THE PRODUCT OVER ALL PRIMES IS $4\pi^2$

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Abstract. *We compute "à la Euler" the regularized product over all prime numbers. The result is* $4\pi^2$.

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Christophe Soulé asks (and the question is proposed in [SABK] p.101) to give a meaning and find a value for the product over all primes, similar to the magic

$$\infty! = 1.2.3.\ldots = \sqrt{2\pi} \ .$$

We carry a computation "à la Euler" in order to provide an answer. We show that

$$\prod_p p = 4\pi^2 \ .$$

Given an increasing sequence $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ one defines the regularized infinite product

$$\prod_{n=1}^{+\infty} \lambda_n = \exp(-\zeta'_\lambda(0))$$

where ζ_λ is the zeta function associated to the sequence (λ_n),

$$\zeta_\lambda(s) = \sum_{n=1}^{+\infty} \lambda_n^{-s} \ .$$

(see [SABK] chapter V, definition 5 page 97). Implicitely this assumes that the zeta function has an analytic extension up to 0, or that we have some way of computing $\zeta'_\lambda(0)$. We will take a "liberal" view on this point, which is necessary since, as observed in [SABK],

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the zeta function associated to the sequence of primes does not extend meromorphically to a neighborhood of 0.

Recall that (this is used in the definition of the Artin-Hasse exponential, see [Ko] chapter IV)

$$\exp(X) = \prod_{n=1}^{+\infty} (1 - X^n)^{-\frac{\mu(n)}{n}} ,$$

where μ is Möebius function.

From this we get

$$e^{p^{-s}} = \prod_{n=1}^{+\infty} (1 - p^{-ns})^{-\frac{\mu(n)}{n}} .$$

We consider now the zeta function associated to the sequence of primes

$$\mathcal{P}(s) = \sum_p \frac{1}{p^s} .$$

So it follows that

$$\begin{aligned} e^{\mathcal{P}(s)} &= \prod_p e^{p^{-s}} \\ &= \prod_p \prod_{n=1}^{+\infty} (1 - p^{-ns})^{-\frac{\mu(n)}{n}} \\ &= \prod_{n=1}^{+\infty} \prod_p (1 - p^{-ns})^{-\frac{\mu(n)}{n}} \\ &= \prod_{n=1}^{+\infty} \zeta(ns)^{\frac{\mu(n)}{n}} \end{aligned}$$

where we have used in the last equality Euler's expansion of Riemann zeta function (and following Euler's tradition we don't justify the product exchange, but is can be done). This formula

$$e^{\mathcal{P}(s)} = \prod_{n=1}^{+\infty} \zeta(ns)^{\frac{\mu(n)}{n}}$$

can be found in [LW] and in [Da], but in this last reference there is a typo in the formula (n in the denominator of the exponent is missing).

Now, taking the logarithmic derivative we get

$$\mathcal{P}'(s) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n} \frac{n\zeta'(ns)}{\zeta(ns)} = \sum_{n=1}^{+\infty} \mu(n) \frac{\zeta'(ns)}{\zeta(ns)} ,$$

thus

$$\mathcal{P}'(0) = \left(\sum_{n=1}^{+\infty} \mu(n)\right) \frac{\zeta'(0)}{\zeta(0)} \ .$$

Now we recall that

$$\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

so

$$\mathcal{P}'(0) = \frac{1}{\zeta(0)} \frac{\zeta'(0)}{\zeta(0)} = -2 \log(2\pi) \ .$$

We conclude

$$\prod_p p = e^{-\mathcal{P}'(0)} = (2\pi)^2 \ .$$

Similarly we get

$$\prod_p p^s = (2\pi)^{2s} \ .$$

Since

$$\zeta(s) = \frac{\prod_p p^s}{\prod_p (p^s - 1)}$$

we get (not in a "regularized" way)

$$\prod_p (p^s - 1) = \frac{(2\pi)^{2s}}{\zeta(s)} \ ,$$

in particular

$$\prod_p (p - 1) = 0 \ ,$$

$$\prod_p (p^2 - 1) = 48\pi^2 \ ,$$

and so on...

A "more convergent" derivation.

We can define the function of two complex variables

$$e^{\mathcal{P}(s,t)} = \prod_{n=1}^{+\infty} \zeta(ns)^{\frac{\mu(n)}{n^t}} \ ,$$

which is meromorphic for $\operatorname{Re} s > 1$ and $\operatorname{Re} t > 1$.

Now we can compute

$$\frac{\partial \mathcal{P}}{\partial s}(s,t) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{t-1}} \frac{\zeta'(ns)}{\zeta(ns)} \ ,$$

which converges for $\mathrm{Re}\, t > 1$ and $\mathrm{Re}\, s > 0$. For a fixed $t \in \mathbf{C}$ with $\mathrm{Re}\, t > 2$, and for $s \to 0$ we get

$$\lim_{\substack{s\to 0\\ \mathrm{Re}\, s>0}} \frac{\partial \mathcal{P}}{\partial s}(s,t) = \left(\sum_{n=1}^{+\infty} \frac{\mu(n)}{n^{t-1}}\right) \frac{\zeta'(0)}{\zeta(0)} = \frac{\zeta'(0)}{\zeta(t-1)\zeta(0)} \ .$$

Taking now the meromorphic extension of this expression and passing to the limit $t \to 1$, we get

$$\lim_{t\to 1} \lim_{\substack{s\to 0\\ \mathrm{Re}\, s>0}} \frac{\partial \mathcal{P}}{\partial s}(s,t) = \frac{\zeta'(0)}{\zeta(0)^2} \ .$$

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