DIMENSIONAL EXACTNESS OF SELF-MEASURES FOR RANDOM COUNTABLE ITERATED FUNCTION SYSTEMS WITH OVERLAPS

EUGEN MIHAILESCU AND MARIUSZ URBAŃSKI

Abstract. We study projection measures for random countable (finite or infinite) conformal iterated function systems with arbitrary overlaps. In this setting we extend Feng's and Hu's result from [6] about deterministic finite alphabet iterated function systems. We prove, under a mild assumption of finite entropy, the dimensional exactness of the projections of invariant measures from the shift space, and we give a formula for their dimension, in the context of random infinite conformal iterated function systems with overlaps. There exist numerous differences between our case and the finite deterministic case. We give then applications and concrete estimates for pointwise dimensions of measures, with respect to various classes of random countable IFS with overlaps. Namely, we study several types of randomized extensions of iterated function systems related to Kahane-Salem sets; also, a random system related to a statistical problem of Sinai; and randomized infinite IFS in the plane, for which the number of overlaps is uniformly bounded from above.

1. Introduction

Let (X, ρ) be a metric space. A finite Borel measure μ on X is called *exact dimensional* if

$$d_\mu(x) := \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{1.1}$$

exists for μ-a.e. $x \in X$ and is equal to a common value denoted by d_μ. Exact dimensionality of the measure μ has profound geometric consequences (for eg [10], [15], [18]).

The question of which measures are exact dimensional attracted the attention at least since the seminal paper of L.S Young [22], where it was proved a formula for the Hausdorff dimension of a hyperbolic measure invariant under a surface diffeomorphism, formula involving the Lyapunov exponents of the measure. As a consequence of that proof, she established what (now) is called the dimensional exactness of such measures. The topic of dimensional exactness was then pursued by the breakthrough result of Barreira, Pesin, and Schmeling who proved in [1] the Eckmann–Ruelle conjecture asserting that any hyperbolic measure invariant under smooth diffeomorphisms is exact dimensional ([4]). Dimensional exactness, without using these words, was also established in the book [11] for all projected invariant measures with finite entropy, in the setting of conformal iterated function systems with countable alphabet which satisfy the Open Set Condition (OSC); in particular for all projected invariant measures if the alphabet is finite and we have OSC.

2010 *Mathematics Subject Classification.* 37H15, 37F40, 28A80, 28D05, 37C45, 37A35.

Key words and phrases. Dimensions of measures, entropy, random iterated function systems, overlaps, Lyapunov exponents, countable alphabet, random fractals, deterministic systems.

The next difficult task to deal with was the case of a conformal iterated function system with *overlaps*, i.e. without assuming the Open Set Condition. For the case of iterated function systems with finite alphabet and having overlaps, this was done by Feng and Hu in [6].

Overlaps in iterated function systems (IFS) are challenging. Our goal in the present paper is to extend Feng's and Hu's result in *two directions*. Firstly, by allowing the alphabet of a conformal iterated function system to be *countable infinite*; and secondly, to consider *random* iterated function systems rather than deterministic IFS. Random IFS's contain a single (deterministic) IFS as a special case.

In general, infinite IFS with overlaps behave **differently** than finite IFS with overlaps (for eg [11], etc). In the infinite case, the limit set is not necessarily compact (by contrast to the finite IFS case), also the diameters of the sets $\phi_i(X)$ converge to 0, etc. In addition, for an infinite IFS $\mathcal{S}$, the boundary at infinity $\partial_\infty(\mathcal{S})$ plays an important role, and we have to take into consideration whether an invariant probability gives measure zero (or not) to $\partial_\infty(\mathcal{S})$ (for eg [11], [12], etc). Even when OSC is satisfied, the Hausdorff dimension of the limit set is not always given as the zero of the pressure of a certain potential. However, a version of Bowen's formula for the Hausdorff dimension still exists; see [11]. For example even when assuming OSC, and unlike in the finite alphabet case, the Hausdorff measure can *vanish* and the packing measure may become *locally infinite at every point*. In addition for infinite systems with overlaps we may have infinitely many basic sets of the system, overlapping at points in the limit set J, or the number of overlaps may be unbounded over J.
In [12], we obtained lower estimates for the Hausdorff dimension of the limit set J of a deterministic infinite IFS with overlaps, by using the pressure function and a preimage counting function, that counts the overlaps at points of J. This preimage counting function plays an important role in general, for iterated function systems with overlaps, and we found also that the Hausdorff dimension of the limit set J takes its "minimal" value exactly when the number of overlaps over *every* point in J is k (assuming that this number of overlaps is everywhere finite, and bounded above by an integer $k \geq 2$). In addition, in [12] we gave several classes of examples of infinite conformal iterated function systems with overlaps.

By extension from the case of infinite IFS with overlaps discussed above, the case of **random infinite IFS with overlaps** presents even more differences and new phenomena, when compared to the case of finite IFS with overlaps. For instance several proofs that used compactness type arguments cannot be applied to random infinite IFS with overlaps. We also have to impose certain conditions on the randomization process $\theta : \Lambda \to \Lambda$ and on the invariant probability measure μ on $\Lambda \times E^{\mathbb{N}}$, etc.

Starting from the general strategy of Feng and Hu paper [6], we will prove under a mild assumption of finite conditional entropy, the dimensional exactness of the projections of invariant measures from the shift space, in the context of random conformal iterated function systems with countable alphabet and having arbitrary overlaps.

Our main result is contained in Theorem 3.13 where we prove *dimensional exactness*, and provide a *formula* for the dimension of typical projection measures, by employing a random projectional entropy and the Lyapunov exponents of the measure with respect to the random countable IFS with overlaps. Also, in Theorem 2.5 we give lower and upper bounds for the random projectional entropy of a measure. In the last Section, we apply these results to *pointwise dimension estimates* for several concrete classes of random countable IFS with overlaps. Our results work for both finite random systems, and for infinite random systems. Randomization allows to have a unitary setting to study limit sets and measures in a family of dynamical systems for *generic* parameter values, which proves useful in cases when a study of individual systems is difficult. Moreover, randomization allows us to obtain *new* types of fractal sets defined with the help of *random series*.
Hence, in Section 3 we introduce and investigate several **classes of examples** of random countable iterated systems with overlaps. First, we will give several ways to randomize countable IFS related to generalizations of Kahane-Salem sets ([8]) and infinite convolutions of Bernoulli distributions. Then, we shall give examples of random infinite conformal IFS with overlaps in the plane, which have a uniformly bounded preimage counting function; we will study the projection measures on the respective limit sets, finding lower and upper bounds for their pointwise dimensions. We will also investigate a randomized finite iterated function system based on a statistical problem of Sinai, and will verify the exact dimensionality of projection measures on its limit set.

We mention that several authors investigated the question of dimension for measures in the context of random dynamical systems or random finite iterated function systems, for eg [2], [7], [14], [16], [19], etc. Our randomization here is different from the one studied in [14].

2. Preliminaries from Random Countable Alphabet Iterated Function Systems.

First let us recall some well-known geometric concepts, see for eg [15], [18]). For a finite Borel measure μ on a metric space (X, ρ), we denote by $\underline{d}_\mu(x)$ and $\overline{d}_\mu(x)$ respectively, the lower and upper limits of $\frac{\log \mu(B(x,r))}{\log r}$, when $r \to 0$. These *lower, and upper pointwise dimensions* of μ are guaranteed to exist at every $x \in X$, in contrast to the limit in (1.1). Now define also the dimensions:

$$\mathrm{HD}_\star(\mu) := \inf\{\mathrm{HD}(Y) : \mu(Y) > 0\} \text{ and } \mathrm{HD}^\star(\mu) = \inf\{\mathrm{HD}(Y) : \mu(X \setminus Y) = 0\}.$$

In the case when $\mathrm{HD}_\star(\mu) = \mathrm{HD}^\star(\mu)$, this common value is called the *Hausdorff dimension of the measure* μ and is denoted by $\mathrm{HD}(\mu)$.

Analogous concepts can be formulated for packing dimension, with respective notation $\mathrm{PD}_\star(\mu)$, $\mathrm{PD}^\star(\mu)$; if $\mathrm{PD}(\mu)$ exists, it is called the *packing dimension of the measure* μ.

The first (and very basic) relations between these concepts are given in the following well–known theorem (see for ex. [18]):

Theorem 2.1 (General properties of dimensions of measures on metric spaces)**.** *(i) If μ is a finite Borel measure on a metric space (X,ρ), then*

$$\mathrm{HD}_\star(\mu)=\operatorname{ess\,inf}\underline{d}_\mu,\ \mathrm{HD}^\star(\mu)=\operatorname{ess\,sup}\underline{d}_\mu,\quad \textit{and}\quad \mathrm{PD}_\star(\mu)=\operatorname{ess\,inf}\overline{d}_\mu,\ \mathrm{PD}^\star(\mu)=\operatorname{ess\,sup}\overline{d}_\mu$$

(ii) If μ is an exact dimensional finite Borel measure on a metric space (X,ρ), then both its Hausdorff dimension and packing dimension are well-defined and

$$\mathrm{HD}(\mu)=\mathrm{PD}(\mu)=d_\mu.$$

Let now X be a compact connected subset of $\mathbb{R}^q$, $q\geq 1$ with $X=\overline{\mathrm{Int}(X)}$. Consider also E to be a countable set (either finite or infinite), called an alphabet.

Definition 2.2. A *random countable conformal iterated function system*

$$\mathcal{S}=(\theta:\Lambda\to\Lambda,\{\lambda\mapsto\varphi_e^\lambda\}_{e\in E})$$

is defined by an invertible ergodic measure-preserving transformation of a complete probability space $(\Lambda,\mathcal{F},m)$, namely

$$\theta:(\Lambda,\mathcal{F},m)\to(\Lambda,\mathcal{F},m),$$

and by a family of injective conformal contractions on X, defined for each $e\in E$ and $\lambda\in\Lambda$,

$$\varphi_e^\lambda:X\to X,$$

all of whose Lipschitz constants do not exceed a common value $0<s<1$. We in fact assume that there exists a bounded open connected set $W\subset\mathbb{R}^q$ containg X, such that all maps $\phi_e^\lambda:X\to X$ extend confomally to (injective) maps from W to W. □

We will denote in the sequel by $E^{\mathbb{N}}$ the space of one-sided infinite sequences $\omega=(\omega_0,\omega_1,\ldots),\omega_i\in E, i\geq 0$; and by E^* the set of all finite sequences $\tau=(\tau_0,\tau_1,\ldots,\tau_k),\tau_i\in E, 0\leq i\leq k, k\geq 1$. We have the usual shift map $\sigma:E^{\mathbb{N}}\to E^{\mathbb{N}}$.

We shall assume in the sequel that the contraction maps $\varphi_e^\lambda:W\to W$ satisfy the following *Bounded Distortion Property* (BDP):

Property 2.3 (BDP)**.** *There exists a function $K:[0,1)\to[1,\infty)$ such that $\lim_{t\searrow 0}K(t)=K(0)=1$, and*

$$\sup\left\{\frac{\left|(\phi_\omega^\lambda)'(y)\right|}{\left|(\phi_\omega^\lambda)'(x)\right|}:e\in E,\ \lambda\in\Lambda,\ x\in X,\ ||y-x||\leq t\cdot\mathrm{dist}(x,\mathbb{R}^q\setminus W)\right\}\leq K(t).$$

We also require some common measurability conditions. Precisely, we assume that for every $e \in E$ and every $x \in X$ the map

$$\Lambda \ni \lambda \mapsto \varphi_e^\lambda(x)$$

is measurable. According to Lemma 1.1 in [3], this implies that, for all $e \in E$, the maps

$$\Lambda \times X \ni (\lambda, x) \mapsto \varphi_e(x, \lambda) := \varphi_e^\lambda(x)$$

are (jointly) measurable. For every finite sequence $\omega \in E^*$, and every $\lambda \in \Lambda$, let us define also the (randomized) composition of contractions

$$\varphi_\omega^\lambda := \varphi_{\omega_1}^\lambda \circ \varphi_{\omega_2}^{\theta(\lambda)} \circ \ldots \circ \varphi_{\omega_{|\omega|}}^{\theta^{|\omega|-1}(\lambda)}$$

This formula exhibits the random aspect of our iterations: we choose consecutive generators $\varphi_{\omega_1}, \varphi_{\omega_2}, \ldots, \varphi_{\omega_n}$ according to a random process governed by the ergodic map $\theta : \Lambda \to \Lambda$. This random aspect is particularly striking if θ is a Bernoulli shift when, in the random composition we choose ϕ_e^λ in an independent identically distributed way.

Given $\omega \in E^{\mathbb{N}}$ and $\lambda \in \Lambda$, we define, analogously to the deterministic case, the singleton

$$\pi_\lambda(\omega) := \bigcap_{n=1}^{\infty} \varphi_{\omega|_n}^\lambda(X),$$

and then the fractal limit set of the random countable IFS, corresponding to $\lambda \in \Lambda$ is:

$$J_\lambda := \pi_\lambda(E^{\mathbb{N}})$$

Let us denote by $\pi_\Lambda : \Lambda \times E^{\mathbb{N}} \to \Lambda$ and $\pi_{E^{\mathbb{N}}} : \Lambda \times E^{\mathbb{N}} \to E^{\mathbb{N}}$, the projections on the first, respectively the second coordinates. And by $\pi_{\mathbb{R}^q} : \Lambda \times E^{\mathbb{N}} \to \mathbb{R}^q$ the projection defining the limit sets $J_\lambda,\ \lambda \in \Lambda$, namely $\pi_{\mathbb{R}^q}(\lambda, \omega) = \pi_\lambda(\omega)$, for $(\lambda, \omega) \in \Lambda \times E^{\mathbb{N}}$.

Let us also denote by ξ the partion of $E^{\mathbb{N}}$ into initial cylinders of length 1; we will work in the sequel with conditional entropies of partitions and of probability measures (see for example [21], [9] for general definitions and properties).

Given a Lebesgue space $(Y, \mathcal{B}, \mu)$ and two measurable partitions of it, η and ζ, we will sometimes write $H_\mu(\eta|\zeta)$ without loss of generality, for the measure-theoretic conditional entropy $H_\mu(\eta|\hat{\zeta})$ of the partition η with respect to the σ-algebra $\hat{\zeta}$ generated by ζ. We will introduce now a notion of *measure-theoretical projectional entropy* for the random infinite system and for a projection measure, which is similar to the projection entropy from [6], but which is adapted to the random setting.

Definition 2.4. Given the random countable iterated function system $\mathcal{S}$ as above, and a $\theta \times \sigma$–invariant probability measure μ on $\Lambda \times E^{\mathbb{N}}$, define the *random projectional entropy* of the measure μ relative to the system $\mathcal{S}$, to be:

$$\mathrm{h}_\mu(\mathcal{S}) := \mathrm{H}_\mu\big(\pi_{E^{\mathbb{N}}}^{-1}(\xi) \big| \pi_\Lambda^{-1}(\varepsilon_\Lambda) \vee (\theta \times \sigma)^{-1}(\pi_{\mathbb{R}^q}^{-1}(\varepsilon_{\mathbb{R}^q}))\big) - \mathrm{H}_\mu\big(\pi_{E^{\mathbb{N}}}^{-1}(\xi) \big| \pi_\Lambda^{-1}(\varepsilon_\Lambda) \vee \pi_{\mathbb{R}^q}^{-1}(\varepsilon_{\mathbb{R}^q})\big),$$

where $\varepsilon_\Lambda, \varepsilon_{\mathbb{R}^q}$ are the point partitions of Λ, respectively $\mathbb{R}^q$.

In the sequel we will consider only those $\theta\times\sigma$–invariant probability measures μ on $\Lambda\times E^{\mathbb{N}}$ whose marginal measure on the parameter space Λ is equal to m, i. e. such that

$$\mu\circ\pi_\Lambda^{-1}=m$$

We denote then by $(\mu_\lambda)_{\lambda\in\Lambda}$ the Rokhlin's disintegration of the measure μ with respect to the fiber partition $(\pi_\Lambda^{-1})_{\lambda\in\Lambda}$. Its elements, $\{\lambda\}\times E^{\mathbb{N}}$, $\lambda\in\Lambda$, will be frequently identified with the set $E^{\mathbb{N}}$ and we will treat each probability measure μ_λ as defined on $E^{\mathbb{N}}$.

The desintegration $(\mu_\lambda)_{\lambda\in\Lambda}$ depending measurably on λ, is uniquely determined by the property that for any μ-integrable function $g:\Lambda\times E^{\mathbb{N}}\to\mathbb{R}$, we have

$$\int_{\Lambda\times E^{\mathbb{N}}} g d\mu=\int_\Lambda\int_{E^{\mathbb{N}}} g d\mu_\lambda\ dm(\lambda)$$

Thus from Lemma 2.2.3 in [2], we have the following equivalent desintegration formula for the random projectional entropy:

$$\mathrm{h}_\mu(\mathcal{S})=\int_\Lambda \mathrm{H}_{\mu_\lambda}\big(\xi\big|\sigma^{-1}(\pi_{\theta(\lambda)}^{-1}(\varepsilon_{J_{\theta(\lambda)}})\big)dm(\lambda)-\int_\Lambda \mathrm{H}_{\mu_\lambda}\big(\xi\big|\pi_\lambda^{-1}(\varepsilon_{J_\lambda})\big)dm(\lambda) \tag{2.1}$$

Using Definition 2.4 and the definitions of conditional entropy and conditional expectations (for eg from [21], etc.), we can then further write:

$$\begin{aligned}\mathrm{h}_\mu(\mathcal{S})=\int_\Lambda\Big[\int_{E^{\mathbb{N}}}&\log E_{\mu_\lambda}\big(\mathbb{1}_{[\omega_1]}\big|\pi_\lambda^{-1}(\varepsilon_{J_\lambda})\big)(\omega)d\mu_\lambda(\omega)-\\&-\int_{E^{\mathbb{N}}}\log E_{\mu_\lambda}\big(\mathbb{1}_{[\omega_1]}\big|(\pi_{\theta(\lambda)}\circ\sigma)^{-1}(\varepsilon_{J_{\theta(\lambda)}})\big)(\omega)d\mu_\lambda(\omega)\Big]dm(\lambda)\end{aligned} \tag{2.2}$$

We will see that there are important differences from the finite deterministic case, since here we have a family $(J_\lambda)_{\lambda\in\Lambda}$ of possibly non-compact limit sets, and a family of boundaries at infinity $(\partial_\infty\mathcal{S}_\lambda)_{\lambda\in\Lambda}$. The λ-*boundary at infinity* of $\mathcal{S}$, denoted by $\mathcal{S}_\lambda(\infty)$, is defined as the set of accumulation points of sequences of type $(\phi_{e_n}^\lambda(x_n))_n$, for arbitrary points $x_n\in X$ and infinitely many different indices $e_n\in E$. Similarly as in the deterministic case [12], we define

$$\mathcal{S}_\lambda^+(\infty):=\bigcup_{\omega\in E^*}\phi_\omega^{\theta(\lambda)}(\mathcal{S}_\lambda(\infty))$$

We give now some results about the relations between the random projectional entropy $h_\mu(\mathcal{S})$ and the measure-theoretical entropy $h(\mu)$ of the $(\theta\times\sigma)$-invariant probability μ on $\Lambda\times E^{\mathbb{N}}$. In this way we get bounds for the random projectional entropy $h_\mu(\mathcal{S})$.

Theorem 2.5. *In the above setting, if $\mathcal{S}$ is a random countable iterated function system and if μ is a $(\theta\times\sigma)$-invariant probability on $\Lambda\times E^{\mathbb{N}}$, we have the following inequalities:*

(a)

$$h_\mu(\mathcal{S})\le h(\mu)$$

(b) Assume that there exists an integer $k \geq 1$, such that for μ-almost every $(\lambda, \omega) \in \Lambda \times E^{\mathbb{N}}$ there exists $r(\lambda, \omega) > 0$ and k indices $e_1, \ldots, e_k \in E$, so that if the ball $B(\pi_\lambda(\omega), r(\lambda, \omega)) \subset \mathbb{R}^q$ intersects a set of type $\phi_e^{\lambda'}(J_{\lambda'})$, $e \in E, \lambda' \in \Lambda$, then e must belong to $\{e_1, \ldots, e_k\}$. Then

$$h_\mu(\mathcal{S}) \geq h(\mu) - \log k$$

Proof. (a) Let us denote by $\mathcal{B}$ the σ-algebra of borelian sets in $\mathbb{R}^q$, and by $\hat{\xi}$ the σ-algebra generated by the partition $\tilde{\xi} = \pi_{E^{\mathbb{N}}}^{-1}\xi$ in $\Lambda \times E^{\mathbb{N}}$. We want to prove first that

$$\hat{\xi} \vee (\theta \times \sigma)^{-1}\pi_{\mathbb{R}^q}^{-1}\mathcal{B} = \hat{\xi} \vee \pi_{\mathbb{R}^q}^{-1} \tag{2.3}$$

But an element of the σ-algebra $\hat{\xi} \vee (\theta \times \sigma)^{-1}\pi_{\mathbb{R}^q}^{-1}\mathcal{B}$ is a set of type

$$\bigcup_{i \in E} (\Lambda \times [i]) \cap (\theta \times \sigma)^{-1}\pi_{\mathbb{R}^q}^{-1} A_i,$$

where $A_i \in \mathcal{B}, i \in E$. Let us take an element $(\lambda, \omega) \in \pi_{\mathbb{R}^q}^{-1}(A_i)$, so $\pi_{\mathbb{R}^q}(\lambda, \omega) \in A_i$, where $\omega = (\omega_1, \omega_2, \ldots)$. Then an element ζ from the preimage set $(\theta^{-1} \times \sigma)^{-1}(\lambda, \omega)$, has the form $(\theta^{-1}\lambda, (\omega_0, \omega_1, \ldots)$, for arbitrary $\omega_0 \in E$; if this element belongs in addition to $\Lambda \times [i]$, then $\omega_0 = i$. Now $\pi_{\mathbb{R}^q}(\zeta) = \phi_i^{\theta^{-1}\lambda}(\pi_{\mathbb{R}^q}(\lambda, \omega) \in \phi_i^{\theta^{-1}\lambda}(A_i)$. Therefore we proved that

$$(\Lambda \times [i]) \cap (\theta \times \sigma)^{-1}\pi_{\mathbb{R}^q}^{-1} A_i = (\Lambda \times [i]) \cap \pi_{\mathbb{R}^q}^{-1}(\phi_i^{\theta^{-1}\lambda}(A_i))$$

Thus $\hat{\xi} \vee (\theta \times \sigma)^{-1}\pi_{\mathbb{R}^q}^{-1}\mathcal{B} \subseteq \hat{\xi} \vee \pi_{\mathbb{R}^q}^{-1}\mathcal{B}$, and after showing also the converse inequality of σ-algebras we obtain (2.3), i.e that $\hat{\xi} \vee (\theta \times \sigma)^{-1}\pi_{\mathbb{R}^q}^{-1}\mathcal{B} = \hat{\xi} \vee \pi_{\mathbb{R}^q}^{-1}\mathcal{B}$.

For an arbitrary integer $n \geq 1$, let us denote the measurable partition $\tilde{\xi}_0^{n-1} := \xi \vee \sigma^{-1}\xi \ldots \vee \sigma^{-n}\xi$. Using now the fact that the measure μ is $(\theta \times \sigma)$-invariant on $\Lambda \times E^{\mathbb{N}}$, and the same type of argument as in Lemma 4.8 of [6], we obtain that for every integer $n \geq 1$,

$$H_\mu(\tilde{\xi}_0^{n-1} | (\theta \times \sigma)^{-n}\pi_{\mathbb{R}^q}^{-1}\mathcal{B}) - H_\mu(\tilde{\xi}_0^{n-1} | \pi_{\mathbb{R}^q}^{-1}\mathcal{B}) = n \cdot \left[H_\mu(\tilde{\xi} | (\theta \times \sigma)^{-1}\pi_{\mathbb{R}^q}^{-1}\mathcal{B}) - H_\mu(\tilde{\xi} | \pi_{\mathbb{R}^q}^{-1}\mathcal{B})\right] \tag{2.4}$$

Hence from formula (2.4) we obtain the following inequality:

$$nh_\mu(\mathcal{S}) = H_\mu(\tilde{\xi}_0^{n-1} | (\theta \times \sigma)^{-1}\pi_{\mathbb{R}^q}^{-1}\mathcal{B}) - H_\mu(\tilde{\xi}_0^{n-1} | \pi_{\mathbb{R}^q}^{-1}\mathcal{B}) \leq H_\mu(\tilde{\xi}_0^{n-1})$$

Therefore, as $h(\mu)$ is the supremum of the limits of $\frac{1}{n} H_\mu\big(\bigvee_0^{n-1} (\theta \times \sigma)^{-i}\tau\big)$ when $n \to \infty$, over all partitions τ of $\Lambda \times E^{\mathbb{N}}$, we obtain the upper bound $h_\mu(\mathcal{S}) \leq h(\mu)$.

(b) We remind that ξ is the partition of $E^{\mathbb{N}}$ into the 1-cylinders $[i] := \{\omega \in E^{\mathbb{N}}, \omega = (\omega_1, \omega_2, \ldots),\ \omega_1 = i\}$, for $i \in E$; and also that for simplicity of notation, given in general 2 measurable partitions η, ζ of a Lebesgue space (Y, ν), we will sometimes write $H_\nu(\eta|\zeta)$ instead of $H_\nu(\eta|\hat{\zeta})$ where $\hat{\zeta}$ is the σ-algebra generated by ζ. We now assume that for μ-almost every $(\lambda, \omega) \in \Lambda \times E^{\mathbb{N}}$, there are at most k indices $e \in E$ so that sets of type $\phi_e^{\lambda'}(J_{\lambda'}), \lambda' \in \Lambda$ intersect the ball $B(\pi_\lambda(\omega), r(\lambda, \omega))$. Let us consider next the partition $\mathcal{P}_n$ of $\mathbb{R}^q$ with sets of type $I_{(i_1, \ldots, i_q)} = [\frac{i_1}{2^n}, \frac{i+1}{2^n}) \times \ldots \times [\frac{i_q}{2^n}, \frac{i_q+1}{2^n})$, for all multi-indices $(i_1, \ldots, i_q) \in \mathbb{Z}^q$.

For m-almost every $\lambda \in \Lambda$ we will now construct the subpartition $\mathcal{R}_n(\lambda) \subseteq \mathcal{P}_n$, which uses only those sets $I_{(i_1, \ldots, i_q)} \in \mathcal{P}_n$ that contain points $\pi_\lambda(\omega) \in J_\lambda, \omega \in E^{\mathbb{N}}$, with $r(\lambda, \omega) > q/2^n$, and where the union of all the remaining cubes $I_{(i_1,\dots,i_q)}$ of $\mathcal{P}_n$ represents just one element of $\mathcal{R}_n(\lambda)$. But we assumed that for μ-almost all $(\lambda,\omega)\in\Lambda\times E^{\mathbb{N}}$, there exists a radius $r(\lambda,\omega)>0$, such that:

$$\text{Card}\{i\in E,\ \exists\lambda'\in\Lambda \text{ s.t } B(\pi_\lambda(\omega),r(\lambda,\omega))\cap\phi_i^{\lambda'}(J_{\lambda'})\neq\emptyset\}\le k \tag{2.5}$$

So using the fact that n was chosen so that any cube $I_{(i_1,\dots,i_q)}\in\mathcal{R}_n(\lambda)$ contains at least a point of type $\pi_\lambda(\omega),\omega\in E^{\mathbb{N}}$ with $r(\lambda,\omega)>\frac{q}{2^n}$, we obtain that any fixed set A from the partition $\pi_\lambda^{-1}(\mathcal{R}_n(\lambda))$ of $E^{\mathbb{N}}$, intersects *at most* k elements of the partition $\xi\vee\pi_\lambda^{-1}(\mathcal{R}_n(\lambda))$ of $E^{\mathbb{N}}$. Recall also that $\mu_\lambda\circ\pi_\lambda^{-1}$ is a σ-invariant probability measure on $E^{\mathbb{N}}$, for $\lambda\in\Lambda$. Hence from above and using [13], [21], it follows that the conditional entropy $H_{\mu_\lambda}\big(\xi|\pi_\lambda^{-1}(\mathcal{R}_n(\lambda))\big)$ satisfies:

$$H_{\mu_\lambda}\big(\xi|\pi_\lambda^{-1}(\mathcal{R}_n(\lambda))\big)=H_{\mu_\lambda}(\xi\vee\pi_\lambda^{-1}\mathcal{R}_n(\lambda))-H_{\mu_\lambda}(\pi_\lambda^{-1}(\mathcal{R}_n(\lambda))\le\log k \tag{2.6}$$

But now, since we known that for μ-almost all $(\lambda,\omega)\in\Lambda\times E^{\mathbb{N}}$ there exists a radius $r(\lambda,\omega)>0$ satisfying condition (2.5), we infer that $\pi_\lambda^{-1}(\mathcal{R}_n(\lambda))\nearrow\pi_\lambda^{-1}(\epsilon_{\mathbb{R}^q})$, when $n\to\infty$; and the same conclusion for the respective σ-algebras generated by these partitions in $E^{\mathbb{N}}$. Therefore from (2.6) and [13], and since $\mu\circ\pi_\Lambda^{-1}=m$, it follows that for m-almost every $\lambda\in\Lambda$, the conditional entropy $H_{\mu_\lambda}(\xi|\pi_\lambda^{-1}\mathcal{B})$ satisfies the inequality

$$H_{\mu_\lambda}(\xi|\pi_\lambda^{-1}(\mathcal{B}))=\lim_{n\to\infty}H_{\mu_\lambda}(\xi|\pi_\lambda^{-1}\mathcal{R}_n(\lambda))\le\log k$$

In addition we have that for m-almost any parameter $\lambda\in\Lambda$,

$$H_{\mu_\lambda}(\xi|\sigma^{-1}(\pi_{\theta(\lambda)}^{-1}\epsilon_{J_{\theta(\lambda)}}))\ge H_{\mu_\lambda}(\xi|\sigma^{-1}(\mathcal{B}(E^{\mathbb{N}})))=h_\sigma(\mu_\lambda),$$

since ξ is a generator partition for μ_λ on $E^{\mathbb{N}}$, and by using section 3-1 of [13]. Therefore, from (2.1) and the last two displayed inequalities, we obtain the required inequality, namely

$$h_\mu(\mathcal{S})\ge\int_\Lambda h_\sigma(\mu_\lambda)dm(\lambda)-\log k=h(\mu)-\log k$$

□

Remark 2.6. We remark that the condition in Theorem 2.5, part (b), implies that there are no points from $\mathcal{S}_\lambda(\infty)$ in any of the limit sets $J_{\lambda'}$ for all $\lambda,\lambda'\in\Lambda$. We shall give an example of such a random infinite system with overlaps in the last section. The difficulty without this condition is that, there may be a variable number of overlaps at points from the possibly non-compact fractal J_λ, and that this number may tend to ∞ even for a given λ, or that it may tend to ∞ when λ varies in Λ; in both of these cases, we cannot obtain however a lower estimate for $h_\mu(\mathcal{S})$ like the one in Theorem 2.5 (b).

3. Pointwise dimension for random projections of measures.

Given a metric space (X,ρ) and a measurable map $H: E^{\mathbb{N}} \to X$, then for every sequence $\omega \in E^{\mathbb{N}}$ and every $r>0$, we shall denote by

$$B_H(\omega,r) := H^{-1}(B_\rho(H(\omega),r)).$$

Throughout this section we keep the setting and notation from the previous section. Our main result in this section is the exact dimensionality of random projections μ_λ on J_λ, of $(\theta\times\sigma)$-invariant probabilities μ from $\Lambda\times E^{\mathbb{N}}$, for m-almost all parameters $\lambda\in\Lambda$. We start the proofs with the following:

Lemma 3.1. *For all integers $k\geq 0$, every $e\in E$ and $\lambda\in\Lambda$, and μ_λ-a.e. $\omega\in E^{\mathbb{N}}$, we have*

$$\lim_{r\to 0}\log\frac{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\cap[e]\big)}{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\big)} = \log E_{\mu_\lambda}\big(1\!\!1_{[e]}\big|(\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q})\big)\big)(\omega).$$

Proof. Fix $e\in E$ and define the following two Borel measures on $\mathbb{R}^q$:

$$\nu_\lambda := \mu_\lambda\circ(\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}, \text{ and} \tag{3.1}$$

$$\nu_\lambda^e(D) := \mu_\lambda\big([e]\cap(\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}(D)\big), \quad D \text{ Borel set in } \mathbb{R}^d. \tag{3.2}$$

Since $\nu_\lambda^e\leq\nu_\lambda$, the measure ν_λ^e is absolutely continuous with respect to ν_λ. Let us then define the Radon-Nikodym derivative of ν_λ^e with respect to ν_λ:

$$g_\lambda^e := \frac{d\nu_\lambda^e}{d\nu_\lambda}$$

Then, by Theorem 2.12 in [10], we have that:

$$g_\lambda^e(x) = \lim_{r\to 0}\frac{\nu_\lambda^e(B(x,r)}{\nu_\lambda(B(x,r)} \tag{3.3}$$

for ν_λ-a.e. $x\in\mathbb{R}^q$. On the other hand, for every set $F\in(\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}\big(\mathcal{B}_{\mathbb{R}^q}\big)$, say $F = (\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}(\tilde{F})$, $\tilde{F}\in\mathcal{B}_{\mathbb{R}^q}$, we have

$$\begin{aligned}
\int_F E_{\mu_\lambda}\big(1\!\!1_{[e]}\big|(\pi_{\theta^k(\lambda)}\circ\sigma)^{-1}(\mathcal{B}_{\mathbb{R}^q})\big)d\mu_\lambda &= \int_F 1\!\!1_{[e]}d\mu_\lambda = \mu_\lambda(F\cap[e]) \\
&= \mu_\lambda\big((\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}(\tilde{F})\cap[e]\big) = \nu_\lambda^e(\tilde{F}) = \int_{\tilde{F}} g_\lambda^e d\nu_\lambda \\
&= \int_{\tilde{F}} g_\lambda^e d(\mu_\lambda\circ(\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}) = \int_{\mathbb{R}^q} 1\!\!1_{\tilde{F}}\, g_\lambda^e d(\mu_\lambda\circ(\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}) \\
&= \int_{E^{\mathbb{N}}} 1\!\!1_{\tilde{F}}\circ(\pi_{\theta^k(\lambda)}\circ\sigma^k)\, g_\lambda^e\circ(\pi_{\theta^k(\lambda)}\circ\sigma^k)d\nu_\lambda \\
&= \int_{E^{\mathbb{N}}} 1\!\!1_F\, g_\lambda^e\circ(\pi_{\theta^k(\lambda)}\circ\sigma^k)d\nu_\lambda \\
&= \int_F g_\lambda^e\circ(\pi_{\theta^k(\lambda)}\circ\sigma^k)d\nu_\lambda.
\end{aligned}$$

Since, in addition, both functions $E_{\mu_\lambda}\big(\mathbb{1}_{[e]}\big|(\pi_{\theta^k(\lambda)}\circ\sigma)^{-1}(\mathcal{B}_{\mathbb{R}^q})\big)$ and $g_\lambda^e\circ(\pi_{\theta^k(\lambda)}\circ\sigma^k)$ are non-negative and measurable with respect to the σ-algebra $(\pi_{\theta^k(\lambda)}\circ\sigma)^{-1}(\mathcal{B}_{\mathbb{R}^q})$, we conclude that

$$g_\lambda^e\circ(\pi_{\theta^k(\lambda)}\circ\sigma)^{-1}(\mathcal{B}_{\mathbb{R}^q})(\omega)=E_{\mu_\lambda}\big(\mathbb{1}_{[e]}\big|(\pi_{\theta^k(\lambda)}\circ\sigma)^{-1}(\mathcal{B}_{\mathbb{R}^q})\big)(\omega)$$

for μ_λ-a.e. $\omega\in E^{\mathbb{N}}$. Along with (3.3) this means that

$$\lim_{r\to 0}\frac{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\cap[e]\big)}{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\big)}=E_{\mu_\lambda}\big(\mathbb{1}_{[\omega_1]}\big|(\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}(\varepsilon_{J_\lambda})\big)(\omega)$$

for μ_λ-a.e. $\omega\in E^{\mathbb{N}}$. Taking logarithms the lemma follows.

$\square$

Corollary 3.2. *For all integers $k\geq 0$, all $\lambda\in\Lambda$, and μ_λ-a.e. $\omega\in E^{\mathbb{N}}$, we have*

$$\lim_{r\to 0}\log\frac{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\cap[\omega_1]\big)}{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\big)}=\log E_{\mu_\lambda}\big(\mathbb{1}_{[\omega_1]}\big|(\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q}))\big).$$

Proof. We have

$$\begin{aligned}\lim_{r\to 0}\log&\frac{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\cap[\omega_1]\big)}{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\big)}=\\ &=\sum_{e\in E}\mathbb{1}_{[e]}(\omega)\lim_{r\to 0}\log\frac{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\cap[e]\big)}{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\big)}\\ &=\sum_{e\in E}\mathbb{1}_{[e]}(\omega)\log E_{\mu_\lambda}\big(\mathbb{1}_{[e]}\big|(\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q})\big)(\omega)\\ &=\log E_{\mu_\lambda}\big(\mathbb{1}_{[\omega_1]}\big|(\pi_{\theta^k(\lambda)}\circ\sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q}))\big)(\omega).\end{aligned}$$

$\square$

Now we shall prove the following.

Lemma 3.3. *If $\mathrm{H}_\mu\big(\pi_{E^{\mathbb{N}}}^{-1}(\xi)|\pi_\Lambda^{-1}(\varepsilon_\Lambda)\big)<\infty$, then the function*

$$g(\lambda,\omega):=-\inf_{r>0}\log\frac{\mu_\lambda\big([\omega_1]\cap B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\big)}{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega,r)\big)}\in\mathbb{R}$$

is integrable with respect to the measure μ, that is it belongs to $L^1(\mu)$.

Proof. Fix $\lambda\in\Lambda$. Fix also $e\in E$. As in the proof of Lemma 3.1 consider measures ν_λ and ν_λ^e defined by (3.1) and (3.2) respectively. By Theorem 2.19 in [10] we have that

$$\begin{aligned}\nu_\lambda^e&\left(\left\{x\in\mathbb{R}^q:\inf_{r>0}\left\{\frac{\nu_\lambda^e(B(x,r)}{\nu_\lambda(B(x,r))}\right\}<t\right\}\right)=\\ &=\nu_\lambda^e\left(\left\{x\in\mathbb{R}^q:\sup_{r>0}\left\{\frac{\nu_\lambda(B(x,r))}{\nu_\lambda^e(B(x,r))}\right\}>1/t\right\}\right)\\ &\leq C_q t\nu_\lambda(\mathbb{R}^q)=C_q t,\end{aligned}$$

where $1 \le C_q < \infty$ is a constant depending only on q. What we obtained means that

$$\mu_\lambda\left(\left\{\omega \in E^{\mathbb{N}} : \inf_{r>0}\left\{\frac{\mu_\lambda\big([e] \cap B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega, r)\big)}{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega, r)\big)}\right\} < t\right\}\right) \le C_q t.$$

Let us define also the function:

$$G^e_\lambda(\omega) := \inf_{r>0}\left\{\frac{\mu_\lambda\big([e] \cap B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega, r)\big)}{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)}\circ\sigma^k}(\omega, r)\big)}\right\}.$$

Then the previous inequality can be rewritten as:

$$\mu_\lambda\big((G^e_\lambda)^{-1}([0,t))\big) \le C_q t.$$

Define now the function $g_\lambda : E^{\mathbb{N}} \to \mathbb{R}$ by $g_\lambda(\omega) = g(\lambda, \omega)$. Thus the following equlity holds:

$$g_\lambda = \sum_{e\in E} -\mathbb{1}_{[e]} \log G^e_\lambda.$$

Noting also that $g_\lambda \ge 0$, we obtain therefore:

$$\begin{aligned}
\int_{E^{\mathbb{N}}} g_\lambda d\mu_\lambda &= \sum_{e\in E} -\int_{[e]} \log G^e_\lambda d\mu_\lambda = \sum_{e\in E}\int_0^\infty \mu_\lambda\big(\{\omega \in [e] : -\log G^e_\lambda(\omega) > s\}\big)ds \\
&= \sum_{e\in E}\int_0^\infty \mu_\lambda\big(\{\omega \in [e] : G^e_\lambda(\omega) < e^{-s}\}\big)ds \\
&= \sum_{e\in E}\int_0^\infty \mu_\lambda\big(\{\omega \in E^{\mathbb{N}} : G^e_\lambda(\omega) < e^{-s}\} \cap [e]\big)ds \\
&\le \sum_{e\in E}\int_0^\infty \min\{\mu_\lambda([e]), Cqe^{-s}\}ds \\
&= \sum_{e\in E}\left(\int_0^{-\log\mu_\lambda([e])+\log C_q} \mu_\lambda([e])ds + \int_{-\log\mu_\lambda([e])+\log C_q} Cqe^{-s}ds\right) \\
&= \sum_{e\in E}\big(-\mu_\lambda([e])\log\mu_\lambda([e]) + \log(C_q)\mu_\lambda([e])) + \mu_\lambda([e])\big) \\
&= 1 + \log(C_q) + \sum_{e\in E}\big(-\mu_\lambda([e])\log\mu_\lambda([e])\big) \\
&= 1 + \log(C_q) + \mathrm{H}_{\mu_l}(\xi)
\end{aligned}$$

Since $\mathrm{H}_\mu\big(\pi^{-1}_{E^{\mathbb{N}}}(\xi)|\pi^{-1}_\Lambda(\varepsilon_\Lambda)\big) < \infty$, it therefore follows from Lemma 2.3 in [2] that

$$\begin{aligned}
\int_{\Lambda\times E^{\mathbb{N}}} g d\mu = \int_\Lambda\int_{E^{\mathbb{N}}} g_\lambda d\mu_\lambda dm(\lambda) &\le 1 + \log(C_q) + \int_\Lambda \mathrm{H}_{\mu_l}(\xi) \\
&= 1 + \log(C_q) + \mathrm{H}_\mu\big(\pi^{-1}_{E^{\mathbb{N}}}(\xi)|\pi^{-1}_\Lambda(\varepsilon_\Lambda)\big) < \infty
\end{aligned}$$

The proof is thus finished.

$\square$

Remark 3.4. We assumed above the finite entropy condition $\mathrm{H}_\mu\big(\pi^{-1}_{E^{\mathbb{N}}}(\xi)|\pi^{-1}_\Lambda(\varepsilon_\Lambda)\big) < \infty$. This is not a restrictive condition, and it is satisfied by many measures and systems. For example, it is clearly satisfied if the alphabet E is finite. More interestingly, it is also satisfied when E is infinite and $\mu = m \times \nu$, where m is an arbitrary θ-invariant probability on Λ, and ν is a σ-invariant probability on $E^{\mathbb{N}}$ satisfying $\nu([i]) = \nu_i, i \in E$ and

$$h(\nu) = -\sum_{i \in E} \nu_i \log \nu_i < \infty$$

Indeed, if $\mathcal{A}$ is the σ-algebra generated in $\Lambda \times E^{\mathbb{N}}$ by the partition $\pi_\Lambda^{-1}(\epsilon_\Lambda)$, and if $\tilde{\xi} := \pi_{E^{\mathbb{N}}}^{-1}\xi$, then $H_\mu(\tilde{\xi}|\mathcal{A}) = \int I_\mu(\tilde{\xi}|\mathcal{A})$, where $I_\mu(\tilde{\xi}|\mathcal{A})$ is the information function

$$I_\mu(\tilde{\xi}|\mathcal{A}) := -\sum_{A \in \tilde{\xi}} \chi_A \cdot \log E_\mu(\chi_A|\mathcal{A})$$

Now, the conditional expectation $E_\mu(\chi_A|\mathcal{A}) =: g_A$ is $\mathcal{A}$-measurable, and $\int_{B \times E^{\mathbb{N}}} g d\mu = \int_{B \times E^{\mathbb{N}}} \chi_A d\mu$, for all sets B measurable in Λ. Hence if $A = \Lambda \times [i]$, then $\int g_A d\mu = \mu(A \cap (B \times E^{\mathbb{N}})) = m(B) \cdot \nu_i$, so $g_A = \nu_i$ and $H_\mu(\tilde{\xi}|\mathcal{A}) = -\sum_{i \in E} \nu_i \log \nu_i$. Therefore, if $h(\nu) < \infty$, then

$$H_\mu(\tilde{\xi}|\mathcal{A}) < \infty$$

□

As an immediate consequence of Lemma 3.3, Corollary 3.2, and Lebesgue's Dominated Convergence Theorem, we get the following:

Lemma 3.5. *If* $\mathrm{H}_\mu\big(\pi_{E^{\mathbb{N}}}^{-1}(\xi)|\pi_\Lambda^{-1}(\varepsilon_\Lambda)\big) < \infty$, *then*

$$\lim_{r \to 0} \log \frac{\mu_\lambda\big([\omega_1] \cap B_{\pi_{\theta^k(\lambda)} \circ \sigma^k}(\omega, r)\big)}{\mu_\lambda\big(B_{\pi_{\theta^k(\lambda)} \circ \sigma^k}(\omega, r)\big)} = \log E_{\mu_\lambda}\big(\mathbb{1}_{[\omega_1]}\big|(\pi_{\theta^k(\lambda)} \circ \sigma^k)^{-1}(\mathcal{B}_{\mathbb{R}^q})\big)\big)(\omega)$$

for μ*-a.e.* $(\lambda, \omega) \in \Lambda \times E^{\mathbb{N}}$, *and the convergence holds also in* $L^1(\mu)$.

Now we shall prove the following:

Lemma 3.6. *For every* $K \geq 1$ *there exists* $R_1 > 0$ *such that*

$$[\omega_1] \cap B_{\pi_\lambda}\big(\omega, K\big|\big(\phi_{\omega_1}^\lambda\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|r\big) \supset [\omega_1] \cap B_{\pi_{\theta(\lambda)} \circ \sigma}(\omega, r)$$

for all $\lambda \in \Lambda$, *all* $\omega \in E^{\mathbb{N}}$, *and all* $r \in [0, R_1]$.

Proof. Let $\tau \in B_{\pi_{\theta(\lambda)} \circ \sigma}(\omega, r)$. Then $\tau_1 = \omega_1$ and $\pi_{\theta(\lambda)}(\sigma(\tau)) \in B(\pi_{\theta(\lambda)}(\sigma(\omega)), r)$. Hence,

$$\begin{aligned}
\pi_\lambda(\tau) &= \phi_{\omega_1}^\lambda\big(\pi_{\theta(\lambda)}(\sigma(\tau))\big) \in \phi_{\omega_1}^\lambda\big(B(\pi_{\theta(\lambda)}(\sigma(\omega)), r)\big) \\
&\subset B\big(\phi_{\omega_1}^\lambda\big(\pi_{\theta(\lambda)}(\sigma(\omega))\big), K\big|\big(\phi_{\omega_1}^\lambda\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|r\big) \\
&= B\big(\pi_\lambda(\omega), K\big|\big(\phi_{\omega_1}^\lambda\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|r\big),
\end{aligned}$$

where, because of the Bounded Distortion Property (BDP), the inclusion sign "$\subset$" holds assuming $r > 0$ to be small enough. This means that

$$\tau \in \pi_\lambda^{-1}\big(B\big(\pi_\lambda(\omega), K\big|\big(\phi_{\omega_1}^\lambda\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|r\big)\big) = B_{\pi_\lambda}\big(\omega, K\big|\big(\phi_{\omega_1}^\lambda\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|r\big)$$

Since also already know that $\tau_1 = \omega_1$, we are thus done.

$\square$

Lemma 3.7. *For every $K \geq 1$ there exists $R_2 > 0$ such that*

$$[\omega_1] \cap B_{\pi_\lambda}\big(\omega, K^{-1}\big|\big(\phi_{\omega_1}^\lambda\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|r\big) \subset [\omega_1] \cap B_{\pi_{\theta(\lambda)}\circ\sigma}(\omega, r)$$

for all $\lambda \in \Lambda$, all $\omega \in E^{\mathbb{N}}$, and all $r \in [0, R_2]$.

Proof. Because of the Bounded Distortion Property (BDP), we have for all $r \geq 0$ small enough, say $0 \leq r \leq R_2$, that

$$\begin{aligned} B_{\pi_\lambda}\big(\omega, K^{-1}\big|\big(\phi_{\omega_1}^\lambda\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|r\big) &= \pi_\lambda^{-1}\big(B\big(\pi_\lambda(\omega), K^{-1}\big|\big(\phi_{\omega_1}^\lambda\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|r\big)\big) \\ &\subset \pi_\lambda^{-1}\big(\phi_{\omega_1}^\lambda\big(B(\pi_{\theta(\lambda)}(\sigma(\omega)), r)\big)\big) \end{aligned}$$

So, fixing $\tau \in [\omega_1] \cap B_{\pi_\lambda}\big(\omega, K^{-1}\big|\big(\phi_{\omega_1}^\lambda\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|r\big)$, we have $\tau_1 = \omega_1$ and

$$\pi_\lambda(\tau) = \phi_{\omega_1}^\lambda\big(\pi_{\theta(\lambda)}(\sigma(\tau))\big)\phi_{\omega_1}^\lambda\big(B(\pi_{\theta(\lambda)}(\sigma(\omega)), r)\big).$$

This means that $\pi_{\theta(\lambda)}(\sigma(\tau)) \in B(\pi_{\theta(\lambda)}(\sigma(\omega)), r)\big)$, or equivalently, $\tau \in B_{\pi_{\theta(\lambda)}\circ\sigma}(\omega, r)$. The required inclusion is thus proved and the proof is complete.

$\square$

Since the measure μ is fiberwise invariant, we have for all $\omega \in E^{\mathbb{N}}$, all $r > 0$, and m-a.e. $\lambda \in \Lambda$ that

$$\begin{aligned} \mu_\lambda\big(B_{\pi_{\theta(\lambda)}\circ\sigma}(\omega), r)\big) &= \mu_\lambda\big((\pi_{\theta(\lambda)} \circ \sigma)^{-1}\big(B(\pi_{\theta(\lambda)} \circ \sigma(\omega), r)\big)\big) \\ &= \mu_\lambda \circ \sigma^{-1}\big(\pi_{\theta(\lambda)}^{-1}\big(B\big(\pi_{\theta(\lambda)}(\sigma(\omega)), r\big)\big)\big) \\ &= \mu_{\theta(\lambda)}\big(B_{\pi_{\theta(\lambda)}}(\sigma(\omega), r)\big) \end{aligned} \tag{3.4}$$

As an immediate consequence of this formula along with Lemma 3.6 and Lemma 3.7, we get the following:

Lemma 3.8. *For every $K > 1$ there exists $R_K > 0$ such that*

$$\frac{\mu_\lambda\big([\omega_1] \cap B_{\pi_\lambda}\big(\omega, K\big|\big(\phi_{\omega_1}^\lambda\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|r\big)\big)}{\mu_{\theta(\lambda)}\big(B_{\pi_{\theta(\lambda)}}(\sigma(\omega), r)\big)} \geq \frac{\mu_\lambda\big([\omega_1] \cap B_{\pi_{\theta(\lambda)}\circ\sigma}(\omega, r)\big)}{\mu_\lambda\big(B_{\pi_{\theta(\lambda)}\circ\sigma}(\omega), r)\big)}$$

and

$$\frac{\mu_\lambda\big([\omega_1] \cap B_{\pi_\lambda}\big(\omega, K^{-1}\big|\big(\phi_{\omega_1}^\lambda\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|r\big)\big)}{\mu_{\theta(\lambda)}\big(B_{\pi_{\theta(\lambda)}}(\sigma(\omega), r)\big)} \leq \frac{\mu_\lambda\big([\omega_1] \cap B_{\pi_{\theta(\lambda)}\circ\sigma}(\omega, r)\big)}{\mu_\lambda\big(B_{\pi_{\theta(\lambda)}\circ\sigma}(\omega, r)\big)}$$

for all $\omega \in E^{\mathbb{N}}$, all $r \in (0, R_K]$, and m-a.e. $\lambda \in \Lambda$.

Lemma 3.9. *We have that*

$$\int_\Lambda \int_{E^{\mathbb{N}}} \log \mu_\lambda\big(B_{\pi_\lambda}(\omega, r)\big) d\mu_\lambda(\omega) dm(\lambda) > -\infty$$

for all $r > 0$.

Proof. Since X is compact there exist finitely many points $z_1, z_2, \ldots, z_l$ in X such that

$$\bigcup_{j=1}^{l} B(z_j, r/2) \supset X.$$

For every $\lambda \in \Lambda$ and every integer $n \geq 0$ define the set of sequences:

$$A_n(\lambda) := \{\omega \in E^{\mathbb{N}} : e^{-(n+1)} < \mu_\lambda\big(B_{\pi_\lambda}(\omega, r)\big) \leq e^{-n}\}.$$

Assume that

$$A_n(\lambda) \cap \pi_\lambda^{-1}(B(z_j, r/2)) \neq \emptyset$$

for some $1 \leq j \leq l$. Fix $\gamma \in A_n(\lambda) \cap \pi_\lambda^{-1}(B(z_j, r/2))$ arbitrary. Then, because of the triangle inequality, $\pi_\lambda^{-1}(B(z_j, r/2)) \subset B_{\pi_\lambda}(\omega, r)$. Therefore,

$$\mu_\lambda\big(A_n(\lambda) \cap \pi_\lambda^{-1}(B(z_j, r/2))\big) \leq \mu_\lambda\big(\pi_\lambda^{-1}(B(z_j, r/2))\big) \leq \mu_\lambda\big(B_{\pi_\lambda}(\omega, r)\big) \leq e^{-n}.$$

However, $\mu_\lambda(A_n(\lambda)) \cap \pi_\lambda^{-1}(B(z_i, r/2))\big) = 0 \leq e^{-n}$ if $A_n(\lambda) \cap \pi_\lambda^{-1}(B(z_i, r/2)) = \emptyset$, for some $1 \leq i \leq l$. Hence, since $\big\{\pi_\lambda^{-1}(B(z_j, r/2))\big\}_{j=1}^{l}$ is a cover of $E^{\mathbb{N}}$, this implies that

$$\mu_\lambda(A_n(\lambda)) \leq l e^{-n}$$

Therefore we obtain,

$$\begin{aligned}
\int_{E^{\mathbb{N}}} -\log \mu_\lambda\big(B_{\pi_\lambda}(\omega, r)\big) d\mu_\lambda(\omega) &= \sum_{n=0}^{\infty} \int_{A_n(\lambda)} -\log \mu_\lambda\big(B_{\pi_\lambda}(\omega, r)\big) d\mu_\lambda(\omega) \\
&\leq \sum_{n=0}^{\infty} (n+1) l e^{-n} \\
&= l \sum_{n=0}^{\infty} (n+1) e^{-n} < \infty.
\end{aligned}$$

Hence, from the above, we can conclude that

$$\int_\Lambda \int_{E^{\mathbb{N}}} \log \mu_\lambda\big(B_{\pi_\lambda}(\omega, r)\big) d\mu_\lambda(\omega) dm(\lambda) \leq l \sum_{n=0}^{\infty} (n+1) e^{-n} < \infty.$$

$\square$

Then employing this lemma and Birkhoff's Ergodic Theorem, we obtain the following:

Lemma 3.10. *For all $r > 0$ and μ-a.e. $(\lambda, \omega) \in \Lambda \times E^{\mathbb{N}}$, we have:*

$$\lim_{n \to \infty} \frac{1}{n} \log \mu_{\theta^n(\lambda)}\big(B_{\pi_{\theta^n(\lambda)}}\big(\sigma^n(\omega), r\big)\big) = 0$$

Now, we shall prove the following:

Lemma 3.11. *If $\mathrm{H}_\mu\big(\pi_{E^{\mathbb{N}}}^{-1}(\xi) | \pi_\Lambda^{-1}(\varepsilon_\Lambda)\big) < \infty$, then for every $K > 1$, all $r \in (0, R_K)$ and μ-a.e. $(\lambda, \omega) \in \Lambda \times E^{\mathbb{N}}$, we have that*

$$\overline{\lim_{n \to \infty}} \frac{1}{n} \log \mu_\lambda\big(B_{\pi_\lambda}\big(\omega, K^{-n} \big| \big(\phi_{\omega|_n}^\lambda\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega))) \big| r\big)\big) \leq -h_\mu(\mathcal{S}), \tag{3.5}$$

and moreover

$$\varliminf_{n\to\infty} \frac{1}{n}\log\mu_\lambda\big(B_{\pi_\lambda}\big(\omega, K^n\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big) \geq -h_\mu(\mathcal{S}). \tag{3.6}$$

Proof. We prove the first inequality by relying on the second inequality of Lemma 3.8. The proof of the second inequality of the lemma is analogous and will be omitted. We have:

$$\begin{aligned}
&T^-_{\lambda,n}(\omega) = \\
&:= \log\mu_\lambda\big(B_{\pi_\lambda}\big(\omega, K^{-n}\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big) \\
&= \sum_{j=0}^{n-1}\log\frac{\mu_{\theta^j(\lambda)}\big(B_{\pi_{\theta^j(\lambda)}}\big(\sigma^j(\omega), K^{-(n-j)}\big|\big(\phi^{\theta^j(\lambda)}_{\sigma^j(\omega)|_{n-j}}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big)}{\mu_{\theta^j(\lambda)}\big(B_{\pi_{\theta^{j+1}(\lambda)}}\big(\sigma^{j+1}(\omega), K^{-(n-(j+1))}\big|\big(\phi^{\theta^{j+1}(\lambda)}_{\sigma^{j+1}(\omega)|_{n-(j+1)}}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big)}+ \\
&\qquad + \log\mu_{\theta^n(\lambda)}\big(B_{\pi_{\theta^n(\lambda)}}\big(\sigma^n(\omega), r\big)\big) \\
&= \sum_{j=0}^{n-1}\log\frac{\mu_{\theta^j(\lambda)}\big([(\sigma^j(\omega))_1]\cap B_{\pi_{\theta^j(\lambda)}}\big(\sigma^j(\omega), K^{-(n-j)}\big|\big(\phi^{\theta^j(\lambda)}_{\sigma^j(\omega)|_{n-j}}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big)}{\mu_{\theta^j(\lambda)}\big(B_{\pi_{\theta^{j+1}(\lambda)}}\big(\sigma^{j+1}(\omega), K^{-(n-(j+1))}\big|\big(\phi^{\theta^{j+1}(\lambda)}_{\sigma^{j+1}(\omega)|_{n-(j+1)}}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big)}- \\
&\quad - \sum_{j=0}^{n-1}\log\frac{\mu_{\theta^j(\lambda)}\big([(\sigma^j(\omega))_1]\cap B_{\pi_{\theta^j(\lambda)}}\big(\sigma^j(\omega), K^{-(n-j)}\big|\big(\phi^{\theta^j(\lambda)}_{\sigma^j(\omega)|_{n-j}}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big)}{\mu_\lambda\big(B_{\pi_{\theta^j(\lambda)}}\big(\sigma^j(\omega), K^{-(n-j)}\big|\big(\phi^{\theta^j(\lambda)}_{\sigma^j(\omega)|_{n-j}}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big)}+ \\
&\qquad + \log\mu_{\theta^n(\lambda)}\big(B_{\pi_{\theta^n(\lambda)}}\big(\sigma^n(\omega), r\big)\big) \\
&\leq \sum_{j=0}^{n-1}\log\frac{\mu_{\theta^j(\lambda)}\big([(\sigma^j(\omega))_1]\cap B_{\pi_{\theta^{j+1}(\lambda)}\circ\sigma}\big(\sigma^{j+1}(\omega), K^{-(n-(j+1))}\big|\big(\phi^{\theta^{j+1}(\lambda)}_{\sigma^{j+1}(\omega)|_{n-(j+1)}}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big)}{\mu_{\theta^j(\lambda)}\big(B_{\pi_{\theta^{j+1}(\lambda)}\circ\sigma}\big(\sigma^{j+1}(\omega), K^{-(n-(j+1))}\big|\big(\phi^{\theta^{j+1}(\lambda)}_{\sigma^{j+1}(\omega)|_{n-(j+1)}}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big)}- \\
&\quad - \sum_{j=0}^{n-1}\log\frac{\mu_{\theta^j(\lambda)}\big([(\sigma^j(\omega))_1]\cap B_{\pi_{\theta^j(\lambda)}}\big(\sigma^j(\omega), K^{-(n-j)}\big|\big(\phi^{\theta^j(\lambda)}_{\sigma^j(\omega)|_{n-j}}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big)}{\mu_\lambda\big(B_{\pi_{\theta^j(\lambda)}}\big(\sigma^j(\omega), K^{-(n-j)}\big|\big(\phi^{\theta^j(\lambda)}_{\sigma^j(\omega)|_{n-j}}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|r\big)\big)}+ \\
&\qquad + \log\mu_{\theta^n(\lambda)}\big(B_{\pi_{\theta^n(\lambda)}}\big(\sigma^n(\omega), r\big)\big) \\
&= \sum_{j=0}^{n-1}W^-_{n-j}((\theta\times\sigma)^j(\lambda,\omega)) - \sum_{j=0}^{n-1}G^-_{n-j}((\theta\times\sigma)^j(\lambda,\omega)) + \log\mu_{\theta^n(\lambda)}\big(B_{\pi_{\theta^n(\lambda)}}\big(\sigma^n(\omega), r\big)\big),
\end{aligned}$$

where for all $i \geq 1$,

$$W^-_i(\lambda,\omega) := \log\frac{\mu_\lambda\big([\omega_1]\cap B_{\pi_{\theta(\lambda)}\circ\sigma}\big(\sigma^{j+1}(\omega), K^{-(i-1)}\big|\big(\phi^{\theta(\lambda)}_{\sigma(\omega)|_{i-1}}\big)'(\pi_{\theta^i(\lambda)}(\sigma^i(\omega)))\big|r\big)\big)}{\mu_\lambda\big(B_{\pi_{\theta(\lambda)}\circ\sigma}\big(\sigma^{j+1}(\omega), K^{-(i-1)}\big|\big(\phi^{\theta(\lambda)}_{\sigma(\omega)|_{i-1}}\big)'(\pi_{\theta^i(\lambda)}(\sigma^i(\omega)))\big|r\big)\big)}$$

and where

$$G^-_i(\lambda,\omega) := \log\frac{\mu_\lambda\big([\omega_1]\cap B_{\pi_\lambda}\big(\omega, K^{-i}\big|\big(\phi^\lambda_{\omega|_i}\big)'(\pi_{\theta^i(\lambda)}(\sigma^i(\omega)))\big|r\big)\big)}{\mu_\lambda\big(B_{\pi_\lambda}\big(\omega, K^{-i}\big|\big(\phi^\lambda_{\omega|_i}\big)'(\pi_{\theta^i(\lambda)}(\sigma^i(\omega)))\big|r\big)\big)}.$$

Now, by virtue of Lemma 3.5 we see that Corollary 1.6, p. 96 in [9], applies to the sequences $(W^-_i)^\infty_{i=1}$ and $(W^-_i)^\infty_{i=1}$. This, in conjunction with Lemma 3.5, Lemma 3.10, the ergodicity of the measure μ with respect to the dynamical system $\theta\times\sigma$, and formula (2.2), gives us the following inequalities:

$$\begin{aligned}\varlimsup_{n\to\infty} T^-_{\lambda,n}(\omega) \le \int_{\Lambda\times E^{\mathbb{N}}} \Big(&\log E_{\mu_\lambda}\big(\mathbb{1}_{[\omega_1]}\big|(\pi_{\theta(\lambda)}\circ\sigma)^{-1}(\mathcal{B}_{\mathbb{R}^q})\big)\big)(\omega) - \\ &- \log E_{\mu_\lambda}\big(\mathbb{1}_{[\omega_1]}\big|(\pi_\lambda^{-1}(\mathcal{B}_{\mathbb{R}^q}))\big)(\omega)\Big) d\mu_\lambda(\omega)dm(\lambda) \\ = -\mathrm{h}_\mu(\mathcal{S}).\end{aligned}$$

This finishes thus the proof.

□

Definition 3.12. In the above setting, let us define the *Lyapunov exponent* of the measure μ with respect to the endomorphism $\theta\times\sigma : \Lambda\times E^{\mathbb{N}} \to \Lambda\times E^{\mathbb{N}}$ and the random countable iterated function system $\mathcal{S}$:

$$\chi_\mu := \int_{\Lambda\times E^{\mathbb{N}}} -\log\big|\big(\phi^\lambda_{\omega_1}\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big| d\mu(\lambda,\omega).$$

Since the above dynamical system is ergodic, then Birkhoff's Ergodic Theorem yields that, for μ-a.e. $(\lambda,\omega)\in\Lambda\times E^{\mathbb{N}}$, we have

$$\lim_{n\to\infty}\frac{1}{n}\log\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big| = \chi_\mu. \tag{3.7}$$

As a consequence of this lemma and Lemma 3.11, we now prove the main result of our paper:

Theorem 3.13. *If* $\mathrm{H}_\mu\big(\pi^{-1}_{E^{\mathbb{N}}}(\xi)|\pi^{-1}_\Lambda(\varepsilon_\Lambda)\big) < \infty$*, then for* μ*-a.e.* $(\lambda,\omega)\in\Lambda\times E^{\mathbb{N}}$*, we have*

$$\lim_{r\to 0}\frac{\log\big(\mu_\lambda\circ\pi_\lambda^{-1}\big(B_{\pi_\lambda}(\omega,r)\big)\big)}{\log r} = \frac{h_\mu(\mathcal{S})}{\chi_\mu}.$$

Proof. What we want to prove is that:

$$\lim_{r\to 0}\frac{\log\mu_\lambda\big(B_{\pi_\lambda}(\omega,r)\big)}{\log r} = \frac{\mathrm{h}_\mu(\mathcal{S})}{\chi_\mu}.$$

Fix $K>1$. Fix also $(\lambda,\omega)\in\Lambda\times E^{\mathbb{N}}$. Consider any $r\in\big(0, K^{-1}\big|\big(\phi^\lambda_{\omega_1}\big)'(\pi_{\theta(\lambda)}(\sigma(\omega)))\big|\big)$. There then exists a largest $n\ge 0$ such that

$$r \le K^{-n}\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big| R_K.$$

Then for $n\ge 1$,

$$B_{\pi_\lambda}(\omega,r)\subset B_{\pi_\lambda}\big(\omega, K^{-n}\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|R_K\big), \quad \text{and}$$

$$r \ge K^{-(n+1)}\big|\big(\phi^\lambda_{\omega|_{n+1}}\big)'(\pi_{\theta^{n+1}(\lambda)}(\sigma^{n+1}(\omega)))\big|R_K.$$

Therefore,

$$\begin{aligned}\frac{\log \mu_\lambda(\big(B_{\pi_\lambda}(\omega,r)\big)}{\log r} &\geq \frac{\log \mu_\lambda(B_{\pi_\lambda}\big(\omega, K^{-n}\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|R_K\big)\big)}{\log r} \\ &\geq \frac{\log \mu_\lambda(B_{\pi_\lambda}\big(\omega, K^{-n}\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|R_K\big)\big)}{-(n+1)\log K + \log\big|\big(\phi^\lambda_{\omega|_{n+1}}\big)'(\pi_{\theta^{n+1}(\lambda)}(\sigma^{n+1}(\omega)))\big| + \log R_K} \\ &= \frac{\frac{1}{n}\log \mu_\lambda(B_{\pi_\lambda}\big(\omega, K^{-n}\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|R_K\big)\big)}{-(1+\frac{1}{n})\log K + \frac{1}{n}\log\big|\big(\phi^\lambda_{\omega|_{n+1}}\big)'(\pi_{\theta^{n+1}(\lambda)}(\sigma^{n+1}(\omega)))\big| + \frac{1}{n}\log R_K}.\end{aligned}$$

Hence, applying formula (3.5) from Lemma 3.11, and also Lemma 3.7, we get

$$\varliminf_{r\to 0}\frac{\log \mu_\lambda(\big(B_{\pi_\lambda}(\omega,r)\big)}{\log r} \geq \frac{\mathrm{h}_\mu(\mathcal{S})}{\log K + \chi_\mu}$$

for all (λ,ω) in some measurable set $\Omega^+_K \subset \Lambda \times E^{\mathbb{N}}$ with $\mu(\Omega^+_K)=1$. Then

$$\mu\left(\Omega^+ := \bigcap_{j=1}^\infty \Omega^+_{\frac{j+1}{j}}\right) = 1$$

and

$$\varliminf_{r\to 0}\frac{\log \mu_\lambda(\big(B_{\pi_\lambda}(\omega,r)\big)}{\log r} \geq \frac{\mathrm{h}_\mu(\mathcal{S})}{\chi_\mu} \tag{3.8}$$

for all $(\lambda,\omega)\in\Omega^+$. For the proof of the opposite direction fix any $K>1$ so small that

$$K^{-1} > \operatorname{ess\,sup}\big\{\big|\big|\big(\phi^\lambda_e\big)'\big|\big| : e\in E, \lambda\in\Lambda\big\}. \tag{3.9}$$

Having $(\lambda,\omega)\in\Lambda\times E^{\mathbb{N}}$ fix any $r\in\big(0, KR_K \operatorname{ess\,sup}\big\{\big|\big|\big(\phi^\lambda_e\big)'\big|\big| : e\in E, \lambda\in\Lambda\big\}$. Because of (3.9) there exists a least $n\geq 1$ such that

$$K^n\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|R_K \leq r.$$

Then, because of our choice of r, we have that $n\geq 2$,

$$K^{n-1}\big|\big(\phi^\lambda_{\omega|_{n-1}}\big)'(\pi_{\theta^{n-1}(\lambda)}(\sigma^{n-1}(\omega)))\big|R_K \leq r,$$

and

$$B_{\pi_\lambda}(\omega,r) \supset B_{\pi_\lambda}\big(\omega, K^n\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|R_K\big).$$

Therefore,

$$\begin{aligned}\frac{\log \mu_\lambda(\big(B_{\pi_\lambda}(\omega,r)\big)}{\log r} &\leq \frac{\log \mu_\lambda(B_{\pi_\lambda}\big(\omega, K^{n}\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|R_K\big)\big)}{\log r} \\ &\geq \frac{\log \mu_\lambda(B_{\pi_\lambda}\big(\omega, K^{n}\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|R_K\big)\big)}{(n-1)\log K + \log\big|\big(\phi^\lambda_{\omega|_{n-1}}\big)'(\pi_{\theta^{n-1}(\lambda)}(\sigma^{n-1}(\omega)))\big| + \log R_K} \\ &= \frac{\frac{1}{n}\log \mu_\lambda(B_{\pi_\lambda}\big(\omega, K^{n}\big|\big(\phi^\lambda_{\omega|_n}\big)'(\pi_{\theta^n(\lambda)}(\sigma^n(\omega)))\big|R_K\big)\big)}{-(1-\frac{1}{n})\log K + \frac{1}{n}\log\big|\big(\phi^\lambda_{\omega|_{n-1}}\big)'(\pi_{\theta^{n-1}(\lambda)}(\sigma^{n-1}(\omega)))\big| + \frac{1}{n}\log R_K}.\end{aligned}$$

Hence, applying formula (3.5) from Lemma 3.11, and also Lemma 3.7, we get

$$\varlimsup_{r\to 0} \frac{\log \mu_\lambda(\big(B_{\pi_\lambda}(\omega, r)\big)}{\log r} \leq \frac{\mathrm{h}_\mu(\mathcal{S})}{-\log K + \chi_\mu}$$

for all (λ, ω) in some measurable set $\Omega_K^- \subset \Lambda \times E^{\mathbb{N}}$ with $\mu(\Omega_K^-) = 1$. Then we have:

$$\mu\left(\Omega^- := \bigcap_{j=k}^{\infty} \Omega^-_{\frac{j+1}{j}}\right) = 1,$$

where $k \geq 1$ is taken to be so large that $\frac{k+1}{k}\operatorname{ess\,sup}\left\{\big|\big|(\phi_e^\lambda)'\big|\big| : e \in E, \lambda \in \Lambda\right\} < 1$. Also,

$$\varlimsup_{r\to 0} \frac{\log \mu_\lambda(\big(B_{\pi_\lambda}(\omega, r)\big)}{\log r} \leq \frac{\mathrm{h}_\mu(\mathcal{S})}{\chi_\mu}$$

for all $(\lambda, \omega) \in \Omega^-$. Along with (3.8) this yields $\mu\big(\Omega^+ \cap \Omega^-\big) = 1$ and moreover,

$$\lim_{r\to 0} \frac{\log \mu_\lambda(\big(B_{\pi_\lambda}(\omega, r)\big)}{\log r} = \frac{\mathrm{h}_\mu(\mathcal{S})}{\chi_\mu}$$

for all $(\lambda, \omega) \in \Omega^+ \cap \Omega^-$, which gives therefore the required dimensional exactness.

□

Therefore, from the above Theorem 3.13 and Theorem 2.1 we obtain the following result, giving the (common) Hausdorff dimension and packing dimension of the projections $\mu_\lambda \circ \pi_\lambda^{-1}$ on the random limit sets J_λ:

Corollary 3.14. *In the above setting if μ is a $\theta \times \sigma$-invariant probability on $\Lambda \times E^{\mathbb{N}}$ whose marginal on Λ is m, and if $\mathrm{H}_\mu\big(\pi_{E^{\mathbb{N}}}^{-1}(\xi)|\pi_\Lambda^{-1}(\varepsilon_\Lambda)\big) < \infty$, then for m-a.e $\lambda \in \Lambda$, we have*

$$\mathrm{HD}(\mu_\lambda \circ \pi_\lambda^{-1}) = \mathrm{PD}(\mu_\lambda \circ \pi_\lambda^{-1}) = \frac{h_\mu(\mathcal{S})}{\chi_\mu}.$$

4. Applications to examples of random countable IFS with overlaps.

In this section we will study several examples of random IFS with overlaps and the projections of $(\theta \times \sigma)$-invariant measures μ from $\Lambda \times E^{\mathbb{N}}$ to respective limit sets.

4.1. Randomizations related to Kahane-Salem sets.

In [8] Kahane and Salem studied the convolution of infinitely many Bernoulli distributions, namely the measure $\mu = B(\frac{x}{r_0}) * B(\frac{x}{r_1}) * \ldots$, where $B(x)$ denotes the Bernoulli probability supported only at the points $-1, +1$ and giving measure $\frac{1}{2}$ to each one of them. The support of μ is the set F of points of the form $\epsilon_0 r_0 + \epsilon_1 r_1 + \ldots$, where ϵ_k is equal to

$+1$ or -1 with equal probabilities. If we assume $\sum\limits_0^\infty r_k = 1$, and if we introduce the sequence $(\rho_n)_{n\geq 0}$ defined by

$$r_0 = 1 - \rho_0, r_1 = \rho_0(1-\rho_1), r_2 = \rho_0\rho_1(1-\rho_2), \dots,$$

then it can be seen that, if $\rho_k > \frac{1}{2}$ for all but finitely many ks, then F contains intervals. If, on the other hand, $\rho_k < \frac{1}{2}$ for all $k \geq 0$, then F is a Cantor set. If in addition to this, $\lim\limits_{k\to\infty} 2^k \rho_0 \dots \rho_{k-1} = 0$, then F has zero Lebesgue measure and μ is singular.
A particular though interesting case is when $r_k = \rho^k, k \geq 0$, for some $\rho \in (0,1)$. Then the corresponding set $F = F_\rho$ is the set of real numbers of type $\pm 1 \pm \rho \pm \rho^2 \pm \dots$. If $\rho < \frac{1}{2}$, then F_ρ has zero Lebesgue measure and $\mu^{(\rho)}$ is singular; if $\rho > \frac{1}{2}$, then F_ρ contains intervals. The convolution $\mu^{(\rho)}$ is equal to the invariant probability of the IFS with two contractions

$$\phi_1(x) = \rho x + 1, \ \phi_2(x) = \rho x - 1,$$

taken with probabilities 1/2, 1/2. This is a conformal system with overlaps, and F_ρ is equal to the limit set J_ρ of this IFS. The measure $\mu^{(\rho)}$ is the projection $\nu_{(1/2,1/2)} \circ \pi^{-1}$ of the probability $\nu_{(1/2,1/2)}$ from $\{1,2\}^{\mathbb{N}}$, through the canonical projection $\pi : \{1,2\}^{\mathbb{N}} \to J_\rho$. In [5] Erdös proved that when $1/\rho$ is a Pisot number (i.e a real algebraic integer greater than 1 so that all its conjugates are less than 1 in absolute value), then the measure $\mu^{(\rho)}$ is singular. In [17] it was shown that its Hausdorff dimension is strictly smaller than 1. In the other direction, B. Solomyak showed in [20] that $\mu^{(\rho)}$ is absolutely continuous for a.e $\rho \in [1/2, 1)$.

Here we will give several ways to extend and randomize the idea of this construction, and will apply our results on pointwise dimensions of projection measures for random infinite IFS with overlaps:

Random system 4.1.1

A type of random IFS can be obtained by fixing numbers $r_1, r_2 \in (0,1)$, letting $\Lambda = \{1,2\}^{\mathbb{Z}}$, $\theta : \Lambda \to \Lambda$ be the shift homeomorphism, and setting $E = \{1,2\}$ so the alphabet is finite in this case. For arbitrary $\lambda = (\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots) \in \Lambda$ and $e \in E$, consider then the affine contractions ϕ_e^λ in one real variable, defined by:

$$\phi_1^\lambda(x) = r_{\lambda_0} x + 1, \quad \phi_2^\lambda(x) = r_{\lambda_0} x - 1 \tag{4.1}$$

Then, for arbitrary $\lambda = (\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots\} \in \{1,2\}^{\mathbb{Z}}$, the corresponding fractal limit set is

$$J_\lambda := \pi_\lambda(E^{\mathbb{N}}) = \{\phi_{\omega_1}^\lambda \circ \phi_{\omega_2}^{\theta(\lambda)} \circ \dots, \ \omega = (\omega_1, \omega_2, \dots) \in E^{\mathbb{N}}\},$$

which can actually be described as a set of type

$$\left\{ \pm 1 + \sum_{i\geq 1} \sum_{(j,k)\in Z_i} \pm \rho_1^k \rho_2^j \right\},$$

where for any pair of positive integers $(j,k) \in Z_i$ we have $j+k=i,\ i \geq 1$, and where the sets Z_i are prescribed by the parameter $\lambda \in \{1,2\}^{\mathbb{Z}}$, while the signs $\pm$ are arbitrary.
We then consider the 1-sided shift space $E^{\mathbb{N}}$, and a Bernoulli measure $\nu = \nu_Q$ on $E^{\mathbb{N}}$ given by a probability vector $Q=(q_1,q_2)$. Let also a Bernoulli measure $m=m_P$ on Λ associated to the probability vector $P=(p_1,p_2)$, and the probability $\mu = m \times \nu$ on $\Lambda \times E^{\mathbb{N}}$. The above random finite IFS is denoted by $\mathcal{S}$.
Next, by desintegrating μ into conditional measures μ_λ, and projecting μ_λ to the limit set $\mathcal{J}_\lambda$, we obtain the projection measure $\mu_\lambda \circ \pi_\lambda,\ \lambda \in \Lambda$. In this case the finiteness condition of entropy from the statement of Theorem 3.13 is clearly satisfied since E is finite, so we obtain the exact dimensionality of the measures $\mu_\lambda \circ \pi_\lambda^{-1}$ on J_λ for m-almost all $\lambda \in \Lambda$. And from Corollary 3.14 and Theorem 2.5, we obtain an upper estimate for the pointwise dimension of the projection measures,

$$d_{\mu_\lambda \circ \pi_\lambda^{-1}}(\pi_\lambda(\omega)) = \frac{h_\mu(\mathcal{S})}{\chi_\mu} \leq \frac{h(m_P)+h(\nu_Q)}{-p_1 \log r_1 - p_2 \log r_2} = \frac{p_1 \log p_1 + p_2 \log p_2 + q_1 \log q_1 + q_2 \log q_2}{p_1 \log r_1 + p_2 \log r_2}$$

Also, a possibility is to take $\mu = m \times \nu$ on $\Lambda \times E^{\mathbb{N}}$, where $m = m_P$ as before and ν is an equilibrium measure of a Hölder continuous potential on the 1-sided shift space $E^{\mathbb{N}}$.

Random system 4.1.2

Consider now a fixed sequence $\bar{\rho} = (\rho_i)_{i \geq 1}$ of numbers in $(0,1)$ which are smaller than some fixed $\rho \in (0,1)$, and let the parameter space $\Lambda = \{1,2,\ldots\}^{\mathbb{Z}}$ with the shift homeomorphism $\theta : \Lambda \to \Lambda$. Let also an infinite probability vector $P=(p_1,p_2,\ldots)$, and the θ-invariant Bernoulli measure m_P on Λ satisfying $m_P([i]) = p_i,\ i \geq 1$, where $[i] := \{\omega = (\ldots,\omega_{-1},\omega_0,\omega_1,\ldots), \omega_0 = i\},\ i \geq 1$, and $h(\nu_P) < \infty$. Let us take then the set $E := \{1,2,\ldots\}$ and a $(\theta \times \sigma)$-invariant probability measure μ on $\Lambda \times E^{\mathbb{N}}$, having its marginal on Λ equal to m_P. For example we can take $\mu = m_P \times \nu_Q$, where $Q = (q_1,q_2,\ldots)$ is a probability vector, and where $\nu_Q([j]) = q_j, j \geq 1$ is a σ-invariant Bernoulli probability on $E^{\mathbb{N}}$; we assume in addition that the entropy of ν_Q is finite, i.e that

$$-\sum_{j \geq 1} q_j \log q_j < \infty$$

We now define infinitely many contractions ϕ_e^λ on a fixed large enough compact interval X, for arbitrary $e \in E, \lambda = (\ldots,\lambda_{-1},\lambda_0,\lambda_1,\ldots) \in \Lambda,\ \lambda_i \in \{1,2\ldots\}, i \in \mathbb{Z}$, by:

$$\phi_n^\lambda(x) = \rho_{\lambda_n} \cdot x + (-1)^{\lambda_0},\ n \geq 1$$

It is clear that ϕ_e^λ are conformal contractions and they satisfy Bounded Distortion Property. We construct thus a random infinite IFS denoted by $\mathcal{S}(\bar{\rho})$, which has overlaps.

For every $\lambda \in \Lambda$, we construct then the fractal limit set $J_\lambda := \pi_\lambda(E^{\mathbb{N}})$, which may be non-compact. The fractal J_λ is the set of points given as $\phi_{\omega_1}^\lambda \circ \phi_{\omega_2}^{\theta(\lambda)} \circ \ldots$, for all $\omega \in E^{\mathbb{N}}$. The main difference from the previous example 4.1.1 is that now, the plus and minus signs in the series giving the points of J_λ are *not arbitrary*, instead they are determined by $\lambda = (\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots) \in \Lambda$. The randomness in the series comes now from the various possibilities to choose the sequences $\omega = (\omega_0, \omega_1, \ldots) \in E^{\mathbb{N}}$. Thus,

$$J_\lambda = \{(-1)^{\lambda_0} + (-1)^{\lambda_1} \rho_{\lambda_{\omega_1}} + (-1)^{\lambda_2} \rho_{\lambda_{\omega_1}} \rho_{\lambda_{\omega_2}} + \ldots + x \cdot \rho_{\lambda_{\omega_1}} \rho_{\lambda_{\omega_2}} \ldots, \text{ for arbitrary } \omega_i \in \mathbb{N}^*, i \geq 0\}$$

Given the $(\theta \times \sigma)$-invariant probability measure $\mu = m_P \times \nu_Q$, we see from Remark 3.4 that the condition $H_\mu(\pi_{E^{\mathbb{N}}}^{-1}(\xi) | \pi_\Lambda^{-1}(\epsilon_\Lambda)) < \infty$ is satisfied. For arbitrary $\lambda \in \Lambda$, we now take the projection measure $\mu_\lambda \circ \pi_\lambda^{-1}$ on J_λ. Therefore, from Theorem 3.13 and Corollary 3.14 we obtain that for m_P-almost all $\lambda \in \Lambda$, the measure $\mu_\lambda \circ \pi_\lambda^{-1}$ is exact dimensional and its pointwise dimension has a common value equal to $h_\mu(\mathcal{S}(\bar\rho))/\chi_\mu$, where in our case the Lyapunov exponent of μ with respect to the random infinite system $\mathcal{S}(\bar\rho)$ is equal to:

$$\begin{aligned} \chi_\mu &= -\int_{\Lambda \times E^{\mathbb{N}}} \log \rho_{\lambda_{\omega_1}} d\mu(\lambda, \omega) = -\sum_{i \geq 1} q_i \int \log \rho_{\lambda_i} dm_P(\lambda) \\ &= -\sum_{i,j \geq 1} p_j q_i \log \rho_j = -\sum_{j \geq 1} p_j \log \rho_j. \end{aligned}$$

Moreover we have from Theorem 2.5 that the random projectional entropy of μ satisfies

$$h_\mu(\mathcal{S}(\bar\rho)) \leq h(\mu) = h(\mu_P) + h(\nu_Q) = -\sum_{i \geq 1} p_i \log p_i - \sum_{j \geq 1} q_j \log q_j.$$

This helps to give a concrete upper estimate for the pointwise dimensions of $\mu_\lambda \circ \pi_\lambda^{-1}$, namely

$$d_{\mu_\lambda \circ \pi_\lambda^{-1}}(\pi_\lambda(\omega)) \leq \frac{\sum\limits_{i \geq 1} p_i \log p_i + \sum\limits_{j \geq 1} q_j \log q_j}{\sum\limits_{j \geq 1} p_j \log \rho_j}$$

Random system 4.1.3

Let us fix a sequence $\bar\rho = (\rho_0, \rho_1, \rho_2, \ldots)$ in $(0, 1)$, and $\Lambda = [1 - \varepsilon, 1 + \varepsilon]$ for some small $\varepsilon > 0$, together with a homeomorphism $\theta : \Lambda \to \Lambda$ which preserves an absolutely continuous probability m on Λ. Let us take also the set $E = \{1, 2, \ldots\}$ and the σ-invariant Bernoulli measure ν on $E^{\mathbb{N}}$ given by $\nu([i]) = \nu_i, i \geq 1$, where $(\nu_1, \nu_2, \ldots)$ is a probability vector. We assume also that $h(\nu) = -\sum\limits_{i \geq 1} \nu_i \log \nu_i < \infty$. For arbitrary $e \in E$ and $\lambda \in \Lambda$, we now define the sequence of parametrized contractions:

$$\phi_{2n+1}^\lambda(x) = \lambda \rho_n x + 1, \quad \phi_{2n+2}^\lambda(x) = \lambda \rho_n x - 1, \ n \geq 0.$$

By considering also the $(\theta \times \sigma)$-invariant probability $\mu = m \times \nu$ we obtain the random infinite IFS with overlaps $\mathcal{S}(\bar\rho)$.

The corresponding limit set $J_\lambda := \pi_\lambda(E^{\mathbb{N}})$ can be thought of as the set determined, for $\lambda \in \Lambda$, in the following way: $J_\lambda = \{\pm 1 \pm \lambda \rho_{i_1} \pm \lambda^2 \rho_{i_1} \rho_{i_2} \pm \ldots, \text{for all sequences of positive integers } \omega = (i_1, i_2, \ldots) \in E^{\mathbb{N}}\}$. The projection $(\pi_\lambda)_* \mu_\lambda = \mu_\lambda \circ \pi_\lambda^{-1}$ of the measure μ_λ, is a probability measure on J_λ. We see that both (2.3) and the entropy condition $H_\mu(\pi_{E^{\mathbb{N}}}^{-1} \xi \ \pi_\Lambda^{-1} \epsilon_\Lambda) < \infty$, are satisfied in this case.

Hence, we can apply Theorem 3.13 and Corollary 3.14, to obtain that for m-almost all parameters $\lambda \in [1-\varepsilon, 1+\varepsilon]$, the projection measure $\mu_\lambda \circ \pi_\lambda^{-1}$ is exact dimensional, and that its Hausdorff dimension has a common value, which is equal to

$$HD(\mu_\lambda \circ \pi_\lambda^{-1}) = \frac{h_\mu(\mathcal{S}(\bar{\rho}))}{\chi_\mu},$$

where the Lyapunov exponent of μ with respect to $\mathcal{S}(\bar{\rho})$ is given by:

$$\chi_\mu = -\int_{\Lambda \times E^{\mathbb{N}}} \log(\lambda \rho_{[\frac{\omega_1 - 1}{2}]})\, d\mu(\lambda, \omega) = -\int_\Lambda \log \lambda \, dm(\lambda) - \sum_{i \geq 0} (\nu_{2i+1} + \nu_{2i+2}) \log \rho_i$$

From Theorem 2.5 we obtain an upper estimate for the random projectional entropy, $h_\mu(\mathcal{S}) \leq h(m) - \sum_i \nu_i \log \nu_i$, and an upper estimate for the pointwise dimension and the Hausdorff dimension of $\mu_\lambda \circ \pi_\lambda^{-1}$; namely for μ-almost every $(\lambda, \omega) \in [1-\varepsilon, 1+\varepsilon] \times E^{\mathbb{N}}$,

$$d_{\mu_\lambda \circ \pi_\lambda^{-1}}(\pi_\lambda(\omega)) = HD(\mu_\lambda \circ \pi_\lambda^{-1}) \leq \frac{h(m) - \sum_{i \geq 1} \nu_i \log \nu_i}{-\int_\Lambda \log \lambda \, dm(\lambda) - \sum_{i \geq 0} (\nu_{2i+1} + \nu_{2i+2}) \log \rho_i}$$

If all the contraction factors ρ_i are equal to some fixed ρ, then J_λ is a perturbation of the set from the beginning of 4.1.

4.2. Randomizations of deterministic infinite IFS with bounded number of overlaps.

In Example 5.11 of [12], we gave an example of a deterministic infinite IFS defined as follows: let $X = \bar{B}(0,1) \subset \mathbb{R}^2$ be the closed unit disk and for $n \geq 1$ take C_n to be the circle centered at the origin and having radius $r_n \in (0,1)$, $r_n \underset{n \to \infty}{\nearrow} 1$. For each $n \geq 1$ we cover the circle C_n with closed disks $D_n(i), i \in K_n$, of the same radius r'_n, where K_n is a finite set and each disk $D_n(i)$ intersects only two other disks of the form $D_n(j), j \in K_n$, and where none of the disks $D_n(i)$ intersects $C_k, k \neq n$. Moreover, we assume that for any $m \neq n, m, n \geq 1$, the families $\{D_m(i)\}_{i \in K_m}$ and $\{D_n(i)\}_{i \in K_n}$ consist of mutually disjoint disks. Consider contraction similarities $\phi_{n,i} : X \to X, i \in K_n, n \geq 0$ whose respective images of X are the above disks $D_n(i), i \in K_n, n \geq 0$. For this deterministic system, the boundary at infinity $\partial_\infty \mathcal{S}$ is contained in ∂X.

Assume now in addition, that there exists $\varepsilon > 0$, such that for $m \neq n$, any disk $(1+\varepsilon)D_n(i), i \in K_n$ does not intersect any disk of type $(1+\varepsilon)D_m(j), j \in K_m$ (where in general for $\beta > 0$, $\beta D_n(i)$ denotes the disk of the same center as $D_n(i)$ and radius equal to $\beta r'_n$), and that any disk $(1+\varepsilon)D_n(i)$ intersects only two other disks $(1+\varepsilon)D_n(j), j \in K_n$.
We take now $\Lambda = [1-\varepsilon, 1+\varepsilon]$ and $\theta : \Lambda \to \Lambda$ a homeomorphism which preserves an absolutely continuous probability measure m on $[1-\varepsilon, 1+\varepsilon]$. Let the following countable alphabet

$$E = \{(n,i),\ i \in K_n, n \geq 0\},$$

which will be our alphabet. Consider also a fixed probability vector $P = (\nu_e)_{e \in E}$, and the associated Bernoulli probability $\nu = \nu_P$ on $E^{\mathbb{N}}$, and let us assume that $h(\nu) < \infty$.
We now define the conformal contraction $\phi^{\lambda}_{(n,i)}(x)$, as being a similarity with image $\phi^{\lambda}_{(n,i)}(X)$ equal to $\lambda D_n(i)$, for $i \in K_n, n \geq 0$ and $\lambda \in \Lambda$; its contraction factor is equal to $\lambda r'_n, n \geq 0$.
Consider now the probability $\mu = m \times \nu$ defined on $\Lambda \times E^{\mathbb{N}}$. We have constructed thus a random conformal infinite IFS with overlaps, denoted by $\mathcal{S}$; and, from Remark 3.4 and since $h(\nu) < \infty$, we obtain also the finite entropy condition $H_\mu(\pi^{-1}_{E^{\mathbb{N}}}\xi | \pi^{-1}_{\Lambda}\epsilon_\Lambda) < \infty$.
The conditions in Theorem 3.13 and Corollary 3.14 are then satisfied, and therefore for Lebesgue-almost all parameters $\lambda \in \Lambda$ and ν-almost all $\omega \in E^{\mathbb{N}}$, the pointwise dimension of the projection $(\pi_\lambda)_* \mu_\lambda = \mu_\lambda \circ \pi_\lambda^{-1}$ on the non-compact limit set $J_\lambda := \pi_\lambda(E^{\mathbb{N}})$, is given by:

$$d_{\mu_\lambda \circ \pi_\lambda^{-1}}(\pi_\lambda(\omega)) = \frac{h_\mu(\mathcal{S})}{\chi_\mu},$$

where the Lyapunov exponent of μ with respect to the random system $\mathcal{S}$ is equal to:

$$\chi_\mu = -\log \lambda - \sum_{e=(n,i) \in E} \nu_e \log r'_n > 0$$

From the construction of the disks $\lambda D_n(i), i \in K_n, n \geq 0, \lambda \in \Lambda$ above, we notice that the condition in Theorem 2.5, part b) is satisfied with $k = 2$. Hence we can obtain a *lower estimate* for the random projectional entropy of μ, namely

$$h_\mu(\mathcal{S}) \geq h(\mu) - \log 2 = h(m) - \sum_{e \in E} \nu_e \log \nu_e - \log 2$$

Therefore by combining the last two displayed formulas and using Theorem 2.5, we obtain that for μ-almost every pair $(\lambda, \omega) \in \Lambda \times E^{\mathbb{N}}$, the pointwise dimension of $\mu_\lambda \circ \pi_\lambda^{-1}$ satisfies:

$$\frac{h(m) - \sum\limits_{e \in E} \nu_e \log \nu_e - \log 2}{-\log \lambda - \sum\limits_{e=(n,i) \in E} \nu_e \log r'_n} \leq d_{\mu_\lambda \circ \pi_\lambda^{-1}}(\pi_\lambda(\omega)) \leq \frac{h(m) - \sum\limits_{e \in E} \nu_e \log \nu_e}{-\log \lambda - \sum\limits_{e=(n,i) \in E} \nu_e \log r'_n}$$

4.3. Constructions based on a problem of Sinai.

Ya. Sinai asked for which parameters $\alpha \in (0,1)$ is the invariant measure of the IFS formed by the the maps $\{1 + (1-\alpha)x, 1 + (1+\alpha)x\}$, with probabilities $(1/2, 1/2)$, absolutely continuous. The latter of the above maps is never a contraction. Nevertheless, this IFS contracts on average, since a composition of n of the above maps, chosen i.i.d with probabilities 1/2, 1/2, contracts by a factor close to $(1-\alpha^2)^{n/2}$. A randomized version was investigated in [14]. We can use our results to study other randomizations of this system, and the associated projection measures, since our proofs can be adapted to random finite conformal IFS which contract on average. Consider then the random system $\mathcal{S}$ given by the parameter space $\Lambda = [1-\varepsilon, 1+\varepsilon]^{\mathbb{Z}}$ and $m = m_0^{\mathbb{Z}}$, where m_0 is the normalized Lebesgue measure; let θ to be the shift homeomorphism on Λ. Given $\lambda = (\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots) \in \Lambda$, let us introduce also two parametrized conformal maps defined by:

$$\phi_1^\lambda(x) = (1 - \alpha\lambda_{-1})x + \lambda_0, \quad \phi_2^\lambda(x) = (1 + \alpha\lambda_1)x + \lambda_2$$

The alphabet $E = \{1, 2\}$ and if we take $\nu = \nu_{(\frac{1}{2},\frac{1}{2})}$ on $E^{\mathbb{N}}$, then $h(\nu) < \infty$. So if we consider the product measure $\mu = m \times \nu$, we obtain from Remark 3.4 the finiteness condition

$$H_\mu(\pi_{E^{\mathbb{N}}}^{-1}(\xi)|\pi_\Lambda^{-1}(\epsilon_\Lambda)) < \infty$$

We see also immediately that, if $\varepsilon(\alpha)$ is chosen small enough, then for $\varepsilon \in (0, \varepsilon(\alpha))$ the maps $\phi_i^\lambda, i \in E$ are conformal and, since the indices $1, 2$ are taken with equal probability, a typical composition of n of them contracts by a factor smaller than $(1 - \frac{\alpha^2}{2})^{n/2}$.

We obtain then the exact dimensionality and the pointwise dimension of the projection measures $\mu_\lambda \circ \pi_\lambda^{-1}$ on J_λ, by using Theorem 3.13 and Corollary 3.14. In our case, the Lyapunov exponent of the probability μ with respect to the random system $\mathcal{S}$, is equal to

$$\chi_\mu = -\int_\Lambda \log\big[(1 - \alpha\lambda_{-1})(1 + \alpha\lambda_1)\big]\,dm(\lambda),$$

with $\lambda = (\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \ldots) \in \Lambda$. If $\varepsilon(\alpha) > 0$ is small enough and if $\varepsilon \in (0, \varepsilon(\alpha))$, then $\chi_\mu > 0$. The pointwise dimension $d_{\mu_\lambda \circ \pi_\lambda^{-1}}(\pi_\lambda(\omega))$ of the projection $\mu_\lambda \circ \pi_\lambda^{-1}$, is then given as the quotient $\frac{h_\mu(\mathcal{S})}{\chi_\mu}$, for μ-almost all $(\lambda, \omega) \in \Lambda \times E^{\mathbb{N}}$.

References

[1] L. Barreira, Ya. Pesin, and J. Schmeling, Dimension and product structure of hyperbolic measures, Ann. of Math. 149 (1999), 755-783. 1

[2] T. Bogenschütz, Entropy, pressure, and variational principle for random dynamical systems, Random Comput. Dyn. 1 (1992/93), 99-116. 1, 2, 3

[3] H. Crauel, *Random Probability Measures on Polish Spaces*, Stochastic Monographs vol. 11, Taylor & Francis, London 2002. 2

[4] J.P Eckmann, D. Ruelle, Ergodic theory of chaos and strange attractors, Rev. Mod. Physics 57, 617-656 (1985). 1

[5] P. Erdös, On a family of symmetric Bernoulli convolutions, Amer. J. Math., 61, 1939, 974-976. 4

[6] D. J. Feng, H. Hu, Dimension theory of iterated function systems, Commun. on Pure and Applied Math., 62 (2009), 1435-1500. (document), 1, 2, 2

[7] J.P Kahane, Sur le modéle de turbulence de Benoit Mandelbrot. C.R. Acad. Sci. Paris 278A, 621-623 (1974). 1

[8] J. P Kahane, R. Salem, Sur la convolution d'une infinité de distributions de Bernoulli, Colloq Math, 6, 193-202, 1958. 1, 4

[9] R. Mané, *Ergodic Theory and Differentiable Dynamics*, Springer 1987. 2, 3

[10] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge University Press, 1995. 1, 3, 3

[11] D. Mauldin, M. Urbański, *Graph Directed Markov Systems: Geometry and Dynamics of Limit Sets*, Cambridge Univ. Press (2003). 1

[12] E. Mihailescu, M. Urbański, Hausdorff dimension of limit sets of countable conformal iterated function systems with overlaps, Contemp. Math., vol. 600, American Math Society, 273-290, 2013. 1, 2, 4

[13] W. Parry, Entropy and generators in ergodic theory, W. A Benjamin, New York, 1969. 2, 2

[14] Y. Peres, K. Simon, B. Solomyak, Absolute continuity for random iterated function systems with overlaps, J. London Math Soc, **74**, 739-756, 2006. 1, 4

[15] Y. Pesin, *Dimension Theory in Dynamical Systems. Contemporary Views and Applications*, Chicago Lectures in Mathematics, Univ. Chicago Press, Chicago, 1997. 1, 2
[16] Y. Pesin, H. Weiss, On the dimension of deterministic and random Cantor-like sets, symbolic dynamics, and the Eckmann-Ruelle Conjecture, Comm. Math. Phys. 182, 105-153 (1996). 1
[17] F. Przytycki, M. Urbański, On Hausdorff dimension of some fractal sets, Studia Math. 93 (1989), 155-186. 4
[18] F. Przytycki, M. Urbański, *Conformal Fractals – Ergodic Theory Methods*, Cambridge University Press (2010). 1, 2
[19] M. Roy, M. Urbański, Random graph directed Markov systems, Discrete Cont. Dynam. Syst., 30 (2011), 261-298. 1
[20] B. Solomyak, On the random series $\sum \pm\lambda^n$ (an Erdös problem), Ann. Math., 142, 611-625, 1995. 4
[21] P. Walters, An introduction to ergodic theory, Springer New York-Berlin, 1982. 2, 2, 2
[22] L. S. Young, Dimension, entropy and Lyapunov exponents, Ergodic Th. Dynam. Syst. 2 (1982), 109-124. 1

Eugen Mihailescu
Institute of Mathematics of the Romanian Academy, P.O Box 1-764, RO 014700, Bucharest, Romania, and
Institut des Hautes Études Sciéntifiques, Le Bois-Marie 35, route de Chartres 91440, Bures-sur-Yvette, France.
E-mail address: `Eugen.Mihailescu@imar.ro`
Web: www.imar.ro/∼mihailes

Mariusz Urbański
Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA.
E-mail address: `urbanski@unt.edu`
Web: http://www.math.unt.edu/∼urbanski