, and Vlatko Vedral, *Quantum correlations which imply causation*, Sci. Rep. **5** (2015), no. 1, 18281, available at 1302.2731. ↑8

[5] Tobias Fritz, *A synthetic approach to Markov kernels, conditional independence and theorems on sufficient statistics*, Adv. Math. **370** (2020), 107239, available at 1908.07021. ↑2, 3

[6] Luca Giorgetti, Arthur J. Parzygnat, Alessio Ranallo, and Benjamin P. Russo, *Bayesian inversion and the Tomita-Takesaki modular group*, 2021. arXiv preprint: 2112.03129 [math.OA]. ↑5, 11, 13

[7] Chris Heunen and Jamie Vicary, *Categories for quantum theory: an introduction*, Oxford University Press, USA, 2019. ↑4

[8] Dominic Horsman, Chris Heunen, Matthew F. Pusey, Jonathan Barrett, and Robert W. Spekkens, *Can a quantum state over time resemble a quantum state at a single time?*, Proc. R. Soc. A **473** (2017), no. 2205, 20170395, available at 1607.03637. ↑2, 4, 5, 6, 7, 8, 15, 16, 18

[9] Andrzej Jamiołkowski, *Linear transformations which preserve trace and positive semidefiniteness of operators*, Rep. Math. Phys. **3** (1972), no. 4, 275–278. ↑4

[10] Arthur J. Parzygnat, *Inverses, disintegrations, and Bayesian inversion in quantum Markov categories*, 2020. arXiv preprint: 2001.08375 [quant-ph]. ↑3, 11, 15, 18

[11] ———, *Conditional distributions for quantum systems*, Proceedings 18th International Conference on Quantum Physics and Logic, 2021, pp. 1–13. ↑2, 3, 18

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