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Constraints and the E_{10} Coset Model

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Abstract: We continue the study of the one-dimensional E_{10} coset model (massless spinning particle motion on $E_{10}/K(E_{10})$) whose dynamics at low levels is known to coincide with the equations of motion of maximal supergravity theories in appropriate truncations. We show that the coset dynamics (truncated at levels $\ell \leq 3$) can be consistently restricted by requiring the vanishing of a set of constraints which are in one-to-one correspondence with the canonical constraints of supergravity. Hence, the resulting constrained σ-model dynamics captures the full (constrained) supergravity dynamics in this truncation. Remarkably, the bosonic constraints are found to be expressible in a Sugawara-like (current $\times$ current) form in terms of the conserved E_{10} Noether current, and transform covariantly under an upper parabolic subgroup $E_{10}^{+} \subset E_{10}$. We discuss the possible implications of this result, and in particular exhibit a tantalising link with the usual affine Sugawara construction in the truncation of E_{10} to its affine subgroup E_9.

1 Introduction

Work on the symmetry structure of maximal supergravity theories has revealed a remarkable link between geodesic motion of a massless spinning particle on an $E_{10}/K(E_{10})$ coset manifold and the dynamics of maximal supergravity theories [1, 2, 3, 4, 5, 6, 7, 8]. In contrast to an earlier proposal [9, 10, 11] aiming for an 11-dimensional covariant formulation of M theory exhibiting E_{11} invariance, the one-dimensional E_{10} coset model corresponds, on the supergravity side, to a (10 + 1)-dimensional gauge-fixed formulation of the supergravity dynamics, as it arises in studies of the near space-like singularity limit [12, 13, 14]. The reformulation of the dynamics as a 'cosmological billiard' facilitates a systematic dynamical treatment, and directly motivates the conjecture [1] that M theory is (holographically) equivalent to a 'one-dimensional' non-linear σ-model living on the infinite-dimensional coset manifold $E_{10}/K(E_{10})$. Ref. [1] showed that the null geodesic motion on $E_{10}/K(E_{10})$, when truncated to low levels, is equivalent to a truncated version of the bosonic dynamical equations of maximal supergravity where only first order spatial gradients are retained. This equivalence was extended by including the fermions (neglecting spatial gradients) in Refs.[5, 6, 8]. Some further evidence for a correspondence between M theory and the E_{10} coset model came from relating M theory one-loop corrections to certain high-level contributions to the coset action [15].

As is well known, in a canonical treatment of gravity and supergravity, where spacetime is foliated into a sequence of spacelike hypersurfaces, the dynamical equations have to be supplemented by *constraint equations* (to be imposed on the initial data). For instance, in the case of pure gravity these are the Hamiltonian and diffeomorphism constraints. In the present contribution we study how such constraint equations, which are necessary for recovering the full supergravity system, can be consistently incorporated into the coset model approach of [1]. As formulated there, this model already incorporates (a close analog of) the Hamiltonian constraint in the form of a null-motion constraint expressing reparametrisation invariance of the worldline. We shall therefore focus here on the other constraints, and study to which extent they are compatible with the Kac–Moody symmetry structure of these models (not manifest in the standard Hamiltonian formulation of gravity). The consistency of the usual supergravity constraints with the dynamical equations in the context of homogeneous cosmological solutions was already studied long ago [16]. Here, we are interested in establishing, *purely within the context of the $E_{10}/K(E_{10})$ coset model*, the consistency of requiring the vanishing of certain *bilinear* quantities in the coset variables, either quadratic in the coset velocities $\mathcal{P}$ (for the bosonic constraints $\mathcal{C}$), or consisting of a product of $\mathcal{P}$ and the fermionic gravitino variables ψ (for the supersymmetry constraint $\mathfrak{S}$). Namely, we shall show that in the same consistent truncation employed for the dynamical equations, one can define bosonic and fermionic

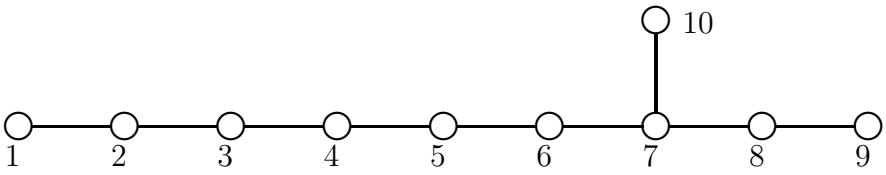

Figure 1: *Dynkin diagram of E_{10} with numbering of nodes.*

constraints of this type (on the massless spinning particle) which are *weakly conserved*[1] along the coset motion, thereby defining a constraint surface in the coset phase space preserved by the geodesic motion. We will spell out the details of this result only for $D = 11$ supergravity [17], but have no doubt that it carries over to the other maximal and non-maximal cases (some of the supergravity constraint equations rewritten in coset variables were already given in [8, 18]). In this way *all* $D = 11$ supergravity equations have been accommodated within the E_{10} model.

In addition to the weak conservation of the constraints we find that the equations describing the time evolution of the constraints exhibit a triangular structure reminiscent of a highest weight representation, cf. (3.5). Studying the tensor structure of the relevant constraints reveals two further structures, namely:

- One can redefine the bosonic constraints $\mathcal{C}$ into an equivalent set $\mathfrak{L}$ of explicitly time-independent (hence *strongly conserved*) 'Sugawara-like' expressions bilinear in the conserved Noether current (or charge) $\mathcal{J}$ associated to the rigid E_{10} symmetry of the $E_{10}/K(E_{10})$ coset action.

- At least for the low A_9 levels considered here, these 'Sugawara-like' constraints $\mathfrak{L}$ transform as a linear representation of the upper parabolic subgroup E_{10}^+ generated by $\mathfrak{gl}(10)$ and the positive-root (raising) generators of E_{10}. In addition, the latter representation can be embedded, at least at the levels considered here, and within the restriction to E_{10}^+, into the integrable highest weight representation $L(\Lambda_1)$ (to be defined below).

A key question at this point concerns the significance and the proper interpretation of the constraints in the context of the E_{10} σ-model. Because the level decomposition of E_{10} w.r.t. any of its regular subgroups gives rise to an exponentially growing spectrum of degrees of freedom, and because this proliferation of states may exceed by far what

[1] We use the word 'weakly' in the (constrained dynamics) sense of 'modulo the constraints'. In other words, a set of constraints $\mathcal{C}$ is weakly conserved iff $d\mathcal{C}/dt$ vanishes *modulo* $\mathcal{C}$.

would be needed to account for the space-time degrees of freedom of the various maximal supergravities, and possibly even M theory, it appears that suitable constraints may be necessary in order to reduce their number to what is appropriate for M theory. Furthermore it seems clear that the emergence of space (and time) along the lines proposed in [1] cannot possibly be explained without a proper understanding of the underlying constraints on the σ-model dynamics.

The tensor structure of the constraints [cf. eqn. (3.1) below] coincides at levels $\ell = 3, 4$ and 5 with the tensor structure of the so-called $L(\Lambda_1)$ representation of E_{10}, while at level $\ell = 6$ it contains only one of the two irreducible Young tableaux contained in $L(\Lambda_1)$. Let us recall that $L(\Lambda_1)$ is an integrable highest weight representation of E_{10} with Dynkin labels $[1\,0\,0\,0\,0\,0\,0\,0\,0\,0]$, where the '1' occurs for the over-extended, hyperbolic node of the E_{10} Dynkin diagram in Fig. 1.[2] We note that the analogous representation for E_{11} had already appeared in previous work [11]. The possible of occurrence of $L(\Lambda_1)$ in the present context might therefore be interpreted as evidence for a covariant formulation along the lines suggested there. However, when properly 'contravariantised' (in a sense to be explained in section 4.2), the constraints transform covariantly only under the upper parabolic subgroup E_{10}^+ leaving invariant the triangular gauge chosen for the representation of the coset manifold; in particular, the putative highest weight state of the representation is *not* annihilated by the relevant raising operators. This is somewhat contrary to what one would expect on the basis of a covariant formulation, as explained in section 4.3. However, the transformations we obtain are fully consistent with a Sugawara-type interpretation of the constraints.

The link between the canonical constraints obtained from supergravity on the one hand, and a kind of Sugawara-like construction based on E_{10} on the other hand, is the most remarkable result of the present paper. It is not clear whether this fact indicates the existence of a 'covariant' set of equations whose gauge-fixed version would give rise to the 'one-dimensional' $E_{10}/K(E_{10})$ σ-model of [1] supplemented by constraints as described in the present paper. What seems clear is that such a putative 'covariant' formulation is likely to be of a rather unconventional type: in a scheme with emergent space-time, the realisation of gauge symmetries must necessarily differ from the standard realisation of gauge symmetries in space and time. This would imply, for instance, that general covariance and other space-time based gauge symmetries might emerge only together with space-time itself, and thus not be fundamental, but only *emergent* properties of the theory.[3]

[2] By definition, the fundamental weights Λ_i are dual to the simple roots of E_{10}, i.e. $\langle \Lambda_i | \alpha_j \rangle = \delta_{ij}$ [19].

[3] In this context we may note that in canonical quantum gravity full general covariance likewise need not necessarily exist prior to the emergence of a classical space-time. In fact, no canonical quantisation of

The evidence for a Sugawara-like construction for E_{10} presented here is also noteworthy on the purely mathematical side. While the existence of the Sugawara construction for affine Lie algebras has been known for a very long time [23, 24, 25, 19], no analog for indefinite Kac–Moody algebras has ever been found. Nevertheless, our results strongly indicate that such a generalisation does exist, although it will certainly exhibit some unexpected features [as already evident from the intricate tensor structure of the pertinent expressions]. Additional evidence for this conjecture derives from the fact that the Sugawara-like structure of the coset constraints reduces to the known one when truncated to the affine E_9 subalgebra of E_{10}. As is well known, in the latter case we also have expressions bilinear in the affine currents $L_m \propto \sum_n j^a_{m-n} j^a_n$ with the current generators j^a_n [23, 24, 25]. In the affine truncation of E_{10} to E_9 (corresponding to a dimensional reduction of maximal supergravity to two space-time dimensions), most of the constraints 'disappear', except for the diffeomorphism constraint, denoted $\mathcal{C}^{(3)}$, an $SL(9)$ singlet. This singlet will be shown to be directly related to the L_{-1} Sugawara generator, which is just the translation generator $(-d/dz)$ w.r.t. the spectral parameter in a current algebra realisation of E_9. Via the linear system of two-dimensional (super-)gravity [26] and its hidden Virasoro symmetries [27], diffeomorphisms in the spectral parameter can be directly related to diffeomorphisms in the spatial coordinate.

In summary, we would thus like to raise the possibility that the Sugawara-like constraints $\mathfrak{L}$ given in section 4.2 constitute the beginnings of a generalisation of the affine Sugawara construction for the hyperbolic Kac–Moody algebra E_{10}, indicating the existence of a so far undiscovered new structure inside E_{10} and its envelopping algebra (and possibly other hyperbolic Kac–Moody algebras) and hinting at the existence of a more concrete realisation of these algebras analogous to the current algebra realisation of affine algebras. In addition, this generalisation might accommodate 10-dimensional spatial diffeomorphisms in a similar way as the ordinary Sugawara construction realises diffeomorphisms on the circle S^1. The present work could thus open new avenues both towards analysing the hyperbolic E_{10} algebra and towards understanding how space (and time) emerge out of the geodesic σ-model of [1].

This article has the following structure. After introducing the necessary notation for the $E_{10}/K(E_{10})$ model in section 2 we propose a set of bosonic constraints and fermionic constraints in section 3. After demonstrating their weak conservation along the geodesic motion we show in section 3.3 that they coincide with the constraint equations of supergravity if the usual E_{10}/supergravity dictionary is used. In section 4 we demonstrate how the bosonic constraints can be reformulated in a Sugawara-like form and that the

gravity is known, in which the full constraint algebra is realised *off shell*, see e.g. [20, 21]. See also [22] for a related discussion.

reduction to E_9 gives the usual Sugawara construction. In this context we also discuss the transformation properties of the constraints and show that a parabolic subgroup E_{10}^+ of E_{10} preserves the constraints. In the concluding section we return to the discussion of the original E_{10} symmetry and the interpretation of our results, including the relation to the Sugawara construction.

2 E_{10} model

In this section we review the formalism of the $E_{10}/K(E_{10})$ coset model and fix our notations and conventions. We restrict attention here to the bosonic fields and treat the fermions in section 3.2.

2.1 Coset variables and transformation

We use the conventions of [3, 8] for the E_{10} commutation relations and for the construction of the dynamics. Therefore, the time-dependent $E_{10}/K(E_{10})$ coset element $\mathcal{V}(t)$ gives rise to Lie algebra elements[4] $\mathcal{Q} \in K(E_{10})$ and $\mathcal{P} \in E_{10} \ominus K(E_{10})$ via the decomposition

$$\partial_t \mathcal{V}\, \mathcal{V}^{-1} = \mathcal{Q} + \mathcal{P}. \tag{2.1}$$

In terms of the generators at low $\mathfrak{sl}(10) = A_9$ levels the 'coset velocity' $\mathcal{P}$ and the 'orthogonal' $K(E_{10})$ gauge connection $\mathcal{Q}$ (which does not enter the coset Lagrangian, see (2.9) below) can be expanded as

$$\begin{aligned} \mathcal{P} &= \frac{1}{2}\overset{(0)}{P}_{ab} S^{ab} + \frac{1}{3!}\overset{(1)}{P}_{a_1a_2a_3} S^{a_1a_2a_3} + \frac{1}{6!}\overset{(2)}{P}_{a_1\ldots a_6} S^{a_1\ldots a_6} \\ &\qquad + \frac{1}{9!}\overset{(3)}{P}_{a_0|a_1\ldots a_8} S^{a_0|a_1\ldots a_8} + \ldots, \end{aligned} \tag{2.2a}$$

$$\begin{aligned} \mathcal{Q} &= \frac{1}{2}\overset{(0)}{Q}_{ab} J^{ab} + \frac{1}{3!}\overset{(1)}{Q}_{a_1a_2a_3} J^{a_1a_2a_3} + \frac{1}{6!}\overset{(2)}{Q}_{a_1\ldots a_6} J^{a_1\ldots a_6} \\ &\qquad + \frac{1}{9!}\overset{(3)}{Q}_{a_0|a_1\ldots a_8} J^{a_0|a_1\ldots a_8} + \ldots. \end{aligned} \tag{2.2b}$$

where the indices $a, b, \ldots = 1, \ldots, 10$ are to be regarded as ('flat') $SO(10)$ vector indices and the bracketed superscripts indicate the A_9 level. The symmetric and anti-symmetric

[4] We use E_{10} (and $K(E_{10})$) to designate both the algebra and the group.

combinations S and J of the E_{10} generators are respectively defined by (on levels $\ell = 0, 1, 2, 3$)

$$S^{ab} = K^a{}_b + K^b{}_a\,, \qquad J^{ab} = K^a{}_b - K^b{}_a\,, \tag{2.3a}$$

$$S^{a_1a_2a_3} = E^{a_1a_2a_3} + F_{a_1a_2a_3}\,, \qquad J^{a_1a_2a_3} = E^{a_1a_2a_3} - F_{a_1a_2a_3}\,, \tag{2.3b}$$

$$S^{a_1\ldots a_6} = E^{a_1\ldots a_6} + F_{a_1\ldots a_6}\,, \qquad J^{a_1\ldots a_6} = E^{a_1\ldots a_6} - F_{a_1\ldots a_6}\,, \tag{2.3c}$$

$$S^{a_0|a_1\ldots a_8} = E^{a_0|a_1\ldots a_8} + F_{a_0|a_1\ldots a_8}\,, \quad J^{a_0|a_1\ldots a_8} = E^{a_0|a_1\ldots a_8} - F_{a_0|a_1\ldots a_8}\,, \tag{2.3d}$$

The elements J generate the maximal compact subgroup $K(E_{10}) \subset E_{10}$ while the elements S span the coset $E_{10} \ominus K(E_{10})$ (which is not a subalgebra); their commutation relations are given in [3, 8].[5]

The coset has the usual non-linear symmetry transformations $\mathcal{V}(t) \to k(t)\mathcal{V}(t)g^{-1}$ with $g \in E_{10}$ a global rotation and $k(t)$ a local gauge transformation, field dependent once a gauge is chosen. Under this transformation one has

$$\mathcal{P} \to k\mathcal{P}k^{-1}\,, \quad \mathcal{Q} \to k\mathcal{Q}k^{-1} + \partial_t k\, k^{-1}\,. \tag{2.4}$$

Note that $\mathcal{P}$ is a $K(E_{10})$-covariant object (coset representation of $K(E_{10})$), while $\mathcal{Q}$ has the typical inhomogeneous transformation law of a gauge connection of $K(E_{10})$. On the components defined in (2.2) this implies for instance the following transformations for infinitesimal $\delta k = \frac{1}{3!}\Lambda^{(1)}_{c_1c_2c_3}J^{c_1c_2c_3} \in K(E_{10})$

$$\delta_{\Lambda^{(1)}} \overset{(0)}{P}_{ab} = \frac{1}{9}\overset{(1)}{\Lambda}_{c_1c_2c_3}\overset{(1)}{P}_{c_1c_2c_3}\delta_{ab} - \overset{(1)}{\Lambda}_{c_1c_2a}\overset{(1)}{P}_{bc_1c_2}, \tag{2.5a}$$

$$\delta_{\Lambda^{(1)}} \overset{(1)}{P}_{a_1a_2a_3} = 3\overset{(1)}{\Lambda}_{ca_1a_2}\overset{(0)}{P}_{a_3c} + \frac{1}{6}\overset{(1)}{\Lambda}_{c_1c_2c_3}\overset{(2)}{P}_{c_1c_2c_3a_1a_2a_3}, \tag{2.5b}$$

$$\delta_{\Lambda^{(1)}} \overset{(2)}{P}_{a_1\ldots a_6} = -20\overset{(1)}{\Lambda}_{a_1a_2a_3}\overset{(1)}{P}_{a_4a_5a_6} + \frac{1}{6}\overset{(1)}{\Lambda}_{c_1c_2c_3}\overset{(3)}{P}_{c_1|c_2c_3a_1\ldots a_6}, \tag{2.5c}$$

$$\delta_{\Lambda^{(1)}} \overset{(3)}{P}_{a_0|a_1\ldots a_8} = -56\big(\overset{(1)}{\Lambda}_{a_0a_1a_2}\overset{(2)}{P}_{a_3\ldots a_8} - \overset{(1)}{\Lambda}_{a_1a_2a_3}\overset{(2)}{P}_{a_4\ldots a_8a_0}\big) + \ldots. \tag{2.5d}$$

In these equations we have employed a notational convention which we will make use of throughout the remainder of this article. Namely, the r.h.s. of the tensor equation is implicitly assumed to be projected onto the same symmetry structure as the l.h.s., that is, the requisite symmetrisations and antisymmetrisations are understood without being written out. For example, the first two lines read in explicit form

$$\delta_{\Lambda^{(1)}} \overset{(0)}{P}_{ab} \equiv \frac{1}{9}\overset{(1)}{\Lambda}_{c_1c_2c_3}\overset{(1)}{P}_{c_1c_2c_3}\delta_{ab} - \overset{(1)}{\Lambda}_{c_1c_2(a}\overset{(1)}{P}_{b)c_1c_2}\,, \tag{2.6a}$$

[5] With an overall minus sign correction in the $[\ell = -3, \ell = 3]$ commutator compared to [3].

$$\delta_{\Lambda^{(1)}} \overset{(1)}{P}_{a_1a_2a_3} \equiv 3\overset{(1)}{\Lambda}_{c[a_1a_2}\overset{(0)}{P}_{a_3]c} + \frac{1}{6}\overset{(1)}{\Lambda}_{c_1c_2c_3}\overset{(2)}{P}_{c_1c_2c_3a_1a_2a_3}\,. \tag{2.6b}$$

For later use we define the $K(E_{10})$ covariant derivative

$$\mathcal{D} \equiv \mathcal{D}_t = \partial_t - \mathcal{Q}\,, \tag{2.7}$$

with the connection term $\mathcal{Q}$ acting in the appropriate representation, e.g. via commutators on $\mathcal{P}$. The level zero generators J^{ab} form an $\mathfrak{so}(10)$ subalgebra of $\mathfrak{sl}(10) \subset E_{10}$ and we will often use the $SO(10)$ covariant derivative

$$D \equiv D_t = \partial_t - \frac{1}{2}\overset{(0)}{Q}_{ab}J^{ab}, \tag{2.8}$$

acting on representations of $SO(10)$; for example, for an $\mathfrak{so}(10)$ vector v_a the covariant derivative evaluates to $Dv_a = \partial_t v_a - Q^{(0)}_{ab}v_b$.

2.2 Equations of motion

The equations of motion of the one-dimensional $E_{10}/K(E_{10})$ σ-model follow from the Lagrangian [1]

$$\mathcal{L} = \frac{1}{4n}\langle \mathcal{P}|\mathcal{P}\rangle \tag{2.9}$$

with the lapse n to ensure invariance under reparametrisations of t. They are given by the geodesic equations

$$\mathcal{D}_t\mathcal{P} = \partial_t\mathcal{P} - [\mathcal{Q},\mathcal{P}] = 0\,, \tag{2.10}$$

and the Hamiltonian constraint

$$\mathcal{H}(\mathcal{P}) \equiv \langle \mathcal{P}|\mathcal{P}\rangle = 0 \tag{2.11}$$

where for convenience we choose the gauge $n = 1$ for the affine parametrisation of the world-line. Imposition of (2.11) requires the coset space geodesic (2.10) to be null.

A major simplification of (2.10) is achieved by adopting the (almost) *triangular gauge*, where $\mathcal{V}$ depends only on the level $\ell \geq 0$ degrees of freedom

$$\mathcal{V}(t) = \mathcal{V}_0(t)\exp\left[\frac{1}{3!}A_{mnp}(t)E^{mnp} + \frac{1}{6!}A_{m_1\ldots m_6}(t)E^{m_1\ldots m_6} + \ldots\right] \tag{2.12}$$

As for instance explained in [1, 3], the first factor on the r.h.s. belongs to the $GL(10)$ subgroup of E_{10}, thus involving only the $\mathfrak{gl}(10)$ generators[6]

$$\mathcal{V}_0(t) \equiv \exp\left(h_m{}^n(t) K^m{}_n\right), \quad e_a{}^m = (e^h)_a{}^m \,. \tag{2.13}$$

One should be careful here not to assign any special transformation properties to $h_m{}^n$ appearing inside the exponential defining $\mathcal{V}_0$, whereas the exponentiated expression, that is $\mathcal{V}_0$ itself, does transform as a zehnbein, i.e. $\mathcal{V}_0 \to k_0 \mathcal{V}_0 g_0^{-1}$ with $k_0 \in SO(10)$ and $g_0 \in GL(10)$. By contrast, the higher level fields appearing in the exponential inside (2.12) do transform as genuine $GL(10)$ tensors after the factor $\mathcal{V}_0$ has been split off. In other words, the indices $m, n, \ldots$ in the second exponent of (2.12) can be thought of as 'world' ($GL(10)$) indices in contrast to the 'flat' ($SO(10)$) indices $a, b, \ldots$ in (2.2). The fields at level one and two, respectively, correspond to the 3-form field of $D = 11$ supergravity, and its magnetic 6-form dual. In this gauge, the 'matrix' $\mathcal{V}(t)$ belongs to a parabolic subgroup of E_{10} which we designate by E_{10}^+. Furthermore, in this gauge, the connection coefficients $Q^{(\ell)}$ appearing in (2.2b) are identified (for $\ell \geq 1$) with the coset coefficients $P^{(\ell)}$ of (2.2a) [3]

$$\overset{(\ell)}{Q} = \overset{(\ell)}{P} \qquad \text{for all } \ell \geq 1 \,. \tag{2.14}$$

The commutators of [3] and the triangular gauge for $\mathcal{V}$ allow us to work out the following expressions for $\mathcal{P}$ and $\mathcal{Q}$ (cf. also [1])

$$\overset{(0)}{Q}_{ab} = e_{m[b}\partial_t e_{a]}{}^m \quad , \quad \overset{(0)}{P}_{ab} = e_{m(b}\partial_t e_{a)}{}^m \,, \tag{2.15a}$$

$$\overset{(1)}{P}_{a_1a_2a_3} = \frac{1}{2} e_{a_1}{}^{m_1} e_{a_2}{}^{m_2} e_{a_3}{}^{m_3} \partial_t A_{m_1m_2m_3} \,, \quad \text{etc.} \tag{2.15b}$$

Here, the matrix $e_m{}^a$ is the inverse of $e_a{}^m$, *viz.* $e_m{}^a e_a{}^n = \delta_m^n$.

The level decomposition allows us to decompose (2.10) into an infinite set of equations, which furthermore can be truncated consistently by setting

$$\overset{(\ell)}{P} = 0 \qquad \text{for } \ell > 3 \,. \tag{2.16}$$

With these gauge choices, and the truncation (2.16), the Hamiltonian and the equations of motion (2.10), respectively, reduce to

$$\langle \mathcal{P}|\mathcal{P}\rangle = \overset{(0)}{P}_{ab}\overset{(0)}{P}_{ab} - \overset{(0)}{P}_{aa}\overset{(0)}{P}_{bb} + \frac{1}{3}\overset{(1)}{P}_{abc}\overset{(1)}{P}_{abc} +$$

[6]The 'dictionary' of section 3.3 associates this coset vielbein to the spatial zehnbein $e_m{}^a(t, \mathbf{x}_0)$ of $D = 11$ supergravity evaluated at a fixed spatial point $\mathbf{x}_0$. Strictly speaking, we should notationally distinguish between the coset zehnbein (considered as a 10-by-10 'submatrix' of $\mathcal{V}(t)$), and the spatial zehnbein of supergravity, but we will refrain from doing so in order not to clutter up the notation.

$$+\frac{1}{4\cdot 5!}\overset{(2)}{P}_{a_1\ldots a_6}\overset{(2)}{P}_{a_1\ldots a_6}+\frac{2}{9!}\overset{(3)}{P}_{a_0|a_1\ldots a_8}\overset{(3)}{P}_{a_0|a_1\ldots a_8}=0 \tag{2.17}$$

and

$$\begin{aligned}
D\overset{(0)}{P}_{ab} &= -\frac{1}{9}\delta_{ab}\overset{(1)}{P}_{c_1c_2c_3}\overset{(1)}{P}_{c_1c_2c_3}+\overset{(1)}{P}_{ac_1c_2}\overset{(1)}{P}_{bc_1c_2}\\
&\quad -\frac{4}{3\cdot 6!}\delta_{ab}\overset{(2)}{P}_{c_1\ldots c_6}\overset{(2)}{P}_{c_1\ldots c_6}+\frac{2}{5!}\overset{(2)}{P}_{ac_1\ldots c_5}\overset{(2)}{P}_{bc_1\ldots c_5}\\
&\quad -\frac{2}{9!}\delta_{ab}\overset{(3)}{P}_{c_0|c_1\ldots c_8}\overset{(3)}{P}_{c_0|c_1\ldots c_8}+\frac{16}{9!}\overset{(3)}{P}_{c_0|ac_1\ldots c_7}\overset{(3)}{P}_{c_0|bc_1\ldots c_7}\\
&\quad +\frac{2}{9!}\overset{(3)}{P}_{a|c_1\ldots c_8}\overset{(3)}{P}_{b|c_1\ldots c_8}, \qquad (2.18a)\\
D\overset{(1)}{P}_{a_1a_2a_3} &= -3\overset{(1)}{P}_{ca_1a_2}\overset{(0)}{P}_{ca_3}-\frac{1}{3}\overset{(1)}{P}_{c_1c_2c_3}\overset{(2)}{P}_{c_1c_2c_3a_1a_2a_3}\\
&\quad -\frac{4}{6!}\overset{(2)}{P}_{c_1\ldots c_6}\overset{(3)}{P}_{c_1|c_2\ldots c_6a_1a_2a_3} \qquad (2.18b)\\
D\overset{(2)}{P}_{a_1\ldots a_6} &= 6\overset{(2)}{P}_{ca_1\ldots a_5}\overset{(0)}{P}_{ca_6}-\frac{1}{3}\overset{(3)}{P}_{c_1|c_2c_3a_1\ldots a_6}\overset{(1)}{P}_{c_1c_2c_3} \qquad (2.18c)\\
D\overset{(3)}{P}_{a_0|a_1\ldots a_8} &= -\overset{(3)}{P}_{c|a_1\ldots a_8}\overset{(0)}{P}_{ca_0}+8\overset{(3)}{P}_{a_0|ca_1\ldots a_7}\overset{(0)}{P}_{ca_8} \qquad (2.18d)
\end{aligned}$$

Here, it is again understood that the r.h.s. is symmetrised in accordance with the symmetries on the l.h.s. of these equations, as explained after (2.5). For the level $\ell = 3$ term in (2.18d) this implies that the r.h.s. vanishes if antisymmetrised over all nine free indices, as required by the Young symmetries of the level three generator. For clarity, we write out the contributions involving $P^{(0)}$ explicitly on the r.h.s., unlike in [3] where these terms were absorbed into the derivative operator on the l.h.s. As we will see below, however, the constraint analysis is simplified considerably by re-absorbing these contributions into the derivative of a suitably redefined l.h.s.

3 Constraints

We next show that the bosonic dynamical equation $\mathcal{D}_t\mathcal{P} = 0$ (truncated at levels $\ell \leq 3$) can be supplemented by certain constraints $\mathcal{C}$ quadratic in the $\mathcal{P}$, such that the equations $\mathcal{C} \approx 0$ are all compatible with the dynamics of the $E_{10}/K(E_{10})$ σ-model. Moreover, these constraints are in one-to-one correspondence with the canonical constraints of supergravity, as we shall see in the next section. Compatibility of constraints with the equations of

motions requires that the time derivatives of the constraints vanish weakly (i.e. modulo the constraints) so that the motion preserves the constraint surface determined by $\mathcal{C}(\mathcal{P}) = 0$. In contrast to the Hamiltonian constraint (2.11) which is an E_{10} singlet, the constraints $\mathcal{C}(\mathcal{P})$ possess a more intricate structure with regard to E_{10}, which we shall now study.

3.1 Bosonic constraints and weak conservation

Motivated by the knowledge of the structure of the supergravity constraints and of their 'translation' in coset variables [16, 8], we wish to study, purely from the viewpoint of the coset dynamics, the possibility of imposing coset constraints $\mathcal{C} \approx 0$ for a 'constraint multiplet' of the general form

$$\overset{(3)}{\mathcal{C}}_{a_1\ldots a_9} = \overset{(0)}{P}_{ca_1}\overset{(3)}{P}_{c|a_2\ldots a_9} + \alpha\overset{(1)}{P}_{a_1a_2a_3}\overset{(2)}{P}_{a_4\ldots a_9}, \tag{3.1a}$$

$$\overset{(4)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1a_2} = \overset{(1)}{P}_{a_1b_1b_2}\overset{(3)}{P}_{a_2|b_3\ldots b_{10}} + \beta\overset{(2)}{P}_{a_1b_1\ldots b_5}\overset{(2)}{P}_{a_2b_6\ldots b_{10}}, \tag{3.1b}$$

$$\overset{(5)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1\ldots a_5} = \overset{(2)}{P}_{a_1\ldots a_4b_1b_2}\overset{(3)}{P}_{a_5|b_3\ldots b_{10}}, \tag{3.1c}$$

$$\overset{(6)}{\mathcal{C}}_{b_1\ldots b_{10}||a_0|a_1\ldots a_7} = \overset{(3)}{P}_{a_0|b_1\ldots b_8}\overset{(3)}{P}_{b_9|b_{10}a_1\ldots a_7}. \tag{3.1d}$$

Let us clarify once more what various antisymmetrisations which are understood here: for instance, all expressions are antisymmetric in the 10-tuple of indices[7] $[b_1 \ldots b_{10}]$, as well as in the indices $a_1, a_2, \ldots$, whereas the index a_0 is to be treated separately (of course, the blocks of ten antisymmetric b indices could be eliminated by means of an ϵ-symbol, but leaving them explicit makes some of the structure more transparent). Thus, to give one more example, the first equation in (3.1) should be read as follows,

$$\overset{(0)}{P}_{ca_1}\overset{(3)}{P}_{c|a_2\ldots a_9} + \alpha\overset{(1)}{P}_{a_1a_2a_3}\overset{(2)}{P}_{a_4\ldots a_9} \equiv \overset{(3)}{P}_{c|[a_2\ldots a_9}\overset{(0)}{P}_{a_1]c} + \alpha\overset{(1)}{P}_{[a_1a_2a_3}\overset{(2)}{P}_{a_4\ldots a_9]}\,. \tag{3.2}$$

The net effect of this prescription is that the l.h.s. and the r.h.s. of all equations have the same symmetries, as it should be. Note also that although $\mathcal{C}^{(6)}$ could *a priori* contain *two* irreducible Young tableaux (of $SL(10)$) in a specific linear combination, the definition of $\mathcal{C}^{(6)}$, together with the $\ell = 3$ irreducibility condition $P^{(3)[a_0|a_1\ldots a_8]} = 0$ implies the algebraic constraint $\mathcal{C}^{(6)}_{b_1\ldots b_{10}||[a_0|a_1\ldots a_7]} = 0$. The ansatz (3.1) is motivated by previous studies of the supersymmetry constraint in [8].

As already mentioned in the introduction the tensor structure of the flat indices appearing in (3.1) is identical [8] with the one of the lowest $SL(10)$ levels appearing in the

[7] The double lines $||$ in the subscripts of the constraints $\mathcal{C}^{(4)}$, $\mathcal{C}^{(5)}$ and $\mathcal{C}^{(6)}$ serve as a mnemonic to separate the 10-tuples $[b_1 \ldots b_{10}]$ from the other $SO(10)$ indices.

[8] Except for the algebraic restriction on $\mathcal{C}^{(6)}$ just mentioned.

$L(\Lambda_1)$ representation of E_{10}, with 3ℓ indices at each A_9 level ℓ [28]. In the form given in (3.1) this is not entirely obvious: we must 'remove' an ϵ-symbol with ten antisymmetric indices, counting the 'missing' index in $\mathcal{C}^{(3)}$ as an extra (really: upper) index. In this way, the index structure of the constraints becomes

$$\overset{(3)}{\mathcal{C}}_a\,,\ \overset{(4)}{\mathcal{C}}_{a_1a_2}\,,\ \overset{(5)}{\mathcal{C}}_{a_1\ldots a_5}\,,\ \overset{(6)}{\mathcal{C}}_{a_0|a_1\ldots a_7}\,,\ldots \tag{3.3}$$

which corresponds to the well known 'central charge representations' of maximal supergravity. The above pattern illustrates that, at least for the low level representations displayed above, the relevant $SL(10)$ Young tableaux are obtained, up to appropriate ϵ tensors, from the low level Young tableaux of the adjoint of E_{10} by removing one box in all possible ways; so, for instance, the 3-form $[a_1a_2a_3]$ at level one becomes a 2-form $[a_1a_2]$, and so on. However, at higher levels there will appear extra representations. Similar representations in the context of very-extended algebras have been studied in [11, 28]. The reason for introducing the surplus antisymmetric indices in (3.1) will be explained in section 4.2, cf. remarks after (4.15).

We find that demanding weak conservation of the constraints above along the coset motion, i.e. using the equations of motion (2.18), *uniquely fixes* the numerical values of the coefficients α, β in (3.1) to be:

$$\alpha = 28\,,\quad \beta = \frac{21}{5}. \tag{3.4}$$

With these special values, the result for the time derivative of the constraints is, using the $SO(10)$-covariant derivative D and setting $n = 1$,

$$\begin{aligned}
D\overset{(3)}{\mathcal{C}}_{a_1\ldots a_9} &= -9\overset{(0)}{P}_{ca_1}\overset{(3)}{\mathcal{C}}_{ca_2\ldots a_9} + 10\overset{(1)}{P}_{c_1c_2c_3}\overset{(4)}{\mathcal{C}}_{a_1\ldots a_9c_1||c_2c_3} \qquad (3.5a)\\
&\quad -\frac{7}{36}\overset{(2)}{P}_{c_1\ldots c_6}\overset{(5)}{\mathcal{C}}_{a_1\ldots a_9c_1||c_2\ldots c_6} + \frac{160}{9!}\overset{(3)}{P}_{c_0|c_1\ldots c_8}\overset{(6)}{\mathcal{C}}_{a_1\ldots a_9c_1||c_0|c_2\ldots c_8}\,,\\
D\overset{(4)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1a_2} &= -10\overset{(0)}{P}_{cb_1}\overset{(4)}{\mathcal{C}}_{cb_2\ldots b_{10}||a_1a_2} - 2\overset{(0)}{P}_{ca_1}\overset{(4)}{\mathcal{C}}_{b_1\ldots b_{10}||ca_2} \qquad (3.5b)\\
&\quad -\frac{5}{6}\overset{(1)}{P}_{c_1c_2c_3}\overset{(5)}{\mathcal{C}}_{b_1\ldots b_{10}||c_1c_2c_3a_1a_2} + \frac{3}{5!}\overset{(2)}{P}_{c_1\ldots c_6}\overset{(6)}{\mathcal{C}}_{b_1\ldots b_{10}||c_1|c_2\ldots c_6a_1a_2}\,,\\
D\overset{(5)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1\ldots a_5} &= -10\overset{(0)}{P}_{cb_1}\overset{(5)}{\mathcal{C}}_{cb_2\ldots b_{10}||a_1\ldots a_5} - 5\overset{(0)}{P}_{ca_1}\overset{(5)}{\mathcal{C}}_{b_1\ldots b_{10}||ca_2\ldots a_5}\\
&\quad -\frac{2}{15}\overset{(1)}{P}_{c_1c_2c_3}\overset{(6)}{\mathcal{C}}_{b_1\ldots b_{10}||c_1|c_2c_3a_1\ldots a_5}\,, \qquad (3.5c)\\
D\overset{(6)}{\mathcal{C}}_{b_1\ldots b_{10}||a_0|a_1\ldots a_7} &= -10\overset{(0)}{P}_{cb_1}\overset{(6)}{\mathcal{C}}_{cb_2\ldots b_{10}||a_0|a_1\ldots a_7} - \overset{(0)}{P}_{ca_0}\overset{(6)}{\mathcal{C}}_{b_1\ldots b_{10}||c|a_1\ldots a_7}
\end{aligned}$$

$$-7\overset{(0)}{P}_{ca_1}\overset{(6)}{\mathcal{C}}_{b_1\ldots b_{10}||a_0|ca_2\ldots a_7}. \tag{3.5d}$$

again with all required symmetrisations implied. Because the time derivatives of the constraints are again proportional to the constraints, the constraints are *weakly conserved* in this truncation, hence the constraints can be imposed to yield a consistent restriction of the dynamics as claimed.

These weak conservation equations exhibit two remarkable structures: (i) the universal appearance of the negative of the zero-level coset velocity $-P^{(0)}_{ab}$ acting (by being contracted) on the r.h.s., on each index of $\mathcal{C}^{(\ell)}_{\ldots b\ldots}$, and (ii) a *triangular structure* of the terms on the r.h.s. involving the $\mathcal{C}^{(\ell')}$'s with ℓ' differing from the level ℓ appearing on the l.h.s.. This triangular structure is reminiscent of a highest (or lowest) weight representation in that the time derivatives $D\mathcal{C}^{(\ell)}$ involve only constraints $\mathcal{C}^{(\ell')}$ with levels $\ell' \geq \ell$, multiplied by $P^{(\ell'-\ell)}$.

We shall show below how these two remarkable structural elements of the above weak-conservation equations are connected to a Sugawara-like reformulation of the constraints. For the time being, we only note that the conservation of the constraints implies that one can consistently constrain null geodesic motion on $E_{10}/K(E_{10})$ beyond the null geodesic constraint, at least in the truncation (2.16) and in triangular gauge. Note also that the Hamiltonian constraint is not required for the above conservation equations to hold.

3.2 The supersymmetry constraint

The results of the preceding sections can be generalised to the case where spin degrees of freedom are added, supplementing the bosonic constraints by a supersymmetry constraint corresponding to local supersymmetry. The inclusion of the fermionic fields of supergravity has already been studied from an E_{10} point of view in [5, 6, 29, 8].

$K(E_{10})$ possesses an unfaithful spinor representation ψ_a of dimension 320 which transforms as a vector-spinor under $SO(10) \subset K(E_{10})$ [5, 6]. The $K(E_{10})$ covariant equation of motion for this representation is the $K(E_{10})$ Dirac equation

$$\begin{aligned}
\mathcal{D}_t\psi_a \;=\; & D_t\psi_a - \frac{1}{12}\overset{(1)}{Q}_{b_1b_2b_3}\Gamma^{b_1b_2b_3}\psi_a - \frac{2}{3}\overset{(1)}{Q}_{ab_1b_2}\Gamma^{b_1}\psi^{b_2} + \frac{1}{6}\overset{(1)}{Q}_{b_1b_2b_3}\Gamma_a{}^{b_1b_2}\psi^{b_3} \\
& - \frac{1}{2\cdot 6!}\overset{(2)}{Q}_{b_1\ldots b_6}\Gamma^{b_1\ldots b_6}\psi_a - \frac{1}{180}\overset{(2)}{Q}_{b_1\ldots b_6}\Gamma_a{}^{b_1\ldots b_5}\psi^{b_6} + \frac{1}{72}\overset{(2)}{Q}_{ab_1\ldots b_5}\Gamma^{b_1\ldots b_4}\psi^{b_5} \\
& - \frac{2}{3\cdot 8!}\left(\overset{(3)}{Q}_{b_0|b_1\ldots b_8}\Gamma_a{}^{b_1\ldots b_8}\psi^{b_0} + 8\overset{(3)}{Q}_{a|b_1\ldots b_8}\Gamma^{b_1\ldots b_7}\psi^{b_8}\right. \\
& \qquad \left. + 2\overset{(3)}{Q}_{c|cb_1\ldots b_7}\Gamma^{b_1\ldots b_7}\psi_a - 28\overset{(3)}{Q}_{c|cb_1\ldots b_7}\Gamma_a{}^{b_1\ldots b_6}\psi^{b_7}\right).
\end{aligned} \tag{3.6}$$

The Γ-matrices are real (32×32)-matrices of $SO(10)$. In the triangular gauge (2.14), which we use throughout, we can replace the connection coefficients $Q^{(\ell)}$ appearing in this $K(E_{10})$ covariant derivative by the coset coefficients $P^{(\ell)}$. Using the dictionary (3.12) one can then verify that – modulo higher order gradients, as always – the above equation coincides with the Rarita Schwinger equation of maximal supergravity [5, 6].

Because one can supplement the bosonic coset dynamics (2.18) by the weakly conserved constraints (3.1) it is natural to ask if similarly a supersymmetry constraint exists in the combined bosonic and fermionic system which is weakly conserved. A candidate constraint was described in [8] where it was derived from supergravity. It has the form $\mathfrak{S} \approx 0$, with

$$\begin{aligned}\Gamma_0\mathfrak{S} &= \frac{1}{2}\left(\overset{(0)}{P}_{ab}\Gamma^a - \overset{(0)}{P}_{cc}\Gamma_b\right)\psi^b + \frac{1}{4}\overset{(1)}{P}_{c_1c_2c_3}\Gamma^{c_1c_2}\psi^{c_3} - \frac{1}{2\cdot 5!}\overset{(2)}{P}_{c_1\ldots c_6}\Gamma^{c_1\ldots c_5}\psi^{c_6} \\ &\quad + \frac{1}{6\cdot 6!}\overset{(3)}{P}_{b|bc_1\ldots c_7}\Gamma^{c_1\ldots c_6}\psi^{c_7} - \frac{1}{3\cdot 8!}\overset{(3)}{P}_{b|c_1\ldots c_8}\Gamma^{c_1\ldots c_8}\psi^b\,. \end{aligned} \tag{3.7}$$

This expression was shown in [8] to coincide with the appropriately truncated supersymmetry constraint of supergravity upon use of (3.12) (and, in fact, can be used to re-derive the dictionary).

We can work out the time derivative of (3.7) using solely the $K(E_{10})$ covariant Dirac equation (3.6) to obtain

$$\begin{aligned}D_t\mathfrak{S} &= \left[\frac{1}{12}\overset{(1)}{P}_{a_1a_2a_3}\Gamma^{a_1a_2a_3} + \frac{1}{2\cdot 6!}\overset{(2)}{P}_{a_1\ldots a_6}\Gamma^{a_1\ldots a_6} + \frac{4}{3\cdot 8!}\overset{(3)}{P}_{b|ba_1\ldots a_7}\Gamma^{a_1\ldots a_7}\right]\cdot\mathfrak{S} \\ &\quad + \frac{3}{8!}\overset{(3)}{\mathcal{C}}_{a_1\ldots a_9}\Gamma^0\Gamma^{a_1\ldots a_8}\psi^{a_9} - \frac{1}{8!}\overset{(4)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1a_2}\Gamma^0\Gamma^{b_1\ldots b_{10}}\Gamma^{a_1}\psi^{a_2} \\ &\quad - \frac{5}{3\cdot 8!}\overset{(4)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1a_2}\Gamma^0\Gamma^{b_1\ldots b_9}\Gamma^{a_1a_2}\psi^{b_{10}} + \frac{5}{8\cdot 9!}\overset{(5)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1\ldots a_5}\Gamma^0\Gamma^{b_1\ldots b_9}\Gamma^{a_1\ldots a_5}\psi^{b_{10}} \\ &\quad - \frac{15}{16\cdot 9!}\overset{(5)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1\ldots a_5}\Gamma^0\Gamma^{b_1\ldots b_{10}}\Gamma^{a_1\ldots a_4}\psi^{a_5} + \ldots \end{aligned} \tag{3.8}$$

Consequently, the supersymmetry constraint is also conserved on the constraint surface where one imposes both the bosonic constraints (3.1) and the supersymmetry constraint (3.7) itself. This result does not depend on the Hamiltonian constraint, and thus provides no extra consistency checks on the latter.

Remarkably, the above conservation equation can be recast in terms of a $K(E_{10})$ covariant derivative acting on $\mathfrak{S}$, suggesting that the supersymmetry constraint behaves as a $K(E_{10})$ Dirac spinor. This is achieved by shifting the first three terms on the r.h.s. of

(3.8) to the l.h.s. In triangular gauge we can then replace $P^{(\ell)}$ by $Q^{(\ell)}$ to obtain

$$\begin{aligned}\mathcal{D}_t\mathfrak{S} &\equiv \left[D_t - \frac{1}{12}\overset{(1)}{Q}_{a_1a_2a_3}\Gamma^{a_1a_2a_3} - \frac{1}{2\cdot 6!}\overset{(2)}{Q}_{a_1\ldots a_6}\Gamma^{a_1\ldots a_6} - \frac{4}{3\cdot 8!}\overset{(3)}{Q}_{b|ba_1\ldots a_7}\Gamma^{a_1\ldots a_7} + \ldots\right]\mathfrak{S} \\ &= \frac{3}{8!}\overset{(3)}{\mathcal{C}}_{a_1\ldots a_9}\Gamma^0\Gamma^{a_1\ldots a_8}\psi^{a_9} - \frac{1}{8!}\overset{(4)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1a_2}\Gamma^0\Gamma^{b_1\ldots b_{10}}\Gamma^{a_1}\psi^{a_2} \\ &\quad -\frac{5}{3\cdot 8!}\overset{(4)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1a_2}\Gamma^0\Gamma^{b_1\ldots b_9}\Gamma^{a_1a_2}\psi^{b_{10}} + \frac{5}{8\cdot 9!}\overset{(5)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1\ldots a_5}\Gamma^0\Gamma^{b_1\ldots b_9}\Gamma^{a_1\ldots a_5}\psi^{b_{10}} \\ &\quad -\frac{15}{16\cdot 9!}\overset{(5)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1\ldots a_5}\Gamma^0\Gamma^{b_1\ldots b_{10}}\Gamma^{a_1\ldots a_4}\psi^{a_5} + \ldots\,, \end{aligned} \tag{3.9}$$

where the dots indicate possible higher level contributions. In this respect the supersymmetry constraint thus behaves like a Dirac spinor representation of $K(E_{10})$ (defined in [29, 5, 8]). However, as already noted in [8] the supersymmetry constraint does not transform properly under $K(E_{10})$ (this observation is analogous to the one which will be made in section 4.3 for the bosonic constraints), so the significance of the appearance of the $K(E_{10})$ covariant derivative in the above equation remains to be clarified.

3.3 Translation to supergravity

In the previous section we have worked entirely within the coset model, except for the fact that we *motivated* the general algebraic structure (without using precise information about the numerical coefficients α and β in (3.1)) of possible constraints by previous knowledge from the supergravity side of the coset/supergravity correspondence. In this section we shall use the 'dictionary' relating the unconstrained σ-model to the dynamical equations of supergravity [1] to compare the constraints (3.1) to the known canonical constraints of supergravity in detail. It is therefore gratifying that we shall re-obtain the uniquely determined coset values (3.4) by matching the supergravity expressions to the coset ones in this way.

All bosonic equations are displayed in table 1. Both the dynamical equations and the constraint equations can be obtained by the standard ADM procedure from the field equations of $D = 11$ supergravity [17], namely the Einstein equations $\mathcal{G}_{AB}$ and the matter (4-form field strength) equations $\mathcal{M}^{BCD}$; in the present conventions the latter read

$$\mathcal{G}_{AB} = R_{AB} - \frac{1}{3}F_{ACDE}F_B{}^{CDE} + \frac{1}{36}\eta_{AB}F_{CDEF}F^{CDEF}\,, \tag{3.10a}$$

$$\mathcal{M}^{BCD} = D_A F^{ABCD} + \frac{1}{576}\epsilon^{BCDE_1\ldots E_4F_1\ldots F_4}F_{E_1\ldots E_4}F_{F_1\ldots F_4}\,, \tag{3.10b}$$

where indices are flat space-time indices $A, B = 0, 1, \ldots, 10$.

Supergravity	Coset	Name
$\mathcal{G}_{ab} = 0$	$\overset{(0)}{\mathcal{D}P}_{ab} = 0$	Einstein dynamical eq.
$\mathcal{M}^{a_1a_2a_3} = 0$	$\overset{(1)}{\mathcal{D}P}_{a_1a_2a_3} = 0$	Matter dynamical eq.
$D_{[0}F_{a_1\ldots a_4]} = 0$	$\epsilon_{a_1\ldots a_4b_1\ldots b_6}\overset{(2)}{\mathcal{D}P}_{b_1\ldots b_6} = 0$	F-Bianchi I
$R_{[0a\,b]c} = 0$	$\epsilon_{bcd_1\ldots d_8}\overset{(3)}{\mathcal{D}P}_{a\|d_1\ldots d_8} = 0$	R-Bianchi I
$\mathcal{G}_{00} = 0$	$\langle\mathcal{P}\|\mathcal{P}\rangle = 0$	Hamiltonian constraint
$\mathcal{G}_{0a} = 0$	$\epsilon_{ac_1\ldots c_9}\overset{(3)}{\mathcal{C}}_{c_1\ldots c_9} = 0$	Momentum constraint
$\mathcal{M}^{0a_1a_2} = 0$	$\epsilon_{b_1\ldots b_{10}}\overset{(4)}{\mathcal{C}}_{b_1\ldots b_{10}\|\|a_1a_2} = 0$	Gauss constraint
$D_{[c_1}F_{c_2\ldots c_5]} = 0$	$\epsilon_{b_1\ldots b_{10}}\epsilon_{a_1\ldots a_5c_1\ldots c_5}\overset{(5)}{\mathcal{C}}_{b_1\ldots b_{10}\|\|a_1\ldots a_5} = 0$	F-Bianchi II
$R_{[c_1c_2\,c_3]a_0} = 0$	$\epsilon_{b_1\ldots b_{10}}\epsilon_{a_1\ldots a_7c_1c_2c_3}\overset{(6)}{\mathcal{C}}_{b_1\ldots b_{10}\|\|a_0\|a_1\ldots a_7} = 0$	R-Bianchi II

Table 1: *Complete list of all (bosonic) coset equations and their corresponding (bosonic) supergravity equations.*

In order to make the comparison of the supergravity equations to the coset equations we need to gauge-fix and truncate the supergravity model. More specifically, using the conventions of ref. [8], the 'dictionary' is specified by assuming a zero-shift gauge of the vielbein, *viz.*

$$E_M{}^A = \begin{pmatrix} N & 0 \\ 0 & e_m{}^a \end{pmatrix} \;; \tag{3.11}$$

we will write $e \equiv \det(e_m{}^a)$ for the determinant of the spatial zehnbein. All indices $a, b, \ldots$ here and in (3.12) below are flat $SO(10)$ indices. Furthermore, all supergravity fields are evaluated at a fixed spatial point $\mathbf{x}_0$, and are truncated by setting spatial frame derivatives of the spin connection, the field strengths and the lapse to zero: $\partial_a\omega_{b\,cd} = \partial_aF_{0bcd} = \partial_aF_{bcde} = \partial_aN \equiv 0$. The coefficients of anholonomy $\Omega_{ab\,c}$ are assumed to be tracefree, i.e. $\Omega_{ab\,b} = 0$. The dictionary is then given by

$$n(t) \;\longleftrightarrow\; Ne^{-1}(t, \mathbf{x}_0)\,, \tag{3.12a}$$

$$\overset{(0)}{Q}_{ab}(t) \;\longleftrightarrow\; -N\omega_{0\,ab}(t, \mathbf{x}_0) \equiv -e_{[a}{}^m\partial_t e_{m|b]}(t, \mathbf{x}_0)\,, \tag{3.12b}$$

$$\overset{(0)}{P}_{ab}(t) \;\longleftrightarrow\; -N\omega_{a\,b0}(t, \mathbf{x}_0) \equiv -e_{(a}{}^m\partial_t e_{m|b)}(t, \mathbf{x}_0)\,, \tag{3.12c}$$

$$\overset{(1)}{P}_{a_1a_2a_3}(t) \quad \longleftrightarrow \quad N F_{0a_1a_2a_3}(t,\mathbf{x}_0)\,, \tag{3.12d}$$

$$\overset{(2)}{P}_{a_1\ldots a_6}(t) \quad \longleftrightarrow \quad -\frac{1}{4!}N\epsilon_{a_1\ldots a_6 b_1\ldots b_4}F_{b_1\ldots b_4}(t,\mathbf{x}_0)\,, \tag{3.12e}$$

$$\overset{(3)}{P}_{a_0|a_1\ldots a_8}(t) \quad \longleftrightarrow \quad \frac{3}{4}N\epsilon_{a_1\ldots a_8 b_1 b_2}\tilde{\Omega}_{b_1 b_2\, a_0}(t,\mathbf{x}_0)\,, \tag{3.12f}$$

where $a, b, \ldots$ are now to be interpreted as ***flat*** spatial indices in ten spatial dimensions.

Substituting the above expressions into the constraints (3.1) with the values (3.4), and contracting with an ϵ-tensor we arrive at

$$N^{-2}\epsilon_{a_1\ldots a_9\, a}\overset{(3)}{\mathcal{C}}_{a_1\ldots a_9} = \frac{1}{2}\,8!\left(3\tilde{\Omega}_{ab\,c}\omega_{b\,c0} + F_{ab_1b_2b_3}F_{0b_1b_2b_3}\right), \tag{3.13a}$$

$$N^{-2}\epsilon_{b_1\ldots b_{10}}\overset{(4)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1a_2} = \frac{3}{2}\,8!\Bigg(F_{0a_1b_2b_2}\tilde{\Omega}_{b_1b_2\,a_2} + \frac{1}{576}\epsilon_{a_1a_2b_1\ldots b_4c_1\ldots c_4}F_{b_1\ldots b_4}F_{c_1\ldots c_4}\Bigg), \tag{3.13b}$$

$$N^{-2}\epsilon_{b_1\ldots b_{10}}\epsilon_{a_1\ldots a_5c_1\ldots c_5}\overset{(5)}{\mathcal{C}}_{b_1\ldots b_{10}||a_1\ldots a_5} = -6!\cdot 8!\;\tilde{\Omega}_{c_1c_2\,d}F_{dc_3c_4c_5}, \tag{3.13c}$$

$$N^{-2}\epsilon_{b_1\ldots b_{10}}\epsilon_{a_1\ldots a_7c_1c_2c_3}\overset{(6)}{\mathcal{C}}_{b_1\ldots b_{10}||a_0|a_1\ldots a_7} = 9\cdot 7!\cdot 7!\cdot 3!\;\tilde{\Omega}_{c_1c_2\,d}\tilde{\Omega}_{c_3d\,a_0}. \tag{3.13d}$$

These expressions correspond to the truncated versions of the supergravity constraints with the truncation as specified above. In the order given in (3.13) the supergravity constraint equations are: the momentum (diffeomorphism) constraint, the Gauss constraint, the F-Bianchi constraint and the R-Bianchi constraint (cyclic identity $R_{[c_1c_2\,c_3]a_0} = 0$). The truncation here amounts to ignoring spatial gradients of the spin connection and field strength terms (for instance, the full momentum constraint would have an extra term $\propto \partial_b\omega_{b\,a0}$, and the Gauss constraint an extra $\partial_c F_{0abc}$). It is a non-trivial fact that the same numerical values (3.4) for α and β ensure the weak conservation of the constraints both w.r.t. the coset dynamics and the supergravity one, because the two Hamiltonians do differ at level 3 (even within the truncation we use on both sides) by a term that could have modified the weak conservation condition (but did not). [See [3] for the precise mismatches in $P^{(3)}$ terms.]

The equations given in table 1 exhaust all bosonic equations of the $D = 11$ supergravity system and we have found appropriate E_{10} counterparts in the present truncation.[9]

[9]The Riemann Bianchi components $R_{[0a\,b]0}$ and $R_{[ab\,c]0}$ vanish identically in our truncation. In the full gravity theory the relation $R_{[abc]0} = -R_{[0a\,b]c}$ holds which seems to be inconsistent with R-Bianchi I. The resolution is that such a relation no longer holds in the truncation appropriate for the σ-model (E_{10} does not know about the Riemann tensor).

4 A Sugawara-like construction for E_{10}?

In this section we investigate in more detail the structure and properties of the bosonic constraints (3.1) and show that they can be equivalently expressed in a Sugawara-like form $\mathcal{J} \otimes \mathcal{J}$ in terms of the E_{10} Noether current $\mathcal{J}$. In the level-3 truncation we will see that the constraints, when written in this form, transform covariantly under a Borel subgroup $E_{10}^{+} \subset E_{10}$. In the last subsection, we will establish the link with the more familiar affine Sugawara construction by considering the embedding $E_9 \subset E_{10}$.

4.1 The E_{10} Noether Current

By Noether's theorem, the E_{10} global symmetry of the coset action implies the existence of an infinite number of exactly conserved quantities, *viz.* [10]

$$\mathcal{J} = n^{-1}\mathcal{V}^{-1}\mathcal{P}\mathcal{V}\,. \tag{4.1}$$

Henceforth (as elsewhere in this paper) we will use the gauge $n = 1$. The current $\mathcal{J}$ takes values in the Lie algebra of E_{10}, and is time-independent, that is, the σ-model equations of motion (2.10) are equivalent to current conservation $\partial_t \mathcal{J} = 0$. Expanding the current according to level and making use of the triangular gauge (2.12) for $\mathcal{V}(t)$, it is straightforward to see that the level truncation condition (2.16) is equivalent to

$$\mathcal{J}^{(\ell)} = 0 \qquad \text{for } \ell = -4, -5, -6, \ldots \tag{4.2}$$

Consequently, we have the expansion

$$\begin{aligned} \mathcal{J} \;=\; & \frac{1}{9!}\overset{(-3)}{J}{}^{m_0|m_1\ldots m_8} F_{m_0|m_1\ldots m_8} + \frac{1}{6!}\overset{(-2)}{J}{}^{m_1\ldots m_6} F_{m_1\ldots m_6} + \frac{1}{3!}\overset{(-1)}{J}{}^{mnp} F_{mnp} + \\ & + \overset{(0)}{J}{}^{n}{}_{m} K^{m}{}_{n} + \frac{1}{3!}\overset{(1)}{J}{}_{mnp} E^{mnp} + \frac{1}{6!}\overset{(2)}{J}{}_{m_1\ldots m_6} E^{m_1\ldots m_6} + \ldots \end{aligned} \tag{4.3}$$

where the ellipses on the right stand for infinitely many non-vanishing positive-level components of $\mathcal{J}$. Expressing the current components in terms of (contravariant) velocities and fields, we obtain at the lowest levels

$$\overset{(-3)}{J}{}^{m_0|m_1\ldots m_8} = P^{(3)m_0|m_1\ldots m_8}\,, \tag{4.4a}$$

$$\overset{(-2)}{J}{}^{m_1\ldots m_6} = P^{(2)m_1\ldots m_6} + \frac{1}{3!}A_{pqr}P^{(3)p|qrm_1\ldots m_6}\,, \tag{4.4b}$$

[10] By abuse of language, we usually refer to $\mathcal{J}$ as the '*(conserved) current*', although one should more properly speak of a conserved charge.

$$\overset{(-1)}{J}{}^{mnp} = P^{(1)mnp} + \frac{1}{3!}A_{rst}P^{(2)rstmnp} + \\ + \left(\frac{2}{3}A_{r_1\dots r_6} + \frac{1}{72}A_{r_1r_2r_3}A_{r_4r_5r_6}\right)P^{(3)r_1|r_2\dots r_6mnp}\,. \tag{4.4c}$$

Three important features here should be noted:

- $\mathcal{J}$ is an E_{10} object, transforming as $\mathcal{J} \to \mathcal{J}' = g\mathcal{J}g^{-1}$ under rigid E_{10} transformations $g \in E_{10}$. This implies in particular that all indices in eq. (4.3) are $GL(10)$ ('world') indices, which are covariant or contravariant according to their position, as indicated in the above formula.

- $\mathcal{J}$ is manifestly inert under $K(E_{10})$. This means that the truncation condition (4.2) is *gauge invariant*, hence does *not* rely on any particular choice of gauge (such as the triangular gauge). In contrast, the truncation condition (2.16) on $\mathcal{P}$ is not gauge invariant.

- Unlike the velocities $P^{(\ell)}$, of which there are only four non-vanishing components (for $\ell = 0, 1, 2, 3$) with the truncation (2.16), there are *infinitely many* non-vanishing components $J^{(\ell)}$ at positive level. As shown in [14] and explicitly exhibited in (4.4) the most negative level component of $\mathcal{J}$ is purely *velocity*- (or *momentum*-)like, while the positive level components contain an increasing dependence on the coset coordinates (or *positions*) $= \{A_{mnp}, A_{m_1\dots m_6}, A_{m_0|m_1\dots m_8}, \dots\}$. An Euclidean-group analog of this situation would be to consider geodesic motion on Euclidean space: the conserved quantities would be (p_i, L_{ij}), where the linear momentum p_i is pure velocity, whereas the angular momentum L_{ij} involves both velocities and positions.

Clearly, the truncation condition (4.2) is preserved only by the parabolic subgroup $E_{10}^+ \subset E_{10}$ which is generated by the level $\ell \geq 0$ generators of E_{10}, that is, by $\mathfrak{gl}(10)$ and the positive level generators $E^{mnp}, E^{m_1\dots m_6}, \dots$. This is the part of E_{10} which transforms the coordinates, but leaves unchanged the coset velocoities (or momenta). This property of E_{10}^+ is one of the reasons why the (presently known) coset constraints will transform only under E_{10}^+; indeed, any negative level transformations will automatically violate (4.3). Under such a (strictly upper triangular) transformation

$$g = \exp\left(\frac{1}{3!}\Lambda^{(1)}_{mnp}E^{mnp} + \frac{1}{6!}\Lambda^{(2)}_{m_1\dots m_6}E^{m_1\dots m_6} + \dots\right) \in E_{10}^+ \tag{4.5}$$

the lowest components of the current transform as

$$\delta\overset{(0)}{J}{}^{m}{}_{n} = \quad = \quad \frac{1}{18}\delta^m_n\Lambda^{(1)}_{pqr}\overset{(-1)}{J}{}^{pqr} - \frac{1}{2}\Lambda^{(1)}_{npq}\overset{(-1)}{J}{}^{mpq}\,, \tag{4.6a}$$

$$\delta \overset{(-1)}{J}{}^{mnp} = \frac{1}{6}\Lambda^{(1)}_{qrs}\overset{(-2)}{J}{}^{qrsmnp}, \tag{4.6b}$$

$$\delta \overset{(-2)}{J}{}^{m_1\ldots m_6} = \frac{1}{6}\Lambda^{(1)}_{qrs}\overset{(-3)}{J}{}^{q|rsm_1\ldots m_6}, \tag{4.6c}$$

$$\delta \overset{(-3)}{J}{}^{m_0|m_1\ldots m_8} = 0\,. \tag{4.6d}$$

Here, we have shown the infinitesimal result when only $\Lambda^{(1)}_{pqr}$ is non-zero. The truncation (4.2) implies that $J^{(-3)}$ is invariant. Infinitesimally we have in general that $\delta J^{(\ell)} = \sum_n \Lambda^{(n)} J^{(\ell-n)}$.

We shall next investigate the relation of the conserved charges $\mathcal{J}$ to the constraints derived in the foregoing section. Before doing so, however, it is useful to recall that there is already one constraint which can be expressed in manifest 'Sugawara form', the Hamiltonian constraint. Namely, from (4.1) it is evident that

$$\langle \mathcal{P}|\mathcal{P}\rangle = 0 \qquad \Longleftrightarrow \qquad \langle \mathcal{J}|\mathcal{J}\rangle = 0\,. \tag{4.7}$$

Furthermore this constraint is obviously an E_{10} singlet.

4.2 Sugawara-like construction of the constraints

As mentioned at the end of section 3.1 above, the weak conservation of the constraints exhibits two remarkable structural features. The first of these is the universal action of the zero-level coset velocity $-P^{(0)}_{ab}$ on the r.h.s. of the weak conservation equations (3.5). Namely, as already observed in [3], this action can be combined with the similar universal action of the $SO(10) \subset K(E_{10})$ gauge connection $-Q^{(0)}_{ab}$ on the l.h.s. by using the formulas (3.12) [11]

$$\overset{(0)}{P}_{ab} - \overset{(0)}{Q}_{ab} = -e_b{}^m\partial_t e_{ma} = +e_{ma}\partial_t e_b{}^m \tag{4.8}$$

whence

$$\partial_t v_a + \big(\overset{(0)}{P}_{ab} - \overset{(0)}{Q}_{ab}\big)v^b = e_{ma}\partial_t v^m\,. \tag{4.9}$$

Here, $v^m \equiv e_a{}^m v^a$ is the *contravariant* 'world' version of the tangent space vector $v^a \equiv v_a$. The (inverse) coset zehnbein $e_m{}^a = (e^{-h})_m{}^a$ is obtained from $\mathcal{V}_0$ as in (2.13). Therefore, the universal structure of the $P^{(0)}_{ab}$ and $Q^{(0)}_{ab}$ contributions in the weak conservation equation (3.5) above is precisely such that they can all be eliminated if one replaces

[11] We always adhere to the conventions and notations of [8].

all the $SO(10)$ 'flat' indices $a, b, \ldots$ by *contravariant* 'world' $GL(10)$ indices $m, n, \ldots$ Accordingly, we can now convert the constraints of the previous section (written in terms of 'flat' indices) to *contravariant form* by means of the coset zehnbein defining

$$\overset{(-3)}{\mathfrak{C}}{}^{m_1\ldots m_9} \equiv e^{a_1 m_1}\cdots e^{a_9 m_9}\overset{(3)}{\mathcal{C}}_{a_1\ldots a_9} \quad , \quad \text{etc.} \tag{4.10}$$

In this 'contravariant' form the constraints $\mathfrak{C}$ are now $GL(10)$ tensors rather than $SO(10)$ ('flat' or 'Lorentz') tensors. The reasons for switching to a labelling with *negative* integers will become apparent shortly. The conversion (4.10) into $GL(10)$ world indices only changes the transformation under the level $\ell = 0$; below we will see that converting all $K(E_{10})$ indices into E_{10} indices is more natural and gives a more unified structure.

The second noteworthy feature was the triangular evolution structure of (3.5), which becomes *strictly upper triangular* with the above redefinitions; that is, the weak conservation equations now take the form, for $\ell \leq 6$,

$$\partial_t \overset{(-\ell)}{\mathfrak{C}}{}^{m_1\ldots m_{3\ell}} \sim \sum_{k\geq 1} \overset{(k)}{P}_{n_1\ldots n_{3k}} \overset{(-\ell-k)}{\mathfrak{C}}{}^{m_1\ldots m_{3\ell} n_1\ldots n_{3k}}, \tag{4.11}$$

with the *covariant* velocities $P^{(k)}_{n_1\ldots n_{3k}}(t) \equiv e_{n_1}{}^{a_1}\cdots e_{n_{3k}}{}^{a_{3k}} P^{(k)}_{a_1\ldots a_{3k}} = \partial_t A_{n_1\ldots n_{3k}} + \ldots$. Index contractions here are only schematic; we do not indicate the various (anti-)symmetrisations required for the pertinent Young tableaux. The important feature of (4.11) is the distinction of contravariant and covariant world indices.

This triangular evolution system can be recursively integrated, in the present truncation starting from $\ell = 6$. Indeed, the above procedure eliminates all the terms on the r.h.s. of the last equation in (3.5), implying that the contravariant constraint $\mathfrak{C}^{(-6)}$ is actually *constant*, and not only weakly constant. Due to the identity (4.4) between $J^{(-3)}$ and the contravariant $P^{(3)}$ we can also rewrite $\mathfrak{C}^{(-6)}$ in current$\times$current form and define

$$\overset{(-6)}{\mathfrak{L}}{}^{m_1\ldots m_{10}||n_0|n_1\ldots n_7} \equiv \overset{(-6)}{\mathfrak{C}}{}^{m_1\ldots m_{10}||n_0|n_1\ldots n_7} = \overset{(-3)}{J}{}^{n_0|m_1\ldots m_8} \overset{(-3)}{J}{}^{m_9|m_{10}n_1\ldots n_7} . \tag{4.12}$$

Here, and in similar formulas below, the same symmetrisations as in (3.1) are understood. This way of writing the constraints makes it plainly evident that $\mathfrak{L}^{(-6)}$ is strongly conserved since the current components are. The notation $\mathfrak{L}$ is chosen in anticipation of a Sugawara-like construction.

Examining then the weak conservation law for the penultimate (contravariant) constraint $\mathfrak{C}^{(-5)}$, one finds that it, too, can be integrated explicitly. More specifically, it is easy to check that the time derivative of

$$\overset{(-5)}{\mathfrak{L}}{}^{m_1\ldots m_{10}||n_1\ldots n_5} \equiv \overset{(-5)}{\mathfrak{C}}{}^{m_1\ldots m_{10}||n_1\ldots n_5} + \frac{1}{15} A_{p_1p_2p_3} \overset{(-6)}{\mathfrak{C}}{}^{m_1\ldots m_{10}||p_1|p_2p_3n_1\ldots n_5} \tag{4.13}$$

is identically zero by virtue of (3.5c) and (2.15). After a little algebra, we find that, remarkably, we can again rewrite this exactly conserved quantity in current $\times$ current form as

$$\overset{(-5)}{\mathfrak{L}}{}^{m_1\ldots m_{10}||n_1\ldots n_5} = \overset{(-2)}{J}{}^{n_1\ldots n_4 m_1 m_2}\overset{(-3)}{J}{}^{n_5|m_3\ldots m_{10}} \tag{4.14}$$

by using (4.4).

The recursive integration can be continued, in principle, for the other constraints $\mathfrak{C}^{(-4)}$ and $\mathfrak{C}^{(-3)}$ and we anticipate that the so obtained ('E_{10} covariantised') constraints $\mathfrak{L}^{(-4)}$ and $\mathfrak{L}^{(-3)}$ can also be expressed as bilinears in current components via

$$\overset{(-4)}{\mathfrak{L}}{}^{m_1\ldots m_{10}||n_1 n_2} = \frac{21}{5}\overset{(-2)}{J}{}^{n_1 m_1\ldots m_5}\overset{(-2)}{J}{}^{n_2 m_6\ldots m_{10}} + \overset{(-1)}{J}{}^{n_1 m_1 m_2}\overset{(-3)}{J}{}^{n_2|m_3\ldots m_{10}}\,, \tag{4.15a}$$

$$\overset{(-3)}{\mathfrak{L}}{}^{n_1\ldots n_9} = 28\overset{(-1)}{J}{}^{n_1 n_2 n_3}\overset{(-2)}{J}{}^{n_4\ldots n_9} + \overset{(0)}{J}{}^{n_1}{}_{p}\overset{(-3)}{J}{}^{p|n_2\ldots n_9}\,. \tag{4.15b}$$

where we have substituted from (3.4) for α and β. We will show that these are the correct expressions below in section 4.3 by obtaining them from a symmetry transformation. The above Sugawara-like, i.e. current $\times$ current, form of the redefined constraints now renders manifest their strong conservation.[12] The reason for switching to a labelling by negative levels for the redefined constraints is now obvious: it follows immediately from the level structure on the current components, and is such that $\mathfrak{L}^{(-\ell)} = \sum_n J^{(-\ell+n)}J^{(-n)}$, in a fashion very similar to the Sugawara construction of the Virasoro generators for affine algebras. This connection will be made more explicit in section 4.4. As is evident already from the few terms in (4.15) a Sugawara construction for E_{10} will be far more intricate than the usual construction for affine algebras since the tensorial structure on the various terms is different whereas in the affine case only the level remains. Without truncation we also expect formally infinite sums as extensions of (4.15).[13]

The above expressions (4.15) at last furnish an explanation why we need to introduce so many indices to parametrise the constraints in (3.1), even though inspection of (3.13) might suggest that a more economical form of the constraints could be obtained

[12] In geometrical terms, strongly conserved (under geodesic motion) quantities which are linear in the velocities (such as $\mathcal{J}$) define 'Killing vectors', while strongly conserved quantities which are quadratic in the velocities define 'Killing tensors'. It is a priori quite possible to have Killing tensors which cannot be expressed in Sugawara form, i.e. as a combination of tensor products of Killing vectors (this is for instance the case for the Carter Killing tensor on a Kerr spacetime). Though we should leave open the possible existence of such non-trivial Killing tensors at higher levels, it seems that 'Sugawara-like', geometrically trivial Killing tensors are sufficient in our problem.

[13] In a quantum version these infinite sums presumably need to be normal ordered such that they become well-defined operators on any finite occupation number state.

by dualizing and contracting out seemingly superfluous indices. Namely, when written in contravariant form (4.15), it is obvious that we would need a metric $g_{mn}(t)$ (rather than merely a Kronecker symbol δ_{ab}) to contract away indices. However, the latter metric is not a proper E_{10} object, and therefore a contracted version of (4.15) cannot possibly transform under E_{10} (or rather, as we will see, E_{10}^{+}) in the proper way; besides, contraction with a time-dependent quantity would spoil strong conservation.

As is evident from the above construction, at the origin $\mathcal{V} = 1$ in coset space the Sugawara-like constraints (4.15) agree with the weakly conserved ones in (3.1), since the coset zehnbein ${e_m}^a = {\delta_m}^a$ and the additional coordinates $A_{m_1m_2m_3} = A_{m_1\ldots m_6} = \ldots = 0$. Away from the origin the identity $\mathcal{J} = \mathcal{P}$ no longer holds and the constraints (3.1) and (4.15) also start to differ. This is captured by the contravariantisation by ${e_m}^a$, turning $\mathcal{C}^{(\ell)}$ into $\mathfrak{C}^{(-\ell)}$, and the additional terms proportional to $A_{m_1m_2m_3}$ etc., turning $\mathfrak{C}^{(-\ell)}$ into the Sugawara-like $\mathfrak{L}^{(-\ell)}$.

4.3 Transformation of the constraints

We now examine the transformation properties of the constraints under the basic E_{10} symmetry. This question can be addressed both for the strongly conserved constraints (4.15) quadratic in the charges $\mathcal{J}$ as well as for the equivalent weakly conserved constraints (3.1) quadratic in the velocities $\mathcal{P}$.

As already mentioned, the tensor structure of the low level constraints is identical to that of the so-called integrable $L(\Lambda_1)$ representation of E_{10}. The highest weight of this E_{10} representation is Λ_1 with Dynkin labels $[1\,0\,0\,0\,0\,0\,0\,0\,0\,0]$, The '1' here occurs for the over-extended, hyperbolic node of the E_{10} Dynkin diagram as shown in Fig. 1 and the definition of the fundamental weights was given in footnote 2. The low level content of the $L(\Lambda_1)$ representation (sometimes also referred to as 'central charge representation') w.r.t. the A_9 subgroup of E_{10} was given in an appendix of [28].[14] Note however, that the constraint $\mathfrak{L}^{(-6)}$ contains only one of the two $GL(10)$ tensors appearing in the level decomposition of $L(\Lambda_1)$ due to the algebraic restriction that it should vanish upon anti-symmetrisation in the indices $n_0, n_1, \ldots, n_7$. The possible occurrence of $L(\Lambda_1)$ of E_{10} in the present context might be interpreted as evidence for a covariant formulation involving E_{11}. In this case, gauge-fixing and a canonical analysis should lead to the replacement of a (presently unknown, and hypothetically E_{11} invariant) set of 'covariant' equations by an E_{10} invariant set of dynamical equations augmented by constraint equations transforming

[14] The analogue of $L(\Lambda_1)$ for E_{11} was proposed in [11] to be responsible for the emergence of space-time. We will here take a different view on emergent space by the unfolding of constraint equations, but note that just like in (4.16) below further E_{11} representations beyond $L(\Lambda_1)$ may be required there as well.

in a representation of E_{10}, whose structure should follow from an E_{10} decomposition of E_{11}. Indeed the $L(\Lambda_1)$ representation is the first in an infinite sequence of integrable highest weight representations of E_{10} arising in the decomposition of E_{11} w.r.t. its natural E_{10} subalgebra [30]:

$$E_{11} = \cdots \oplus L(\Lambda_3)^* \oplus L(\Lambda_1)^* \oplus (E_{10} \oplus \mathbb{R}\kappa) \oplus L(\Lambda_1) \oplus L(\Lambda_3) \oplus \cdots , \qquad (4.16)$$

where we have also indicated the dual lowest weight representations corresponding to the positive step operators (the fundamental weights for E_{11} are defined in a completely analogous fashion as for E_{10}). κ is a 'level counting operator' which commutes with E_{10} (and is analogous to the central charge in the decomposition of E_{10} under E_9). If (4.16) were indeed the correct way of splitting the looked-for E_{11}-covariant equations into dynamical E_{10} equations and constraints, one would accordingly expect an infinite set of (separately infinite) towers of constraints, of which the known supergravity constraints would just be the lowest lying members. The piece associated to κ is E_{10} invariant and would correspond to the Hamiltonian constraint.

In order to verify the E_{10} transformation properties of the constraints it is better to work with the strongly conserved version $\mathfrak{L}^{(-\ell)}$ since the constituent charges transform directly under E_{10} according to (4.6). In contrast the weakly conserved constraints $\mathcal{C}^{(\ell)}$ of (3.1) only transform under the induced $K(E_{10})$ transformation and we will study them below as a second step.

The truncation condition (4.2) is only maintained by the parabolic subgroup E_{10}^+. Let us start then by considering the effect of an E_{10}^+ transformation on the contravariantised constraints. Using the transformation (4.6) in (4.12), (4.14) and (4.15) we obtain under an infinitesimal $\Lambda^{(1)}$ transformation of E_{10}

$$\delta \overset{(-3)}{\mathfrak{L}}{}^{m_1\ldots m_9} = -5\overset{(1)}{\Lambda}_{pqr} \overset{(-4)}{\mathfrak{L}}{}^{m_1\ldots m_9 p||qr} , \qquad (4.17a)$$

$$\delta \overset{(-4)}{\mathfrak{L}}{}^{m_1\ldots m_{10}||n_1 n_2} = \frac{5}{12}\overset{(1)}{\Lambda}_{pqr} \overset{(-5)}{\mathfrak{L}}{}^{m_1\ldots m_{10}||pqrn_1n_2} , \qquad (4.17b)$$

$$\delta \overset{(-5)}{\mathfrak{L}}{}^{m_1\ldots m_{10}||n_1\ldots n_5} = \frac{1}{15}\overset{(1)}{\Lambda}_{pqr} \overset{(-6)}{\mathfrak{L}}{}^{m_1\ldots m_{10}||p|qrn_1\ldots n_5}, \qquad (4.17c)$$

$$\delta \overset{(-6)}{\mathfrak{L}}{}^{m_1\ldots m_{10}||n_0|n_1\ldots n_7} = 0 , \qquad (4.17d)$$

where the last relation is again due to the truncation condition (4.2). The result (4.17) exhibits two remarkable features: (i) the contravariantised constraints transform as a linear representation (which was not at all guaranteed by their definition), and (ii) the linear transformations exhibited in (4.17) are the same as one would find for the $L(\Lambda_1)$ representation of E_{10} restricted to E_{10}^+. We have already mentioned that $\mathfrak{L}_{-6}$ contains only one of

the two possible Young tableaux. The set of constraints $\mathfrak{L}^{(-3)}, \ldots, \mathfrak{L}^{(-6)}$ here furnish an unfaithful representation of E_{10}^{+} contained in $L(\Lambda_1)$ of E_{10}. However, it is not clear that this relation between $L(\Lambda_1)$ and the constraints continues to hold when the truncation is relaxed. We stress that the nice transformation laws (4.17) do *not* mean that the constraints constitute a highest weight representation of the full E_{10}. In fact one can show that the combination of $F_{[m_1 m_2 m_3} \otimes F_{m_4 \ldots m_9]}$ and $F_{n|[m_1 \ldots m_8} \otimes K^{n}{}_{m_9]}$ of E_{10} generators underlying $\mathfrak{L}^{(-3)}$ is not annihilated by the raising operator E^{pqr}. This suggests that the tensor product of two adjoint representations $\mathcal{J}$ of E_{10} does not contain $L(\Lambda_1)$ as a subrepresentation. A similar result is known in the affine case E_9 [33].[15]

The difficulties with the transformation under full E_{10} also become apparent when studying the way the weakly conserved constraints (3.1) change under the action of E_{10}. The infinitesimal variation is determined by the induced $K(E_{10})$ transformation, involving both positive and negative step operators of E_{10}. For simplicity we restrict to the transformations (2.5) (with parameter $\Lambda^{(1)}$); the action of this transformation on the constraint $\overset{(3)}{\mathcal{C}}_{a_1 \ldots a_9}$ is

$$\begin{aligned} \delta_{\Lambda^{(1)}} \overset{(3)}{\mathcal{C}}_{a_1 \ldots a_9} &= -5 \overset{(1)}{\Lambda}_{c_1 c_2 c_3} \overset{(4)}{\mathcal{C}}_{a_1 \ldots a_9 c_1 || c_2 c_3} - \frac{1}{2} \overset{(1)}{\Lambda}_{a_1 c_1 c_2} \overset{(1)}{P}_{c_1 c_2 c_3} \overset{(3)}{P}_{c_3 | a_2 \ldots a_9} \\ &\quad + 28 \overset{(1)}{\Lambda}_{a_1 a_2 c} \overset{(0)}{P}_{c a_3} \overset{(2)}{P}_{a_4 \ldots a_9} + 56 \overset{(1)}{\Lambda}_{a_1 a_2 a_3} \overset{(0)}{P}_{c a_4} \overset{(2)}{P}_{c a_5 \ldots a_9} \, . \end{aligned} \tag{4.18}$$

In general, to have covariance under the basic $K(E_{10})$ transformation $\delta_{\Lambda^{(1)}} \mathcal{C}^{(\ell)}$ would need to be equal to the sum of two terms $\Lambda^{(1)} \mathcal{C}^{(\ell+1)}$ and $\Lambda^{(1)} \mathcal{C}^{(\ell-1)}$. In the case $\ell = 3$, as $\mathcal{C}^{(2)}$ does not (seem to) exist, we would therefore like to have $\delta_{\Lambda^{(1)}} \mathcal{C}^{(3)} \propto \Lambda^{(1)} \mathcal{C}^{(4)}$. The first term on the r.h.s. of (4.18) ($\sim \Lambda^{(1)} \mathcal{C}^{(4)}$) is thus the expected covariant transformation of constraint into constraint and naturally agrees with the corresponding one in (4.17). However, the other three terms (containing $P^{(1)} P^{(3)}$ or $P^{(0)} P^{(2)}$) do not rearrange into combinations of constraints. It is not inconceivable that, when relaxing the truncation to levels $\ell \leq 3$ and considering a non-zero value of $P^{(4)}$, the term containing $P^{(1)} P^{(3)}$ might cancel against a term coming from the variation of $P^{(4)}$ in a possible additional contribution $\sim P^{(1)} P^{(4)}$ to the definition of $\mathcal{C}^{(3)}$. However, this type of argument does not seem to apply for the two last offending terms, of the type $P^{(0)} P^{(2)}$, in (4.18) which arise because of the F_{pqr} piece in the $K(E_{10})$ transformation. Concerning the latter problematic terms we note, however, that they arise because of the restricted Young symmetry of $P^{(3)}$ — if there was an additional anti-symmetric piece on level $\ell = 3$, equivalent to an additive modification of $P^{(3)}$, these terms could superficially be made to vanish. This remark could be indicative of a possible Borcherds extension of E_{10}. We will discuss this idea further in

[15] However, in this case one can formally construct something like a highest weight vector.

the conclusions. Evidently, the constraints $\mathcal{C}$ are invariant under $g \in E_{10}^{+}$ transformations since these do not induce any non-trivial $K(E_{10})$ transformations on $\mathcal{P}$.

Even though the transformation properties are identical in either form used for the constraints (strongly conserved or weakly conserved), the geometrical status of the constraints is somewhat clearer when considering them in Sugawara-like form $\mathfrak{L}$. The parabolic subgroup E_{10}^{+} has a transitive action on the coset $E_{10}/K(E_{10})$ (this action is even essentially simply transitive, if we neglect the minor ambiguity linked to the negative-root part of $GL(10)$). Therefore we can use E_{10}^{+} as a group of 'translations' over the coset $E_{10}/K(E_{10})$. Similarly to the notion of 'Clifford translations' and 'Clifford parallelism' in 3-dimensional elliptic space, we can use E_{10}^{+} to translate, 'in a parallel manner', the bundle of geodesics issued from the origin (i.e. the unit element) to a different (and arbitrary) point in coset space. This symmetry argument allows us to complete the proof (given in section 4.2 only for $\mathfrak{L}^{(-6)}$ and $\mathfrak{L}^{(-5)}$) that the Sugawara-like $\mathfrak{L}^{(-\ell)}$ constraints are equivalent to the $\mathcal{C}^{(\ell)}$ ones. Indeed it suffices to parallelly transport the $\mathfrak{L}^{(-\ell)}$ back to the origin where they agree with the weakly conserved $\mathcal{C}^{(\ell)}$. The E_{10}^{+} covariance of the constraints means that this translation operation maps 'good' (i.e. satisfying the constraints) geodesics stemming from a point to other good geodesics stemming from a different (and arbitrary) point in coset space. On the other hand, the apparent non-covariance under the full E_{10} of the constraints means that the set of good geodesics stemming from (say) the origin is not invariant under the isotropy group leaving the origin fixed (which is the group $K(E_{10})$). We will comment on this (unresolved) puzzle of partial loss of symmetry in the concluding section.

4.4 Affine E_9 truncation and standard Sugawara construction

We now show how the Sugawara-like form of the constraints (4.15) relates to the well-known Sugawara construction of an associated Virasoro algebra which exists for any affine Kac-Moody (current) algebra [23, 24, 25]. This is done by reducing to the affine $E_9 \subset E_{10}$. If the Fourier modes of the (left or right) current E_9 are denoted j_n^a (where a is an E_8 Lie algebra index), the Fourier modes of the associated Virasoro generators are of the form $L_m \sim \sum_n j_{m-n}^a j_n^a$.

Let us consider the reduction of the E_{10} conserved current $\mathcal{J}$, and of the E_{10} constraints, to E_9. We discussed above the covariance of our E_{10} Sugawara construction under 'translations' by the transitive parabolic subgroup E_{10}^{+}. We can exploit this translation-covariance to limit ourselves to considering a null geodesic starting (say at $t = 0$) from the coset origin (i.e. the unit element of the group). In this case we have (at $t = 0$) $\mathcal{J} = \mathcal{P}$ and $\mathcal{J}$ is therefore symmetric.

The reduction of E_{10} to its natural affine subalgebra E_9 (as visible on their Dynkin diagrams fig. 1 by separating out the first, leftmost node labelled 1) corresponds to letting all indices only range over $2, 3, \ldots, 10$. Then the number of 2s is related to the affine level in E_9, see [31] and below. In this truncation all of the constraints (3.1) identically vanish due to the presence of the antisymmetric 10-tuples $[b_1 \ldots b_{10}]$, *except* for the lowest one, $\mathcal{C}^{(3)}_{a_1 \ldots a_9}$, which becomes an $SL(9)$ singlet. Using the notation of [31] for the E_9 current algebra, we find [16]

$$\begin{aligned}
\mathcal{J} \;=\; & \frac{3}{9!}\overset{(3)}{P}_{i|k_1\ldots k_8}\epsilon^{k_1\ldots k_8}(Z^i_{(0)} + Z^{(0)}_i) + \frac{3\cdot 8}{9!}\overset{(3)}{P}_{i|k_1\ldots k_7 2}\epsilon^{jk_1\ldots k_7}(G^i_{(1)j} + G^j_{(-1)i}) \\
& + \frac{3\cdot 8}{9!}\overset{(3)}{P}_{2|2k_1\ldots k_7}\epsilon^{k_1\ldots k_7 i}(Z^{(2)}_i + Z^i_{(-2)}) + \frac{1}{6!}\overset{(2)}{P}_{i_1\ldots i_6}(Z^{i_1\ldots i_6}_{(0)} + Z^{(0)}_{i_1\ldots i_6}) \\
& + \frac{1}{3!\cdot 5!}\overset{(2)}{P}_{2k_1\ldots k_5}\epsilon_{k_1\ldots k_5 i_1 i_2 i_3}(Z^{(1)}_{i_1 i_2 i_3} + Z^{i_1 i_2 i_3}_{(-1)}) + \frac{1}{3!}\overset{(1)}{P}_{i_1 i_2 i_3}(Z^{i_1 i_2 i_3}_{(0)} + Z^{(0)}_{i_1 i_2 i_3}) \\
& + \frac{1}{2\cdot 6!}\overset{(1)}{P}_{2k_1k_2}\epsilon_{k_1k_2 i_1\ldots i_6}(Z^{(1)}_{i_1\ldots i_6} + Z^{i_1\ldots i_6}_{(-1)}) + \frac{1}{2}\overset{(0)}{P}_{ij}(G^i_{(0)j} + G^j_{(0)i}) \\
& + \overset{(0)}{P}_{2i}(Z^i_{(-1)} + Z^{(1)}_i) + \left(\overset{(0)}{P}_{22} + \overset{(0)}{P}_{ii}\right) d - \left(\overset{(0)}{P}_{11} + \overset{(0)}{P}_{ii}\right) c\,. \qquad (4.19)
\end{aligned}$$

The notation here is such that $i, j, \ldots$ are $SL(8)$ vector indices and take $3, 4, \ldots, 10$; the index value 2 corresponds to the affine E_9 extension of E_8, and at the same time labels the remaining spatial coordinate x^2 in the dimensional reduction. The bracketed sub- and superscripts on the E_8 generators Z give the affine level. E_8 itself is written in $SL(8)$ level decomposition as

$$\begin{array}{rcccccccccccc}
E_8 & = & \bar{\mathbf{8}} & \oplus & \mathbf{28} & \oplus & \mathbf{5\bar{6}} & \oplus & (\mathbf{63}\oplus\mathbf{1}) & \oplus & \mathbf{56} & \oplus & \mathbf{\bar{28}} & \oplus & \mathbf{8} \\
 & & Z_i & & Z_{i_1\ldots i_6} & & Z_{i_1 i_2 i_3} & & G^i{}_j & & Z^{i_1 i_2 i_3} & & Z^{i_1\ldots i_6} & & Z^i
\end{array}$$

Using the current algebra basis, we expand the E_9 valued conserved current (4.19) as

$$\begin{aligned}
\mathcal{J} \;=\; & \sum_{m\in\mathbb{Z}}\big[J^{(m)}_i Z^i_{(m)} + J^i_{(m)} Z^{(m)}_i + \frac{1}{6!}J^{(m)}_{i_1\ldots i_6}Z^{i_1\ldots i_6}_{(m)} + \frac{1}{6!}J^{i_1\ldots i_6}_{(m)}Z^{(m)}_{i_1\ldots i_6} \\
& + \frac{1}{3!}J^{(m)}_{i_1 i_2 i_3}Z^{i_1 i_2 i_3}_{(m)} + \frac{1}{3!}J^{i_1 i_2 i_3}_{(m)}Z^{(m)}_{i_1 i_2 i_3} + J^j_{(m)i}G^i_{(m)j} + J_d d + J_c c\big]\,. \qquad (4.20)
\end{aligned}$$

[16] The additional factor of 3 for the level 3 terms comes from changing to canonical normalisation of E_8. The central extension is $c = -K^1{}_1$ and the derivation $d = K^2{}_2$, see [31] for details. Since we are working with the identity vielbein, the distinction between flat and curved $E_{10}/K(E_{10})$ indices is not necessary here.

In the present situation (4.19) only components up to affine level $|m| \leq 2$ are non-zero.

The naïve Sugawara construction[17] of the Virasoro generator L_{-1} gives with the standard normalisations (in terms of charges and up to an overall factor)

$$\begin{aligned} L_{-1} &= J_i^{(-2)} J^i_{(1)} + J_i^{(-1)} J^i_{(0)} + J^i_{(-1)j} J^j_{(0)i} + J^i_{(-1)i} J^j_{(0)j} \\ &\quad + \frac{1}{3!} J^{(-1)}_{i_1 i_2 i_3} J^{i_1 i_2 i_3}_{(0)} + \frac{1}{6!} J^{(-1)}_{i_1 \ldots i_6} J^{i_1 \ldots i_6}_{(0)} + \ldots , \end{aligned} \tag{4.21}$$

where the dots vanish identically in the present truncation.

Substituting the above expansion (4.19) into expression (4.21) gives

$$\begin{aligned} \frac{1}{3}\epsilon_{k_1 \ldots k_8} L_{-1} &= \frac{1}{9} \overset{(0)}{P}_{2i} \overset{(3)}{P}_{i|k_1 \ldots k_8} + \frac{8}{9} \overset{(0)}{P}_{2k_1} \overset{(3)}{P}_{2|k_2 \ldots k_8 2} + \frac{8}{9} \overset{(0)}{P}_{k_1 i} \overset{(3)}{P}_{i|k_2 \ldots k_8 2} \\ &\quad - \frac{8}{9} \overset{(0)}{P}_{ii} \overset{(3)}{P}_{2|k_1 \ldots k_8} + 28 \cdot \frac{6}{9} \overset{(2)}{P}_{2k_1 \ldots k_5} \overset{(1)}{P}_{k_6 k_7 k_8} + 28 \cdot \frac{3}{9} \overset{(1)}{P}_{2k_1 k_2} \overset{(2)}{P}_{k_3 \ldots k_8} . \end{aligned} \tag{4.22}$$

This is to be compared with the reduction of $\mathfrak{L}^{(-3)}$ to E_9 which gives

$$\begin{aligned} \overset{(-3)}{\mathfrak{L}}_{2k_1 \ldots k_8} &= \frac{1}{9} \overset{(0)}{P}_{2i} \overset{(3)}{P}_{i|k_1 \ldots k_8} + \frac{8}{9} \overset{(0)}{P}_{2k_1} \overset{(3)}{P}_{2|k_2 \ldots k_8 2} + \frac{8}{9} \overset{(0)}{P}_{k_1 i} \overset{(3)}{P}_{i|k_2 \ldots k_8 2} + \frac{1}{9} \overset{(0)}{P}_{22} \overset{(3)}{P}_{2|k_1 \ldots k_8} \\ &\quad + 28 \cdot \frac{6}{9} \overset{(1)}{P}_{k_1 k_2 k_3} \overset{(2)}{P}_{k_4 \ldots k_8 2} + 28 \cdot \frac{3}{9} \overset{(1)}{P}_{2k_1 k_2} \overset{(2)}{P}_{k_3 \ldots k_8} . \end{aligned} \tag{4.23}$$

where the normalisation is the same as in (3.1). Remarkably, the coefficients of most terms in (4.22) and (4.23) are identical, so that at least to the order considered, the momentum constraint $\mathcal{C}^{(3)}$ (generating translations along the residual spatial coordinate x^2 [32]) appears to be related to the L_{-1} generator (generating translations in the spectral parameter in a current algebra realisation of E_9). The term which does not agree involves the contribution $P^{(2)}_{22}$ which in $\mathcal{J}$ of (4.19) is only multiplied by d and therefore cannot occur in L_{-1}.

This near-perfect agreement between the highest-level Sugawara-like E_{10} coset constraints (4.15), and the standard Sugawara construction of L_{-1} in the affine subalgebra E_9 suggests that the hyperbolic Sugawara-like definition of the E_{10} coset constraints (obtained above only when assuming a truncation to levels ≤ 3) should extend to the exact, untruncated coset model, and that the expected infinite tower of $L(\Lambda_1)$-like E_{10} constraints (of levels $-3, -4, -5, \ldots$), possibly with additional towers, should be somehow

[17] 'Naïve' here means without normal ordering although this is superfluous anyway for the symmetric expansion used here. Ignoring normal ordering also makes the affine Sugawara generators invariant under the action of the affine group, in agreement with the observation in footnote 15.

analogous to the infinite tower of Virasoro generators L_m. In other words, the gauge symmetry of M-theory in its E_{10} formulation would be contained in a vast generalization of the Sugawara construction of the Virasoro constraints (encoding the conformal symmetry of a gauge-fixed string action). Moreover, we expect that the supersymmetry constraint (3.7) arises as the part of the construction of the fermionic current $G(z)$ as in string theory.

5 Discussion

In this concluding chapter we summarise our findings and outline some interesting avenues of further research suggested by our results.

In this paper we have shown that it is possible to further constrain the coset motion on $E_{10}/K(E_{10})$ by a set of weakly conserved constraints (3.1) which precisely correspond to the appropriate truncations of the supergravity canonical constraints. We have also shown that these constraints can be equivalently re-expressed in a Sugawara-like manner (4.15), which is interestingly linked to the standard Sugawara construction of a Virasoro algebra for the natural affine subalgebra E_9 of E_{10}. We have investigated these coset constraints under the assumption of a (consistent) dynamical truncation where the coset velocities $\mathcal{P}$ (in triangular gauge) of levels 4 and higher, or more invariantly the conserved charges $\mathcal{J}$ of levels -4 and below, vanish. We conjecture that if one relaxes this truncation by admitting non-zero coset charges down to level k it will be possible to extend both the definition of the existing constraints (e.g. by adding new terms, with total grading -3, to $\mathfrak{L}^{(-3)}$ up to $\mathcal{J}^{(-3+k)}\mathcal{J}^{(-k)}$, etc.), and the number of constraints (by including lower-level constraints, down to the level $\mathfrak{L}^{(-2k)}$). In the limit $k \to +\infty$ where the truncation is removed, one would end up with (at least) one infinite tower of constraints $\mathfrak{L}^{(-3-n)}$, with $n \in \mathbb{N}$. Such an infinite number of constraints might be conjectured to be needed to reduce the potentially problematic 'exponentially infinite' number of degrees of freedom in the hyperbolic $E_{10}/K(E_{10})$ coset model to a more manageable size, hopefully compatible with the physically expected M-theory degrees of freedom. On the other hand, this also means that the set of constraints should not grow in size faster (when the level increases) than the number of generators in E_{10}^{+}.[18]

However, it seems that the (say Sugawara-defined) coset constraints are only covariant under a parabolic subgroup E_{10}^{+} of E_{10}. The constraint surface therefore seems to partially break the original E_{10} symmetry of the coset model. By contrast, let us remark that, in the E_9-invariant (two-dimensional) reduction of supergravity (which is closely related to

[18] From this point of view, one hopes that the apparent coincidence between the algebraic structure of the low-level constraints and the E_{10} highest-weight representation $L(\Lambda_1)$ does not persist to all levels, because this representation grows in size faster than E_{10}^{+}.

the E_9 reduction of the E_{10} coset model), E_9 *does* map solutions to solutions and so respects the constraint surface. This point deserves further investigation, as well as the relation between the hyperbolic and affine Sugawara constructions, which might open a new perspective on the E_{10} structure.

We see several possible 'resolutions' of this partial loss of symmetry. First, one might think of keeping the full E_{10} Kac-Moody symmetry by enlarging the set of constraints, i.e. by treating the offensive terms in the transformation of the constraints (4.18) as additional constraints that need to be imposed. Whereas we partially checked that this will lead to new consistent conditions (of level -2 in Sugawara form) on the geodesic motion, the transformation of these new constraints again does not close covariantly but necessitates yet again new terms (of level -1), etc.. Anticipating that this phenomenon persists indefinitely it is not clear to us whether any non-trivial solutions to the geodesic equations remain in this process, given the apparently very large, and maybe infinite[19], total set of constraints. In addition, the new constraints do not appear to have a good interpretation in supergravity.

A second possible resolution of the symmetry problem might turn out to be that E_{10} is only an auxiliary symmetry of the theory, which is broken by the constraints. An analogy with, say, bosonic string theory, might suggest how this could be the case. Indeed, the usual, conformal-gauge-fixed dynamics of bosonic string theory consists of two separate elements: *(i)* the conformal-gauge-fixed action $S_{gf}[X^\mu(\tau,\sigma)]$, and *(ii)* the Virasoro constraints $L_n = 0$. The symmetry of $S_{gf}[X^\mu(\tau,\sigma)]$ include *both* conformal transformations of worldsheet coordinates (τ,σ), *and* the following (active) transformations of the target field X^μ: $\delta X^\mu = \epsilon^\mu_n \exp(-in(\tau \pm \sigma))$, for arbitrary n (and arbitrary choice of sign $\pm$ if one discusses the closed string; for the open string one must combine terms of the two signs). The Noether conserved current associated to the second symmetry of $S_{gf}[X^\mu(\tau,\sigma)]$ is the worldsheet current $j^\mu(\tau \pm \sigma) = \partial_\pm X^\mu$. The Fourier modes of this Noether current are the usual $j^\mu_n = \alpha^\mu_n$. The current algebra is abelian (w.r.t. the 'Lie algebra index' μ) and contains only the (level-1) anomaly term $[j^\mu_m, j^\mu_n] = m\eta^{\mu\nu}\delta_{m+n}$. The Virasoro constraints are obtained by the standard Sugawara construction from the Noether current: $L_m = \frac{1}{2}\sum_n \eta^{\mu\nu} j^\mu_{m-n} j^\nu_n$. Now, the crucial point we wish to make is that the constraints $L_m = 0$ are *not covariant* under the current symmetry $j^\mu_n = \alpha^\mu_n$ of the gauge-fixed action $S_{gf}[X^\mu(\tau,\sigma)]$. Indeed, the commutation relation $[j^\mu_m, L_n] = m j^\mu_{m+n}$ expresses the covariance of the current under conformal transformations, but exhibits the non-covariance of the constraints under the symmetry associated to the current (only the

[19]There is, however, the possibility that, under our usual truncation, the above-defined set of new constraints at levels $-2, -1, \ldots$ does terminate at level $+3$. Actually, the fact that, at level -6, $\mathfrak{L}^{(-6)}$ seems already to contain only one of the two independent corresponding objects of $L(\Lambda_1)$ might be the first sign of such a reduction of $L(\Lambda_1)$ to a smaller (possibly irreducible) representation of E^+_{10}.

Poincaré symmetry under the zero-mode j_0^μ is present). Let us also recall that the string gauge-fixed Hamiltonian is proportional to the level-0 Virasoro constraint L_0 (or $L_0 + \bar{L}_0$ in the closed string case), and that this dissymetric role of L_0 w.r.t. the other L_n's reflects the specific gauge used to fix the worldsheet diffeomorphism symmetry. By analogy, it might happen that the E_{10} coset action is a gauge-fixed version of an underlying gauge-invariant action, and that the gauge fixing has the effect of, both, selecting one specific constraint ($\mathcal{C}_0 = \langle \mathcal{P}|\mathcal{P}\rangle = \langle \mathcal{J}|\mathcal{J}\rangle$) as Hamiltonian, and introducing an auxiliary symmetry which does not preserve the constraints. We might, however, expect that (as in the string case where the symmetry generators, $j_n^\mu = \alpha_n^\mu$ generate the full spectrum) the auxiliary symmetry be a spectrum generating symmetry. And in the case where the analogy should be taken with the light-cone-gauge-fixed string action, the auxiliary symmetry might be similar to a DDF algebra.

An alternative explanation of the apparent restriction to E_{10}^+, one might need to modify the underlying E_{10} Kac-Moody symmetry (e.g. into a related Borcherds symmetry, see previous subsection). We finally come back to the possibility of a Borcherds extension mentioned in section 4.3 in connection with the 'missing' full E_{10} covariance of the constraint transformations. Supposing that this implies that E_{10} may not be the correct full symmetry structure of the complete supergravity system (and of M-theory) one should look for a modification of E_{10} preserving the remarkable features of the E_{10} model. Arguably the simplest modification of E_{10} is an extension of E_{10} by additional simple generators. Adjoining new *imaginary simple roots* leads to a *Borcherds extension* of E_{10}. (For introductory literature to Borcherds algebras see for example [34].) Introducing a new anti-symmetric nine index generator corresponding to salvaging the transformation properties of (4.18) can be achieved by means of adding a single null imaginary simple root which attaches with a single line at the hyperbolic node of the E_{10} Dynkin diagram. (In supergravity terms such a new generator relates to the spatial trace of the spin connection.) However, the transformation of the associated component of $\mathcal{P}$ under an infinitesimal $\Lambda_{a_1a_2a_3}$ transformation as in (2.5) does not give rise (as would be needed to cancel the offending terms $\propto P^{(0)}P^{(2)}$) to a new contribution to $\delta P^{(3)}$ in (2.5d) precisely since the new root is simple and not composite. Therefore such a new root does not appear to correct the transformation properties of the constraints. Another possible Borcherds extension is the vertex operator algebra obtained from a lattice construction on the E_{10} root lattice (see for example [35]). This Borcherds algebra also contains E_{10} as a proper subalgebra but has additional imaginary simple roots. The first such imaginary simple root is time-like and as a root vector identical to Λ_2 and therefore occurs at level $\ell = 6$, much too high a level than needed to correct the constraint transformation (4.18). We note, however, that a Borcherds modification of E_{10} (maybe involving several independent copies of the null imaginary root mentioned above) could also help to produce

additional negative definite terms in the Hamiltonian constraint $\langle \mathcal{P}|\mathcal{P}\rangle = 0$ in order to improve agreement with supergravity [3].

To further investigate the situation, it will be important to compute the algebra generated by the constraints. This should give access to the underlying gauge symmetry of the full model. From the supergravity correspondence (proven at low levels only), we expect that the bosonic coset constraints will contain a generalization of both diffeomorphism invariance, and the gauge invariance of the 3-form. It would be interesting to see whether a richer algebraic structure, maybe appropriate to a theory in which both spacetime and its general covariance are expected to be emergent properties, comes out of a group-theoretical analysis of the constraint algebra.

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