# EQUIVARIANT OPERATIONAL CHOW RINGS OF SPHERICAL VARIETIES AND T-LINEAR VARIETIES

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Abstract. We stablish localization theorems for the equivariant operational Chow rings (or equivariant Chow cohomology) of singular spherical varieties and $T$-linear varieties. Our main results provide a GKM description of these rings in the case of singular spherical varieties admitting a BB-decomposition into *algebraic* rational cells. Our description extends certain topological results to intersection theory on singular varieties.

## 1. Introduction and statement of the main results

Let $G$ be a connected reductive group defined over an algebraically closed field $k$ of characteristic zero. Let $B$ be a Borel subgroup of $G$ and $T \subset B$ be a maximal torus of $G$. An algebraic variety $X$, equipped with an action of $G$, is *spherical* if it contains a dense orbit of $B$. (Usually spherical varieties are assumed to be normal but this condition is not needed here). Spherical varieties have been extensively studied in the works of Brion [Br1], [Br2], Knop [Kn1] and Luna-Vust [L-V]. If $X$ is spherical, then it has a finite number of $B$-orbits, and thus, also a finite number of $G$-orbits (see e.g. [Vin], [Kn2]). In particular, $T$ acts on $X$ with a finite number of fixed points. These properties make spherical varieties particularly suited for applying the methods of Goresky-Kottwitz-MacPherson (GKM theory) [GKM] in the topological setup, and Brion's extension of GKM theory [Br3] to the algebraic setting of equivariant Chow groups, as defined by Totaro, Edidin and Graham [EG].

Examples of spherical varieties include $G \times G$-equivariant embeddings of $G$ (e.g., toric varieties are spherical) and the regular symmetric varieties of De Concini-Procesi [DP-1]. The equivariant cohomology and equivariant Chow groups of smooth projective spherical varieties have been studied by De Concini-Procesi [DP-2], [BCP], De Concini-Littelmann [LP], Brion [Br3] and Brion-Joshua [BJ-2]. In these cases, there are many comparison results relating equivariant cohomology with equivariant Chow groups. As for the study of the equivariant Chow groups of possibly singular spherical varieties, some progress has been made by Danilov [D], Brion [Br3], Payne [P] and the author [G2], [G3].

The problem of developing intersection theory on singular varieties comes from the fact that the Chow groups do not admit, in general, a natural ring structure or intersection product. But when singularities are mild, e.g. when $X$ is a quotient of a smooth variety $Y$ by a finite group $F$, then $A_*(X) \otimes \mathbb{Q} \simeq (A_*(Y) \otimes \mathbb{Q})^F$, and so $A_*(X) \otimes \mathbb{Q}$ inherits the ring structure of $A_*(Y) \otimes \mathbb{Q}$. This happens, for instance, in the case of simplicial toric varieties.

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In order to study more general singular spaces, Fulton and MacPherson [Fu] introduced the notion of operational Chow groups or Chow cohomology. Similarly, Edidin and Graham have defined the Equivariant Operational Chow groups [EG], which we briefly recall. Let $X$ be a $G$-variety. The $G$-equivariant operational Chow groups of $X$, denoted $A^i_G(X)$, are defined as operations $c(Y \to X) : A^G_*(Y) \to A^G_{*-i}(Y)$ for every $G$-map $Y \to X$. As for ordinary operational Chow groups ([Fu], Chapter 17), these operations should be compatible with the operations on equivariant Chow groups (pull-back for l.c.i. morphisms, proper push-forward, etc.). From this definition it is clear that for any $X$, $A^*_G(X)$ has a ring structure. The ring $A^*_G(X)$ is graded, and $A^i_G(X)$ can be non-zero for any $i \geq 0$. Equivariant operational Chow groups come with cap products, making $A^G_*(X)$ into an $A^*_G(X)$-module. Moreover, the equivariant Chern classes are elements of the equivariant operational Chow ring, see [EG] for more details. It follows from [EG] that $A^i_G(X)$ can be identified with the operational Chow group $A^i(X_G)$, where $X_G$ is a finite approximation of the Borel construction. This implies, using a result of Vistoli [Vis-2], that $A^*_G(X) \simeq A^*_T(X)^W$. Throughout this paper, we consider *rational* operational Chow groups $A^*_G(X) \otimes \mathbb{Q}$, so we drop $\mathbb{Q}$ from the notation in the sequel.

In [FMSS], Fulton, MacPherson, Sottile and Sturmfels succeed in describing the non-equivariant operational Chow groups of complete spherical varieties. Indeed, they show that the Kronecker duality homomorphism

$$\mathcal{K} : A^i(X) \longrightarrow \mathrm{Hom}(A_i(X), \mathbb{Q}), \qquad \alpha \mapsto (\beta \mapsto \deg(\beta \cap \alpha))$$

is an isomorphism for complete spherical varieties. Here deg is the degree homomorphism $A_0(X) \to \mathbb{Q}$. Moreover, they show that $A_*(X)$ is finitely generated by the $B$-orbit closures, and with the aid of the map $\mathcal{K}$, they provide a combinatorial description of $A^*(X)$ and the structure constants of the cap and cup products [FMSS]. Although we stated their result in the case of spherical varieties, it holds more generally for complete varieties with a finite number of orbits of a solvable group. In particular, the results of [FMSS] hold for Schubert varieties. This result is quite marvelous in that it gives a presentation of a very abstract ring, namely $A^*(X)$, in a very combinatorial manner. This is the result the motivated the main results of this article.

The aim of this paper is to extend the results of [FMSS] to the setting of equivariant operational Chow rings, for two natural classes of algebraic varieties with group actions: spherical varieties and $T$-linear varieties: briefly, a $T$-linear variety is a variety with a $T$-action that can be obtained by an inductive procedure starting with a finite dimensional $T$-representation, in such a way that the complement of a linear variety equivariantly embedded in affine space is also a linear variety, and any $T$-variety which can be stratified as a finite disjoint union of linear varieties is a linear variety. See Definition 2.1 for a formal definition. $T$-varieties have been studied by Jannsen [J], Totaro [T], and Joshua-Krishna [J-K].

We also extend the localization techniques from equivariant Chow groups [Br3] to equivariant Chow cohomology. Our description falls within the context of GKM theory. The main applications of our theory are to the study of possibly singular spherical varieties and to Schubert varieties. Our results give a large class of singular spaces for which localization holds in equivariant Chow cohomology. Previously, this was known to be the case for smooth spherical varieties, by the work of Brion [Br3], and toric varieties, by work of Payne [P].

For smooth varieties our results are not new. In fact, for smooth spherical varieties they are due to Brion [Br3]. The importance of these paper resides on the fact that it can be applied to a large class of singular spaces, e.g., rationally smooth projective embeddings of reductive groups, Schubert varieties and $\mathbb{Q}$-filtrable complex spherical varieties (Sections 3 to 6). Our results here complement Brion's deepest results [Br3].

This article is organized as follows. The first two sections briefly review the results from [Br3], [Ki] and [FMSS] needed in our study. It turns out that Brion's localization theorem (see Proposition 2.8) can be slightly modified to serve our purposes. From work of Kimura [Ki] we know that equivariant operational Chow groups are more computable. We end Section 2 with a discussion as to why it is reasonable to expect a description of $A^*_T(-)$ as a ring of piecewise polynomial functions.

Section 3 is the conceptual core of this article. We start by defining *Equivariant Kronecker duality spaces*. These spaces are complete $T$-varieties $X$ which satisfy two conditions: (i) $A^T_*(X)$ is finitely generated over $S = A^T_*(pt)$, and (ii) the equivariant Kronecker duality map

$$\mathcal{K}_{\mathcal{T}} : A^*_T(X) \longrightarrow Hom_S(A^T_*(X), S) \qquad \alpha \mapsto (\beta \mapsto p_{X*}(\beta \cap \alpha))$$

is an isomorphism of $S$-modules. Here $p_{X*} : A^T_*(X) \to S$ is the map induced by pushforward to a point. As an example, we show that this class includes all $T$-linear varieties (Proposition 3.5), a result that follows almost immediately from the work of Joshua and Krishna [J-K]. Later in that section we prove our first main result, namely, the Localization Theorem for equivariant Kronecker duality spaces.

**Theorem 3.6** *Let $X$ be a complete $T$-variety satisfying Kronecker duality. Let $T' \subset T$ be a subtorus of $T$ and let $i_{T'} : X^{T'} \to X$ be the inclusion of the fixed point subvariety. If $X^{T'}$ also satisfies $T$-equivariant Kronecker duality, then the injective morphism*

$$i^*_{T'} : A^*_T(X) \to A^*_T(X^{T'})$$

*becomes an isomorphism after inverting finitely many characters of $T$ that restrict non-trivially to $T'$.*

Let $X$ be an equivariant Kronecker duality space. We say that $X$ satisfies the *strong version* of Kronecker duality if the fixed point varieties $X^T$ and $X^{T'}$, where runs over all codimension one subtori $T'$ of $T$, satisfy Kronecker duality. This notion has indeed very important implications, in particular, there is a precise version of the localization theorem.

**Theorem 3.8.** *Let $X$ be a complete $T$-variety satisfying the strong equivariant Kronecker duality. If the $S$-module $A^*_T(X)$ is free, then the image of the injective map*

$$i^*_T : A^*_T(X) \to A^*_T(X^T)$$

*is the intersection of the images of the maps*

$$i^*_{T,T'} : A^*_T(X^{T'}) \to A^*_T(X^T),$$

*where $T'$ runs over all subtori of codimension one of $T$.*

From here we deduce GKM theory for equivariant operational Chow rings. Recall that a $T$-variety is called $T$-skeletal if $T$ acts with a finite number of fixed points and invariant curves.

**Theorem 3.11.** *Let $X$ be a normal projective $T$-skeletal variety. If $X$ satisfies equivariant Kronecker duality and the $A_T^*(pt)$-module $A_*^T(X)$ is free, then the restriction mapping*

$$A_T^*(X) \longrightarrow A_T^*(X^T) = \bigoplus_{x_i \in X^T} A_T^*$$

*is injective, and its image is the subalgebra $PP_T^*(X)$ of piecewise polynomial functions on the GKM-graph of $X$.*

Section 4 is devoted to spherical varieties. After introducing a few key structural properties, we show that they are *strong* Equivariant Kronecker duality spaces and, when $T$-skeletal, GKM theory holds.

**Theorem 4.3, 4.9.** *Let $X$ be a complete spherical $G$-variety. Then $X$ satisfies the strong Equivariant Kronecker duality. Moreover, if the $S$-module $A_T^*(X)$ is free, then the image of the injective map*

$$i_T^* : A_T^*(X) \to A_T^*(X^T)$$

*is the intersection of the images of the maps*

$$i_{T,T'}^* : A_T^*(X^{T'}) \to A_T^*(X^T),$$

*where $T'$ runs over all subtori of codimension one of $T$.*

Hence, we describe not only the module structure of the equivariant operational Chow groups, but also their ring structure in the case of spherical varieties and $T$-linear varieties.

From our previous results, it follows that the freeness of the equivariant Chow group $A_*^T(X)$, of a spherical $G$-variety $X$, is a key condition for applying our results. Hence, in the second part of this article we provide a way for computing $A_T^*(X)$ for $\mathbb{Q}$-filtrable varieties: normal projective spherical varieties whose associated BB-decomposition consists of algebraic rational cells (Definition 5.5). Concisely, an affine $T$-variety $X$ with an attractive fixed point $x$ is called an algebraic rational cell if the associated link

$$\mathbb{P}(X) := [X \setminus \{0\}]/\mathbb{G}_m$$

satisfies $A_*(\mathbb{P}(X)) \simeq A_*(\mathbb{P}^{n-1})$, where $n = \dim(X)$. The notion of algebraic rational cell is inspired from our previous work on the topology of possibly singular group embeddings [G1]. It turns out to be very well suited for the study of Chow groups on singular varieties. The main result of Section 5 is presented next.

**Theorem 5.4, 5.14.** *Let $X$ be a $\mathbb{Q}$-filtrable $T$-variety. Then the $T$-equivariant Chow group of $X$ is a free module of rank $|X^T|$. In fact, it is freely generated by the classes of the closures of the cells $W_i$. Consequently, $A_*(X)$ is also freely generated by the classes of the cell closures $W_i$.*

In Section 6, we provide some applications. We show that the equivariant operational Chow rings of Schubert varieties are, not surprisingly, piecewise polynomial functions defined on the Bruhat graph (Theorem 6.2). Finally, in the second half of Section 6 we compare our results here with the topological results of [G2] and describe the Chow groups and operational Chow rings of complex rationally smooth group embeddings (Theorem 6.3)

The theoretical results of Section open the way for a systematic, characteristic free, study of the equivariant Chow groups of spherical singular varieties, by using the notion of equivariant multiplicities. This will be pursued in a subsequent paper, in the case of projective group embeddings of reductive groups [G3].

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## 2. Preliminaries

**Notation and conventions:** Throughout this paper, we fix an algebraically closed field $k$ of characteristic zero. All algebraic varieties and algebraic groups are assumed to be defined over $k$. An algebraic variety is a separated reduced scheme of finite type over $k$. Observe that varieties need not be irreducible. We denote by $G$ a connected reductive algebraic group. A variety $X$ provided with an algebraic action of $G$ is called a *$G$-variety*.

In this paper, group actions are assumed to be locally linear, i.e. the varieties we consider are covered by invariant quasi-projective open subsets (and hence by invariant affine open subsets in the case of torus actions). This assumption is fulfilled, e.g., for $G$-stable subvarieties of normal $G$-varieties [Su].

Let $X$ be a $G$-variety. We denote by $A_*^G(X)$ the equivariant Chow group of $X$ with *rational* coefficients. The equivariant Chow groups are defined in [EG], using Totaro's finite approximation to the Borel construction. When $X$ is smooth, $A_*^G(X)$ admits an intersection pairing and we denote by $A_G^*(X)$ the corresponding ring graded by codimension. For general $X$, we denote by $A_G^*(X)$ the equivariant operational Chow cohomology of $X$, as defined in [EG].

Let $T$ be a maximal torus of $G$. Let $W$ be the Weyl group of $(G,T)$. Denote by $p_{X,T}: X_T \to BT$ the structural map (similarly for $p_{X,G}: X_G \to BG$). Let $S = A_T^*(pt)$ and $S^W = A_G^*(pt)$ be the corresponding equivariant Chow groups of a point. It follows from [EG], using the structural map, that $A_*^T(X)$ (resp. $A_*^G(X)$) are modules over $S$ (resp. $S^W$). We denote by $p_{X,T*}$ the proper pushforward (also denoted by $\int_X$).

**Definition 2.1.** Let $T$ be an algebraic torus and let $X$ be a $T$-variety.

(1) We say that $X$ is *$T$-equivariantly $0$-linear* if it is either empty or isomorphic to a finite-dimensional rational representation of $T$.
(2) For a positive integer $n$, we say that $X$ is *$T$-equivariantly $n$-linear* if there exists a family of $T$-varieties $\{U, Y, Z\}$, such that $Z \subset Y$ is a $T$-invariant closed subvariety with $U$ its complement, $Z$ and one of the varieties $U$ or $Y$ are $T$-equivariantly $(n-1)$-linear and $X$ is the other member of the family $\{U, Y, Z\}$.

(3) We say that $X$ is *$T$-equivariantly linear* (or simply, *$T$-linear*) if it is $T$-equivariantly $n$-linear for some $n \geq 0$.

It is immediate from the above definition that if $T \to T'$ is a morphism of algebraic tori, then every $T'$-equivariantly linear variety is also $T$-equivariantly linear.

The following is recorded in [J-K].

**Proposition 2.2.** *Let $T$ be an algebraic torus and let $T'$ be a quotient of $T$. Let $T$ act on $T'$ via the quotient map. Then the following hold:*

*(1) $T'$ is $T$-linear.*
*(2) A toric variety with dense torus $T$ is $T$-linear.*
*(3) A $T$-cellular scheme is $T$-linear.*
*(4) Every $T$-variety with finitely many $T$-orbits is $T$-linear.* $\square$

It follows from the previous result and the Bruhat decomposition that flag varieties, partial flag varieties and Schubert varieties are all $T$-linear, since they come with a paving by affine spaces (i.e., they are $T$-cellular). Moreover, if $X$ is a smooth projective spherical $G$-variety, then it is $T$-linear. Indeed, $X$ comes with a $BB$-decomposition into affine spaces [B2] (because $X^T$ is finite and $X$ smooth). Since this $BB$-decomposition is filtrable [B1], $X$ is $T$-cellular. We do not know if all spherical varieties are $T$-linear.

Next we state here Brion's description [Br3] of the equivariant Chow groups of $T$-varieties.

**Theorem 2.3.** *Let $X$ be a $T$-variety. Then the $S$-module $A_*^T(X)$ is defined by generators $[Y]$ where $Y$ is an invariant subvariety of $X$ and relations $[\mathrm{div}_Y(f)] - \chi[Y]$ where $f$ is a rational function on $Y$ which is an eigenvector of $T$ of weight $\chi$. Moreover, the map $A_*^T(X) \to A_*(X)$ vanishes on $MA_*^T(X)$, and it induces an isomorphism*

$$A_*^T(X)/MA_*^T(X) \to A_*(X).$$

$\square$

The result below is due to Brion [Br3]. It is a refinement of Theorem 2.3 for varieties with a torus action which extends to an action of a larger group.

**Theorem 2.4.** *Let $X$ be a variety with an action of a connected solvable linear algebraic group $\Gamma$, and let $T$ be a maximal torus of $\Gamma$.*

*(i) The equivariant Chow group $A_*^T(X)$ is generated as an $S$-module by the classes $[Y]$ where $Y \subset X$ is a $\Gamma$-invariant subvariety.*
*(ii) If moreover the $S$-module $A_*^T(X)$ is free, then the $S$-module of relations between these classes is generated by the $[\mathrm{div}_Y(f)] - \chi[Y]$ where $Y \subset X$ is a $\Gamma$-invariant subvariety, and where $f$ is a rational function on $Y$ which is an eigenvector of $\Gamma$ of weight $\chi$.*

$\square$

**Remark 2.5.** It follows from Theorem 2.4 that if $X$ has an action of a solvable group $\Gamma$ with finitely many orbits, then $A_*^T(X)$ is generated by the closures of the $\Gamma$-orbits, and so $A_*^T(X)$ is a finitely generated $S$-module. In particular, if $X$ is a spherical $G$-variety, then $A_T^*(X)$ is finitely generated over $S$.

As for $T$-linear varieties, one has the following result.

**Lemma 2.6.** *Let $X$ be a $T$-linear variety. Then $A_*^T(X)$ is finitely generated $S$-module.*

*Proof.* This is a consequence of the inductive definition of $T$-linear varieties and the fact that for 0-linear varieties, i.e., the $T$-equivariant linear representations $\mathbb{A}^n$ of $T$, one has $A_T^*(\mathbb{A}^n) \simeq S$ (by homotopy invariance). Now we argue by induction. Assume the result for $T$-equivariantly $(n-1)$-linear varieties. Let $X$ be a $n$-linear variety. By definition, two localization sequences can occur:

$$A_*^T(Z) \to A_*^T(X) \to A_*^T(U) \to 0,$$

where $Z$ and $U$ are $(n-1)$-linear. By the inductive hypothesis, the terms on both ends are finitely generated, hence so is $A_*^T(X)$. In the second case

$$A_*^T(Z) \to A_*^T(Y) \to A_*^T(X) \to 0,$$

where $Z$ and $Y$ are $(n-1)$-linear. Clearly, in this case, it follows that $A_*^T(X)$ is also finitely generated. □

The following is the localization theorem for equivariant Chow groups. See [Br3].

**Theorem 2.7.** *Let $i : X^T \to X$ be the inclusion of the fixed point variety. Then the $S$-linear map $i_* : A_*^T(X^T) \to A_*^T(X)$ is an isomorphism after inverting all non-zero elements of $M$.* □

For later use, we record here a slightly more general version of the previous localization theorem.

**Proposition 2.8.** *Let $X$ be a $T$-variety. Let $H$ be a subtorus of $T$. Then the induced morphism of equivariant Chow groups*

$$i_* : A_*^T(X^H) \to A_*^T(X)$$

*becomes an isomorphism after inverting finitely many characters of $T$ that restrict non-trivially to $H$.*

Before proving this proposition, let us recall the following fundamental facts.

**Lemma 2.9.** *Let $T$ be an algebraic torus and let $X$ be an affine $T$-variety. Denote by $X^T$ the fixed point variety. Then the ideal of $X^T$ is generated by all regular functions on $X$ which are eigenvectors of $T$ with a non-trivial weight.*

*Proof.* Let $x \in X^T$ and let $f$ be an eigenvector of $T$ with non-trivial weight, say $\chi$. Then, for all $t \in T$, we have $f(x) = f(tx) = \chi(t)f(x)$. But $\chi$ is a non-trivial character, so $f(x) = 0$. For the converse, simply recall that if $x \notin X^T$, then there exists a function $f$, eigenvector of $T$, such that $f(x) \neq 0$. □

This lemma could be generalized as follows. Let $H$ be a subtorus of $T$. Our aim is to describe $X^H$, the fixed point set of $H$.

**Lemma 2.10.** *The ideal of $X^H$ is generated by all regular functions on $X$ which are eigenvectors of $T$ with a weight that restricts non-trivially to $H$.*

*Proof.* Let $x \in X^H$ and let $f$ be an eigenvector of $T$ with a weight $\chi$ such that $\chi|_H \neq 1$. Then, for all $h \in H$, we have $f(x) = f(hx) = \chi(h)f(x)$. Now pick $h_0 \in H$ so that $\chi(h_0) \neq 1$. Hence the identity $f(x) = \chi(h_0)f(x)$ implies $f(x) = 0$.

Before proving the converse, let us fix some notation. Denote by $\mathcal{J}_H^T$ the set of all regular functions on $X$ which are eigenvectors of $T$ with a weight that restricts non-trivially to $H$. In contrast, write $\mathcal{I}_H$ for the set of all regular functions on $X$ which are eigenvectors of $H$ with a non-trivial weight. Clearly,

$$\mathcal{J}_H^T \subset \mathcal{I}_H.$$

Denote by $Z(\mathcal{J}_H^T)$ and $Z(\mathcal{I}_H)$ the subvarieties of $X$ defined by the ideals generated by $\mathcal{J}_H^T$ and $\mathcal{I}_H$ respectively.

With this notation, we ought to prove that

$$\text{if } x \in Z(\mathcal{J}_H^T), \text{ then } x \in Z(\mathcal{I}_H)$$

(the consequent being equivalent to $x \in X^H$ by the previous lemma).

So fix $x' \in Z(\mathcal{J}_H^T)$, and let $g \in \mathcal{I}_H$ with non-trivial $H$-weight $\chi_g$. Since $T$ is acting on $X$, we can write

$$g = \sum_i f_i,$$

where $f_i$ are eigenvectors of $T$ of weight $\chi_i$. We can further split this sum into two parts

$$g = \sum_{i \in N} f_i + \sum_{i \notin N} f_i,$$

where $i \in N$ if and only if $H \subset \ker(\chi_i)$. It follows from our construction that $\chi_i(h) = 1$ for all $h \in H$ and all $i \in N$. In constrast, $f_i(x) = 0$ for all $i \notin N$; that is, evaluating $g$ at $x'$ yields $g(x') = \sum_{i \in N} f_i(x')$.

Now let $h \in H$ and let us evaluate $g(hx')$. We know that

$$g(hx') = \sum_{i \in N} f_i(hx') + \sum_{i \notin N} f_i(hx') = \sum_{i \in N} \chi_i(h) f_i(x') + \sum_{i \notin N} \chi_i(h) f_i(x'),$$

which in turn reduces to

$$g(hx') = \sum_{i \in N} \chi_i(h) f_i(x') + 0 = \sum_{i \in N} f_i(x') = g(x')$$

by our earlier assumptions. Hence,

$$g(hx') = g(x').$$

But then,

$$\chi_g(h) g(x') = g(hx') = g(x'),$$

for all $h \in H$ (recall that $g$ is a $H$-eigenvector of non-trivial $H$-weight $\chi_g$). This readily implies $g(x') = 0$. □

*Proof of Proposition 2.8.* By virtue of Lemma 2.10, our proof is an adaptation, almost word for word, of Brion's proof of Corollary 2.3.2 in [Br3]. So we provide only a sketch of the crucial points. From Theorem 2.3 we know that $A_*^T(X)$ is generated by the classes of $T$-invariant subvarieties $Y \subset X$. Moreover, by assumption, $X$ is a finite union of $T$-stable affine open subsets $X_i$. Now Lemma 2.10 implies that the ideal of each fixed point variety $X_i^H$ is generated by all regular functions on $X_i$ which are eigenvectors of $T$ with a weight that restricts non-trivially to $H$. We can choose a finite set of such generators $(f_{ij})$, with respective weights $\chi_{ij}$.

Now let $Y \subset X$ be a $T$-invariant subvariety of positive dimension. If $Y$ is not fixed pointwise by $H$, then one of the $f_{ij}$ defines a non-zero rational function on $Y$. Then, in the Chow group, we have $\chi_{ij}[Y] = [\mathrm{div}_Y f_{ij}]$. So after inverting $\chi_{ij}$, we get $[Y] = \chi_{ij}^{-1}[\mathrm{div}_Y f_{ij}]$. Arguing by induction on the dimension of $Y$, we obtain that $i_*$ becomes surjective after inverting the $\chi_{ij}$'s. A similar argument, using these

$\chi_{ij}$'s in Brion's proof of Corollary 2.3.2 in [Br3], shows that $i^*$ is injective after localization. $\square$

When $X$ is projective and smooth [Br3], the localization theorem yields a GKM description of the image of the natural map $i_T^* : A_T^*(X) \to A_T^*(X^T)$. Our purpose is to extend this picture, as much as possible, to possibly singular varieties (Section 3). Before proceeding to the core part of this paper, we state a few results from Kimura [Ki] and Edidin-Graham that will be useful in our task.

Recall ([Fu], Definition 18.3) that an envelope $p : \tilde{X} \to X$ is a proper map such that for any subvariety $W \subset X$ there is a subvariety $\tilde{W}$ mapping birationally to $W$ via $p$. In the case of group actions, we will say that $p : \tilde{X} \to X$ is an *equivariant envelope* if $p$ is $G$-equivariant, and if we can take $\tilde{W}$ to be $G$-invariant for $G$-invariant $W$. If there is an open set $X^0 \subset X$ over which $p$ is an isomorphism, then we say that $p : \tilde{X} \to X$ is a *birational* envelope. This properties are compatible with Totaro's algebraic approximation to the Borel construction. For details see [EG], Section 2.6.

**Lemma 2.11.** *Let $G$ be a connected linear algebraic group. Let $X$ be a $G$-variety. Then there exists a $G$-equivariant smooth envelope $p : \tilde{X} \to X$.*

*Proof.* By equivariant resolution of singularities, there is a resolution $\pi : X' \to X$ such that $\pi$ is an isomorphism outside some invariant subvariety $S \subset X$. By Noetherian induction, we may assume that we have constructed an equivariant envelope $\tilde{S} \to S$. Now set $\tilde{X} = X' \sqcup \tilde{S}$ (disjoint union). The claim follows. $\square$

Kimura's computation of Chow cohomology implies that $A_T^*(X)$ of a singular variety $X$ injects into $A_T^*(\tilde{X})$ of a smooth resolution (which is the usual Chow ring of a smooth variety) with an explicit cokernel.

**Theorem 2.12.** [Ki] *Let $p : \tilde{X} \to X$ be a smooth equivariant envelope. Then the induced map $p^* : A_G^*(X) \to A_G^*(\tilde{X})$ is injective.* $\square$

**Lemma 2.13.** *Let $X$ be a complete $T$-variety and let $i : X^T \to X$ be the inclusion of the fixed point set. Then the induced map $i_T^* : A_T^*(X) \to A_T^*(X^T)$ is injective.*

*Proof.* By the previous lemma, there exists a $T$-equivariant smooth envelope $p : \tilde{X} \to X$. It follows that $p^* : A_T^*(X) \to A_T^*(\tilde{X})$ is injective [Ki]. Now notice that $\tilde{X}$ is smooth projective, and so the induced map $i^* : A_T^*(\tilde{X}) \to A_T^*(\tilde{X}^T)$ is injective. Thus, the commutative diagram below

$$\begin{tikzcd} A_T^*(X) \arrow[r, "p^*"] \arrow[d, "i^*"] & A_T^*(\tilde{X}) \arrow[d, "i^*"] \\ A_T^*(X^T) \arrow[r, "p^*"] & A_T^*(p^{-1}(X^T)). \end{tikzcd}$$

renders $i^* : A_T^*(X) \to A_T^*(X^T)$ injective. Indeed, because $p^{-1}(X^T)$ contains $\tilde{X}^T$ and $\tilde{X}$ is smooth, then the vertical map on the right is injective. Also, $p^*$ is injective by [Ki]. Thus, we conclude that $i^*$ on the left is also an injection. $\square$

**Corollary 2.14.** *Let $X$ be a complete $T$-variety. Let $Y$ be a $T$-invariant subvariety containing $X^T$. Denote by $j : Y \to X$ the natural inclusion. Then the induced map $j^* : A_T^*(X) \to A_T^*(Y)$ is injective.*

*Proof.* Simply notice that $j : Z \to X$ fits into the commutative triangle

$$\begin{tikzcd} & Y \arrow[dr, "j"] & \\ X^T \arrow[ur, "i_{T,Y}"] \arrow[rr, "i_T"] & & X \end{tikzcd}$$

Thus the induced map $i_T^* : A_T^*(X) \to A_T^*(X^T)$ factors as $j^* : A_T^*(X) \to A_T^*(Y)$ followed by $i_{T,Y}^* : A_T^*(Y) \to A_T^*(X^T)$. By Lemma 2.13, $i_T^*$ is injective. Hence, $j^*$ is also injective. $\square$

We wish to describe the image of the restriction map

$$i_T^* : A_T^*(X) \to A_*^T(X^T).$$

For this, let $T' \subset T$ be a subtorus of codimension one. Observe that $i_T : X^T \to X$ factors as $i_{T,T'} : X^T \to X^{T'}$ followed by $i_{T'} : X^{T'} \to X^T$. Thus, the image of $i_T^*$ is contained in the image of $i_{T,T'}^*$. In symbols,

$$\mathrm{Im}[i_T^* : A_T^*(X) \to A_T^*(X^T)] \subseteqq \bigcap_{T' \subset T} \mathrm{Im}[i_{T,T'}^* : A_T^*(X^{T'}) \to A_T^*(X^T)],$$

where the intersection runs over all codimension-one subtori of $T$. This observation will lead, as in the classical case, to a complete description of the image of $i_T^*$, if the $A_T^*$-module $A_T^*(X)$ is free.

## 3. Equivariant Kronecker duality, localization and GKM theory for equivariant Chow cohomology

### 3.1. Equivariant Kronecker duality spaces.

**Definition 3.1.** Let $X$ be a complete $T$-variety. We say that $X$ satisfies **$T$-equivariant Kronecker duality** (or Kronecker duality for short) if the following hold:

(i) $A_*^T(X)$ is a finitely generated $S$-module.
(ii) The equivariant Kronecker duality map

$$\mathcal{K}_{\mathcal{T}} : A_T^*(X) \longrightarrow Hom_S(A_*^T(X), S) \qquad \alpha \mapsto (\beta \mapsto p_{X*}(\beta \cap \alpha))$$

is an isomorphism of $S$-modules.

**Remark 3.2.** Notice that the equivariant Kronecker duality map is functorial for morphisms between complete varieties. Indeed, let $p : \tilde{X} \to X$ be an equivariant (proper) morphism of complete varieties. It is important to notice that

$$\int_{\tilde{X}} p^*(\xi) \cap z = \int_X p_*(p^*(\xi) \cap z) = \int_X (\xi \cap p_*(z)),$$

due to the projection formula [Fu]. This formula implies the commutativity of the diagram

$$\begin{tikzcd} A_T^*(X) \arrow[r, "p^*"] \arrow[d, "\mathcal{K}_{\mathcal{T}}"'] & A_T^*(\tilde{X}) \arrow[d, "\mathcal{K}_{\mathcal{T}}"] \\ \mathrm{Hom}_S(A_*^T(X), S) \arrow[r, "(p_*)^t"] & \mathrm{Hom}_S(A_*^T(\tilde{X}), S), \end{tikzcd}$$

where $(p_*)^t$ is the transpose of $p_* : A_*^T(\tilde{X}) \to A_*^T(X)$.

Not all smooth varieties with a torus action satisfy Equivariant Kronecker duality. For instance, by taking the trivial action of $T$ on a projective smooth curve, one sees that $\mathcal{K}_{\mathcal{T}}$ is an extension of the usual non-equivariant Kronecker duality map $\mathcal{K}$. As pointed out in [FMSS], the kernel of $\mathcal{K}$ in degree one is the Jacobian of the curve.

**Lemma 3.3.** *Let $X$ be a smooth projective $T$-variety. Then $X$ satisfies equivariant Kronecker duality if and only if it satisfies the non-equivariant Kronecker duality, i.e. $\mathcal{K} : A^i(X) \to Hom(A_i(X), \mathbb{Q})$ is an isomorphism for all $i$.*

*Proof.* Since $X$ is smooth and projective, then both $A^*_T(X)$ and $A^T_*(X)$ are free $S$-modules. The claim now follows from the Graded Nakayama lemma and the commutativity of the diagram below:

$$\begin{tikzcd}
A^*_T(X) \arrow[r, "\mathcal{K}_{\mathcal{T}}"] \arrow[d] & \mathrm{Hom}_S(A^T_*(X), S) \arrow[d] \\
A^*(X) = A^*_T(X)/MA^*_T(X) \arrow[r, "\mathcal{K}"] & \mathrm{Hom}(A^T_*(X)/MA^T_*(X), \mathbb{Q}) = \mathrm{Hom}(A_*(X), \mathbb{Q}).
\end{tikzcd}$$

□

Next we show that equivariant Kronecker duality is satisfied for a large class of varieties, namely, $T$-linear varieties. The main ingredient is the following result, due to Joshua and Krishna [J-K] in the equivariant setting, and to Totaro [T] and Jannsen [J] in the non-equivariant setting.

**Proposition 3.4.** [J-K] *If $X$ is a $T$-linear variety and $Y$ is any $T$-variety, then*

$$A^T_*(X) \otimes_S A^T_*(Y) \simeq A^T_*(X \times Y).$$

□

In other words, $T$-linear varieties satisfy the Equivariant Künneth Decomposition. Next we state a result due essentially to Joshua and Krishna (though it is not stated in [J-K]). Its proof can be derived formally from the previous Proposition, as it is done in [FMSS], Theorem 3.

**Proposition 3.5.** *If $X$ is a complete $T$-variety such that*

$$A^T_*(X) \otimes_S A^T_*(Y) \simeq A^T_*(X \times Y)$$

*for all $T$-varieties $Y$, then the equivariant Kronecker map*

$$\mathcal{K}_{\mathcal{T}} : A^*_T(X) \to \mathrm{Hom}_S(A^T_*(X), S)$$

*is an isomorphism.*

We only outline the proof of Proposition 3.5, which is based on one of the main results of Fulton, MacPherson, Sottile, and Sturmfels [FMSS]. It suffices to construct a formal inverse to $\mathcal{K}_{\mathcal{T}}$. Thus, for every $T$-map $f : Y \to X$ and $m \geq i$, we have to construct a homomorphism from $A^T_m(Y)$ to $A^T_{m-i}(Y)$. This homomorphism is defined to be the composite

$$\begin{aligned} A^T_m(Y) \longrightarrow A^T_m(X \times Y) = \bigoplus (A^T_j(X) \otimes A^T_{m-j}(Y)) \longrightarrow A^T_i(X) \otimes A^T_{m-i}(Y) \\ \longrightarrow S \otimes A^T_{m-i}(Y) \longrightarrow A^T_{m-i}(Y). \end{aligned}$$

The maps here are clear except for the first one, which is induced by the inclusion of $Y$ into $X \times Y$ via the graph of $f : Y \to X$. One checks that these maps (for different $Y$'s) satisfy the compatibility conditions of [Fu], Chapter 17, to give an element of $A_T^i(X)$, and that every element of $A_T^i(X)$ is so obtained. We refer the reader to [FMSS] for the details.

The main feature of equivariant Kronecker duality spaces, from the viewpoint of algebraic torus actions on varieties, is that they supply a theoretical background for stablishing Localization Theorems on the equivariant operational Chow groups.

**Theorem 3.6.** *Let $X$ be a complete $T$-variety satisfying Kronecker duality. Let $T' \subset T$ be a subtorus of $T$ and let $i_{T'} : X^{T'} \to X$ be the inclusion of the fixed point subvariety. If $X^{T'}$ also satisfies $T$-equivariant Kronecker duality, then the injective morphism*

$$i_{T'}^* : A_T^*(X) \to A_T^*(X^{T'})$$

*becomes an isomorphism after inverting finitely many characters of $T$ that restrict non-trivially to $T'$.*

*Proof.* Because of the (usual) localization theorem for equivariant Chow groups (Proposition 2.8) we know that the localized map $(i_{T'}*)_{\mathcal{F}} : A_*^T(X^{T'})_{\mathcal{F}} \to A_*^T(X)_{\mathcal{F}}$ is an isomorphism, where $\mathcal{F}$ is a finite family of characters of $T$ that restrict non-trivially to $T'$.

Now consider the commutative diagram

$$\begin{tikzcd}
A_T^*(X) \arrow[r, "i_{T'}^*"] \arrow[d] & A_T^*(X^{T'}) \arrow[d] \\
\mathrm{Hom}_S\,(A_*^T(X), S) \arrow[r, "(i_{T'*})^t"] & \mathrm{Hom}_S\,(A_*^T(X^{T'}), S),
\end{tikzcd}$$

where $(i_{T'*})^t$ represents the transpose of $i_{T'*} : A_*^T(X^{T'}) \to A_*^T(X)$ (commutativity follows from Remark 3.2, because $i_{T'}$ is proper). By our assumptions on $X$ and $X^{T'}$, both vertical maps are isomorphisms. Moreover, after localization at $\mathcal{F}$, the above commutative diagram becomes

$$\begin{tikzcd}
A_T^*(X)_{\mathcal{F}} \arrow[r, "(i_{T'}^*)_{\mathcal{F}}"] \arrow[d] & A_T^*(X^{T'})_{\mathcal{F}} \arrow[d] \\
(\mathrm{Hom}_S\,(A_*^T(X), S))_{\mathcal{F}} \arrow[r, "((i_{T'*})^t)_{\mathcal{F}}"] & \mathrm{Hom}_{S_{\mathcal{F}}}\,(A_*^T(X^{T'})_{\mathcal{F}}, S_{\mathcal{F}}),
\end{tikzcd}$$

Since $A_*^T(X)$ is a finitely generated $S$-module (Remark 2.5), localization commutes with formation of Hom (see [Ei], Prop. 2.10, p. 69), and so

$$A_T^*(X)_{\mathcal{F}} \simeq (\mathrm{Hom}_S\,(A_*^T(X), A_T^*))_{\mathcal{F}} \simeq \mathrm{Hom}_{S_{\mathcal{F}}}(A_*^T(X)_{\mathcal{F}}, S_{\mathcal{F}}).$$

Similarly, for $X^{T'}$ we obtain

$$A_T^*(X^{T'})_{\mathcal{F}} \simeq \mathrm{Hom}_{S_{\mathcal{F}}}(A_*^T(X^{T'})_{\mathcal{F}}, S_{\mathcal{F}}).$$

In other words, the bottom map in the diagram above fits in the square diagram

$$\begin{tikzcd}
(\mathrm{Hom}_S\,(A_*^T(X),S))_{\mathcal{F}} \arrow[r, "((i_{T'*})^t)_{\mathcal{F}}"] \arrow[d] & \mathrm{Hom}_{S_{\mathcal{F}}}\,(A_*^T(X^{T'})_{\mathcal{F}}, S_{\mathcal{F}}) \arrow[d] \\
\mathrm{Hom}_{S_{\mathcal{F}}}(A_*^T(X)_{\mathcal{F}}, S_{\mathcal{F}}) \arrow[r, "((i_{T'*})_{\mathcal{F}})^t"] & \mathrm{Hom}_{S_{\mathcal{F}}}(A_*^T(X^{T'})_{\mathcal{F}}, S_{\mathcal{F}}),
\end{tikzcd}$$

where the vertical maps are natural isomorphisms ([Ei], Prop. 2.10, p. 69). But we already know that $(i_{T'*})_{\mathcal{F}}$ is an isomorphism, hence so are $((i_{T'*})_{\mathcal{F}})^t$, $((i_{T'*})^t)_{\mathcal{F}}$ and $(i^*_{T'})_{\mathcal{F}}$. We are done. $\square$

Let $X$ be $T$-variety. We know from Lemma 2.13 that $i^*_T : A^*_T(X) \to A^*_T(X^T)$ is injective. We wish to describe the image of the restriction map

$$i^*_T : A^*_T(X) \to A^T_*(X^T).$$

For this, let $T' \subset T$ be a subtorus of codimension one. Observe that $i_T : X^T \to X$ factors as $i_{T,T'} : X^T \to X^{T'}$ followed by $i_{T'} : X^{T'} \to X^T$. Thus, the image of $i^*_T$ is contained in the image of $i^*_{T,T'}$. This observation will lead, as in the classical case, to a complete description of the image of $i^*_T$, if the $A^*_T$-module $A^*_T(X)$ is free and $X$ is an equivariant Kronecker duality space.

**Definition 3.7.** Let $X$ be a $T$-variety. We say that $X$ satisfies **the strong equivariant Kronecker duality** if $X$ as well as $X^T$ and $X^{T'}$, for all codimension one subtori $T' \subset T$, satisfy $T$-equivariant Kronecker duality.

Notice that there is only finitely many codimension one subtori $T'$ of $T$ for which $X^{T'} \neq X^T$. This can be seen by linearizing the action around a fixed point $x \in X^T$.

We now state a precise version of the Localization Theorem for $T$-varieties satisfying the strong equivariant Kronecker duality.

**Theorem 3.8.** *Let $X$ be a complete $T$-variety satisfying the strong equivariant Kronecker duality. If the $S$-module $A^*_T(X)$ is free, then the image of the injective map*

$$i^*_T : A^*_T(X) \to A^*_T(X^T)$$

*is the intersection of the images of the maps*

$$i^*_{T,T'} : A^*_T(X^{T'}) \to A^*_T(X^T),$$

*where $T'$ runs over all subtori of codimension one of $T$.*

*Proof.* We follow Brion's proof of Theorem 6 in [Br4] almost verbatim, interchanging equivariant cohomology with equivariant Chow cohomology in virtue of Theorem 3.6. We already know that $i^*_T$ is injective. Moreover, it becomes an isomorphism after inverting a finite family $\mathcal{F}$ of non-trivial characters of $T$ (by Theorem 3.6 applied to $X$ and $X^T$). It remains to show that the intersection of the images of the $i^*_{T,T'}$ is contained in the image of $i^*_T$.

Choose a basis $(e_j)_{j\in J}$ of the free $S$-module $A^*_T(X)$. For any $j \in J$, let

$$e^*_j : A^*_T(X) \to S$$

be the corresponding coordinate function. Then there exists a $S$-linear map

$$f_j : A^*_T(X^T) \to S[1/\chi]_{\chi\in\mathcal{F}}$$

such that $f_j \circ i^*_T = e^*_j$.

We may assume that each $\chi \in \mathcal{F}$ is primitive, i.e., not divisible in $\Xi(T)$. Then its kernel $\mathrm{Ker}(\chi) \subset T$ is a subtorus of codimension one. Let $u$ be in the image of $i^*_{T,\mathrm{Ker}(\chi)}$; write

$$u = i^*_{T,\ker(\chi)}(v)$$

where $v \in A^*_T(X^{\ker(\chi)})$. By Theorem 3.6 applied to $\Gamma = \ker(\chi)$, there exists a product $P_\chi$ of weights of $T$ which are not multiples of $\chi$, such that $P_\chi v$ is in the image of $i^*_{\ker(\chi)}$. It follows that $P_\chi u$ is in the image of $i^*_T$. Applying $f_j$, we obtain $P_\chi f_j(u) \in S$. Thus, the denominator of $f_j(u)$ is not divisible by $\chi$.

If $u \in A^*_T(X^T)$ is in the intersection of the images of the $i^*_{T,\ker(\chi)}$ for all $\chi \in \mathcal{F}$, then $f_j(u) \in S[1/\chi]_{\chi\in\mathcal{F}}$, but the denominator of $f_j(u)$ is not divisible by any element of $\mathcal{F}$; whence $f_j(u) \in S$. It follows that $u = i^*_T(\sum_{j\in J} f_j(u)e_j)$ is in the image of $i^*_T$. $\square$

From the inductive definition of $T$-linear varieties it follows that the fixed point set $X^H$ of any subtorus $H \subset T$ is $T$-equivariantly $n$-linear if $X$ is $T$-equivariantly $n$-linear. Hence we are led to the following.

**Corollary 3.9.** *Let $X$ be a complete $T$-linear variety. Then $X$ satisfies the strong equivariant Kronecker duality. If moreover $A^T_*(X)$ is a free $S$-module, then then the image of the injective map*

$$i^*_T : A^*_T(X) \to A^*_T(X^T)$$

*is the intersection of the images of the maps*

$$i^*_{T,T'} : A^*_T(X^{T'}) \to A^*_T(X^T),$$

*where $T'$ runs over all subtori of codimension one of $T$.* $\square$

3.2. **GKM theory.** GKM theory is a relatively recent tool that owes its name to the work of Goresky, Kottwitz and MacPherson [GKM]. This theory encompasses techniques that date back to the early works of Atiyah, Segal, Borel and Chang-Skjelbred. It has been extremely useful in the description of the equivariant cohomology of singular spherical varieties (e.g. [G2]). This theory has been extended to the equivariant Chow groups in the work of Brion [Br3].

Here we propose yet another generalization of GKM theory to the study of possibly singular Kronecker duality spaces and their equivariant operational Chow rings. Our motto is that equivariant Kronecker duality spaces are the analogue of the equivariantly formal spaces of Goresky, Kottwitz, MacPherson in the setting of operational Chow cohomology.

We start by recalling a few definitions from [GKM] and [G1].

**Definition 3.10.** Let $X$ be a projective $T$-variety. Let $\mu : T \times X \to X$ be the action map. We say that $\mu$ is a ***T*-skeletal action** if

(1) $X^T$ is finite, and
(2) The number of one-dimensional orbits of $T$ on $X$ is finite.

In this context, $X$ is called a ***T*-skeletal variety**.

Let $X$ be a normal projective $T$-skeletal variety. Then $X$ has an equivariant embedding into a projective space with a linear action of $T$ ([Su], Theorem 1).

Moreover, it is possible to define a ring $PP^*_T(X)$ of **piecewise polynomial functions**. Indeed, let $R = \bigoplus_{x\in X^T} R_x$, where $R_x$ is a copy of the polynomial algebra $A^*_T$. We then define $PP^*_T(X)$ as the subalgebra of $R$ defined by

$$PP^*_T(X) = \{(f_1, ..., f_n) \in \bigoplus_{x\in X^T} R_x \mid f_i \equiv f_j \; mod(\chi_{i,j})\}$$

where $x_i$ and $x_j$ are the two *distinct* fixed points in the closure of the one-dimensional $T$-orbit $\mathcal{C}_{i,j}$, and $\chi_{i,j}$ is the character of $T$ associated with $\mathcal{C}_{i,j}$. The character $\chi_{i,j}$ is uniquely determined up to sign (permuting the two fixed points changes $\chi_{i,j}$ to its opposite).

**Theorem 3.11.** *Let $X$ be a normal projective $T$-skeletal variety. If $X$ satisfies equivariant Kronecker duality and the $S$-module $A^T_*(X)$ is free, then the restriction mapping*

$$A^*_T(X) \longrightarrow A^*_T(X^T) = \bigoplus_{x_i\in X^T} A^*_T$$

*is injective, and its image is the subalgebra $PP^*_T(X)$.*

To derive our GKM theorem from Theorem 3.8 we need a few technical lemmas.

**Lemma 3.12.** *Let $X$ be a normal projective $T$-skeletal variety. If $X$ satisfies equivariant Kronecker duality, then it also satisfies the strong version of equivariant Kronecker duality.*

*Proof.* Since $X^T$ is finite, it obviously satisfies Kronecker duality. Thus, we only need to check that, for any subtorus $T' \subset T$, $X^{T'}$ satisfies Kronecker duality. For this we argue as follows. First, notice that if $X^{T'} \neq X^T$, then $X^{T'}$ is a finite union of fixed points and $T$-invariant curves. Moreover, because the action of $T$ on $X$ is linearizable, each irreducible component $Y$ of $X^{T'}$ is either a point or a $T$-invariant curve with exactly two fixed points. Notice that the normalization of each one of these curves is isomorphic to $\mathbb{P}^1$. Moreover, $T$ acts on each irreducible component with a dense orbit. It follows that $X^{T'}$ is a union of one-dimensional projective toric varieties with at most quotient singularities. Now Theorem 4.3 yields the assertion of the lemma. $\square$

*Proof of Theorem 3.11.* Observe that a codimension one subtorus of $T$ is the kernel of a primitive (i.e. indivisible) character of $T$. Such character is uniquely defined up to sign.

Let $\pi$ be a primitive character of $T$. Notice $X^{\ker(\pi)}$ is a union of fixed points $x_1, \ldots, x_m$ and $T$-invariant curves $C_1, \ldots, C_p$ with exactly two fixed points. If this is a disjoint union, then arguing on each curve $C_i$ separately we get that

$$A^*_T(C_i) = \{(f, g) \in S \times S \,|\, f \cong g \mod \chi\},$$

where $T$ acts on $C_i$ through the weight $\chi$ (a multiple of $\pi$), see [Br3], Theorem 3.4.

If it is the case that we have two curves $C$ and $C'$ intersecting at a fixed point, the following sequence ([Ki] Proposition 3.4)

$$0 \to A^*_T(C \cup C') \to A^*_T(C) \oplus A^*_T(C') \to A^*_T(C \cap C')$$

yields the statement. Now arguing by induction on the length of the connected component of $C$ in $X^{\ker(\pi)}$ concludes the proof, in view of Theorem 3.8. $\square$

**Example 3.13.** The following are examples of smooth $T$-skeletal varieties satisfying equivariant Kronecker duality: smooth projective embeddings of reductive groups [BCP], [LP]. More generally, smooth wonderful symmetric varieties of minimal rank and regular compactifications of symmetric varieties of minimal rank. The Chow rings of the latter class have been computed in [BJ-2].

**Example 3.14.** Examples of singular $T$-skeletal varieties are Schubert varieties and rationally smooth group embeddings of reductive groups. The former class are $T$-linear varieties (as they come with a paving by affine cells), and their equivariant cohomology is described in [C]. The latter class are singular spherical varieties and their equivariant cohomology has been described by the author in [G2]. In Section 6 we extend both descriptions to their corresponding equivariant operational Chow rings.

**Remark 3.15.** It was proved in [P], that for a toric variety $X$, one can take $X_T$ to be a toric variety for a larger torus $T'$. In that case, Theorem 4.3 follows directly from [FMSS]. For a general spherical $G$-variety $X$, we do not know if we can choose $X_G$ so that it is $G'$-spherical, for a larger $G'$.

## 4. Equivariant Kronecker duality for spherical varieties

In this section we show that spherical varieties form a natural class of varieties for which Equivariant Kronecker duality holds. We start by stating a few crucial properties from the theory of spherical varieties that will be relevant to our study. For a complete treatment of the subject, the reader is cordially invited to consult [Br1], [Kn1], [L-V] and [BLV].

Let $G$ be a connected reductive group, with Borel subgroup $B$ and maximal torus $T \subset B$. A $G$-variety $X$ is called *spherical* if it contains a dense orbit of $B$. We say that $X$ is simply-connected spherical (or *scs* for short) if, in addition, the $B$-isotropy group of this dense orbit is connected.

Any spherical $G$-variety contains only finitely many $G$-orbits; as a consequence, it contains only finitely many fixed points of a maximal torus $T$ of $G$.

Next we recall that any spherical $G$-variety $X$ admits an equivariant resolution of singularities, i.e., there exists a smooth $G$-variety $\tilde{X}$ together with a proper birational $G$-equivariant morphism $\pi : \tilde{X} \to X$. Then the $G$-variety $\tilde{X}$ is also spherical; if moreover $X$ is complete, we may arrange so that $\tilde{X}$ is projective.

Notice that a resolution of singularities need not be an equivariant envelope. The following is an important class of spherical varieties for which equivariant resolutions *are* equivariant envelopes. We thank M. Brion for leading us to the following proof.

**Proposition 4.1.** *Let $X$ be a normal simply-connected spherical $G$-variety. Let $f : \tilde{X} \to X$ be a proper birational morphism. Then $f$ is an equivariant envelope.*

*Proof.* Let $p : \tilde{X} \to X$ be a toroidal resolution of $X$. It suffices to show that every $B$-orbit in $X$ is the isomorphic image via $p$ of a $B$-orbit in $\tilde{X}$. So let $\mathcal{O} = (B){\cdot}x = B/B_x$ be an orbit in $X$. It follows from [BJ] that $\mathcal{O}$ has a connected isotropy group. The preimage $p^{-1}(\mathcal{O}) \subset \tilde{X}$ is of the form $B \times^{B_x} F$, where $F$ denotes the fiber $p^{-1}(x)$. Since $F$ is connected and complete (by Zariski's main theorem), it contains a fixed point $y$ of the connected solvable group $B_x$. Then the orbit $B \cdot y$ in $\tilde{X}$ is mapped isomorphically to $B \cdot x$. $\square$

**Remark 4.2.** Examples of scs varieties include all normal $G \times G$-equivariant embeddings of $G$. In particular, all toric varieties are scs. This fact is used crucially in Payne's proof that the equivariant operational Chow ring of *any* toric variety is isomorphic to the ring of piecewise polynomial functions defined on the associated fan [P].

As before, set $S := A_T^*(pt)$ and $S^W := A_G^*(pt)$.

**Theorem 4.3.** *Let $X$ be a complete $G$-variety with a finite number of $B$-orbits. Then the equivariant Kronecker map*

$$\mathcal{K}_{\mathcal{T}} : A_T^*(X) \longrightarrow Hom_S(A_*^T(X), S) \qquad \alpha \mapsto (\beta \mapsto p_{X*}(\beta \cap \alpha))$$

*is an $S$-module isomorphism. Furthermore, the $G$-equivariant Kronecker duality map $\mathcal{K}_{\mathcal{G}}$ is also an isomorphism.*

*Proof.* First, we consider the case when $X$ is projective and smooth. Then, because $T$ acts on $X$ with finitely many fixed points, $X$ has a cellular decomposition (see [B1]). It follows that $A_*^T(X)$ is a free $S$-module ([Br3], Corollary 3.2.1). Moreover, $A_*^T(X)$ carries an intersection product making it into a graded ring $A_T^*(X)$ (graded by codimension). Let $M$ be the character group of $T$. Observe that $A_T^*(X)/MA_T^*(X) \simeq A^*(X)$ ([Br3], Corollary 2.3). Moreover, the freeness of $A_*^T(X)$ yields the identifications

$$\mathrm{Hom}_S(A_*^T(X), S)/M \cdot \mathrm{Hom}_S(A_*^T(X), S) \simeq \mathrm{Hom}_{\mathbb{Q}}(A_*(X), \mathbb{Q}).$$

Now, consider the commutative diagram

$$\begin{tikzcd}
A_T^*(X) \arrow[r, "\mathcal{K}_{\mathcal{T}}"] \arrow[d] & \mathrm{Hom}_S(A_*^T(X), S) \arrow[d] \\
A^*(X) \arrow[r, "\mathcal{K}"] & \mathrm{Hom}_{\mathbb{Q}}(A_*(X), \mathbb{Q}).
\end{tikzcd}$$

In order to prove that $\mathcal{K}_{\mathcal{T}}$ is an isomorphism, it suffices, by the graded Nakayama lemma, to show that the non-equivariant Kronecker duality map $\mathcal{K}$ is an isomorphism, but this has already been stablished in [FMSS].

In the general case, we start by constructing a $G$-equivariant smooth envelope $p : \tilde{X} \to X$, where $\tilde{X}$ is projective. For this, we proceed as follows. Let $\pi : X' \to X$ be an equivariant resolution of $X$, i.e. $X'$ is a disjoint union of equivariant resolutions of the irreducible components of $X$. By our previous remarks, $X'$ is projective. Bear in mind that this map is not necessarily a $G$-equivariant envelope, but $\pi$ is an isomorphism outside some $G$-invariant subvariety $S \subset X$. By Noetherian induction, we may assume that we have constructed an equivariant smooth envelope $\tilde{S} \to S$, where both $\tilde{S}$ and $S$ have a finite number of $B$-orbits. Now set $\tilde{X} = X' \sqcup \tilde{S}$. It follows that $p : \tilde{X} \to X$ is an equivariant smooth envelope ([EG]).

Therefore, we have a fibre square

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "p"] \\
S \arrow[r, "i"'] & X,
\end{tikzcd}$$

where $i$ is the inclusion of $S$ into $X$ and $E = p^{-1}(S)$. Kimura [Ki] assigns to this fibre sequence an exact sequence of operational Chow rings, which, in the equivariant case, yields

$$0 \to A_G^*(X) \to A_G^*(S) \oplus A_G^*(\tilde{X}) \to A_G^*(E).$$

Notice that $S$ has codimension at least one. Hence, arguing by induction on the dimension of $X$, and using the fact that $\tilde{X}$ is smooth and that the Kronecker duality map $\mathcal{K}_{\mathcal{T}}$ is functorial for proper morphisms (Remark 3.2) we obtain the result. It is clear that replacing $T$ by $G$ in the statement of the Theorem does not alter the argument, for $A_*^G(X)/S_+^W A_*^G(X) \simeq A_*(X)$ by [Br3], Corollary 6.7.1. □

**Corollary 4.4.** *Under the hypothesis of the Theorem 4.3,*

$$A_T^*(X) \simeq A_G^*(X) \otimes_{S^W} S.$$

*Proof.* Simply notice that

$$\mathrm{Hom}_S(A_*^G(X) \otimes_{S^W} S, S^W \otimes_{S^W} S) \simeq S \otimes_{S^W} \mathrm{Hom}_{S^W}(A_*^G(X), S^W),$$

because $S$ is free over $S^W$ and $A_G^*(X)$ is finitely generated. Now observe that the term on the left hand side identifies in turn to $A_T^*(X)$, due to our previous result and the fact that $A_*^T(X) \simeq A_*^G(X) \otimes_{S^W} S$ ([Br3]). Finally, notice that the right hand side corresponds to $S \otimes_{S^W} A_G^*(X)$ by the previous theorem too. □

**Corollary 4.5.** *Under the assumptions of Theorem 4.3, if $A_*^T(X)$ is $S$-free, then restriction to the fiber is surjective and $A^*(X) \simeq A_T^*(X)/MA_T^*(X)$.*

*Proof.* Theorem 4.3 together with freeness of $A_*^T(X)$ yield

$$A_T^*(X)/MA_T^*(X) \simeq \mathrm{Hom}_{\mathbb{Q}}(A_*^T(X)/MA_*^T(X), \mathbb{Q}).$$

Furthermore, by Theorem 2.3, the term on the right hand side above corresponds to $\mathrm{Hom}_{\mathbb{Q}}(A_*(X), \mathbb{Q})$, which, in turn, is isomorphic to $A^*(X)$, due to the non-equivariant version of Kronecker duality ([FMSS], Theorem 3). □

**Remark 4.6.** It is worth noting that unlike the case of equivariant Chow groups (Theorem 2.3), the map $i^* : A_T^*(X) \to A^*(X)$ is not surjective in general. Even worse, its kernel is not necessarily generated in degree one, see e.g. [PK]. In Section 6 we provide the theory with a large class of singular spherical varieties satisfying Corollary 4.5, namely $\mathbb{Q}$-filtrable spherical varieties. This class includes all rationally smooth projective equivariant embeddings of reductive groups [G2].

In light of Theorem 4.3, it is natural to ask whether spherical varieties actually satisfy the strong version of equivariant Kronecker duality. If so, it would be possible to apply GKM theory in various situations of interest (Theorem 3.8). We will show that this is in fact the case.

Recall that a subtorus $T' \subset T$ is *regular* if its centralizer $C_G(T')$ is equal to $T$; otherwise $T'$ is singular. A subtorus of codimension one is singular if and only if it is the kernel of some positive root $\alpha$. Then $\alpha$ is unique, and the group $C_G(T')$ is the product of $T'$ with a subgroup $\Gamma$ isomorphic to $SL_2$ or to $PSL_2$. Notice that the fixed point set of $T'$ in any $G$-variety inherits an action of the group $C_G(T')/T'$, a quotient of $\Gamma$. The following is an important structural result due to Brion and Luna [Br3].

**Proposition 4.7.** *Let $X$ be a spherical $G$-variety. Let $T' \subset T$ be a subtorus of codimension one. Then each irreducible component of $X^{T'}$ is a spherical $C_G(T')$-variety. Moreover,*

*(1) If $T'$ is regular, then $X^{T'}$ is at most one-dimensional.*
*(2) If $T'$ is singular, then $X^{T'}$ is at most two-dimensional. If moreover $X$ is complete and nonsingular, then any two-dimensional connected component of $X^{T'}$ is (up to a finite, purely inseparable equivariant morphism) either a rational ruled surface*

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$$

*where $C_G(T')$ acts through the natural action of $SL(2)$, or the projective plane where $C_G(T')$ acts through the projectivization of a non-trivial $SL(2)$-module of dimension three.* □

**Corollary 4.8.** *Let $X$ be a complete spherical $G$-variety. Let $H \subset T$ be a subtorus of codimension at most one and let $i_H : X^H \to X$ be the inclusion of the fixed point set. Then the $S$-linear map $i_H^* : A_T^*(X) \to A_T^*(X^H)$ is injective and it becomes an isomorphism after inverting finitely many characters of $T$ that restrict non-trivially to $H$.*

*Proof.* This follows directly from Proposition 4.7, Theorem 4.3 and Theorem 3.6. □

**Corollary 4.9.** *Let $X$ be a complete spherical $G$-variety. Then $X$ satisfies the strong Equivariant Kronecker duality. Moreover, if the $S$-module $A_T^*(X)$ is free, then the image of the injective map*

$$i_T^* : A_T^*(X) \to A_T^*(X^T)$$

*is the intersection of the images of the maps*

$$i_{T,T'}^* : A_T^*(X^{T'}) \to A_T^*(X^T),$$

*where $T'$ runs over all subtori of codimension one of $T$.*

*Proof.* Immediate from the fact that $X^T$ is finite, Proposition 4.7 and Theorem 3.8. □

Hence, if $X$ is a $T$-skeletal spherical variety such that $A_*^T(X)$ is free, then $A_T^*(X) \simeq PP_T^*(X)$ (Theorem 3.11).

For smooth projective spherical varieties, the results above are due to Brion [Br3]. It is also worth noting in such case, Proposition 4.7 yields a complete description of $A_T^*(X)$ in terms of congruences involving pairs, triples or quadruples of $T$-fixed points. For more details, see Theorem 7.3 of [Br3].

The importance of the theoretical results of this sections is their applicability to the study of singular spherical varieties and singular $T$-linear varieties. This is the topic of our next section.

## 5. Equivariant Chow groups of $\mathbb{Q}$-filtrable spherical varieties

It is clear from the previous sections that we need to find conditions under which the equivariant Chow group of singular varieties is free. But first, we recall a few notions from [Br3].

**Definition 5.1.** Let $X$ be a $T$-variety with a fixed point $x$.

(1) We say that $x$ is a *non-degenerate fixed point* if all weights of $T$ in the tangent space $T_xX$ are non-zero.
(2) We say that $x$ is an *attractive fixed point* if there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ and a Zariski neighborhood $U$ of $x$, such that $\lim_{t\to 0} \lambda(t)\cdot y = x$ for all points $y$ in $U$.

To study possibly singular varieties (e.g. Schubert varieties), Brion [Br3] develop a notion of *equivariant multiplicity* at non-degenerate fixed points. The main features of this notion are as follows.

**Theorem 5.2** ([Br3])**.** *Let $X$ be a $T$-variety with an action of $T$, let $x \in X$ be a non-degenerate fixed point and let $\chi_1, \ldots, \chi_n$ be the weights of $T_xX$.*

*(i) There exists a unique $S$-linear map*
$$e_{x,X} : A_*^T(X) \longrightarrow \frac{1}{\chi_1 \cdots \chi_n} S$$
*such that $e_{x,X}[x] = 1$ and that $e_{x,X}[Y] = 0$ for any $T$-invariant subvariety $Y \subset X$ which does not contain $x$.*

*(ii) For any $T$-invariant subvariety $Y \subset X$, the rational function $e_{x,X}[Y]$ is homogeneous of degree $-\dim(Y)$ and it coincides with $e_{x,Y}[Y]$.*

*(iii) The point is nonsingular in $X$ if and only if*
$$e_x[X] = \frac{1}{\chi_1 \cdots \chi_n}.$$
□

For any $T$-invariant subvariety $Y \subset X$, we set $e_{x,X}[Y] := e_x[Y]$, and we call $e_x[Y]$ the *equivariant multiplicity of $Y$ at $x$*.

**Remark 5.3.** It follows from [Br6] that if $X$ is an affine $T$-variety with an attractive fixed point $x$, then
$$X = \{y \in X \mid \lim_{t\to 0} \lambda(t)\, y = x_0\},$$
for a suitable one-parameter subgroup $\lambda$. Notably, $\{x_0\}$ is the unique closed $T$-orbit in $X$, and $X$ admits a closed $T$-equivariant embedding into $T_xX$. For convenience, we say that $(X, x)$ is an *attractive cell* in this situation.

The technical result on attractive cells will be of importance in the sequel. Let $\mathcal{Q}$ be the quotient field of $S$.

**Lemma 5.4.** *Let $(X, x)$ be an attractive cell of dimension $n$. If $A_*^T(X)$ is free, then the equivariant multiplicity morphism $e_x : A_*^T(X) \to \mathcal{Q}$ is injective.*

*Proof.* It follows from [Br3], Prop. 4.1 that the map $i_* : A_*^T(x) \to A_*^T(X)$ is injective. Moreover, the image of $i_*$ contains $\chi_1 \cdots \chi_n A_*^T(X)$, where $\chi_i$ are the $T$-weights of $T_xX$. Now recall that $e_x$ is defined as follows: given $\alpha \in A_*^T(X)$, we can form the product $\chi_1 \cdots \chi_n\alpha$. Thus, there exists $\beta \in S$ such that $i_*(\beta) = \chi_1 \cdots \chi_n\alpha$. Now let $e_x(\alpha) = \frac{\beta}{\chi_1\cdots\chi_n}$. It is clear from the construction that if $A_*^T(X)$ is free, then $e_x$ is injective. □

Let $(X, x)$ be an attractive cell. In this case, the geometric quotient

$$\mathbb{P}(X) := [X \setminus \{x\}]/\mathbb{G}_m$$

exists and we call it the *link* at $x$. This is a projective variety since $X$ is assumed to be affine.

In [G1] we studied the links of *complex* rationally smooth cells. Recall that a complex algebraic variety $X$, of dimension $n$, is called *rationally smooth* if

$$H^m(X, X - \{y\}) = (0) \text{ if } m \neq 2n, \quad \text{and } H^{2n}(X, X - \{y\}) = \mathbb{Q}.$$

for all $x \in X$. Such varieties satisfy Poincaré duality with rational coefficients. If $(X, x)$ is a complex rational cell, then $\mathbb{P}(X)$ is a rational cohomology complex projective space. Many important results on the equivariant cohomology of $T$-varieties admitting a paving by rational cells are provided in [G1], for instance, such varieties have no cohomology in odd degrees and their equivariant cohomology is a free $S$-module. Our goal in this section is to provide analogues of these notions in the context of Chow groups. To do so, we give need an extra ingredient.

We thank M. Brion for leading us to the following definition.

**Definition 5.5.** Let $(X, x)$ be an attractive cell of dimension $n$. We say that $X$ is an algebraic rational cell if and only if

$$A_*(\mathbb{P}(X)) \simeq A_*(\mathbb{P}^{n-1}).$$

Some consequences of this definition appear next.

**Lemma 5.6.** *Let $(X, x)$ be an algebraic rational cell. Then*

$$A_k(X) = \begin{cases} \mathbb{Q} & \text{if} \quad k = n \\ 0 & \text{if} \quad k \neq n \end{cases}$$

*Moreover, $A_*^{\mathbb{G}_m}(X) \simeq A_*^{\mathbb{G}_m}(pt)$.*

*Proof.* Recall that we have a short exact sequence

$$0 \to A_*^{\mathbb{G}_m}(x) \to A_*^{\mathbb{G}_m}(X) \to A_*^{\mathbb{G}_m}(X \setminus \{x\}) \to 0,$$

which steems from the fact that the $T$-fixed point $x$ is non-degenerate ([Br3], Proposition 4.1). Moreover, there exists a $\mathbb{G}_m$-equivariant finite surjective map $\pi : X \to \mathbb{A}^n$ such that $\pi^{-1}(0) = x$ ([Br6], Proposition A3). This map induces the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A_*^{\mathbb{G}_m}(x) \arrow[r, "i_*"] \arrow[d, "\pi^*"] & A_*^{\mathbb{G}_m}(X) \arrow[r, "j^*"] \arrow[d, "\pi^*"] & A_*^{\mathbb{G}_m}(X - \setminus\{x\}) \arrow[r] \arrow[d, "\pi^*"] & 0 \\
0 \arrow[r] & A_*^{\mathbb{G}_m}(0) \arrow[r, "i^*"] & A_*^{\mathbb{G}_m}(\mathbb{A}^n) \arrow[r, "j^*"] & A_*^{\mathbb{G}_m}(\mathbb{A}^n - \setminus\{0\}) \arrow[r] & 0
\end{tikzcd}$$

By [EG], the vertical map on the right represents

$$\pi^* : A_*(\mathbb{P}(X)) \to A_*(\mathbb{P}^{n-1}),$$

because $A_*^{\mathbb{G}_m}(X \setminus \{x\}) \simeq A^*(\mathbb{P}(X))$ and we are working with rational coefficients. This fact, together with our assumptions on $\mathbb{P}(X)$, imply that both the right and left vertical maps are isomorphisms; hence, so is the middle one. Therefore, $A_*^{\mathbb{G}_m}(X) \simeq A_*^{\mathbb{G}_m}(\mathbb{A}^n)$. Finally, Theorem 2.3 gives $A_*(X) \simeq A_*^{\mathbb{G}_m}(\mathbb{A}^n)/MA_*^{\mathbb{G}_m}(\mathbb{A}^n) \simeq \mathbb{Q}$, where $M$ is the maximal graded ideal generated by homogeneous elements of positive degree. This amounts to $A_n(X) = \mathbb{Q}$ and $A_k(X) = 0$, for $k \neq n$. $\square$

The previous lemmas hint to a more general structural property of (algebraic) rational cells with respect to the $T$-action.

**Proposition 5.7.** *Let $(X, x)$ be an attractive cell. Set $n = \dim X$. Then the following conditions are equivalent.*

*(i)* $A_*(\mathbb{P}(X)) \simeq A_*(\mathbb{P}^{n-1})$.

*(ii)* $A_*(X) \simeq A_*(\mathbb{A}^n)$.

*(iii)* $A_*^T(X) \simeq A_*^T(pt) = S$.

*Proof.* (i) $\Longrightarrow$ (ii) follows from Lemma 5.6.

(ii) $\Longrightarrow$ (i) By the graded Nakayama lemma (which can be applied because the degrees in $A_*^S(X)$ are at most the dimension of $X$) and the assumption on the usual Chow group of $X$, we have that $A^*_{\mathbb{G}_m}(X)$ is a $A_*^{\mathbb{G}_m}(pt)$-module generated by one element, i.e. there is a surjective module map $r : A_*^{\mathbb{G}_m}(pt) \twoheadrightarrow A_*^{\mathbb{G}_m}(X)$. Moreover, using the surjective map $\pi_* : A_*^{\mathbb{G}_m}(X) \twoheadrightarrow A^{\mathbb{G}_m}(\mathbb{A}^n)$ from the proof of Lemma 5.6, we get a commutative triangle

$$\begin{tikzcd} & A_*^{\mathbb{G}_m}(pt) \arrow[dl, "r"'] \arrow[dd, "\pi_* \circ r"] \\ A_*^{\mathbb{G}_m}(X) \arrow[dr, "\pi_*"'] & \\ & A_*^{\mathbb{G}_m}(\mathbb{A}^n), \end{tikzcd}$$

hence $\pi_* \circ r$ is surjective map of free $A_*^{\mathbb{G}_m}$-modules of rank one, and thus it is an isomorphism, which implies that both $r$ and $\pi_*$ are isomorphisms (since they are already surjective).

(ii) $\Longrightarrow$ (iii) If $(X, x)$ satisfies $A_*(X) \simeq A_*(\mathbb{C}^n)$, then, by the Graded Nakayama Lemma, there is a surjective $S$-module morphism $\varphi : S \twoheadrightarrow A_*^T(X)$. This map descends to a surjective map of localized modules $\tilde{\varphi} : \mathcal{Q} \to A_*^T(X) \otimes \mathcal{Q}$. But, by the localization theorem, the latter is isomorphic to $\mathcal{Q}$, and hence $\varphi$ is an isomorphism. Now, in the commutative diagram

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d] & A_*^T(X) \arrow[d] \\ \mathcal{Q} \arrow[r, "\tilde{\varphi}"] & A_*^T(X) \otimes \mathcal{Q}, \end{tikzcd}$$

the first vertical map is injective and thus $\varphi$ is injective as well.

(iii) $\Longrightarrow$ Clear from Theorem 2.3. This concludes the proof. □

Combining Lemma 5.4 with the previous Proposition yields

**Corollary 5.8.** *Let $X$ be a $T$-variety with an attractive fixed point $x$. Suppose that $(X, x)$ is an algebraic rational cell. Then the equivariant multiplicity morphism $e_{X,x} : A_*^T(X) \to \mathcal{Q}$ is injective.* □

Before introducing the notion of $\mathbb{Q}$-filtrable varieties, we recall here the *Bialynicki-Birula decomposition.* Let $X$ be a projective algebraic variety with a $\mathbb{G}_m$-action and a finite number of fixed points $x_1, \ldots, x_m$. Consider the associated **BB-decomposition** $X = \bigsqcup_i W_i$, where each cell is defined as follows

$$W_i = \{x \in X \mid \lim_{t \to 0} t \cdot x = x_i\}.$$

**Remark 5.9.** In general, the $BB$-decomposition of a projective variety is not a stratification; that is, it may happen that the closure of a cell is not the union of cells, even if we assume our variety to be smooth. For a justification of this claim, see [B2].

**Definition 5.10.** Let $X$ be an algebraic variety endowed with a $\mathbb{G}_m$-action and a finite number of fixed points. A BB-decomposition $\{W_i\}$ is said to be **filtrable** if there exists a finite increasing sequence $X_0 \subset X_1 \subset \ldots \subset X_m$ of closed invariant subvarieties of $X$ such that:

a) $X_0 = \emptyset$, $X_m = X$,

b) For each $j = 1, \ldots, m$, the "stratum" $X_j \setminus X_{j-1}$ is a cell of the decomposition $\{W_i\}$.

The following result is due to Bialynicki-Birula ([B2]).

**Theorem 5.11.** *Let $X$ be a normal projective variety with $\mathbb{G}_m$-action and a finite number of fixed points. Then the BB-decomposition is filtrable.* $\square$

In the present section, we show that algebraic rational cells are a good substitute for the notion of affine space in the study of equivariant Chow groups of singular varieties. We aim at an inductive description of the equivariant Chow groups of filtrable $T$-varieties in the case when the cells are all rational. Our results provide purely algebraic analogues of the topological results in our earlier paper [G1].

Let $T$ be an algebraic torus acting on a variety $X$. A one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ is called *generic* if $X^{\mathbb{G}_m} = X^T$, where $\mathbb{G}_m$ acts on $X$ via $\lambda$. Generic one-parameter subgroups always exist. Note that the $BB$-cells of $X$, obtained using $\lambda$, are $T$-invariant. Our results in this section suggest the following definition.

**Definition 5.12.** Let $X$ be a variety equipped with a $T$-action. We say that $X$ is **$\mathbb{Q}$-filtrable** if the following hold:

(1) X is normal,
(2) the fixed point set $X^T$ is finite, and
(3) there exists a generic one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ for which the associated $BB$-decomposition of $X$ is filtrable and consists of $T$-invariant *algebraic rational cells*.

If, moreover, the algebraic rational cells are isomorphic to affine spaces, then $X$ is called **$T$-cellular**.

Let $X$ be a $\mathbb{Q}$-filtrable algebraic variety with a $T$-action. Then, by assumption, there is a closed cell $F = (X_1, x_1)$ (using the induced order of the fixed points), and moreover $U = X \setminus F$ is filtrable. We describe $A_*^T(X)$ in terms of $A_*^T(F)$ and $A_*^T(U)$.

**Proposition 5.13.** *Let $X$ be a $\mathbb{Q}$-filtrable $T$-variety. Let $F$ be the closed cell and $U = X \setminus F$ be the filtrable open complement. Then the maps $j_{F*} : A_*^T(F) \to A_*^T(X)$ and $j_U^* : A_*^T(X) \to A_*^T(U)$ fit into the exact sequence*

$$0 \to A_*^T(F) \to A_*^T(X) \to A_*^T(U) \to 0.$$

*Proof.* The proof reduces to show that $j_{F*}$ is injective. But this follows from the fact that $e_{x,F} = e_{x,X} \circ j_{F*}$. Because $e_{x,F}$ is injective (Corollary 5.8), then $j_{F*}$ is injective. □

Arguing by induction on the length of the filtration leads to the following result.

**Corollary 5.14.** *Let $X$ be a $\mathbb{Q}$-filtrable $T$-variety. Then the $T$-equivariant Chow group of $X$ is a free module of rank $|X^T|$. In fact, it is freely generated by the classes of the closures of the cells $W_i$. Consequently, $A_*(X)$ is also freely generated by the classes of the cell closures $W_i$.* □

Hence, by Theorem 3.6, Theorem 3.11 and Corollary 5.14, if $X$ is a $\mathbb{Q}$-filtrable spherical $G$-variety with a finite number of $T$-invariant curves, then

$$A_T^*(X) \simeq PP_T^*(X).$$

This remark will be applied in the next subsection to complex projective rationally smooth group embeddings.

Analogously, if $X$ is a $T$-cellular projective variety, then each filtered piece $X_i$ is also $T$-cellular. Thus, Corollary 5.14 together with Corollary 3.9 and Theorem 3.11 yield the applicability of GKM theory, in its Chow cohomology version, at each step of the filtration

$$\emptyset = X_0 \subset X_1 \subset \ldots \subset X_m = X.$$

Notice that the filtered pieces $X_i$ need not be smooth (e.g. Schubert varieties).

## 6. Examples

6.1. **Schubert varieties.** Let $G$ be a connected reductive group with Borel subgroup $B$ and maximal torus $T \subset B$. Denote by $W$ the Weyl group. Recall that there is a natural order on $W$, the Bruhat order. The homogeneous space $G/B$ is called the **flag variety** of $G$. It is a projective variety. Notice that $T$ acts on $G/B$ with a finite number of fixed points, namely $(G/B)^T \simeq W$. It follows from the Bruhat decomposition, $G = \sqcup_{w\in W} BwB$, that the flag variety $G/B$ admits a paving by affine cells of the form $B[w] = BwB/B$, indexed over $w \in W$. Each one of these cells is isomorphic to an affine space $\mathbb{A}^{\ell(w)}$, where $\ell(w)$ is the length of $w$. Moreover, $G/B$ is a smooth $T$-skeletal variety, so to describe $A_T^*(G/B)$, it suffices to collect the necessary $GKM$-data [Br3]. Here we do more, by showing that the usual GKM picture holds for the operational Chow rings $A_T^*(X(w))$, where $X(w) = \overline{BwB/B}$ is a Schubert variety.

**$T$-invariant curves and the Bruhat graph.** The Weyl group is generated by reflections $\{s_\alpha\}_{\alpha\in\Phi}$, where $s_\alpha$ corresponds to reflection with respect to the hyperplane defined by $\alpha$. Let $\mathcal{G}_{s_\alpha}$ denote the copy of $SL(2)$ in $G$ generated by $U_\alpha$ and $U_{-\alpha}$. The following is a result of Carrell ([C]).

**Proposition 6.1.** *The flag variety $G/B$ is a $GKM$-variety. In fact, every closed $T$-invariant curve in $G/B$ has the form $\mathcal{G}_{s_\alpha}w$, for some $w$ in $W$ and reflection $s_\alpha$. Consequently, every $T$-invariant curve is non-singular. Moreover, $(\mathcal{G}_{s_\alpha}w)^T = \{w, s_\alpha w\}$, so $\mathcal{G}_{s_\alpha}x \subset X(w)$ if and only if $x, s_\alpha x \leq w$.* □

What follows is an extension of the usual picture of $A_T^*(G/B)$ to the operational Chow ring of the Schubert varieties $X(w)$. Denote by $I_w$ the Bruhat interval $[1, w] = \{x \in W \,|\, x \leq w\}$. Notice that $X(w)^T = I_w$. Let $i : (X(w))^T \to X(w)$ be the inclusion of the fixed point set. Let $S = A_T^*(pt)$. Then, $A_T^*(X(w)^T) =$

$\oplus_{x\in I_w} S$, a subalgebra of $S[W]$. Since $X(w)$ is $T$-cellular, Corollary 5.14 together with Theorem 3.11 yield the next result.

**Theorem 6.2.** *If $X(w) = \overline{BwB/B}$ is a Schubert variety, then $X(w)$ satisfies the strong Equivariant Kronecker duality and $A_T^*(X(w))$ is $S$-free. Moreover, the image of*

$$i^* : A_T^*(X(w)) \to \oplus_{x\in I_w} S$$

*consists of all $\sum_{x\in I_w} f_x x$ such that $f_x \cong f_{s_\alpha x}(\mathrm{mod}\,\alpha)$, whenever (i) $s_\alpha$ is a reflection of $W$ and (ii) $x, s_\alpha x \in I_w$.* □

6.2. **Rationally smooth group embeddings.** In this subsection we work over the complex numbers. Our purpose is to contrast the results of this paper with those of [G1] and [G2]. In those articles, a projective $T$-variety is called *topologically $\mathbb{Q}$-filtrable* if $X$ is normal, $X^T$ is finite, and there exists a generic one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ for which the associated $BB$-decomposition of $X$ consists of $T$-invariant *rationally smooth cells.* Our purpose is to compare this definition with Definition 5.12 in the case of spherical varieties.

Let $G$ be a connected reductive group with Borel subgroup $B$ and maximal torus $T$. Let $X$ be a projective spherical $G$-variety. We say that $X$ is *compatibly $\mathbb{Q}$-filtrable* if the generic-one parameter subgroup $\lambda$ above can be chosen so that $G(\lambda) = B$. Recall that

$$G(\lambda) = \{g \in G \mid \lambda(t)g\lambda(t)^{-1}\,\mathrm{has\,a\,limit\,as}\,t \to 0\}.$$

It is well-known that $G(\lambda)$ is a parabolic subgroup of $G$. Moreover, if we choose $\lambda$ inside the Weyl chamber of $T$, then both $X^\lambda = X^T$ and $G(\lambda) = B$. From now on, we assume such $\lambda$ to be fixed.

Let $X$ be a compatibly $\mathbb{Q}$-filtable spherical variety. It follows from our assumptions that the associated cells

$$W_i = \{x \in X \mid \lim_{t\to 0} t \cdot x = x_i\}.$$

are also $B$-invariant. Because $B$ acts on $X$ with finitely many orbits, so it does on $W_i$. The following was proved by Totato [T] in the general setup of linear varieties.

**Theorem 6.3.** *For any $B$-variety $Y$ with a finite number of orbits, the natural map*

$$A_i(Y) \otimes \mathbb{Q} \longrightarrow W_{-2i}H_{2i}^{BM}(Y, \mathbb{Q}),$$

*from the Chow groups into the smallest subspace of Borel-Moore homology with respect to the weight filtration is an isomorphism.* □

Since $B$-acts on each cell $W_i$ with a finite number of orbits, we conclude that

$$A_i(Y) \simeq W_{-2i}H_{2i}^{BM}(W_i, \mathbb{Q}) \simeq H_c^{2i}(Y, \mathbb{Q}) \simeq \mathbb{Q}$$

where the latter identification steems from the fact that each $W_i$ is a rational cell. Induction on the length of the filtration yields

**Theorem 6.4.** *Let $X$ be a spherical $G$-variety with a compatible $\mathbb{Q}$-filtration. Then both cycle maps,*

$$cl_X : A_*(X) \to H_*(X)$$

*and*

$$cl_X^T : A_*^T(X) \otimes \mathbb{Q} \to H_*^T(X)$$

*are isomorphisms. In particular, $A_T^*(X)$ is a free $S$-module and the operational Chow ring $A^*_T(X)$ is isomorphic to $H^*_T(X)$. If moreover $X$ is $T$-skeletal, then $A^*_T(X) \simeq PP^*_T(X)$.*

*Proof.* It only remains to show the statement about the operational Chow rings $A^*_T(X)$. But this easily follows from the fact that $X$ satisfies equivariant Kronecker duality, by Theorem 4.3. If $X$ is $T$-skeletal, then Theorem 3.11 implies the last assertion and finishes the proof. □

Theorem 5.8 is compatible with our earlier results on equivariant cohomology [G1]. Moreover, it follows from Theorem 5.18 that the $\mathbb{Q}$-filtrable varieties of Definition 5.12 are yet a larger class of varieties where one can obtain the freeness of the equivariant Chow groups, without using equivariant cohomology, and apply the results of this paper (for $T$-varieties satisfying the strong version of equivariant Kronecker duality).

*Question:* Are all rationally smooth spherical varieties $\mathbb{Q}$-filtrable? If $X$ is smooth, then [B2] implies that each cell $C_i$ is isomorphic to some affine space $\mathbb{A}^{n_i}$. If $X$ is a rationally smooth projective group embedding, then it is $\mathbb{Q}$-filtrable, by [G2]. But it would be interesting to know the answer to this question in full generality.

Let $X$ be a projective group embedding of $G$. That is, $X$ is a $G \times G$-variety with an open orbit isomorphic to $G \simeq (G \times G)/\Delta(G)$. It follows from the Bruhat decomposition that $X$ is $G \times G$-spherical. Moreover, all projective group embeddings $X$ can be obtained as projectivizations of linear algebraic monoids (see [R1], [Ti]), i.e. $X = [M \setminus \{0\}]/\mathbb{G}_m$, for some affine reductive monoid $M$. In [G1] and [G2] we show that rationally smooth projective group embeddings are $\mathbb{Q}$-filtrable and $T$-skeletal. Moreover, we obtain explicitly all the GKM data needed to describe their equivariant cohomology as a ring of piecewise polynomial functions on the associated Renner monoid of $M$. Theorem 5.8 above shows that, over the complex numbers, our earlier findings translate verbatim to the operational Chow rings of rationally smooth group embeddings. But this is just a sample of a more general phenomenon that occurs in *arbitrary characteristic*. In a forthcoming paper [G3], we study algebraic $\mathbb{Q}$-filtrations on group embeddings in a purely algebraic way, using the richer structure of Chow groups and the finer combinatorial structure of algebraic monoids. The results of [G3] are *independent* of Totaro's topological Theorem 5.17. Furthermore, we characterize algebraic rational cells (Definition 5.5) in very combinatorial terms in the case of algebraic monoids. This characterization is inspired in Renner's classification of rationally smooth algebraic monoids [R2]. The results will appear elsewhere.

## 7. Further remarks

1. The results of Section 3 are clearly extendable to Kronecker duality spaces (e.g. $T$-linear varieties and spherical varieties) in any equivariant operational theory $\mathrm{op}\mathcal{H}^*_T(-)$ (à la Fulton-MacPherson). It would be interesting to characterize all $\mathbb{Q}$-filtrable, $T$-skeletal, spherical varieties. This class includes all projective group embeddings with mild singularities ([R2], [G2], [G3]) and all projective embeddings of spherical varieties of minimal rank [BJ-2].

2. *Description of the image of restriction to the fiber* $i^* : A^*_T(X) \to A^*(X)$ *by using equivariant multiplicities.* This has been carried out for toric varieties by Payne and Katz [PK]. Unlike the case of Chow groups, $i^*$ is in general not surjective and its kernel is not necessarily generated in degree one. See [PK] for an illustration of these claims.

3. *Understand the action of* $PP^*_T(X)$ *on* $A^T_*(X)$ *for* $T$*-skeletal spherical varieties*, in view of Brion's description of the intersection pairing between curves and divisors on spherical varieties [Br2]. This will be pursued in a subsequent paper.

4. *Vistoli's Alexander schemes.* Let $(X, x)$ be an algebraic rational cell. If $\mathbb{P}(X)$ is smooth, then it follows from [Ki] that $X$ is an Alexander scheme, i.e. the natural map

$$A^k(X) \to A_{n-k}(X),$$

given by the cap product with the fundamental class $[X]$, is an isomorphism. Alexander schemes have been studied by Vistoli [Vis]. They are the most natural class of schemes that behave like smooth schemes from the viewpoint of intersection theory with rational coefficients. Indeed, any Alexander scheme $X$ carries an intersection product on $A_*(X)$, derived formally from the ring structure on its operational Chow ring $A^*(X)$. From a conceptual viewpoint, it would be interesting to determine which singular $\mathbb{Q}$-filtrable spherical varieties are Alexander schemes.

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