TENSOR STRUCTURE FROM SCALAR FEYNMAN MATROIDS

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Abstract. We show how to interpret the scalar Feynman integrals which appear when reducing tensor integrals as scalar Feynman integrals coming from certain nice matroids.

1. Introduction

Feynman integrals arise as evaluations of Feynman graphs by Feynman rules. In momentum space, these integrals $\int I_\Gamma$ are integrals over rational functions I_Γ of internal loop momenta, with unit numerator in case of a scalar field theory.

The tensor integrals appearing in the general case can be reduced to scalar integrals as well. A basis of such scalar integrals sufficient to compute a desired amplitude to a given order is often called a set of master integrals [15]. Such master integrals are not straightforwardly related to graphs.

Indeed, consider a Feynman integral arising from some graph, with a denominator given by a product of scalar propagators $P_{k_e} = 1/(k_e^2 + m_e^2)$ for each edge e. With tensor structure, we will have scalar products $k_e \cdot k_j$ in the numerator. Assuming that propagators $P_{k_e}, P_{k_j}, P_{k_f}$, $k_f = k_e - k_j$, appear in the denominator, we could resolve

$$2k_e \cdot k_j = P_{k_e} + P_{k_j} - P_{k_f} - m_e^2 - m_j^2 + m_f^2, \tag{1}$$

eliminating the scalar product in favor of scalar integrals with possibly fewer propagators. The latter then correspond to graphs where an edge is contracted to a point.

But those propagators might not be present in the denominator. Then, the combinatorial interpretation in terms of graphs is missing. Nevertheless, general products of propagators, in the denominator or numerator, even more generally with arbitrary complex powers, have proven useful in practical computations [11, 10]. Indeed, any tensor integral can be reduced to a scalar integral on the expense of having a sufficiently general product of propagators at hand. But the combinatorial interpretation alluded to above in terms of graphs is not available when the product under consideration does not configure a graph.

If it does, we have the important tools of parametric representations via Kirchhoff polynomials available. Being compatible with the raising of propagators to non-integer powers, these polynomials allow for systematic insights into the algebraic geometric properties of Feynman amplitudes and a satisfying mathematical understanding of these periods and functions [5, 6, 7, 1].

Here, we answer the question what replaces such graph polynomials in the general case, when the product of propagators to start with does not configure a graph. We will see that the notion of a graph is replaced by more general notion of matroid, with the notion of one-particle irreducibility most crucially still being intact. Hence, a systematic way to cope

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with tensor structure of Feynman graphs is to switch to what we might dub scalar Feynman matroids.

2. Matroids

2.1. **Definitions.** There are many equivalent ways to define matroids. The most useful for our purposes is the circuit definition. A standard reference for matroid theory is [12].

Definition 1. A *matroid* consists of a finite set E and a set $\mathcal{C}$ of subsets of E satisfying

(1) $\emptyset \notin \mathcal{C}$
(2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$.
(3) If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there is a $C_3 \in \mathcal{C}$ with $C_3 \subseteq (C_1 \cup C_2) \smallsetminus e$.

The set E corresponds to the set of edges of a graph. The elements of $\mathcal{C}$ are called the *circuits* of the matroid and correspond to cycles (with no repeated vertices) of a graph. Thus, in the case of graphs, the first two axioms state the obvious facts that the empty set is not a cycle and that no cycle contains a smaller cycle. The third axiom is the interesting one and for graphs says that if two cycles share an edge, then together but without the edge they must form or contain a cycle.

Consequently, a graph defines a matroid, called the *cycle matroid* of the graph, and any matroid which is the cycle matroid of some graph is called a *graphic matroid*.

Another perspective is through the incidence matrix of a graph. Take a graph G with e edges and v vertices, and direct the edges. The incidence matrix of G is the $v \times e$ matrix $B = (b_{ij})$ with

$$b_{ij} = \begin{cases} -1 & \text{edge } j \text{ begins at vertex } i \\ 1 & \text{edge } j \text{ ends at vertex } i \\ 0 & \text{otherwise} \end{cases}$$

For an example see equation 5. To view this matrix as a matroid, let E be the set of columns and let a circuit be a set of columns which is linearly dependent but with every proper subset linearly independent. Any matrix can be viewed as a matroid in this way, and such matroids are called *representable* (over the real numbers). Note that all graphic matroids are representable.

Note also that elementary row operations on the matrix do not change the matroid, nor does removing any rows of all zeroes or scaling columns by nonzero scalars. If we label the columns then we can also swap columns along with their labels without changing the matroid.

One very nice property of matroids is that every matroid has a dual which generalizes the graph dual for planar graphs. It will suffice for us to describe how to calculate the dual for representable matroids[1]. Take a representable matroid with matrix M. Row reduce M swapping columns and removing zero rows until it has the form

$$(I_n|D)$$

where I_n is the $n \times n$ identity matrix and D is $n \times m$. Then the dual matroid is represented by the matrix

$$(-D^T|I_m).$$

[1]see for instance [12] chapter 2, for the general definition and the proofs that the matroid dual generalized the planar dual of graphs and that the given representable calculation is well defined.

Matroids which are the duals of graphic matroids are called *cographic* matroids.

The matroid of a graph cannot distinguish between a graph which is disconnected and one with the same components connected only at a vertex. This is very natural for quantum field theory since the Feynman integral also can't make this distinction. Despite this ambiguity, the notion of one-particle irreducible is well defined for matroids; a matroid $M = (E, \mathcal{C})$ is 1PI if it is *bridgeless*, that is every $e \in E$ is in at least one circuit.

2.2. **Regular matroids and unique representability.** As noted above matrices which differ by elementary row operations, by nonzero column scalings and column swaps, and by rows of all zeroes represent the same matroid[2]. However, given a representable matroid M there may be matrices which represent M but which are *inequivalent* using the above operations. It is only in very nice cases that M is *uniquely representable*, that is, it has no inequivalent representations. Fortunately some of these nice cases are central to our applications.

A matroid is *regular* if it can be represented by a *totally unimodular* matrix, that is a matrix for which every submatrix has determinant -1, 0, or 1. Graphic matroids are examples of regular matroids since the incidence matrix is totally unimodular. Cographic matroids are also regular. Another characterization of regular matroids is that they are representable over any field, in fact any totally unimodular representation will do this.

Some results on unique representability.

Proposition 2. *Let I_r be the $r \times r$ identity matrix.*

- *Regular matroids are uniquely representable over every field.*
- *If M is a matroid which is representable over $\mathbb{Z}/2\mathbb{Z}$ and over a field F then M is uniquely representable over F.*
- *Let F be a field and let $(I_r|D_1)$ and $(I_r|D_2)$, matrices over F with labelled columns, represent the same matroid with the same labelling. Suppose that every entry of D_1 and D_2 is 0 or ± 1, then $(I_r|D_1)$ and $(I_r|D_2)$ are equivalent representations (Brylawski and Lucas 1976, see* [12] *Theorem 10.1.1).*
- *If M is a matroid which is 3-connected and representable over $\mathbb{Z}/2\mathbb{Z}$ and a field F but not $\mathbb{Z}/4\mathbb{Z}$ then M is uniquely representable over F* [16].

The analogue of a spanning tree for a matroid is called a *base* or *basis*. A base of $M = (E, \mathcal{C})$ is a maximal subset of E which contains no cycle. In terms of a matrix of a representable matroid a base is, as expected, a basis of the column space of the matrix. One of the alternate characterizations of matroids is in terms of the set of bases instead of the set of circuits, which shows that the bases carry the full information of the matroid.

Since we have a notion analogous to the spanning tree, we can hope to form the Kirchhoff polynomial or first Symanzik polynomial. However, there is one subtlety. In view of the matrix-tree theorem we have two characterizations of the first Symanzik polynomial Ψ_G of a connected graph G with edge variable a_e associated to edge e, namely

$$\sum_{\substack{T \text{ spanning} \\ \text{tree of } G}} \prod_{e \notin T} a_e = \Psi_G = \det \begin{pmatrix} \Lambda & \widetilde{B}^T \\ -\widetilde{B} & 0 \end{pmatrix} \tag{2}$$

[2]We can also add to this list the action of any field automorphism, should we be working over a field which, unlike $\mathbb{R}$, has nontrivial automorphisms.

where Λ is the diagonal matrix of the a_e and $\widetilde{B}$ is the incidence matrix of G with any one row removed. Alternate ways to write the second equality include

$$\Psi_G = (\prod_{e\in G} a_e)\left(\det \widetilde{B}\Lambda\widetilde{B}^T|_{a_e\leftarrow\frac{1}{a_e},e\in G}\right) = \det \widetilde{B^*}\Lambda\widetilde{B^*}^T \tag{3}$$

where $\widetilde{B^*}$ is as $\widetilde{B}$ but for the dual of G.

For matroids the first equality of (2) is always available. However, the usefulness of the Kirchhoff polynomial for quantum field theory comes from the second equality of (2) which is one way to convert momentum space integrals to Schwinger parametric integrals. Thus, unlike in [3], we will look to the second equality (equivalently to (3)) in order to define the first Symanzik polynomial for the matroids of interest to us.

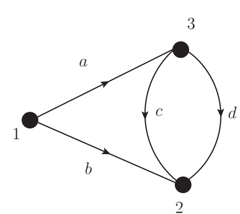
Note that the incidence matrix of a connected graph has one more row than its rank, and so, to translate to matroids, $\widetilde{B}$ should correspond to a representing matrix of full rank. For uniquely representable matroids, we have a unique (up to unproblematic transformations) matrix to use for $\widetilde{B}$. However the subdeterminants which, when squared, become the coefficients of the monomials in the determinant are no longer necessarily ± 1, and so we will obtain a variant of (2),

$$\Psi_G = \sum_T w_T \prod_{e\notin T} a_e \tag{4}$$

with positive weights w_T on the terms. In the case of regular matroids we have a totally unimodular matrix, and so we retain exactly the identity (2).

Likewise, adding in external edges to carry the external momenta we can form the second Symanzik polynomial by forming the same polynomial from this larger matrix and then taking the terms which are quadratic in the external momenta.

Let us pause here and compare incidence matrices with configuration polynomials [13, 2]. Let us actually be very explicit and start with the Dunce's cap graph, where we label oriented edges $e \in \{a, b, c, d\}$ and vertices $v \in \{1, 2, 3\}$ as follows.

$\Gamma =$ 

We then have the incidence matrix

$$B_\Gamma = \begin{array}{c} \begin{matrix} -\ell & \ell+q & -\ell+k & -k \end{matrix} \\ \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \end{pmatrix} \end{array}. \tag{5}$$

Let Ca, C_b, C_c, C_d be the columns. There are three circuits, any two of them forming a basis for the first Betti homology. For example, $-C_a + C_b - C_d = 0 = -C_a + C_b - C_d$ determine two circuits as solutions of

$$\sum_{e\in\{a,b,c,d\}} w_e C_e = 0,$$

with coefficients in $\{-1, 0, 1\}$, as it befits a proper graph.

We then get the first graph polynomial as the determinant of the two by two matrix

$$\det \begin{pmatrix} a+b+c & a+b \\ a+b & a+b+d \end{pmatrix} = (a+b)(c+d) + cd.$$

Similarly, the second graph polynomial is a configuration polynomial which can be obtained as the Pfaffian norm N_{rp} or Moore determinant of the matrix [2]

$$N := \begin{pmatrix} a+b+c & a+b & a\mu_a + b\mu_b + c\mu_c \\ a+b & a+b+d & a\mu_a + b\mu_b + d\mu_d \\ a\bar{\mu}_a + b\bar{\mu}_b + c\bar{\mu}_c & a\bar{\mu}_a + b\bar{\mu}_b + d\bar{\mu}_d & a\bar{\mu}_a\mu_a + b\bar{\mu}_b\mu_b + c\bar{\mu}_c\mu_c + d\bar{\mu}_d\mu_d \end{pmatrix},$$

with

$$\begin{aligned} N_{rp}(N) &= ((c+d)\underbrace{\overline{(\mu_a - \mu_b)}(\mu_a - \mu_b)}_{=q_1^2} b \\ &\quad + \underbrace{\overline{(\mu_a - \mu_c - \mu_d)}(\mu_a - \mu_c - \mu_d)}_{=q_3^2} cd)a + \underbrace{\overline{(\mu_b - \mu_c - \mu_d)}(\mu_b - \mu_c - \mu_d)}_{=q_2^2} bcd, \end{aligned}$$

writing four-momenta q_i entering at vertices i as quaternionic matrices [2].

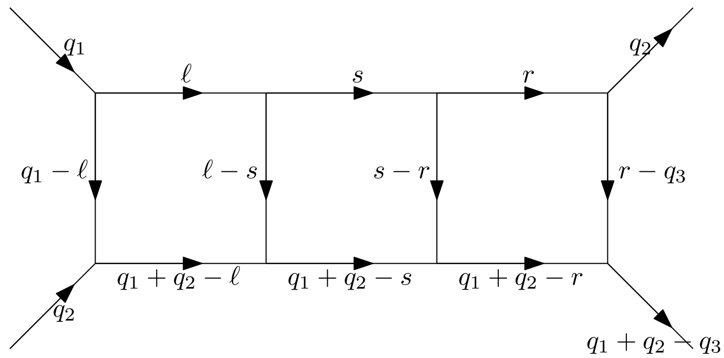
3. Encoding tensor structure with matroids

3.1. **From Feynman integral to matroid.** For mathematical physics, the value of a mathematical structure must be in what it can *do*. There are three main ways in which matroids are better than graphs for representing Feynman integrals.

First, and most trivially, Feynman graphs are redundant in that non-isomorphic graphs can give the same Feynman integral. For instance, graphs which differ by splitting at two vertices and rejoining with a twist have the same Feynman integral. Matroids remove this redundancy exactly: graphs corresponding to the same integrand give the same matroid and vice versa, see [3].

Second, if we allow matroids then every graph has a dual. This should be useful wherever the planar dual is currently useful. In particular it should be useful for the question of understanding the periods of high loop, massless, primitive ϕ^4 graphs where the tools, [4, 14, 5, 7], can predominantly be translated over to matroids.

Third, and the topic of this paper, we can use matroids to give an easy and natural way to represent Feynman integrals with arbitrary numerator structure as scalar matroid integrals with appropriate powers of the propagators appearing. Consider as an example a Feynman graph with the following topology.



The Feynman integral is of the form

$$\int \frac{d^4\ell d^4 s d^4 r}{\ell^2(\ell-s)^2 s^2 (s-r)^2 r^2 (q_1-\ell)^2 (q_1+q_2-\ell)^2 (q_1+q_2-s)^2 (q_1+q_2-r)^2 (r-q_3)^2}$$

If a similar topology is realized with quantum fields which do not sit in the trivial representation of the Poincaré group, we get tensor structure in the denominator as discussed in the introduction. For example, assume there is a $l \cdot r$ in the numerator. We are then left with integrals of the form

(6)

$$\int \frac{d^4\ell d^4 s d^4 r}{\ell^2(\ell-s)^2 s^2 (s-r)^2 r^2 (q_1-\ell)^2 (q_1+q_2-\ell)^2 (q_1+q_2-s)^2 (q_1+q_2-r)^2 (r-q_3)^2 (\ell-r)^2}$$

The denominator does not correspond to any graph. However, it does correspond to a matroid.

We can see this by simply writing down a matrix which gives the desired matroid. Each factor of the denominator of (6) is a generalized edge, that is an element of E, and thus a column in the matrix. In the incidence matrix of a graph, the rows correspond to vertices. If we modify the matrix by elementary row operations then the rows no longer correspond to vertices, but they remain sets of edges with momentum conservation. For a matroid, vertices are no longer a well-defined concept, so we can only say that the rows are sets which conserve momentum. With this in mind we can construct a matrix for the matroid M of the denominator of (6).

(7)

$$\begin{array}{c}
\begin{array}{ccccccccccccccc} \ell & \ell-s & s & s-r & r & q_1-\ell & q-\ell & q-s & q-r & r-q_3 & q_1 & q_2 & q_3 & q-q_3 & \ell-r \end{array} \\
\begin{pmatrix}
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}$$

where $q = q_1 + q_2$. The original graph corresponds to removing the final row and column. This says that the original graph is M with the new element corresponding to $(\ell - r)^2$

contracted. The process of reversing a contraction in a matroid is called *coextension*; it is not unique, but one element coextensions are completely characterized (see [8] pp156–158).

Further consider the circuits of M. Label the factors of (6) from left to right $a_1, \ldots, a_{11}$. Take any cycle C of the original graph. Change variables in the internal momenta so that C is indexed by one of the new internal momentum variables, t. $C \cup \{a_{11}\}$ is a circuit of M if $\ell - r$ depends on t when written in the new variables, and C itself is a circuit of M otherwise. This gives the circuits

$$\{a_1, a_2, a_6, a_7, a_{11}\}, \{a_2, a_3, a_4, a_8\}, \{a_4, a_5, a_9, a_{10}, a_{11}\}, \tag{8}$$

$$\{a_1, a_3, a_4, a_6, a_7, a_8, a_{11}\}, \{a_2, a_3, a_5, a_8, a_9, a_{10}, a_{11}\}, \{a_1, a_3, a_5, a_6, a_7, a_8, a_9, a_{10}\} \tag{9}$$

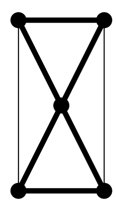
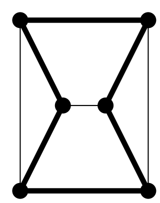
Comparing this to the original graph

$$\{a_1, a_2, a_6, a_7\}, \{a_2, a_3, a_4, a_8\}, \{a_4, a_5, a_9, a_{10}, \},$$
$$\{a_1, a_3, a_4, a_6, a_7, a_8\}, \{a_2, a_3, a_5, a_8, a_9, a_{10}\}, \{a_1, a_3, a_5, a_6, a_7, a_8, a_9, a_{10}\}$$

a_{11} has simply been removed, which is another way to see that the original graph is M with a_{11} contracted.

To include the information from the external momenta we can further put the external edges e_1, e_2, e_3, e_4 into E and include as circuits (cycles through infinity) the momentum flow of the external momenta, and other possible external momentum flows coming from changes of variables. a_{11} will be included or not included in the circuits of M by the same criterion as above.

This still does not give all the circuits of M. Other circuits can come from pairs of cycles in the original graph which are joined in the coextension. The following small graph example illustrates the situation. Let

$G =$  $H =$ 

Then H is a coextension of G, and the pair of cycles in G indicated by the fat lines become a single cycle in H. Fortunately, given (8), the axioms determine the remaining circuits, as if C_1 and C_2 are circuits of M with $C_1 \cap C_2 = \{a_{11}\}$. Then $D = C_1 \cup C_2 \smallsetminus \{a_{11}\}$, which is a pair of cycles in the original graph, must contain a circuit of M. If either cycle of D is itself a circuit of M then D cannot itself be a circuit of M by the second axiom. If neither cycle of D is itself a circuit of M then by the third axiom D must be a circuit of M. In the above case we add only the circuit

$$\{a_1, a_2, a_6, a_7, a_4, a_5, a_9, a_{10}\}$$

Together this is all the circuits of M.

Note that we never used any graph specific properties of the original graph, and so we can iterate this procedure to remove all dot product factors from the numerator. We keep the new matrices nice in the following sense.

Proposition 3. *Let G be a graph and P a set of pairs of edges of the graph (these are the pairs whose momenta appear dotted together in the numerator). Applying the above* *construction with appropriate choices, we obtain a matrix row equivalent to*

$$\begin{pmatrix} I_{\mathrm{rk}G} & 0 & C \\ 0 & I_r & D \end{pmatrix} \tag{10}$$

where $(I_{\mathrm{rk}G}\, C)$ *represents the matroid of* G, $r \leq |P|$, *and all entries of* C *and* D *are* 0 *or* ± 1.

Proof. We need a few facts first. If A is a matrix with entries in $\{-1, 0, 1\}$ which represents a matroid M both over $\mathbb{R}$ and over $\mathbb{Z}/2\mathbb{Z}$ then A has no 2×2 subdeterminant equal to ± 2. This is because otherwise M would have a minor isomorphic to the matroid $U_{2,4}$, which is the matroid represented by

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

But this is not possible for a matroid representable over $\mathbb{Z}/2\mathbb{Z}$.

Next, if A is a matrix with entries in $\{-1, 0, 1\}$ which represents a matroid M both over $\mathbb{R}$ and over $\mathbb{Z}/2\mathbb{Z}$, and R_1, R_2 are two rows of A which both have a nonzero entry in column i then the linear combination of R_1 and R_2 with zero entry in column i has all entries 0 or ± 1, and replacing R_1 by this new row still represents M over both $\mathbb{R}$ and $\mathbb{Z}/2\mathbb{Z}$.

To see this, suppose otherwise. Write $R_i = (r_{i,1}, \dots, r_{i,m})$. The only possible other entry is ± 2. If ± 2 appears in column j then $r_{1,j} r_{2,i} - r_{2,j} r_{1,i} = \pm 2$ and the 2×2 matrix made of columns i and j of R_1 and R_2 has determinant ± 2 which is impossible. Replacing R_1 by this new row only involves scaling by ± 1 and so this new matrix still represents M both over $\mathbb{R}$ and over $\mathbb{Z}/2\mathbb{Z}$.

Now returning to the original problem, row reduce the adjacency matrix of G using the above tools and remove any zero rows to obtain a matrix of the form $(I_{\mathrm{rk}G}\, C)$ with the entries of C in $\{0, \pm 1\}$ and which represents the matroid of G over $\mathbb{R}$ and $\mathbb{Z}/2\mathbb{Z}$.

We inductively coextend for each element of P. The base case is above. Suppose we have a matrix of the form

$$A = \begin{pmatrix} I_{\mathrm{rk}G} & 0 & C \\ 0 & I_r & D \end{pmatrix}$$

and a pair of edges (e_1, e_2) of G. If any edge of G or new edge which we have already added from P is already a linear combination with both weights nonzero of the momenta of e_1 and e_2 then we don't need a new edge and so just discard this pair.

Otherwise, e_1 and e_2 correspond to two column indices i and j which are either in the $I_{\mathrm{rk}G}$ part or the C part of A. If i and j are both in the C part then coextend A to

$$\begin{pmatrix} I_{\mathrm{rk}G} & 0 & 0 & C \\ 0 & I_r & 0 & D \\ 0 & 0 & 1 & v \end{pmatrix}$$

where v has either ± 1 in positions i and j and zero elsewhere.

If i is in the $I_{\mathrm{rk}G}$ part and j is in the C part then let ϵ be the i, j entry of A. Build the new matrix

$$A' = \begin{pmatrix} I_{\mathrm{rk}G} & 0 & 0 & C \\ 0 & I_r & 0 & D \\ v_1 & 0 & 1 & v_2 \end{pmatrix}$$

where v_1 has -1 in position i and 0 elsewhere and v_2 has $\begin{cases} \pm 1 & \text{if } \epsilon = 0 \\ -\epsilon & \text{otherwise} \end{cases}$ in position j and 0 elsewhere. Then we can row reduce A' by adding row i to the last row and obtain a matrix in the form of (10).

Finally if both i and j are in the $I_{\mathrm{rk}G}$ part then we will make a matrix of the form

$$A' = \begin{pmatrix} I_{\mathrm{rk}G} & 0 & 0 & C \\ 0 & I_r & 0 & D \\ v & 0 & 1 & 0 \end{pmatrix}$$

with v nonzero only in entries i and j. Call these entries $v(i)$ and $v(j)$. Let R_i and R_j be rows i and j of C. If there is no column for which R_i and R_j are both nonzero then any choice of ± 1 is fine for $v(i)$ and $v(j)$. If R_i and R_j are both nonzero in column k then let $v(i) = r_{i,k}$ and $v(j) = -r_{j,k}$. Then row reducing A' to clear entries $v(i)$ and $v(j)$ puts the row $-r_{i,k}R_i + r_{j,k}R_j$ below D, but by the facts we initially observed this also has all entries 0 or ± 1 and so we again obtain a matrix in the required form. □

3.2. **From matroid to Feynman integral.** We can go from a matrix like (7), or any matrix of the form (10), (along with the information of which edges are external) back to a scalar Feynman integral by assigning a momentum variable for each edge and then applying Feynman rules in the sense that each internal edge contributes a factor $1/k^2$ and each row of the matrix contributes a delta function corresponding to reading across the row. The rows generate the set of momentum preserving subsets of edges, and so the counting works exactly as in the graph case even though we have no notion of vertices. Alternately, we can build parametric Feynman integrals from the first and second Symanzik polynomials in the usual way.

In order to convert a real-representable matroid back into a Feynman integral, first notice that row operations do not change the Feynman integral since matrices which differ only by row operations simply lead to different but equivalent products of delta functions. Rows of all zeros simply contribute nothing at all. Column scalings and column swaps simply scale or rename the internal momenta and so also do not change the Feynman integral. Thus equivalent representations lead to the same Feynman integral.

By Proposition 3 the matroids we are interested always have a representation in the form $(I|D)$ with D having entries 0 or ± 1, and by the equivalent representation result of Brylawski and Lucas (see Proposition 2) and such representations are equivalent given that we have fixed labels for the edges. So even if our matroids are not uniquely representable they have favoured representations which are all equivalent. So we can build our Feynman integrals in either momentum space or parametric space out of these representations. Thus we can speak of Feynman integrals associated to such matroids, and we can decompose tensor integrals into scalar integrals of matroids.

In the big example the matroid, which is represented by (7), is cographic, so in particular it is regular and so uniquely representable.

Finally, note that weights in the base expansion of the first Symanzik polynomial, as in (4), really do come up in physically relevant cases. Consider the complete bipartite graph

$K_{3,3}$. It is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \end{pmatrix}$$

Row reduced

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Pick edges 7 and 8, which do not share a vertex. There are two possibilities for the coextension which are consistent with the preceding discussion.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{pmatrix}$$

In the first case the submatrix will all rows and with columns 2,5,6,8,9,10 has determinant 2 while in the second case the submatrix with all rows and columns 2,3,4,5,8,9 has determinant 2. So in either case a coefficient of 4 will appear in the first Symanzik polynomial which can be verified by direct computation. Both of these matroids are thus not regular. Computing with Macek [9] we can check that these are representable over $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{R}$ but not $\mathbb{Z}/4\mathbb{Z}$ and hence by results of Whittle (see Proposition 2) are none-the-less uniquely representable.

4. Discussion

Generalizing Feynman graphs to matroids gives us combinatorial interpretations for scalar master integrals, without sacrificing the graph based tools and definitions we need, such as 1PI, duality, contraction, and deletion. In fact it improves these tools in that duality becomes defined for all graphs.

Moving to matroids also suggests a hierarchy of difficulty. Planar graphs are most straightforward; both they and their duals are graphs. General graphs and cographs come next; cographs are less familiar but they behave very much as graphs do. Cographs are a natural next term for any series which begins with a planar piece and doesn't continue with graphs themselves. Very slightly more than this is the class of regular matroids. Regular matroids are nice in many ways, notably they are uniquely representable over every field and they have a matrix-tree theorem identical to that of graphs; typically one's graph based intuition is valid. Finally, the most general matroids we need are more subtle, but do always have a nice representation in the form $(I|D)$ with D having entries $0, 1, -1$.

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