Detweiler's gauge-invariant redshift variable: analytic determination of the nine and nine-and-a-half post-Newtonian self-force contributions

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Continuing our analytic computation of the first-order self-force contribution to Detweiler's redshift variable we provide the exact expressions of the ninth and ninth-and-a-half post-Newtonian terms.

I. INTRODUCTION

The prospect of soon detecting the gravitational-wave signals emitted by coalescing compact binaries motivates a renewed study of the general relativistic two-body problem. One of the useful lines of attack on this problem is the gravitational self-force program, which considers large-mass-ratio binary systems ($m_1 \ll m_2$), and uses an expansion in powers of the mass ratio $q \equiv m_1/m_2 \ll 1$. Within this program, Detweiler [1] has emphasized the importance of focussing on the computation of gauge-invariant quantities, and he gave (for the case of circular motions) the example of the function relating the redshift $U^t = dt/ds$ along the worldline of the small mass m_1 to the orbital frequency Ω. To first order in q this gives rise to the gauge-invariant function $U_1^t(y)$, where $U^t(y) = (1-3y)^{-1/2} + q\, U_1^t(y) + O(q^2)$ and where $y \equiv (Gm_2\Omega/c^3)^{2/3}$ denotes a dimensionless parameter related to the orbital frequency. Note that y can be considered as measuring (in a gauge-invariant way) the dimensionless gravitational potential $Gm_2/(c^2 R_\Omega)$, with R_Ω denoting the invariant radius canonically associated with Ω via Kepler's law around the large mass: $Gm_2 = \Omega^2 R_\Omega^3$. In the following, we shall denote the mathematical argument of the first-order self-force function $U_1^t(.)$ by u (evoking a gravitational potential) rather than y. [This notation is purely a matter of choice.] We henceforth also often set $G = c = 1$.

Detweiler [1] has shown that $U_1^t(u)$ could be computed in terms of the first-order metric perturbation of a Schwarzschild metric of mass m_2, say $\delta g_{\mu\nu} = g_{\mu\nu}(x^\lambda; m_1, m_2) - g^{\rm Schw}_{\mu\nu}(x^\lambda; m_2) \equiv q\, h_{\mu\nu}(x^\lambda) + O(q^2)$ as

$$U_1^t(u) = \frac{1}{2(1-3u)^{3/2}} h^R_{kk}(u)\,, \tag{1.1}$$

where

$$h^R_{kk}(u) := \left[h_{\mu\nu}(x^\lambda)\right]^R k^\mu k^\nu\,. \tag{1.2}$$

Here k^μ denotes the Killing vector $k = \partial_t + \Omega\partial_\varphi$, and the superscript R denotes the *regularized* value of $h_{\mu\nu}(x^\lambda)$ on the world line of the small mass m_1. [When evaluating the first-order quantity h^R_{kk} along a circular orbit of (coordinate) radius R_0 it is enough to use the approximation $R_0 \approx R_\Omega = m_2/u$.]

The beginning of the post-Newtonian (PN) expansion of the (first-order) self-force contribution $U_1^t(u)$ was analytically derived in Ref. [1], namely :

$$U_1^t(u) = -u - 2u^2 - 5u^3 - \cdots \tag{1.3}$$

Here, the first term $(-u)$ is of Newtonian order, so that the second $(-2u^2)$ and third $(-5u^3)$ respectively represent 1PN and 2PN contributions. More generally, a term $\propto u^{n+1}$ corresponds to the nPN level in $U_1^t(u)$.

The 3PN term was analytically derived (using full PN theory) by Blanchet et al. [2]. In 2013, we [3] showed how to analytically compute the 4PN term by a combined use of Regge-Wheeler-Zerilli (RWZ) formalism for the Schwarzschild perturbations together with the hypergeometric-expansion analytical solutions of the RWZ radial equation obtained by Mano, Suzuki and Takasugi [4] (MST). We then progressively extended the analytical knowledge of the PN expansion of h^R_{kk} up to the 8.5 PN level [5, 6]. [See the latter works for references to other related analytical studies.]

Parallely to these analytical studies, Detweiler's redshift variable was numerically computed in Refs. [1, 2, 7], and these numerical data were used to extract numerical estimates of several higher-order (then unknown) PN expansion coefficients [1, 2]. A breakthrough in this extraction of PN coefficients from numerical self-force calculations was accomplished by Shah, Friedman and Whiting [8] who numerically evaluated the MST hypergeometric-expansion of $U_1^t(u)$ to one part in 10^{225} for orbital radii extending up to $10^{30} Gm_2/c^2$. This extremely high numerical accuracy on $U_1^t(u)$ for extremely small values of the argument u allowed them to numerically extract PN coefficients up to the 10.5 PN level, and also to provide educated guesses for the exact analytical form of several high-order PN coefficients.

We have shown in [5, 6] that the results of Shah, Friedman and Whiting [8] agreed with our (fully) analytical ones up to the highest PN level we had then computed, namely the 8.5PN level. The aim of the present short note is to report on an extension of our analytical computation to the 9.5 PN level (using the techniques explained in our previous papers), and on its comparison with the results of Shah et al.

II. NEW TERMS IN THE PN EXPANSION OF $U_1^t(u)$ AT THE 9 AND 9.5 PN LEVELS

Following the notation of Eq. (21) in Ref. [8], we write

$$U_1^t(u) = U_1^t(u)|_{8.5\mathrm{PN}} + (\alpha_9 - \beta_9 \ln u + \gamma_9 \ln^2 u)u^{10} + (\alpha_{9.5} - \beta_{9.5} \ln u)u^{21/2} \,, \tag{2.1}$$

where $U_1^t(u)|_{8.5\mathrm{PN}}$ is known from [6]. Here (α_9, β_9, γ_9), and ($\alpha_{9.5}$, $\beta_{9.5}$) are the coefficients of, respectively, the 9PN (u^{10}) and 9.5PN ($u^{21/2}$) terms, which we have now analytically derived. Our results for these coefficients are:

$$\begin{aligned}
\alpha_9 &= -\frac{10480362137370508214933}{2044301131372500} + \frac{5921855038061194}{442489422375}\gamma - \frac{2076498568312502}{442489422375}\ln(2) \\
&\quad + \frac{10221088}{2835}\zeta(3) - \frac{16110330832}{9823275}\gamma^2 + \frac{32962327798317273}{549755813888}\pi^4 \\
&\quad - \frac{11665762236240841}{226072985600}\pi^2 + \frac{33662546992}{9823275}\ln(2)^2 - \frac{27101981341}{100663296}\pi^6 \\
&\quad - \frac{58533203125}{15567552}\ln(5) - \frac{96889010407}{277992000}\ln(7) + \frac{9647042994387}{392392000}\ln(3) - \frac{94770}{49}\ln(3)^2 \\
&\quad - \frac{10020829088}{9823275}\ln(2)\gamma - \frac{189540}{49}\ln(2)\ln(3) - \frac{189540}{49}\gamma\ln(3) \\
&= \boxed{-32239.627595092556412367706034592096152340561204299} \\
\beta_9 &= \frac{16110330832}{9823275}\gamma - \frac{2921280466785797}{442489422375} + \frac{5010414544}{9823275}\ln(2) + \frac{94770}{49}\ln(3) \\
&= \boxed{-3176.92918115396920639233883269266608882}22379193868 \\
\gamma_9 &= -\frac{4027582708}{9823275} \\
\alpha_{9.5} &= -\frac{30185191523470507}{12236744520000}\pi - \frac{410021764}{385875}\ln(2)\pi - \frac{198373004}{1157625}\pi\gamma - \frac{1055996}{11025}\pi^3 + \frac{246402}{343}\pi\ln(3) \\
&= \boxed{-10864.6255867062440752457674}4350668665810584498692 0 \\
\beta_{9.5} &= \frac{99186502}{1157625}\pi
\end{aligned} \tag{2.2}$$

Our analytically derived results for γ_9 and $\beta_{9.5}$ agree with the numerical-based analytical expressions previously obtained for these two particular coefficients by Shah et al. [8]. Concerning the other (newly analytically computed) coefficients, namely α_9, β_9, and $\alpha_{9.5}$ we have indicated by boxes in the above equations the extent to which our results agree with the numerical estimates given by Shah et al. [8]. More precisely, the boxes above include one more digit than those given in Table I of [8]. In all cases, the agreement is perfect modulo possible rounding effects on the last digit quoted in [8].

III. CONCLUDING REMARKS

The analytic computation of the post-Newtonian expansion of the first-order self-force contribution $U_1^t(u)$ to Detweiler's redshift function $U^t(\Omega)$ has been raised here to the nine and nine-and-a-half PN level, thereby providing the exact analytical expressions of terms which were previously obtained only numerically by Shah et al. [8].

Let us finally note that, using the results of Refs. [9, 10], our results can be translated into the computation of the nine and nine-and-a-half PN contributions to the linear-in-mass-ratio piece of the main radial potential $A(u;\nu)$ of the effective one-body formalism [11, 12]. Denoting, $A(u;\nu) = 1 - 2u + \nu a(u) + O(\nu^2)$ (where

$\nu \equiv m_1 m_2/(m_1+m_2)^2 = q/(1+q)^2$, we have

$$a(u) = a_{8.5\mathrm{PN}}(u) + (a_{10}^{c} + a_{10}^{\ln}\ln u + a_{10}^{\ln^2}\ln^2 u)u^{10} + (a_{10.5}^{c} + a_{10.5}^{\ln}\ln u)u^{21/2}\,, \tag{3.1}$$

where $a_{8.5\mathrm{PN}}(u)$ was given in [6], and where the newly derived 9PN and 9.5PN coefficients are:

$$\begin{aligned}
a_{10}^{c} &= \frac{18605478842060273}{7079830758000}\ln(2) - \frac{1619008}{405}\zeta(3) - \frac{21339873214728097}{1011404394000}\gamma \\
&\quad + \frac{27101981341}{100663296}\pi^6 - \frac{6236861670873}{125565440}\ln(3) + \frac{360126}{49}\ln(2)\ln(3) + \frac{180063}{49}\ln(3)^2 \\
&\quad - \frac{121494974752}{9823275}\ln(2)^2 - \frac{24229836023352153}{549755813888}\pi^4 + \frac{1115369140625}{124540416}\ln(5) + \frac{96889010407}{277992000}\ln(7) \\
&\quad + \frac{75437014370623318623299}{18690753201120000} - \frac{60648244288}{9823275}\ln(2)\gamma + \frac{200706848}{280665}\gamma^2 \\
&\quad + \frac{11980569677139}{2306867200}\pi^2 + \frac{360126}{49}\gamma\ln(3) \\
a_{10}^{\ln} &= -\frac{21275143333512097}{2022808788000} + \frac{200706848}{280665}\gamma - \frac{30324122144}{9823275}\ln(2) + \frac{180063}{49}\ln(3) \\
a_{10}^{\ln^2} &= \frac{50176712}{280665} \\
a_{10.5}^{c} &= -\frac{185665618769828101}{24473489040000}\pi + \frac{377443508}{77175}\ln(2)\pi + \frac{2414166668}{1157625}\pi\gamma - \frac{5846788}{11025}\pi^3 - \frac{246402}{343}\pi\ln(3) \\
a_{10.5}^{\ln} &= \frac{1207083334}{1157625}\pi\,.
\end{aligned} \tag{3.2}$$

The corresponding numerical values are (consistently with, but more accurately than in, Eqs. (27), (28) in [6])

$$\begin{aligned}
a_{10}^{c} &= 4845.870557019444177347393421579822176656222929365\ldots \\
a_{10}^{\ln} &= -8207.441915171961061495913671985378064622493890134\ldots \\
a_{10}^{\ln^2} &= 178.777945237204496463755723014982274241533500792\ldots \\
a_{10.5}^{c} &= -28324.307465213628065671194515396169336328722651715\ldots \\
a_{10.5}^{\ln} &= 3275.813959067119914181314855451516851587807063565\ldots\,.
\end{aligned} \tag{3.3}$$

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[1] S. L. Detweiler, "A Consequence of the gravitational self-force for circular orbits of the Schwarzschild geometry," Phys. Rev. D **77**, 124026 (2008) [arXiv:0804.3529 [gr-qc]].

[2] L. Blanchet, S. L. Detweiler, A. Le Tiec and B. F. Whiting, "Post-Newtonian and Numerical Calculations of the Gravitational Self-Force for Circular Orbits in the Schwarzschild Geometry," Phys. Rev. D **81**, 064004 (2010) [arXiv:0910.0207 [gr-qc]].

[3] D. Bini and T. Damour, "Analytical determination of the two-body gravitational interaction potential at the fourth post-Newtonian approximation," Phys. Rev. D **87**, no. 12, 121501 (2013) [arXiv:1305.4884 [gr-qc]].

[4] S. Mano, H. Suzuki and E. Takasugi, "Analytic solutions of the Regge-Wheeler equation and the post-Minkowskian expansion," Prog. Theor. Phys. **96**, 549 (1996) [gr-qc/9605057].

[5] D. Bini and T. Damour, "High-order post-Newtonian contributions to the two-body gravitational interaction potential from analytical gravitational self-force calculations," Phys. Rev. D **89**, no. 6, 064063 (2014) [arXiv:1312.2503 [gr-qc]].

[6] D. Bini and T. Damour, "Analytic determination of the eight-and-a-half post-Newtonian self-force contributions to the two-body gravitational interaction potential," Phys. Rev. D **89**, no. 10, 104047 (2014) [arXiv:1403.2366 [gr-qc]].

[7] S. Akcay, L. Barack, T. Damour and N. Sago, "Gravitational self-force and the effective-one-body formalism between the innermost stable circular orbit and the light ring," Phys. Rev. D **86**, 104041 (2012) [arXiv:1209.0964 [gr-qc]].

[8] A. G. Shah, J. L. Friedman and B. F. Whiting, "Finding high-order analytic post-Newtonian parameters from a high-precision numerical self-force calculation," Phys. Rev. D **89**, no. 6, 064042 (2014) [arXiv:1312.1952 [gr-qc]].

[9] A. Le Tiec, L. Blanchet and B. F. Whiting, "The First Law of Binary Black Hole Mechanics in General Relativity and Post-Newtonian Theory," Phys. Rev. D **85**, 064039 (2012) [arXiv:1111.5378 [gr-qc]].

[10] E. Barausse, A. Buonanno, and A. Le Tiec, "The complete non-spinning effective-one-body metric at linear order in the mass ratio," Phys. Rev. D **85**, 064010 (2012) [arXiv:1111.5610 [gr-qc]].

[11] A. Buonanno and T. Damour, "Effective one-body approach to general relativistic two-body dynamics," Phys. Rev. D **59**, 084006 (1999) [gr-qc/9811091].

[12] T. Damour, P. Jaranowski and G. Schaefer, "On the determination of the last stable orbit for circular general relativistic binaries at the third postNewtonian approximation," Phys. Rev. D **62**, 084011 (2000) [gr-qc/0005034].