Fractal tube formulas and a Minkowski measurability criterion for compact subsets of Euclidean spaces

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Received *****; accepted after revision +++++

Presented by

Abstract

We establish pointwise and distributional fractal tube formulas for a large class of compact subsets of Euclidean spaces of arbitrary dimensions. These formulas are expressed as sums of residues of suitable meromorphic functions over the complex dimensions of the compact set under consideration (i.e., over the poles of its fractal zeta function). Our results generalize to higher dimensions (and in a significant way) the corresponding ones previously obtained for fractal strings by the first author and van Frankenhuijsen. They are illustrated by several examples and applied to yield a new Minkowski measurability criterion.

To cite this article: M. L. Lapidus, G. Radunović, D. Žubrinić, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Résumé

Formules de tubes fractales et un critère de Minkowski mesurabilité pour des sous-ensembles compacts de l'espace euclidien. Nous obtenons des "formules de tubes fractales" pour une large classe de compacts de l'espace euclidien en dimension arbitraire. Plus précisément, la formule de tube fractale est exprimée comme une somme de résidus (de fonctions méromorphes convenables) prise sur l'ensemble des dimensions complexes (i.e., les poles de la fonction zêta fractale associée) du compact considéré. Nos résultats généralisent à des dimensions quelconques (et de façon significative) les résultats correspondants obtenus pour des cordes fractales par le premier auteur et van Frankenhuijsen. Nous les utilisons pour obtenir un critère de Minkowski mesurabilité et les illustrons à l'aide de plusieurs exemples. *Pour citer cet article : M. L. Lapidus, G. Radunović, D. Žubrinić, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

1. Introduction

We begin by stating some definitions and results from the research monograph [8] that will be needed in this article, as well as by recalling some well-known notions. Given a bounded subset A of $\mathbb{R}^N$ (always assumed to be nonempty in this paper), we denote its *δ-neighborhood* by $A_\delta := \{x \in \mathbb{R}^N : d(x,A) < \delta\}$. Here, $d(x,A) := \inf\{|x-y| : y \in A\}$ is the Euclidean distance between the point x and the set A.[3] Furthermore, for a compact subset A of $\mathbb{R}^N$ and $r \geq 0$, we define its *upper r-dimensional Minkowski content*, $\overline{\mathcal{M}}^r(A) = \limsup_{t\to 0^+} t^{r-N}|A_t|$, and its *upper box dimension*, $\overline{\dim}_B A = \inf\{r \geq 0 : \overline{\mathcal{M}}^r(A) = 0\}$. The value $\underline{\mathcal{M}}^r(A)$ of the *lower r-dimensional Minkowski content* of A, is defined analogously as $\overline{\mathcal{M}}^r(A)$, except for a lower instead of an upper limit, and similarly for the *lower box dimension* $\underline{\dim}_B A$. If $\underline{\dim}_B A = \overline{\dim}_B A$, this common value is called the *Minkowski* (or *box*) *dimension* of A and denoted by $\dim_B A$. If $0 < \underline{\mathcal{M}}^D(A)(\leq)\overline{\mathcal{M}}^D(A) < \infty$, for some $D \geq 0$, the set A is said to be *Minkowski nondegenerate*. It then follows that $\dim_B A$ exists and is equal to D. Moreover, if $\mathcal{M}^D(A)$ exists and is different from 0 and ∞ (in which case $\dim_B A$ exists and then necessarily, $D = \dim_B A$), the set A is said to be *Minkowski measurable*.

We will now introduce the notions of distance and tube zeta functions of compact sets and state their basic properties. These definitions have enabled us in [8–10] to develop a higher-dimensional extension of the theory of complex dimensions of fractal strings ([12]), valid for arbitrary compact sets.

Definition 1.1 (Fractal zeta functions, [8]) Let A be a compact subset of $\mathbb{R}^N$ and fix $\delta > 0$. We define the *distance zeta function* ζ_A of A and the *tube zeta function* $\widetilde{\zeta}_A$ of A by the following Lebesgue integrals, respectively, for some $\delta > 0$ and for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large:

$$\zeta_A(s;\delta) := \int_{A_\delta} d(x,A)^{s-N}\mathrm{d}x \quad \text{and} \quad \widetilde{\zeta}_A(s;\delta) := \int_0^\delta t^{s-N-1}|A_t|\,\mathrm{d}t. \tag{1}$$

It is not difficult to show that the distance and tube zeta functions of a compact subset A of $\mathbb{R}^N$ satisfy the following functional equation, which is valid on any connected open set $U \subseteq \mathbb{C}$ to which any of the two zeta functions has a meromorphic continuation (see [8, §2.2]):

$$\zeta_A(s;\delta) = \delta^{s-N}|A_\delta| + (N-s)\widetilde{\zeta}_A(s;\delta). \tag{2}$$

Furthermore, in the above definition (see Eq. (1)), the dependence of the zeta functions on the parameter $\delta > 0$ is inessential, from the point of view of the theory of complex dimensions (see Def. 1.4 below). Indeed, it is shown in [8] that the difference of two distance (or tube) zeta functions of the same compact set A, and corresponding to any two different values of the parameter δ, is an entire function.

Let us briefly summarize the main properties of the distance and tube zeta functions (see [8, Ch. 2]):

If A is a compact subset of $\mathbb{R}^N$, then the tube zeta function $\widetilde{\zeta}_A(\,\cdot\,;\delta)$ is holomorphic in the half-plane $\{\operatorname{Re} s > \overline{\dim}_B A\}$ and $\overline{\dim}_B A$ coincides with the abscissa of (absolute) convergence of $\widetilde{\zeta}_A(\,\cdot\,;\delta)$. Furthermore, if the box (or Minkowski) dimension $D := \dim_B A$ exists and $\underline{\mathcal{M}}^D(A) > 0$, then $\widetilde{\zeta}_A(s;\delta) \to +\infty$ as $s \in \mathbb{R}$ converges to D from the right. The above statements are also true if we replace $\widetilde{\zeta}_A$ by ζ_A

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1 . The research of Michel Lapidus was supported by the U.S. National Science Foundation (NSF) under the grants DMS-0707524 and DMS-1107750, as well as by the Institut des Hautes Études Scientifiques (IHES).

2 . The research of Goran Radunović and Darko Žubrinić was supported by the Croatian Science Foundation under the grant IP-2014-09-2285 and by the Franco–Croatian PHC-COGITO project.

3. Without loss of generality, we may replace A by its closure, $\overline{A}$, and hence assume from now on that A is compact.

and in the preceding sentence assume, in addition, that $D < N$. Finally, we have the *scaling property*; that is, if for a $\lambda > 0$ we let $\lambda A := \{\lambda x : x \in A\}$, then $\zeta_{\lambda A}(s; \lambda\delta) = \lambda^s \zeta_A(s;\delta)$ and $\widetilde{\zeta}_{\lambda A}(s;\lambda\delta) = \lambda^s \widetilde{\zeta}_A(s;\delta)$.

If A is a Minkowski nondegenerate subset of $\mathbb{R}^N$ (so that $D := \dim_B A$ exists), and for some $\delta > 0$ there exists a meromorphic extension of $\widetilde{\zeta}_A(\,\cdot\,;\delta)$ to a neighborhood of D, then D is a simple pole of $\widetilde{\zeta}_A(\,\cdot\,;\delta)$, and $\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,;\delta), D)$ is independent of δ. Furthermore, we have $\underline{\mathcal{M}}^D(A) \le \operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,;\delta), D) \le \overline{\mathcal{M}}^D(A)$. In particular, if A is Minkowski measurable, then $\operatorname{res}(\widetilde{\zeta}_A(\,\cdot\,;\delta), D) = \mathcal{M}^D(A)$. If, additionally, $D < N$, the analogous statement and conclusion is true for the distance zeta function ζ_A and we have $(N-D)\underline{\mathcal{M}}^D(A) \le \operatorname{res}(\zeta_A(\,\cdot\,;\delta), D) \le (N-D)\overline{\mathcal{M}}^D(A)$. Moreover, if A is Minkowski measurable, then $\operatorname{res}(\zeta_A(\,\cdot\,;\delta), D) = (N-D)\mathcal{M}^D(A)$.

Let us now introduce some additional definitions, which are adapted from [12] to the present, much more general, context of compact subsets of an arbitrary Euclidean space, $\mathbb{R}^N$ (with $N \ge 1$):

The *screen* S is the graph of a bounded, real-valued, Lipschitz continuous function $S(\tau)$, with the horizontal and vertical axes interchanged: $S := \{S(\tau) + \mathbb{i}\tau : \tau \in \mathbb{R}\}$. The Lipschitz constant is denoted by $\|S\|_{\mathrm{Lip}}$. Furthermore, we let $\sup S := \sup_{\tau\in\mathbb{R}} S(\tau) \in \mathbb{R}$. For a compact subset A of $\mathbb{R}^N$, we always assume that the screen S lies to the left of the *critical line* $\{\operatorname{Re} s = \overline{D}\}$, i.e., that $\sup S \le \overline{D}$. Moreover, the *window* W is defined as $W := \{s \in \mathbb{C} : \operatorname{Re} s \ge S(\operatorname{Im} s)\}$. The set A is said to be *admissible* if its tube (or distance) zeta function can be meromorphically extended to an open connected neighborhood of some window W.

Definition 1.2 (d-languid set; adapted from [12, Def. 5.2]) An admissible compact subset A of $\mathbb{R}^N$ is said to be *d-languid* if there exists a $\delta > 0$ such that $\zeta_A(s;\delta)$ satisfies the following growth conditions: There exist real constants κ and $C > 0$ and a two-sided sequence $(T_n)_{n\in\mathbb{Z}}$ of real numbers such that $T_{-n} < 0 < T_n$ for $n \ge 1$, $\lim_{n\to\infty} T_n = +\infty$ and $\lim_{n\to\infty} T_{-n} = -\infty$, satisfying the following two hypotheses, **L1** and **L2**:

L1 *There exists $c > N$ such that $|\zeta_A(\sigma + \mathbb{i}T_n;\delta)| \le C(|T_n|+1)^\kappa$, for all $n \in \mathbb{Z}$ and all $\sigma \in (S(T_n), c)$.*

L2 *For all $\tau \in \mathbb{R}$, with $|\tau| \ge 1$, we have that $|\zeta_A(S(\tau) + \mathbb{i}\tau;\delta)| \le C|\tau|^\kappa$.*

Definition 1.3 (Strongly d-languid set; adapted from [12, Def. 5.3]) A compact subset A of $\mathbb{R}^N$ is said to be *strongly d-languid* if for some $\delta > 0$, $\zeta_A(s;\delta)$ satisfies **L1** with $S(\tau) \equiv -\infty$ in condition **L1**; i.e., for every $\sigma < c$ and, additionally, there exists a sequence of screens $S_m(\tau)\colon \tau \mapsto S_m(\tau) + \mathbb{i}\tau$ for $m \ge 1$, $\tau \in \mathbb{R}$ with $\sup S_m \to -\infty$ as $m \to \infty$ and with a uniform Lipschitz bound, $\sup_{m\ge1} \|S_m\|_{\mathrm{Lip}} < \infty$, such that

L2' There exist $B, C > 0$ such that $|\zeta_A(S_m(\tau) + \mathbb{i}\tau;\delta)| \le CB^{|S_m(\tau)|}(|\tau|+1)^\kappa$, for all $\tau \in \mathbb{R}$ and $m \ge 1$.

Definition 1.4 (Complex dimensions, [8]) Let A be an admissible compact subset of $\mathbb{R}^N$. Then, the set of *visible complex dimensions of A (with respect to U)* is defined as $\mathcal{P}(\zeta_A(\,\cdot\,;\delta), U) := \{\omega \in U : \omega \text{ is a pole of } \zeta_A(\,\cdot\,;\delta)\}$. If $U = \mathbb{C}$, we say that $\mathcal{P}(\zeta_A(\,\cdot\,;\delta), \mathbb{C})$ is the set of *complex dimensions* of A.[4]

2. Pointwise and distributional tube formulas and a criterion for Minkowski measurability

In this section, we state and sketch the proof of our main results, the pointwise and distributional tube formulas, valid for a large class of compact subsets of $\mathbb{R}^N$ (see Thms. 2.1 and 2.2 below), along with an associated Minkowski measurability criterion (see Thm. 2.3). These results extend to higher dimensions the corresponding tube formulas and Minkowski measurability criterion obtained for fractal strings in [12],

4. Clearly, $\mathcal{P}(\zeta_A(\,\cdot\,;\delta), U)$ is a discrete subset of $\mathbb{C}$ and is independent of δ; hence, so is $\mathcal{P}(\zeta_A(\,\cdot\,;\delta), \mathbb{C})$. Therefore, we will often write $\mathcal{P}(\zeta_A, U)$ or $\mathcal{P}(\zeta_A, \mathbb{C})$ instead.

§8.1 and §8.3, respectively. We point out that the detailed proofs of our main results (stated in a much more general form and within the broader context of relative fractal drums) can be found in the long paper corresponding to this note, [11]. Moreover, we note that in light of (2), Thms. 2.1, 2.2 and 2.3 have an obvious analog for tube (instead of distance) zeta functions. Also, the exact tube formula stated in Thm. 2.1 has a counterpart with error term (much as in Thm. 2.2). Finally, we refer to [11] and [12, §13.1] for many additional references on tube formulas in various settings, including, [1–3, 5–8, 13, 16].

The key observation in deriving Thms. 2.1 and 2.2 below is the fact that the tube zeta function of a compact set A in $\mathbb{R}^N$ is equal to the Mellin transform of its modified tube function $f(t) := \chi_{(0,\delta)}(t)t^{-N}|A_t|$, where χ_E denotes the characteristic function of the set E. More precisely, one has that $\widetilde{\zeta}_A(s;\delta) = \{\mathfrak{M}f\}(s) := \int_0^{+\infty} t^{s-1}f(t)\,\mathrm{d}t$, where $\mathfrak{M}$ denotes the Mellin transform. One then applies the Mellin inversion theorem (see, e.g., [15, Thm. 28]) to deduce that $|A_t| = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} t^{N-s}\widetilde{\zeta}_A(s;\delta)\,\mathrm{d}s$, for all $t\in(0,\delta)$, where $c > \overline{\dim}_B A$ is arbitrary. One then proceeds in a similar manner as in [12, Ch. 5] for the case of fractal strings. More precisely, one works with a k-th primitive function of $t\mapsto|A_t|$ in order to be able to represent the above integral as a sum over the complex dimensions contained in the window W. Here, $k\in\mathbb{N}$ is taken large enough to ensure pointwise convergence of this sum. From this result, one then derives the distributional tube formula for every value of k (even for $k\in\mathbb{Z}$), and, in particular, for $k=0$. In this way, we obtain the fractal tube formulas expressed in terms of the tube zeta function and then use the functional equation (2) in order to translate them in terms of the distance zeta function.

Theorem 2.1 (Pointwise tube formula) *Let A be a compact subset of $\mathbb{R}^N$ such that $\overline{\dim}_B A < N$. Furthermore, assume that there exists a constant $\lambda > 0$ such that λA is strongly d-languid for some $\delta > 0$ and $\kappa < 1$. Then, for every $t\in(0,\lambda^{-1}\min\{1,\delta,B^{-1}\})$ the following exact pointwise tube formula is valid (where B is the constant appearing in* **L2'** *of Def. 1.3 above) :* [5]

$$|A_t| = \sum_{\omega\in\mathcal{P}(\zeta_A,\mathbb{C})} \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_A(s),\omega\right). \tag{3}$$

In the case when $\kappa\in\mathbb{R}$, we usually only have a distributional tube formula. Furthermore, if A is only d-languid, we will have a distributional error term, with information about its asymptotic order given in the sense of [12, §5.4]. Namely, the distribution $\mathcal{R}\in\mathcal{D}'(0,\delta)$ is said to be of *asymptotic order at most* t^α (resp., *less than* t^α) as $t\to0^+$ if when applied to a test function $\varphi\in\mathcal{D}(0,\delta)$,[6] we have that $\langle\mathcal{R},\varphi_a\rangle = O(a^\alpha)$ (resp., $\langle\mathcal{R},\varphi_a\rangle = o(a^\alpha)$), as $a\to0^+$, where $\varphi_a(t) := a^{-1}\varphi(t/a)$ (and the implicit constants may depend on φ). We then write that $\mathcal{R}(t) = O(t^\alpha)$ (resp., $\mathcal{R}(t) = o(t^\alpha)$) as $t\to0^+$.

Theorem 2.2 (Distributional tube formula) *Let A be a d-languid compact subset of $\mathbb{R}^N$, for some $\delta>0$ and $\kappa\in\mathbb{R}$. Furthermore, assume that $\overline{\dim}_B A < N$ and denote by $\mathcal{V}(t)$ the distribution generated by $t\mapsto|A_t|$. Then, we have the following distributional equality:*

$$\mathcal{V}(t) = \sum_{\omega\in\mathcal{P}(\zeta_A,W)} \operatorname{res}\left(\frac{t^{N-s}}{N-s}\zeta_A(s),\omega\right) + \mathcal{R}(t). \tag{4}$$

More precisely, the action of $\mathcal{V}(t)$ on a test function $\varphi\in\mathcal{D}(0,\infty)$ is given by

$$\langle\mathcal{V},\varphi\rangle = \sum_{\omega\in\mathcal{P}(\zeta_A,W)} \operatorname{res}\left(\frac{\{\mathfrak{M}\varphi\}(N-s+1)}{N-s}\zeta_A(s),\omega\right) + \langle\mathcal{R},\varphi\rangle. \tag{5}$$

5. We write here and in Thm. 2.2 below $\zeta_A(s)$ instead of $\zeta_A(s;\delta)$ since the residues in the formula do not depend on the parameter δ in any way.

6. Here, $\mathcal{D}(0,\delta) := C_c^\infty(0,\delta)$ is the standard space of infinitely differentiable test functions with compact support.

In Eq. (4)*, the distributional error term* $\mathcal{R}(t)$ *is* $O(t^{N-\sup S})$ *as* $t \to 0^+$*. Moreover, if* $S(\tau) < \sup S$ *for all* $\tau \in \mathbb{R}$*, then* $\mathcal{R}(t)$ *is* $o(t^{N-\sup S})$ *as* $t \to 0^+$*. If, in addition,* λA *is strongly d-languid for some* $\lambda > 0$*, then, for test functions in* $\mathcal{D}\big(0, \lambda^{-1}\min\{1, \delta, B^{-1}\}\big)$*, we have that* $\mathcal{R} \equiv 0$ *and* $W = \mathbb{C}$*; hence, we obtain an* exact *tube formula in that case.*

One of the applications of the above results is a Minkowski measurability criterion for a compact d-languid subset of $\mathbb{R}^N$ (see Thm. 2.3 below), which generalizes [12, Thm. 8.15] to higher dimensions. In the proof of Thm. 2.3, one direction is a consequence of the distributional tube formula (Thm. 2.2 above) and the uniqueness theorem for almost periodic distributions (see [14, §VI.9.6, p. 208]). The other direction follows from a generalization of the classic Wiener–Ikehara Tauberian theorem (see [4]).

Theorem 2.3 (Minkowski measurability criterion) *Let* A *be a compact subset of* $\mathbb{R}^N$ *such that* $D := \dim_B A$ *exists and* $D < N$*. Furthermore, assume that* A *is d-languid for a screen passing between the critical line* $\{\mathrm{Re}\, s = D\}$ *and all the complex dimensions of* A *with real part strictly less than* D*. Then, the following statements are equivalent*:

(*a*) *A is Minkowski measurable.*

(*b*) D *is the only pole of the distance zeta function* ζ_A *located on the critical line* $\{\mathrm{Re}\, s = D\}$*, and it is simple.*

There exist d-languid compact sets (and even fractal strings, see [12, Exple. 5.32]) which do not satisfy the hypothesis of Thm. 2.3 concerning the screen. We point out that Thms. 2.1 and 2.2 can be applied to obtain tube formulas for a variety of well-known fractal sets, as is illustrated by the following examples. Furthermore, Exple. 2 below shows how our results can be applied to derive the tube formula of a self-similar fractal set in $\mathbb{R}^3$. We also note that fractal tube formulas can be obtained for examples of higher-dimensional fractal sets that are not self-similar, such as "fractal nests" and "geometric chirps"; see [8] for the definitions of these notions. In such examples, we will generally obtain a distributional (or pointwise) tube formula with an error term; see [11] for details.

Example 1 Let A be the Sierpiński gasket in $\mathbb{R}^2$, constructed in the usual way inside the unit triangle. Then, for $\delta > 1/4\sqrt{3}$, the distance zeta function ζ_A is given for all $s \in \mathbb{C}$ by

$$\zeta_A(s;\delta) = \frac{6(\sqrt{3})^{1-s}2^{-s}}{s(s-1)(2^s-3)} + 2\pi\frac{\delta^s}{s} + 3\frac{\delta^{s-1}}{s-1},$$

which is meromorphic on the whole complex plane (see [8, §3.2]). In particular, $\mathcal{P}(\zeta_A, \mathbb{C}) = \{0\} \cup \big(\log_2 3 + \frac{2\pi}{\log 2}\mathbb{i}\mathbb{Z}\big)$ and by letting $\omega_k := \log_2 3 + \mathbf{p}k\mathbb{i}$ and $\mathbf{p} := 2\pi/\log 2$, we have that $\mathrm{res}(\zeta_A(\,\cdot\,;\delta), \omega_k) = 6(\sqrt{3})^{1-\omega_k}/\big(4^{\omega_k}(\log 2)\omega_k(\omega_k - 1)\big)$ (for all $k \in \mathbb{Z}$) and $\mathrm{res}(\zeta_A(\,\cdot\,;\delta), 0) = 3\sqrt{3} + 2\pi$. One can easily check, by using the scaling property of the distance zeta function, that λA is strongly d-languid, for any $\lambda \geq 2\sqrt{3}$ with $\kappa = -1$. Hence, we can apply Thm. 2.1 in order to obtain the following exact pointwise tube formula, valid for all $t \in (0, 1/2\sqrt{3})$, and which coincides with the one obtained in [5–7] and also, more recently, in [1]: [7]

$$|A_t| = \sum_{\omega \in \mathcal{P}(\zeta_A, \mathbb{C})} \mathrm{res}\left(\frac{t^{2-s}}{2-s}\zeta_A(s;\delta), \omega\right) = \frac{6\sqrt{3}\, t^{2-\log_2 3}}{\log 2} \sum_{k=-\infty}^{\infty} \frac{(4\sqrt{3})^{-\omega_k} t^{-\mathbf{p}k\mathbb{i}}}{(2-\omega_k)(\omega_k - 1)\omega_k} + \left(\frac{3\sqrt{3}}{2} + \pi\right) t^2.$$

Example 2 Let A be the three-dimensional analog of the Sierpiński carpet. More precisely, we construct A by dividing the closed unit cube of $\mathbb{R}^3$ into 27 congruent cubes and remove the open middle cube, then we iterate this step with each of the 26 remaining smaller closed cubes; and so on, ad infinitum. By choosing $\delta > 1/6$, we deduce that ζ_A is meromorphic on $\mathbb{C}$ and given for all $s \in \mathbb{C}$ by (see [11])

7. By Thm. 2.3 (and in accord with [5–7]), it follows that the Sierpiński gasket is not Minkowski measurable.

$$\zeta_A(s;\delta)=\frac{48\cdot 2^{-s}}{s(s-1)(s-2)(3^s-26)}+\frac{4\pi\delta^s}{s}+\frac{6\pi\delta^{s-1}}{s-1}+\frac{6\delta^{s-2}}{s-2}.$$

In particular, $\mathcal{P}(\zeta_A,\mathbb{C})=\{0,1,2\}\cup\big(\log_3 26+\mathbf{p}\mathbb{i}\mathbb{Z}\big)$, where $\mathbf{p}:=2\pi/\log 3$. Furthermore, we have that $\operatorname{res}(\zeta_A(\,\cdot\,;\delta),0)=4\pi-24/25$, $\operatorname{res}(\zeta_A(\,\cdot\,;\delta),1)=6\pi+24/23$, $\operatorname{res}(\zeta_A(\,\cdot\,;\delta),2)=96/17$ and, by letting $\omega_k:=\log_3 26+\mathbf{p}k\mathbb{i}$, (for all $k\in\mathbb{Z}$), $\operatorname{res}(\zeta_A(\,\cdot\,;\delta),\omega_k)=24/\big(13\cdot 2^{\omega_k}\omega_k(\omega_k-1)(\omega_k-2)\log 3\big)$. One easily checks that the hypotheses of Thm. 2.1 are satisfied, and thus we obtain the following exact pointwise tube formula, valid for all $t\in(0,1/2)$:

$$|A_t|=\frac{24\,t^{3-\log_3 26}}{13\log 3}\sum_{k=-\infty}^{\infty}\frac{2^{-\omega_k}t^{-\mathbf{p}k\mathbb{i}}}{(3-\omega_k)(\omega_k-1)(\omega_k-2)\omega_k}+\left(6-\frac{6}{17}\right)t+\left(3\pi+\frac{12}{23}\right)t^2+\left(\frac{4\pi}{3}-\frac{8}{25}\right)t^3.$$

In particular, we conclude that $\dim_B A=\log_3 26$ and, by Thm. 2.3, that the three-dimensional Sierpiński carpet is not Minkowski measurable (as expected). Note also that the part $6t+3\pi t^2+4\pi t^3/3$ from the above equation is exactly equal to $|I_t|-|I|$, where I is the unit cube of $\mathbb{R}^3$.

We conclude this note by pointing out that, in a precise way, the above results generalize the corresponding ones obtained for fractal strings in [12, §8.1 & §8.3]. Namely, this can be seen from the fact that for the geometric zeta function $\zeta_{\mathcal{L}}$ of a nontrivial fractal string $\mathcal{L}=(l_j)_{j\geq 1}$ and the distance zeta function of the set $A_{\mathcal{L}}:=\{a_k:=\sum_{j\geq k}l_j : k\geq 1\}$, we have that $\zeta_{A_{\mathcal{L}}}(s;\delta)=s^{-1}2^{1-s}\zeta_{\mathcal{L}}(s)+2s^{-1}\delta^s$, where $\delta>l_1/2$, and this identity holds on any subdomain U of $\mathbb{C}$ to which any of the two zeta functions has a meromorphic continuation; see [8, §2.1]. Hence, if $U\subseteq\mathbb{C}\setminus\{0\}$, then $\zeta_{\mathcal{L}}$ and $\zeta_{A_{\mathcal{L}}}$ have the same visible complex dimensions in U.

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