Quasi-exact-solvability of the A_2 Elliptic model: algebraic form, $sl(3)$ hidden algebra, polynomial eigenfunctions

Vladimir V Sokolov

Landau Institute for Theoretical Physics, Moscow Region, Russia

and

Institut des Hautes Etudes Scientifique, Bures-sur-Yvette 91440, France

vsokolov@landau.ac.ru

and

Alexander V Turbiner

Instituto de Ciencias Nucleares, UNAM, México DF 04510, Mexico

and

Institut des Hautes Etudes Scientifique, Bures-sur-Yvette 91440, France

turbiner@nucleares.unam.mx, turbiner@ihes.fr

(Dated: September 21, 2014)

Abstract

The potential of the A_2 quantum elliptic model (3-body Calogero elliptic model) is defined by the pairwise three-body interaction through Weierstrass $\wp$-function and has a single coupling constant. A change of variables has been found, which are A_2 elliptic invariants. In those, the potential becomes a rational function, while the flat space metric as well as its associated vector are polynomials in two variables. It is shown the model possesses the hidden sl_3 algebra - the Hamiltonian is an element of the universal enveloping algebra U_{sl_3} for arbitrary coupling constant - being equivalent to sl_3-quantum top. The integral in a form of the third order differential operator with polynomial coefficients is constructed explicitly, being also an element of the universal enveloping algebra U_{sl_3}. It is shown that there exists a discrete sequence of coupling constants for which a finite number of polynomial eigenfunctions up to a (non-singular) gauge factor occur.

The A_2 elliptic model (3-body elliptic Calogero model, see e.g. [1]) describes three particles on the line with pairwise interaction given by the Weierstrass $\wp$-function. It is characterized by the Hamiltonian

$$\mathcal{H}^{(e)}_{A_2} \ = \ -\frac{1}{2}\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} + \nu(\nu-1)\left(\wp(x_1-x_2) + \wp(x_2-x_3) + \wp(x_3-x_1)\right) \ \equiv \ -\frac{1}{2}\Delta^{(3)}+V \ , \quad (1)$$

where $\Delta^{(3)}$ is three-dimensional Laplace operator, $\kappa \equiv \nu(\nu-1)$ is coupling constant. The Weierstrass function $\wp(x) \equiv \wp(x|g_2, g_3)$ (see e.g. [2]) is defined as

$$(\wp'(x))^2 \ = \ 4\ \wp^3(x) - g_2\ \wp(x) \ - \ g_3 \ = \ 4(\wp(x)-e_1)(\wp(x)-e_2)(\wp(x)-e_3), \qquad (2)$$

where $g_{2,3}$ are its invariants and $e_{1,2,3}$ are roots, usually, it is chosen $e \equiv e_1+e_2+e_3 = 0$. If in (2) the trigonometric limit is taken, $\Delta \equiv g_2^3 + 27g_3^3 = 0$, with one of periods going to infinity, the Hamiltonian of A_2 trigonometric/hyperbolic model (3-body Sutherland model) occurs. If both invariants $g_2 = g_3 = 0$ we arrive at A_2-rational (3-body Calogero) model. For future convenience we parameterize the invariants as follows

$$g_2 = 12(\tau^2-\mu) \ , \qquad g_3 = 4\tau(2\tau^2-3\mu) \ , \qquad (3)$$

where τ, μ are parameters.

The Hamiltonian (1) is translation-invariant, thus, it makes sense to introduce center-of-mass coordinates

$$Y = \sum_{1}^{3} x_i \ , \ y_i = x_i - \frac{1}{3}Y \ , \qquad (4)$$

with a condition $\sum_1^3 y_i = 0$. Laplacian $\Delta^{(3)} \equiv \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ in these coordinates takes the form,

$$\Delta^{(3)} \ = \ 3\ \partial_Y^2 \ + \ \frac{2}{3}\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2}\right) \ .$$

Separating out center-of-mass coordinate Y two-dimensional Hamiltonian arises

$$\mathcal{H}_{A_2} \ = \ -\frac{1}{3}\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2}\right) + \nu(\nu-1)\left(\wp(y_1-y_2) \ + \ \wp(2y_1+y_2) \ + \ \wp(y_1+2y_2)\right) \ . \qquad (5)$$

Since we will be interested by general properties of the operator $\mathcal{H}_{A_2}$, without a loss of generality we assume that the operator (5) is defined on real plane, $y_{1,2} \in \mathbf{R}^2$ while the fundamental domain of the Weierstrass function $\wp(x)$ is not fixed. The symmetry of the Hamiltonian (5) is $S^2 \oplus \mathbb{Z}_2 \oplus (T_r)^2 \oplus (T_c)^2$. It consists of permutation $S^2(y_1 \leftrightarrow y_2)$, reflection

$\mathbb{Z}_2(y_{1,2} \leftrightarrow -y_{1,2})$ and four translations $T_{r,1(2)}\,:\; y_{1(2)} \to y_{1(2)}+1$ and $T_{c,1(2)}\,:\; y_{1(2)} \to y_{1(2)}+i\,\tau_c$ (periodicity). Perhaps, $S^2 \oplus (T_r)^2 \oplus (T_c)^2$ can make sense as a double-affine A_2 Weyl group.

Let us consider a formal eigenvalue problem

$$\mathcal{H}_{A_2}\Psi \;=\; E\Psi\ , \tag{6}$$

without posing concrete boundary conditions. Assume $f(x)$ be a non-constant solution of the equation

$$f'(x)^2 = 4f(x)^3 - 12\tau f(x)^2 + 12\mu f(x)\ . \tag{7}$$

Thus, it can be written as

$$f(x) \;=\; \wp(x|g_2, g_3) + \tau\ ,$$

cf. (2),(3). Now let us introduce new variables

$$x = \frac{f'(y_1) - f'(y_2)}{f(y_1)f'(y_2) - f(y_2)f'(y_1)}\,, \qquad y = \frac{2(f(y_1) - f(y_2))}{f(y_1)f'(y_2) - f(y_2)f'(y_1)}\ , \tag{8}$$

which have a property

$$x(-y_1, -y_2) \;=\; x(y_1, y_2)\ ,\ y(-y_1, -y_2) \;=\; -y(y_1, y_2)\ .$$

They are invariant with respect to the partial symmetry of the Hamiltonian (5): $S^2 \oplus (T_r)^2 \oplus (T_c)^2$. It can be shown that in rational limit $\tau = \mu = 0$ where the 3-body Calogero model emerges the variables x, y coincide with those found in Rühl-Turbiner [3]

$$x = -(y_1^2 + y_2^2 + y_1 y_2), \qquad y = -y_1 y_2(y_1 + y_2)\ , \tag{8.1}$$

as well as ones in trigonometric limit $\mu = 0$ where the 3-body Sutherland model emerges [3]

$$x \;=\; \frac{1}{\alpha^2}\,[\cos(\alpha y_1) + \cos(\alpha y_2) + \cos(\alpha(y_1 + y_2)) - 3]\ ,$$

$$y \;=\; \frac{2}{\alpha^3}\,[\sin(\alpha y_1) + \sin(\alpha y_2) - \sin(\alpha(y_1 + y_2))]\ , \tag{8.2}$$

here α is parameter such that $\tau = \alpha^2/12$. After cumbersome calculations it can be found that the elliptic Calogero potential (see (1), (5)) in new variables takes a rational form,

$$V(x, y) \;=\; \frac{3\nu(\nu - 1)}{4}\,\frac{\left(x + 2\tau x^2 + \mu x^3 - 6(\mu - \tau^2)y^2 + 3\mu\tau x y^2\right)^2}{D}\ , \tag{9}$$

where

$$
12D(x,y) \;=\; 9\mu^2x^4y^2+54\tau\mu^2x^2y^4+27\mu^2(3\tau^2-4\mu)y^6-12\mu x^5-72\tau\mu x^3y^2- \tag{10}
$$

$$
108\mu(\tau^2-2\mu)xy^4-12\tau\, x^4-18(4\tau^2+5\mu)x^2y^2-54\tau(2\tau^2-3\,\mu)y^4-4x^3-108\tau xy^2-27y^2\ .
$$

It is worth noting that the potential (9) is symmetric in y, $V(x,y)=V(x,-y)$ as well as $D(x,y)=D(x,-y)$. Furthermore, 2D Laplacian (5) becomes the Laplace-Beltrami operator

$$
\Delta_g(z_1,z_2) \;=\; g^{-1/2}\ \sum_{ij}\frac{\partial}{\partial z_i}g^{1/2}g^{ij}\frac{\partial}{\partial z_j} \;=\; g^{ij}\frac{\partial^2}{\partial z_i\partial z_j}+\sum\frac{g^{ij}_{,i}}{2}\ \frac{\partial}{\partial z_j}\ ,
$$

which in (x,y)-coordinates looks explicitly as

$$
\Delta_g(x,y;\tau,\mu)=3\Big(\frac{x}{3}+\tau x^2+\mu x^3+(\mu-\tau^2)y^2-\mu\tau xy^2-\mu^2x^2y^2\Big)\frac{\partial^2}{\partial x^2}\ +
$$

$$
y\Big(3+8\tau x+7\mu x^2-3\mu\tau y^2-6\mu^2xy^2\Big)\frac{\partial^2}{\partial x\partial y}\ +\ \Big(-\frac{x^2}{3}+3\tau y^2+4\mu xy^2-3\mu^2y^4\Big)\frac{\partial^2}{\partial y^2}\ + \tag{11}
$$

$$
\Big(1+4\tau x+5\mu x^2-3\mu\tau y^2-6\mu^2xy^2\Big)\frac{\partial}{\partial x}\ +2y\Big(2\tau+3\mu x-3\mu^2y^2\Big)\frac{\partial}{\partial y}\ .
$$

Thus, the flat contravariant metric, defined by the symbol of the Laplace-Beltrami operator in these coordinates, becomes polynomial in x,y. The Hamiltonian is the sum of Laplace-Beltrami operator (11) with polynomial coefficients and rational potential (9). Taking in the Laplace-Beltrami operator (11) in the rational limit $\tau=\mu=0$, we arrive at the Laplace-Beltrami operator $\Delta_g^{(rat)}$ of the 3-body Calogero model [3]. If we take the trigonometric limit $\mu=0$, the Laplace-Beltrami operator $\Delta_g^{(trig)}$ of the 3-body Sutherland model emerges [3].

The denominator D in (9) turns out to be equal to the determinant of the contravariant metric $D=\mathrm{Det}(g^{ij})=\frac{1}{g}$. It is worth noting some properties of the determinant D: in rational case $D^{1/2}$ is the zero mode of the Laplace-Beltrami operator

$$
\Delta_g^{(rat)}D^{1/2}\ =\ 0\ .
$$

In trigonometric case

$$
\Delta_g^{(trig)}D^{1/2}\ =\ -12\tau D^{1/2}\ ,
$$

and in general case,

$$
\Delta_g(x,y;\tau,\mu)D^{1/2}\ =\ -12\tau\left(1-\mu(2x-3\mu y^2)\right)D^{1/2}\ .
$$

It easy to verify that the determinant $D(x,y)$ given by formula (10) can be written as

$$D(x,y) \;=\; \frac{1}{12}W^2\ , \tag{12}$$

where the function

$$W = \frac{\partial y}{\partial y_2}\frac{\partial x}{\partial y_1} - \frac{\partial x}{\partial y_2}\frac{\partial y}{\partial y_1}\ , \tag{13}$$

is the Jacobian associated with the change of variables $(y_1, y_2) \to (x, y)$. Perhaps, the equation $w^2 = 12\,D(x,y)$ can be considered as the equation for the elliptic surface [11]. One can verify that W admits a representation in factorized form,

$$W(y_1,y_2) = \frac{\sigma(y_1-y_2)\,\sigma(y_1+2y_2)\,\sigma(y_2+2y_1)}{\sigma_1^3(y_1)\,\sigma_1^3(y_2)\,\sigma_1^3(y_1+y_2)}\ . \tag{14}$$

Here the Weierstrass σ-function [2] has the parameters g_i given by (3) and $e = -\tau$ is a root of the $\wp-$Weierstrass function, $\wp'(-\tau) = 0$. The function σ_1 is the σ-function associated with the half-period ω corresponding to the root $-\tau$, thus, $\wp(\omega) = -\tau$. Then by definition (see [2]),

$$\sigma_1(x) = \frac{\sigma(x+\omega)}{\sigma(\omega)}\ \exp\Big(-\frac{\sigma'(\omega)}{\sigma(\omega)}\,x\Big)\ .$$

There are two essentially different degenerations of the $\wp-$Weierstrass function to trigonometric case: (I) when $e = -\tau$ is double root, thus, $e = 2\tau$ is the simple root and then $\mu = 0$, and (II) when $e = -\tau$ is a simple root and $\mu = \frac{3}{4}\tau^2$. In both cases

$$\wp(x) \to \frac{\alpha^2}{4\sin^2\frac{\alpha x}{2}} - \frac{\alpha^2}{12}$$

but in the case (I) $\tau = \frac{\alpha^2}{12}$ whereas for the second case (II) $\tau = -\frac{\alpha^2}{6}$. For the first degeneration the Jacobian

$$W(y_1,y_2) \;=\; \frac{8}{\alpha^3}\ \sin\frac{\alpha(y_1-y_2)}{2}\ \sin\frac{\alpha(y_1+2y_2)}{2}\ \sin\frac{\alpha(2y_1+y_2)}{2} \tag{14.1}$$

and for the second one the Jacobian is factorized as follows

$$W(y_1,y_2) = \frac{8}{\alpha^3}\ \frac{\sin\frac{\alpha(y_1-y_2)}{2}\ \sin\frac{\alpha(y_1+2y_2)}{2}\ \sin\frac{\alpha(2y_1+y_2)}{2}}{\cos^3\frac{\alpha y_1}{2}\ \cos^3\frac{\alpha y_2}{2}\ \cos^3\frac{\alpha(y_1+y_2)}{2}}\ . \tag{14.2}$$

where α is a parameter such that $\tau = \alpha^2/12$. The factorization of the case (I) cannot be generalized to the elliptic case where, in general, we have no multiple roots.

Surprisingly, the gauge rotation of (5) with determinant D (10) as a gauge factor

$$h \;=\; -3D^{-\frac{\nu}{2}}\,(\mathcal{H}_{\mathrm{A}_2} - E_0)\,D^{\frac{\nu}{2}}\ , \tag{15}$$

where $E_0 = 3\nu(3\nu+1)\tau$, transforms the Hamiltonian $\mathcal{H}_{\mathrm{A}_2} - E_0$ into the algebraic operator(!),

$$
\begin{aligned}
h \ = \ & \Big(x + 3\tau x^2 + 3\mu x^3 + 3(\mu - \tau^2)y^2 - 3\mu\tau x y^2 - 3\mu^2 x^2 y^2\Big)\frac{\partial^2}{\partial x^2} + \\
& y\Big(3 + 8\tau x + 7\mu x^2 - 3\mu\tau y^2 - 6\mu^2 x y^2\Big)\frac{\partial^2}{\partial x \partial y} + \\
& \frac{1}{3}\Big(- x^2 + 9\tau y^2 + 12\mu x y^2 - 9\mu^2 y^4\Big)\frac{\partial^2}{\partial y^2} + \\
& (1+3\nu)\Big(1 + 4\tau x + 5\mu x^2 - 3\mu\tau y^2 - 6\mu^2 x y^2\Big)\frac{\partial}{\partial x} \ + 2(1+3\nu)y\Big(2\tau + 3\mu x - 3\mu^2 y^2\Big)\frac{\partial}{\partial y} + \\
& 3\nu(1+3\nu)\mu\Big(2x - 3\mu y^2\Big) \ .
\end{aligned}
\tag{16}
$$

Note the important $\mathbb{Z}_2$ symmetry property of this gauge-rotated Hamiltonian h,

$$
h(x,y) = h(x,-y) \ .
$$

It implies that in the variables $(u = x, v = y^2)$ the operator h remains algebraic,

$$
\begin{aligned}
h(u,v) \ = \ & \Big(u + 3\tau u^2 + 3\mu u^3 + 3(\mu - \tau^2)v - 3\mu\tau u v - 3\mu^2 u^2 v\Big)\frac{\partial^2}{\partial u^2} + \\
& 2v\Big(3 + 8\tau u + 7\mu u^2 - 3\mu\tau v - 6\mu^2 u v\Big)\frac{\partial^2}{\partial u \partial v} \ + 4v\Big(- \frac{u^2}{3} + 3\tau v + 4\mu u v - 3\mu^2 v^2\Big)\frac{\partial^2}{\partial v^2} + \\
& (1+3\nu)\Big(1 + 4\tau u + 5\mu u^2 - 3\mu\tau v - 6\mu^2 u v\Big)\frac{\partial}{\partial u} + \\
& 2\bigg(- \frac{u^2}{3} + \tau(7+12\nu)v + 2\mu(5+9\nu)uv - 9\mu^2(1+2\nu)v^2\bigg)\frac{\partial}{\partial v} + \\
& 3\nu(1+3\nu)\mu\Big(2u - 3\mu v\Big) \ .
\end{aligned}
\tag{17}
$$

It is an alternative algebraic form of the gauge-rotated operator (15). Note that the variables u, v are invariants with respect to the total symmetry of the Hamiltonian (5): $S^2 \oplus \mathbb{Z}_2 \oplus (T_r)^2 \oplus (T_c)^2$.

The operator $h(x,y)$ has also a certain property of self-similarity: the gauge-rotated operator $\tilde{h} = D^{-m} h D^{m}$ with $m = (\frac{1}{2} - \nu)$ has polynomial coefficients as well as the corresponding gauge-rotated operator $\tilde{k}_{\mathrm{A}_2} = D^{-m} k_{\mathrm{A}_2} D^{m}$ (see below). It is easy to verify that

$$
\tilde{h}_{\nu} = h_{4-3\nu} - 12(1-2\nu)\tau \ .
$$

Evidently, the operator $\tilde{h}_{\nu}$ has the same functional form of the potential (9) as h_{ν}.

Let

$$J_1 = \frac{\partial}{\partial x}\ ,\ J_2 = \frac{\partial}{\partial y}\ ,\ J_3 = x\frac{\partial}{\partial x}\ ,\ J_4 = y\frac{\partial}{\partial x}\ ,\ J_5 = x\frac{\partial}{\partial y}\ ,\ J_6 = y\frac{\partial}{\partial y}\ ,$$
$$J_7 = x(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 3\nu) \quad , \quad J_8 = y(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 3\nu)\ . \tag{18}$$

Notice that these formulas define a representation $(-3\nu, 0)$ of the Lie algebra $sl(3)$ in differential operators of first order (see e.g. [3]). If spin of representation

$$-3\nu = n$$

takes integer value, a finite-dimensional representation appears: the space

$$\mathcal{P}_n \ = < x^p y^q \mid 0 \le p+q \le n >\ , \quad \dim \mathcal{P}_n = \frac{(n+2)(n+1)}{2}\ , \tag{19}$$

is preserved by J's. It can be easily shown that for any ν the operator h (16) can be rewritten in terms of $sl(3)$ generators,

$$\begin{aligned} h \ = \ & (1+3\nu)J_1J_3 - 3\nu J_3J_1 + 3J_1J_6 + 3\tau J_3^2 + 6\tau(1-4\nu)J_3J_6\ + 3(\mu-\tau^2)J_4^2 + \\ & \tau(1+12\nu)(J_4J_5 + J_5J_4) + 2(1+3\nu)\mu J_3J_7 - 3\mu\tau J_4J_8 - \frac{1}{3}J_5^2 + 3\tau J_6^2 + \\ & 4\mu J_6J_7 + \mu(1-6\nu)J_7J_3 - 3\mu^2 J_8^2\ . \end{aligned} \tag{20}$$

Thus, the gauge-rotated Hamiltonian h describes $sl(3)$-quantum top in a constant magnetic field. Hence, 3-body elliptic Calogero model with arbitrary coupling constant is equivalent to $sl(3)$-quantum top in a constant magnetic field. If coupling constant in (1) takes discrete values

$$\kappa \ = \ \frac{n}{9}\ (n+3)\ ,\ n = 0,1,2,\dots\ , \tag{21}$$

the Hamiltonian h has finite-dimensional invariant subspace $\mathcal{P}_n$ as well as the Hamiltonian (5). Hence, there may exist a finite number of analytic eigenfunctions of the form

$$\Psi_{n,i} \ = \ P_{n,i}(x,y)\ D^{\frac{\nu}{2}}\ , \quad i = 1,\dots, \frac{(n+2)(n+1)}{2}\ , \tag{22}$$

where polynomial $P_{n,i}(x,y) \in \mathcal{P}_n$, see (19). For example, for $n=0$ (at zero coupling),

$$E_{0,1} = 0\ ,\ P_{0,1} = 1\ .$$

For $n=1$ at coupling

$$\kappa \ = \ \frac{4}{9}\ ,$$

the operator h has three-dimensional kernel (three zero modes) of the type $(a_1x + a_2y + b)$. The first non-trivial solutions appear for $n = 2$ and

$$\kappa \ = \ \frac{10}{9}\ .$$

Eigenvalues are given by the roots of the algebraic equation of degree 6,

$$(E^2 + 4\tau E + 4\mu)(E^2 + 8\tau E + 4\mu + 12\tau^2)(E^2 + 12\tau E + 4\mu + 16\tau^2) \ = \ 0\ ,$$

given by

$$E^{(1)}_{\pm} = -2\tau \pm 2\sqrt{\tau^2 - \mu}\ , \quad E^{(2)}_{\pm} = -4\tau \pm 2\sqrt{\tau^2 - \mu}\ , \quad E^{(3)}_{\pm} = -6\tau \pm 2\sqrt{5\tau^2 - \mu}\ .$$

The corresponding eigenfunctions are of the form $(a_1x^2 + a_2xy + a_3y^2 + b_1x + b_2y + c)$. Using formulas (8) and (15), one can construct the corresponding eigenfunctions for operator (1) in an explicit form.

Observation: Let us construct the operator

$$i^{(n)}_{par}(x, y) \ = \ \prod_{j=0}^{n}(\mathcal{J}^0(n) + j)\ ,$$

where $\mathcal{J}^0(n) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - n$ is the Euler-Cartan generator of the algebra sl_3 (18). It can be immediately shown that the algebraic operator h (16) at integer n commutes with $i^{(n)}_{par}(x, y)$,

$$[h(x, y)\ ,\ i^{(n)}_{par}(x, y)] :\ \mathcal{P}_n \ \rightarrow \ 0\ ,$$

Hence, $i^{(n)}_{par}(x, y)$ is the particular integral [5] of the A_2 elliptic model (5).

It is known (see [1]) that A_2 elliptic model is (completely)-integrable having a certain 3rd order differential operator k_{A_2} as the integral. Perhaps, the most easy way to find this integral is to look for it in a form of algebraic differential operator of the 3rd order, $[h, k_{\mathrm{A}_2}] = 0$. In the explicit form it is given by the following expression

$$k_{\mathrm{A}_2} = -2\nu(1 + 3\nu)(2 + 3\nu)\,\mu\, y\,(2\tau + 3\mu x - 3\mu^2y^2) \tag{23}$$

$$+\ \frac{1}{3}(1\ +\ 3\nu)(2\ +\ 3\nu)y(\mu\ +\ 8\tau^2\ +\ 28\mu\tau x\ +\ 21\mu^2x^2\ -\ 9\mu^2\tau y^2\ -\ 18\mu^3xy^2)\frac{\partial}{\partial x}$$

$$-\ \frac{2}{9}(1\ +\ 3\nu)(2\ +\ 3\nu)\,(1\ +\ 4\tau\, x\ +\ 6\mu\, x^2\ -\ 24\mu\,\tau y^2\ -\ 36\mu^2xy^2\ +\ 27\mu^3y^4)\frac{\partial}{\partial y}$$

$$
\begin{aligned}
&+ (2+3\nu)y\Big(3\,\tau + 4(2\tau^2+\mu)x + 17\mu\tau x^2 + 8\mu^2 x^3 \\
&\qquad\qquad + 3\mu(\tau^2-2\mu)y^2 - 6\mu^2\tau x y^2 - 6\mu^3 x^2 y^2\Big)\frac{\partial^2}{\partial x^2} \\
&- \frac{2}{3}\,(2+3\nu)\Big(x + 4\tau x^2 + 5\mu\, x^3 + 3(\mu-4\tau^2)y^2 - 27\mu^2 x^2 y^2 - \\
&\qquad\qquad 33\mu\,\tau x y^2 + 9\mu^2\tau y^4 + 18\mu^3 x y^4\Big)\frac{\partial^2}{\partial x\partial y} \\
&- (2+3\nu)y(1 + \frac{8}{3}\tau\, x + 3\mu\, x^2 - 7\mu\tau y^2 - 10\mu^2 x y^2 + 6\mu^3 y^4)\frac{\partial^2}{\partial y^2} \\
&+ y\Big(1 + 5\tau x + 2(2\mu+3\tau^2)x^2 + 3\mu(\tau^2-2\mu)x y^2 + 9\mu\tau x^3 \\
&\qquad\qquad - \tau(3\mu-2\tau^2)y^2 + 3\mu^2 x^4 - 3\mu^2\tau x^2 y^2 - 2\mu^3 x^3 y^2\Big)\frac{\partial^3}{\partial x^3} \\
&+ \Big(-\frac{2}{3}x^2 + 2(5\tau^2+\mu)x y^2 - 2\tau x^3 + 3\tau y^2 - 2\mu\, x^4 + 3\mu(\tau^2-2\mu)y^4 + 19\mu\,\tau x^2 y^2 \\
&\qquad\qquad - 6\mu^3 x^2 y^4 + 10\mu^2 x^3 y^2 - 6\mu^2\tau x y^4\Big)\frac{\partial^3}{\partial x^2\partial y} \\
&- y\Big(x + \frac{10}{3}\tau\, x^2 + \frac{11}{3}\,\mu\, x^3 - 13\mu\,\tau x y^2 + 3(\mu-2\tau^2)y^2 - 11\mu^2 x^2 y^2 \\
&\qquad\qquad + 3\mu^2\tau y^4 + 6\mu^3 x y^4\Big)\frac{\partial^3}{\partial x\partial y^2} \\
&- \Big(y^2 + \frac{2}{27}x^3 + 2\tau\, x y^2 - 3\mu\,\tau y^4 + \frac{5}{3}\mu\, x^2 y^2 - 4\mu^2 x y^4 + 2\mu^3 y^6\Big)\frac{\partial^3}{\partial y^3} \quad .
\end{aligned}
$$

It is invariant with respect to $y \to -y$,

$$
k_{\mathrm{A}_2}(x,y) \ = \ k_{\mathrm{A}_2}(x,-y)\ ,
$$

similarly to the gauge rotated Hamiltonian $h(x,y)$ (see (16)). Thus, after the change of variables $(x,y) \to (u = x, v = y^2)$ the operator $k_{\mathrm{A}_2}(u,v)$ remains algebraic. Let us note for $(2+3\nu) = 0$ or, saying differently, for $n = 2$ the operator k_{A_2} becomes a 3rd order homogeneous differential operator, it contains 3rd derivatives only. This operator can be rewritten in terms of $sl(3)$-generators,

$$
k_{\mathrm{A}_2} \ = \ J_1^2 J_4 + 3(2+3\nu)\tau J_1 J_3 J_4 - \frac{2}{9}(1+3\nu)(2+3\nu)J_1 J_3 J_5 + \tag{24}
$$

$$3\tau J_1J_4J_6+\nu(2+3\nu)J_1J_5J_3-3\nu J_1J_6J_5-(1+9\nu)\tau J_3J_1J_4\ +$$
$$\frac{1}{3}\big(12\mu+12\tau^2-(1+3\nu)(11\mu+16\tau^2)+(1+3\nu)^2(\mu+8\tau^2)\big)J_3^2J_4-\frac{8}{9}(1+3\nu)(2+3\nu)\tau J_3^3J_5\ +$$
$$4(2+3\nu)(1-3\nu)\mu\tau J_3^2J_8\ +$$
$$\frac{2}{3}\big(3\tau^2+(1+3\nu)(5\mu+4\tau^2)-(1+3\nu)^2(\mu+8\tau^2)\big)J_3J_4J_3+\big(\mu+8\tau^2+2(1+3\nu)(\mu-4\tau^2)\big)J_3J_4J_6\ +$$
$$\frac{2}{9}(1+36\nu+72\nu^2)\tau J_3J_5J_3-(1-3\nu)J_3J_6J_2-\frac{4}{3}(1+6\nu)\tau J_3J_6J_5+2(2+3\nu)\mu^2J_3J_7J_8\ +$$
$$-4(1+3\nu)\mu\tau J_3J_8J_6+\frac{1}{3}(1+3\nu)(2+3\nu)(\mu+8\tau^2)J_4J_3^2-(\mu(1+6\nu)-2(5+12\nu)\tau^2)J_4J_3J_6\ -$$
$$\frac{4}{3}(1+3\nu)(2+3\nu)\mu\tau J_4J_3J_7-\tau(3\mu-2\tau^2)J_4^3-3\mu(2\mu-\tau^2)J_4^2J_8\ -\ 3(\mu-2\tau^2)J_4J_6^2\ +$$
$$2(7+6\nu)\mu\tau J_4J_6J_7-3\mu^2\tau J_4J_8^2\ -\ \frac{1}{9}(2+9\nu^2)J_5J_3J_1-\frac{4}{9}(1+18\nu^2)\tau J_5J_3^2\ -$$
$$\frac{4}{3}(2+3\nu)\mu J_5J_3J_7-\frac{2}{27}J_5^3\ +\frac{2}{3}(1+6\nu)\mu J_5J_7J_3-J_6J_2J_6-2(1-4\nu)\tau J_6J_5J_3\ -$$
$$-2\tau J_6J_5J_6-\frac{5}{3}\mu J_6J_5J_7\ -\frac{1}{3}\mu\tau\big(5-72\nu^2\big)J_7J_3J_4\ -\ \mu^2(1+6\nu)\mu^2\big)J_7J_3J_8+$$
$$4\mu^2J_7J_8J_6+12\mu\tau J_8J_6^2-9\mu\tau J_6J_8J_6-2\mu^3J_8^3\ .$$

It is evident that if $-3\nu = n$ the operator (23) has the space $\mathcal{P}_n$ as a finite-dimensional invariant subspace. It seems natural to assume that the gauge-rotated integral k_{A_2} written in variables x_1, x_2, x_3,

$$K_{\mathrm{A}_2}\ =\ D^{\frac{\nu}{2}}\,k_{\mathrm{A}_2}\,D^{-\frac{\nu}{2}}\ ,$$

should coincide with the integral found recently by Oshima [6].

An important observation about a connection of the determinant (10) $D \equiv D(\tau,\mu)$ with discriminants should be made. It can be shown that D being written in Cartesian coordinates has the factorized form,

$$D(0,0)\ =\ 4x^3+27y^2\ \sim\ (y_1-y_2)^2(y_1-y_3)^2(y_2-y_3)^2\ ,$$

so, it is the discriminant of cubic equation;

$$D(\tau,0)\ =\ 12\tau\,x^4+4x^3+72\tau^2x^2y^2+108\tau xy^2+27y^2+108\tau^3y^4\ \sim$$
$$\sin^2\alpha(y_1-y_2)\sin^2\alpha(y_1-y_3)\sin^2\alpha(y_2-y_3)\ , \qquad (25)$$

is a trigonometric discriminant, where $\tau = \frac{\alpha^2}{3}$. In general, $D(\tau,\mu) = \frac{W^2(\tau,\mu)}{12}$, where (cf. (14))

$$W(\tau,\mu) \ \sim \frac{\sigma(y_1-y_2)\,\sigma(y_2-y_3)\,\sigma(y_3-y_1)}{\sigma_1^3(y_1)\,\sigma_1^3(y_2)\,\sigma_1^3(y_3)}\ , \tag{26}$$

and $\sigma(x)$ and $\sigma_1(x)$ are the Weierstrass σ functions (see [2]), might be an elliptic discriminant.

It has to be noted that the operator $h(u,v)$ (see (17)) (as well as $k_{\mathrm{A}_2}(u,v)$) can be rewritten in terms of the generators of the algebra $g^{(2)}$: the infinite-dimensional, eleven generated algebra of differential operators [9]. It can have a finite-dimensional representation space,

$$\mathcal{Q}_n \ = < u^p v^q \mid 0 \le p + 2q \le n > \ . \tag{27}$$

This algebra is the hidden algebra of the G_2 rational and trigonometric models. It may remain the hidden algebra of the G_2 elliptic model.

In this paper we demonstrate that A_2 elliptic model belongs to two-dimensional quasi-exactly-solvable (QES) problems [7, 8]. We show the existence of an algebraic form of the A_2 elliptic Hamiltonian, which is the second order polynomial element of the universal enveloping algebra U_{sl_3}. We construct explicitly the integral - commuting with the Hamiltonian - as the third order polynomial element of the universal enveloping algebra U_{sl_3}. If this algebra appears in a finite-dimensional representation those elements possess a finite-dimensional invariant subspace. This phenomenon occurs for a discrete sequence of coupling constants (21) for which polynomial eigenfunctions may occur. It looks very much similar to the case of A_1 elliptic model (the Lame Hamiltonian, see [4]), where the new variable making the A_1 elliptic Hamiltonian the algebraic operator is $x = \frac{1}{\wp(y_1)}$. A generalization of developed approach to A_n elliptic models for $n > 2$ seems straightforward. It is worth noting that a certain algebraic form for a general BC_n elliptic model was found some time ago in [10] (see also [4]). It was shown also the existence of the $sl(n)$ hidden algebra structure and it was shown that it is equivalent to $sl(n)$ quantum top.

Note added. When the present study was completed, based on the transformation (8), the following has been formulated

Conjecture (M. Matushko, August 2014). The analog of transformation (8) for arbitrary n is given by the solution of the linear system

$$M\mathbf{u} = \mathbf{e},$$

where $\mathbf{u} = (u_1, \ldots, u_n)^t$, $\mathbf{e} = (1, 1, \ldots, 1)^t$ with

$$M^i_j = \frac{d^{j-1}\wp(y_i)}{dy_i^{j-1}} \ .$$

It is evidently correct for $n = 1$. We plan to check validity of this conjecture elsewhere.

Acknowledgments

The authors are grateful to Maxim Kontsevich for the interest to the present work and valuable discussions, V.V.S. thanks Masha Matushko for interest to the work and useful discussions, A.V.T. thanks Alberto Grunbaum for the important remark. From the side of A.V.T. his work was supported in part by the University Program FENOMEC, and by the PAPIIT grant **IN109512** and CONACyT grant **166189** (Mexico).

---

[1] M.A. Olshanetsky and A.M. Perelomov, *Quantum integrable systems related to Lie algebras*, *Phys. Repts.* **94** (1983) 313-393

[2] E.T. Whittaker and G.N. Watson, *A Course in Modern Analysis*
4th edition, Cambridge University Press, 1927

[3] Rühl W. and Turbiner A.V., *Exact solvability of the Calogero and Sutherland models*, *Mod. Phys. Lett.* **A10**, 2213–2222 (1995) `hep-th/9506105`

[4] A.V. Turbiner,
Lame Equation, sl(2) and Isospectral Deformation,
Journ.Phys. **A22**, L1-L3 (1989);
The BC_1 Elliptic model: algebraic forms, hidden algebra $sl(2)$, polynomial eigenfunctions,
arXiv: 1408.1610, pp.9 (August 2014)

[5] A.V. Turbiner,
Particular Integrability and (Quasi)-exact-solvability,
Journal of Physics **A 46** (2013) 025203 (9pp)

[6] T. Oshima,
Completely integrable systems associated with classical root systems, *SIGMA* **3** (2007), 061, 50 pages

[7] A.V. Turbiner,

Quasi-Exactly-Solvable Problems and the $SL(2,R)$ Group,

Comm.Math.Phys. **118**, 467-474 (1988)

[8] A.V. Turbiner,

Lie-algebras and Linear Operators with Invariant Subspaces,

in *Lie Algebras, Cohomologies and New Findings in Quantum Mechanics*,

AMS 'Contemporary Mathematics' **160** (1994) 263-310

[9] A.V. Turbiner,

From quantum A_N (Sutherland) to E_8 trigonometric model: space-of-orbits view,

SIGMA **9** (2013) 003 (25pp)

[10] D. Gomez-Ullate, A. Gonzalez-Lopez and M.A. Rodriguez,

Exact solutions of a new elliptic Calogero-Sutherland Model,

Phys.Lett. **B511**, 112-118 (2001)

[11] In the case of A_1 elliptic model the variable x, which is the invariant with respect to the symmetry of the A_1 Hamiltonian $\mathbb{Z}_2 \oplus (T_r) \oplus (T_c)$, is equal to Weierstrass function, $x = \wp(y)$ (see [4]), the function $W = \frac{\partial x}{\partial y}$ is the Jacobian and the determinant $D(x)$ is a cubic polynomial $D = P_3(x)$; the equation analogous to (12) defines the elliptic curve, $w^2 = P_3(x)$.