Bayesian inversion and the Tomita–Takesaki modular group

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Abstract

We show that conditional expectations, optimal hypotheses, disintegrations, and adjoints of unital completely positive maps, are all instances of Bayesian inverses. We study the existence of the latter by means of the Tomita–Takesaki modular group and we provide extensions of a theorem of Takesaki as well as a theorem of Accardi and Cecchini to the setting of not necessarily faithful states on finite-dimensional C^*-algebras.

Contents

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1 Introduction

There have been many recent advancements in the categorical approach towards probability theory and statistics. For example, the Kolmogorov zero-one law, Basu's theorem, Fisher–Neymann factorization, and de Finetti's theorem have all been proved synthetically using the framework of Markov categories [20, 22, 21]. An immediate question is whether or not such techniques could be used to derive new results in quantum probability. To answer this, the second author constructed generalizations of Markov categories to allow both classical and quantum probabilistic concepts [40]. Among the many additional axioms possible for Markov categories, some of them, such as positivity, causality, and a.e. modularity, were proved for quantum operations in [40]. Other, more subtle, axioms such as the existence of disintegrations, the existence of Bayesian inverses, or the existence of conditionals were studied in [43], [44], and [41], respectively, the first two of which were accomplished by the second and fourth authors of the present paper, and the third of which was done by the second.

The categorical approach towards quantum probability parallels the algebraic approach towards quantum probability, but a closer inspection comparing and contrasting the two approaches has not yet been carried out in detail other than the preliminary results in [40]. In this article, we bring many such connections, most importantly including Tomita–Takesaki modular theory [55]. For example, we prove an equivalence between the existence of Bayesian inverses of unital completely positive maps and an intertwining condition between the modular groups due to Accardi–Cecchini [1] and Anantharaman-Delaroche [4]. The usage of modular theory in recent years in quantum information theory [11, 29, 30, 15, 54, 38] and quantum field theory [61, 16, 32, 31, 13] indicates its importance. In addition, the Bayesian inverses, in the special case of faithful states given by the vacuum state, have been recently used as a notion of inversion for generalized global gauge symmetries of subfactors and local quantum field theories [6, 7, 8] in the algebraic setting [26].

Besides contributing to the dictionary between the categorical and algebraic approaches towards quantum probability, we also develop several new results and applications to quantum information theory and quantum probability. For example, we work with not necessarily faithful states, where the modular group no longer exists but is replaced with a semigroup. Although one naively might think that any statement said about faithful states can be immediately extended to non-faithful ones by looking at the support algebras, we show that this is not always the case. For example, although we prove that disintegrations for non-faithful states exist if and only if they exist on the underlying support algebras, this is not the case for Bayesian inverses. Since non-faithful states and their evolution along (noisy) quantum channels are of relevance to quantum information theory, these results are important

for the reversibility of quantum operations involving non-faithful states (such as pure states). Finally, we illustrate how the non-commutative Bayes' theorem of [44] generalizes a recent result of Carlen and Vershynina [11], which itself is a partial generalization of a result of Nakamura, Takesaki, and Umegaki [37].

The paper is organized as follows. In Section 2, we recall the basic definitions of conditional expectation, state-preserving unital completely positive map, and disintegration in the context of C^*-algebras. A unital $*$-homomorphism is interpreted as a *deterministic* quantum channel, while an honest unital completely positive map is interpreted as a *stochastic* one. We also review the Tomita–Takesaki modular operators and the modular automorphism group in the special case of finite-dimensional C^*-algebras, also called multi-matrix algebras, that we shall mainly deal with in this work. We prove the equivalence between state-preserving conditional expectations and disintegrations on matrix algebras and multi-matrix algebras, both with respect to faithful and non-faithful states. In Section 3, we recall the definition of Bayesian inverse of a state-preserving unital completely positive map and we compare it with the notion of adjoint due to Accardi and Cecchini in the context of operator algebras and quantum probability and with the notion of Petz recovery map in quantum information theory. In the case of faithful states, we characterize the existence of Bayesian inverses on matrix algebras and multi-matrix algebras by means of the modular group. For arbitrary states on multi-matrix algebras, we show that Bayesian inverses generalize disintegrations in the same way as Accardi–Cecchini adjoints generalize state-preserving conditional expectations. In Section 4, we generalize Takesaki's theorem [56], which characterizes the existence of state-preserving conditional expectations by means of the modular group, to non-faithful states on matrix algebras and multi-matrix algebras. In particular, we find an additional necessary condition for the existence of such state-preserving conditional expectations, which is also sufficient together with the usual modular group condition on the support algebras. In Section 5, as a further generalization of Takesaki's theorem, we study the existence of Bayesian inverses with respect to non-faithful states on matrix algebras. Appendix A provides a review of Carlson's theorem from complex analysis. Appendix B provides an explicit characterization (without the aid of modular theory) of the states on multi-matrix algebras (expressed by means of density matrices and classical probability distributions on the centers) that admit a state-preserving conditional expectation onto a given subalgebra.

2 Conditional expectations, disintegrations, and the modular group

2.1 A brief review of definitions

Notation 2.1. In this work, all C^*-algebras will be unital, and all C^*-subalgebras of a (unital) C^*-algebra will be assumed to have the same unit as the larger C^*-algebra unless specified otherwise. The notation $\mathcal{B} \subseteq \mathcal{A}$ will be used to express that a C^*-algebra $\mathcal{B}$ is such a C^*-subalgebra of a C^*-algebra $\mathcal{A}$. A linear map from a C^*-algebra $\mathcal{B}$ to a C^*-algebra $\mathcal{A}$ will be written as $\mathcal{B} \rightsquigarrow \mathcal{A}$, while a $*$-homomorphism will be written as $\mathcal{B} \to \mathcal{A}$. The letters U, C, and P will be used to abbreviate unital, completely, and positive. For example, a UCP map is a unital completely positive map. If $A \in \mathcal{A}$, then $\mathrm{Ad}(A)$ will denote the CP map that sends $A' \in \mathcal{A}$ to $AA'A^*$. In calculations, this map may also be written as Ad_A. Inner products will be denoted with angular brackets as $\langle \cdot , \cdot \rangle$ and will be linear in the right variable and conjugate linear in the left variable. See [45] for background.

Definition 2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. A linear map $F : \mathcal{B} \rightsquigarrow \mathcal{A}$ is ***$*$-preserving*** iff $F(b^*) = F(b)^*$ for all $b \in \mathcal{B}$.

On occasion, we will need to restrict domains and codomains of functions to particular subsets in order to slightly redefine functions and make rigorous sense of certain compositions.

As such, we include the following notation.

Notation 2.3. Let $F : \mathcal{B} \rightsquigarrow \mathcal{A}$ be a linear map of C^*-algebras. Let S and T be *subsets* of $\mathcal{A}$ and $\mathcal{B}$, respectively. The notation $F_{\restriction S} : S \rightsquigarrow \mathcal{A}$ will be used to denote the ***restriction*** of F to the subset S. If the image $F(\mathcal{B})$ of F is contained in T, then the notation ${}_{T\downharpoonright}F : \mathcal{B} \rightsquigarrow T$ will be used to denote the ***corestriction*** of F, i.e., the unique function such that the composite $B \overset{{}_{T\downharpoonright}F}{\rightsquigarrow} T \hookrightarrow \mathcal{A}$ equals F. The notation ${}_{T\downharpoonright}F_{\restriction S} : S \rightsquigarrow T$ is used to combine restriction and corestriction.

Definition 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, with $\mathcal{B} \subseteq \mathcal{A}$. A ***conditional expectation*** is a linear map $E : \mathcal{A} \rightsquigarrow A$ such that

1. E is a ***projection*** onto $\mathcal{B}$, i.e., $E(b) = b$ for all $b \in \mathcal{B}$,
2. E is ***left $\mathcal{B}$-modular***, i.e., $E(ba) = bE(a)$ for all $b \in \mathcal{B},\ a \in \mathcal{A}$, and
3. E is positive.

If $\omega : \mathcal{A} \rightsquigarrow \mathbb{C}$ is a ***state*** on $\mathcal{A}$, i.e., a positive unital functional, an ***ω-preserving conditional expectation*** is a conditional expectation E as above such that $\omega \circ E = \omega$.

Remark 2.5. Since a conditional expectation E is positive, it is $*$-preserving. As such, left $\mathcal{B}$-modularity of E implies right $\mathcal{B}$-modularity. Indeed, if E is left $\mathcal{N}$-modular, then $E(ab) = E(b^*a^*)^* = (b^*E(a^*))^* = E(a)b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Hence, E is ***$\mathcal{B}$-bimodular*** in the sense that $E(b_1ab_2) = b_1E(a)b_2$ for all $b_1, b_2 \in \mathcal{B},\ a \in \mathcal{A}$. Since the unit of $\mathcal{A}$ belongs also to $\mathcal{B}$ by our standing assumption on $\mathcal{B} \subseteq \mathcal{A}$, the map E is unital and it has operator norm equal to 1.

Theorem 2.6 (Tomiyama [57])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with $\mathcal{B} \subseteq \mathcal{A}$. Every projection of norm 1 from $\mathcal{A}$ to $\mathcal{B}$ is a conditional expectation and vice versa.*

Proof. See also [52, Section 9.1]. □

Remark 2.7. A conditional expectation is automatically completely positive by a theorem of Nakamura, Takesaki, and Umegaki [37]. In particular, it is a unital ***Schwarz map***, meaning that $E(a^*a) \geq E(a)^*E(a)$ for all $a \in \mathcal{A}$. We will discuss generalizations of this result for finite-dimensional C^*-algebras later in this manuscript.

Another concept that appears in this work is that of a disintegration. To define it, we first recall the notion of a.e. equivalence and a.e. determinism [40].

Definition 2.8. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, let $\omega : \mathcal{A} \rightsquigarrow \mathbb{C}$ be a state on $\mathcal{A}$, and let $F, G : \mathcal{B} \rightsquigarrow \mathcal{A}$ be $*$-preserving maps. Then F is said to be ***ω-a.e. equivalent to*** G iff any of the following equivalent conditions hold. [1]

i. $\omega(AF(B)) = \omega(AG(B))$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

[1] The equivalence between these two conditions is proved in [40, Theorem 5.12].

ii. $F(B) - G(B) \in N_\omega$, where

$$N_\omega := \{A \in \mathcal{A} \,:\, \omega(A^*A) = 0\}$$

denotes the ***nullspace*** of ω.

In this case, the notation $F \underset{\omega}{=} G$ will be used.

Definition 2.9. A ***non-commutative/quantum probability space***[2] is a pair $(\mathcal{A}, \omega)$ consisting of a C^*-algebra $\mathcal{A}$ and a state $\omega : \mathcal{A} \rightsquigarrow \mathbb{C}$. The state ω is said to be ***faithful*** whenever its nullspace N_ω consists of just the zero vector. Otherwise, ω is said to be ***non-faithful***. The quantum probability space $(\mathcal{A}, \omega)$ is called ***non-degenerate*** (resp., ***degenerate***) whenever ω is faithful (resp., not faithful).

Definition 2.10. Let $(\mathcal{A}, \omega)$ and $(\mathcal{B}, \xi)$ be quantum probability spaces. Let $F : \mathcal{B} \rightsquigarrow \mathcal{A}$ be a UCP ***state-preserving map***, i.e., $\omega \circ F = \xi$.

A ***disintegration***[3] of (F, ω) is a UCP map $G : \mathcal{A} \rightsquigarrow \mathcal{B}$ such that G is state-preserving, i.e., $\xi \circ G = \omega$, and $G \circ F \underset{\xi}{=} \mathrm{id}_{\mathcal{B}}$.

Remark 2.11. Using the notation of Definition 2.9, if G is state-preserving and satisfies the stronger condition $G \circ F = \mathrm{id}_{\mathcal{B}}$, then G is called an *optimal hypothesis* (e.g., [5, 42]).

When dealing with degenerate quantum probability spaces, it will be absolutely necessary to generalize the notion of a $*$-homomorphism to allow for an almost everywhere version of it. This is called a.e. determinism and is defined explicitly for C^*-algebras in the following [40, Section 6].

Definition 2.12. Let $(\mathcal{A}, \omega)$ be a quantum probability space, let $\mathcal{B}$ be a C^*-algebra, and let $F : \mathcal{B} \rightsquigarrow \mathcal{A}$ be a positive unital map. Then F is said to be ***ω-a.e. deterministic*** iff

$$F(B_1B_2) - F(B_1)F(B_2) \in N_\omega \qquad \forall\, B_1, B_2 \in \mathcal{B}.$$

If F is ω-a.e. deterministic, it is not necessarily the case that it is ω-a.e. equivalent to a $*$-homomorphism [40, Example 6.5]. Nevertheless, for von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$, it is equivalent to the condition $F(B_1B_2)P_\omega = F(B_1)F(B_2)P_\omega$ for all $B_1, B_2 \in \mathcal{B}$, where P_ω is the support projection of ω [40, Example 6.4].

Lastly, we review the Tomita–Takesaki modular operator and the modular automorphism group for normal faithful states [55]. Other standard references include [10, Ch. 2.5] and [18, Sections 9.1 and 9.2]. For shorter reviews, we recommend [53] and [61, Section III.A.]. However, we simplify the following presentation by specializing to the finite-dimensional setting.

Lemma 2.13. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a unital $*$-subalgebra and assume that there exists a cyclic and separating vector $\Omega \in \mathcal{H}$ for $\mathcal{M}$, i.e., $\mathcal{H} = \mathcal{M}\Omega$ and $A\Omega = 0$ for any $A \in \mathcal{M}$ implies $A = 0$. Then the assignment*

$$\begin{aligned} \mathcal{H} &\xrightarrow{S_\Omega} \mathcal{H} \\ A\Omega &\mapsto A^*\Omega \end{aligned} \tag{2.14}$$

is a conjugate-linear operator.

[2] In this case "non-commutative" should be read as "not necessarily commutative." Furthermore, "quantum" could also be hybrid quantum/classical.

[3] The motivation for the terminology is discussed in [43, Appendix A] and [40, Example 7.5].

Proof. The cyclic and separating condition guarantees well-definedness of S_Ω. The map is manifestly conjugate-linear. □

Definition 2.15. Let $S_\Omega = J_\Omega \Delta_\Omega^{1/2}$ be the polar decomposition of (2.14), where Δ_Ω is positive definite and J_Ω is an antiunitary involution.[4] The maps $\Delta_\Omega = S_\Omega^* S_\Omega$ and J_Ω are called respectively the ***modular operator*** and ***modular conjugation*** of $(\mathcal{M}, \Omega)$.

Lemma 2.16. *Let $\mathcal{H}$, $\mathcal{M}$, and Ω be as in Lemma 2.13. Then*

$$\Delta_\Omega \Omega = \Omega, \qquad J_\Omega^2 = \mathrm{id}_\mathcal{H}, \quad \textit{and} \qquad J_\Omega \mathcal{M} J_\Omega = \mathcal{M}',$$

where $\mathcal{M}' \subseteq \mathcal{B}(\mathcal{H})$ is the commutant of $\mathcal{M}$ inside $\mathcal{B}(\mathcal{H})$. Furthermore, Δ_Ω^{it} is unitary for all $t \in \mathbb{R}$ and the assignment

$$\mathbb{R} \ni t \mapsto \mathfrak{m}^t_{(\mathcal{M},\Omega)} := \mathrm{Ad}(\Delta_\Omega^{it})$$

is a one-parameter group of $*$-automorphisms of $\mathcal{M}$. Finally, the induced state

$$\mathcal{M} \ni m \overset{\omega}{\mapsto} \frac{\langle \Omega, m\Omega \rangle}{\|\Omega\|^2}$$

is a faithful state on $\mathcal{M}$ satisfying

$$\omega \circ \mathfrak{m}^t_{(\mathcal{M},\Omega)} = \omega \qquad \forall\, t \in \mathbb{R}.$$

Proof. See [10, Thm. 2.5.14]. □

Definition 2.17. The one-parameter automorphism group constructed in Lemma 2.16 is called the ***modular automorphism group*** of $(\mathcal{M}, \Omega)$ inside $\mathcal{B}(\mathcal{H})$. More generally, let $\mathcal{M}$ be a unital finite-dimensional C^*-algebra and let ω be a normal faithful state on $\mathcal{M}$. The ***modular automorphism group*** of $(\mathcal{M}, \omega)$ is the modular automorphism group of $(\mathcal{M}, \Omega)$, where Ω is any pure state representing ω (such as from the GNS representation). Since it only depends on $\mathcal{M}$ and ω, this automorphism group will be denoted by $\mathbb{R} \ni t \mapsto \mathfrak{m}^t_{(\mathcal{M},\omega)} \in \mathrm{Aut}(\mathcal{M})$.

In what follows, we illustrate what the modular automorphism group looks like for faithful states on matrix and multi-matrix algebras over $\mathbb{C}$. In the terminology of [25, Ch. 2], a *multi-matrix algebra* is a finite direct sum of matrix algebras, i.e., an arbitrary finite-dimensional C^*-algebra up to $*$-isomorphism.

Notation 2.18. For the matrix algebra $\mathbb{M}_m(\mathbb{C})$, we denote by tr the *unnormalized trace* (so that $\mathrm{tr}(\mathbb{1}_m) = m$, where $\mathbb{1}_m$ is the identity matrix). When multiple matrix algebras appear in the same formula, the size m of the trace tr will be clear by the matrices it is evaluated on.

Lemma 2.19. *Let ρ be an invertible density matrix on $\mathcal{M} := \mathbb{M}_m(\mathbb{C})$ with associated faithful state $\omega := \mathrm{tr}(\rho\, \cdot\,)$. Then $\log(\rho)$ is a strictly negative operator, i.e., all its eigenvalues are strictly below 0, and the modular automorphism group of $(\mathcal{M}, \omega)$ is given by*

$$\mathbb{R} \ni t \mapsto \mathfrak{m}^t_{(\mathcal{M},\omega)} = \mathrm{Ad}(\rho^{it}) = \mathrm{Ad}(e^{it \log(\rho)}).$$

Proof. See [10, Example 2.5.16]. □

[4] This follows from the fact that S_Ω is involutive, namely $S_\Omega^2 = \mathrm{id}_\mathcal{H}$. Note that the adjoint S_Ω^* of the conjugate-linear map S_Ω is defined by the equation $\langle S_\Omega^* x, y \rangle = \overline{\langle x, S_\Omega y \rangle} \equiv \langle S_\Omega y, x \rangle$ for all $x, y \in \mathcal{H}$.

Lemma 2.20. *Let $\omega = \sum_{x\in X} p_x \operatorname{tr}(\rho_x \,\cdot\,)$ be a faithful state on $\mathcal{A} := \bigoplus_{x\in X} \mathbb{M}_{m_x}(\mathbb{C}) \equiv \bigoplus_{x\in X} \mathcal{A}_x$, where X is a finite set, $m_x \in \mathbb{N}$, $(p_x)_{x\in X}$ defines a nowhere vanishing probability measure on X, and each $\rho_x \in \mathbb{M}_{m_x}(\mathbb{C})$ is an invertible density matrix. Then the modular group of $(\mathcal{A},\omega)$ is given by*

$$\mathbb{R} \ni t \mapsto \mathfrak{m}^t_{(\mathcal{A},\omega)} = \bigoplus_{x\in X} \operatorname{Ad}(\rho_x^{it}) \equiv \bigoplus_{x\in X} \operatorname{Ad}\big(e^{it\log(\rho_x)}\big).$$

Proof. It follows as in the matrix case [10, Example 2.5.16] by checking that the KMS condition with respect to ω is fulfilled by $\mathbb{R} \ni t \mapsto \bigoplus_{x\in X} \operatorname{Ad}(\rho_x^{it})$. □

Remark 2.21. Note that, in the multi-matrix case, the states ω corresponding to different choices of $p_x > 0$, $\sum_{x\in X} p_x = 1$ give the same modular automorphism group of $\mathcal{A}$.

For future reference, we also state and prove some lemmas that will be needed later. First, we recall a general representation formula for conditional expectations between type I factors [58, Prop. 2.4]. Given two von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ realized on the same Hilbert space, we denote by $\mathcal{A}\vee\mathcal{B}$ the von Neumann algebra generated by $\mathcal{A}$ and $\mathcal{B}$, i.e., the smallest von Neumann algebra inside the von Neumann algebra of bounded operators containing both $\mathcal{A}$ and $\mathcal{B}$.

Lemma 2.22. *Let $\mathcal{N}\subseteq\mathcal{M}$ be a type I subfactor. Namely, $\mathcal{N}\cong\mathbb{M}_n(\mathbb{C})$, $\mathcal{M}\cong\mathbb{M}_m(\mathbb{C})$ and $\mathbb{1}_k\otimes\mathbb{M}_n(\mathbb{C})\cong\mathcal{N}\subseteq\mathcal{M}\cong\mathbb{M}_k(\mathbb{C})\otimes\mathbb{M}_n(\mathbb{C})$, with $m=nk$, for some $n,m,k\in\mathbb{N}$ or $n,m=\infty$. Every normal (not necessarily faithful) conditional expectation $E:\mathcal{M}\rightsquigarrow\mathcal{M}$ onto $\mathcal{N}$ can be represented as a partial trace. Namely, there is a (not necessarily invertible) density matrix $\tau\in\mathbb{M}_k(\mathbb{C})$ such that $E=\operatorname{tr}(\tau\,\cdot\,)\otimes\mathrm{id}_n$, where $\mathrm{id}_n:\mathbb{M}_n(\mathbb{C})\to\mathbb{M}_n(\mathbb{C})$ is the identity map.*

Proof. For type I subfactors, $\mathcal{M}=(\mathcal{N}'\cap\mathcal{M})\vee\mathcal{N}\cong(\mathcal{N}'\cap\mathcal{M})\otimes\mathcal{N}$ and $\mathcal{N}'\cap\mathcal{M}$ is a type I factor. The restriction $E_\restriction:\mathcal{N}'\cap\mathcal{M}\rightsquigarrow\mathcal{N}'\cap\mathcal{N}\cong\mathbb{C}\mathbb{1}_m$ is a normal state on the relative commutant. Thus $E_\restriction$ is represented by a unique positive, not necessarily invertible, trace one operator $\tau\in\mathcal{N}'\cap\mathcal{M}$ by the formula

$$E(A\otimes\mathbb{1}_n)=\operatorname{tr}(\tau A)\mathbb{1}_m$$

where $A\in\mathcal{N}'\cap\mathcal{M}$ and tr is the trace on $\mathcal{N}'\cap\mathcal{M}\cong M_k(\mathbb{C})$. The representation formula has a unique extension to simple tensors in $\mathcal{M}$ by $\mathcal{N}$-bimodularity, namely

$$E(A\otimes B)=E((A\otimes\mathbb{1}_n)(\mathbb{1}_k\otimes B))=\operatorname{tr}(\tau A)\mathbb{1}_k\otimes B$$

for $A\in\mathcal{N}'\cap\mathcal{M}$, $B\in\mathcal{N}$, thus to $\mathcal{M}$. □

2.2 Disintegrations on matrix algebras

In the first proposition below, we prove several equivalent conditions for disintegrations to exist on matrix algebras equipped with *faithful* states. We prove this directly using only methods of linear algebra and complex analysis (as opposed to the full power of Takesaki's theorem and modular theory [56]) because the techniques used here will also be used later in this work.

Proposition 2.23. *Let $F:\mathbb{M}_n(\mathbb{C})\to\mathbb{M}_{nk}(\mathbb{C})\cong\mathbb{M}_k(\mathbb{C})\otimes\mathbb{M}_n(\mathbb{C})$ be given by $F(A):=\mathbb{1}_k\otimes A$ and let $\omega\equiv\operatorname{tr}(\rho\,\cdot\,)$ be a faithful state on $\mathbb{M}_{nk}(\mathbb{C})$ that pulls back to $\xi=\operatorname{tr}(\sigma\,\cdot\,)$ along F. Let $\mathfrak{m}^t_{(\mathbb{M}_{nk}(\mathbb{C}),\omega)}$, $t\in\mathbb{R}$, denote the modular group associated to $(\mathbb{M}_{nk}(\mathbb{C}),\omega)$. Then, the following conditions are equivalent.*

i. The pair (F, ω) admits a disintegration.

ii. There exists an ω-preserving conditional expectation $E : \mathbb{M}_{nk}(\mathbb{C}) \rightsquigarrow \mathbb{M}_{nk}(\mathbb{C})$ onto the subalgebra $\mathbb{1}_k \otimes \mathbb{M}_n(\mathbb{C})$.

iii. There exists an invertible density matrix $\tau \in \mathbb{M}_k(\mathbb{C})$ such that $\rho = \tau \otimes \sigma$.

iv. The modular group $\mathfrak{m}^t_{(\mathbb{M}_{nk}(\mathbb{C}),\omega)}$ leaves the subalgebra $F(\mathbb{M}_n(\mathbb{C})) = \mathbb{1}_k \otimes \mathbb{M}_n(\mathbb{C}) \subseteq \mathbb{M}_{nk}(\mathbb{C})$ invariant for every $t \in \mathbb{R}$.

Proof. Throughout this proof, set $\mathcal{A} := \mathbb{M}_{nk}(\mathbb{C})$ and $\mathcal{B} := \mathbb{M}_n(\mathbb{C})$.

$(i \Rightarrow ii)$ Let $G : \mathcal{B} \rightsquigarrow \mathcal{A}$ be a disintegration of F. Then $G \circ F = \mathrm{id}_{\mathcal{B}}$ by faithfulness of ω and the fact that F is injective. Hence, $E := F \circ G$ is a UCP map such that $E^2 = E$ and is therefore a conditional expectation onto the C^*-subalgebra $F(\mathbb{M}_n(\mathbb{C}))$ by Tomiyama's theorem (Theorem 2.6).[5] The state-preserving condition follows from $\omega \circ E = \omega \circ F \circ G = \xi \circ G = \omega$ because G is a disintegration.

$(i \Leftarrow ii)$ Given such a conditional expectation E, set

$$G := \left(\mathcal{A} \overset{\mathbb{1}_k \otimes \mathcal{B} | E}{\rightsquigarrow} \mathbb{1}_k \otimes \mathcal{B} \xrightarrow{(\mathbb{1}_k \otimes \mathcal{B} | F)^{-1}} \mathcal{B} \right).$$

Then it immediately follows that G is a disintegration of (F, ω).

$(i \Leftrightarrow iii)$ This follows from [43, Theorem 4.3] and the fact that ρ is invertible implies τ is invertible.

$(iii \Rightarrow iv)$ Suppose there exists a (necessarily invertible) density matrix $\tau \in M_k(\mathbb{C})$ such that $\rho = \tau \otimes \sigma$. In what follows, we will first prove $\rho^z = \tau^z \otimes \sigma^z$ for all $z \in \mathbb{C}$ using Carlson's theorem (Theorem A.4). Afterwards, we will express the modular group in terms of this result and prove it fixes the subalgebra $\mathcal{N} := \mathbb{1}_k \otimes M_n(\mathbb{C})$. Fix arbitrary vectors $v, w \in \mathbb{C}^{nk}$ and define $f : \mathbb{C} \to \mathbb{C}$ by

$$\mathbb{C} \ni z \mapsto f(z) := \langle v, \rho^z w \rangle - \langle v, \tau^z \otimes \sigma^z w \rangle.$$

Then f is entire since $\rho^z = e^{z \log(\rho)}$, and similarly for τ and σ, which are all well-defined by the faithfulness assumption. For any $z \in \mathbb{C}$,

$$\begin{aligned}
|f(z)| &\leq \left(\|\rho^z\| + \|\tau^z\| \|\sigma^z\| \right) \|v\| \|w\| && \text{by Cauchy–Schwarz} \\
&\leq \left(\left\|e^{z \log(\rho)}\right\| + \left\|e^{z \log(\tau)}\right\| \left\|e^{z \log(\sigma)}\right\| \right) \|v\| \|w\| \\
&= \left(e^{|z| \|\log(\rho)\|} + e^{|z| \|\log(\tau)\|} e^{|z| \|\log(\sigma)\|} \right) \|v\| \|w\| && \text{since } \left\|e^B\right\| \leq e^{\|B\|} \\
&\leq 2 \|v\| \|w\| e^{-\log(\rho_0)|z|}, && (2.24)
\end{aligned}$$

where ρ_0 is the smallest eigenvalue of ρ (and equals the product of the smallest eigenvalues of τ and σ). Secondly, since ρ is self-adjoint, $\rho^{it} = e^{it \log(\rho)}$ is unitary for all $t \in \mathbb{R}$ and similarly for τ and σ. Hence, $\|\rho^{it}\| = \|\tau^{it}\| = \|\sigma^{it}\| = 1$ for all $t \in \mathbb{R}$ and

$$|f(it)| \leq 2 \|v\| \|w\| \qquad \forall\, t \in \mathbb{R}$$

[5] The norm of a UCP map H is always 1 since it equals $\|H(1)\| = \|1\| = 1$.

by the first line in (2.24). Thirdly, for any $l \in \mathbb{N}$,

$$\rho^l = \tau^l \otimes \sigma^l \implies f(l) = 0.$$

Hence by Carlson's theorem (setting $C := 2\|v\|\|w\|, \gamma := -\log(\rho_0)$, and $\gamma' := 0$), $f \equiv 0$. Since this calculation was independent of $v, w \in \mathbb{C}^{nk}$, we have $\rho^z = \tau^z \otimes \sigma^z$ for all $z \in \mathbb{C}$. Therefore, by Lemma 2.19,

$$\mathfrak{m}^t_{(\mathbb{M}_{nk}(\mathbb{C}),\omega)}(\mathbb{1}_k \otimes A) = \mathrm{Ad}_{\rho^{it}}(\mathbb{1}_k \otimes A) = (\tau^{it} \otimes \sigma^{it})(\mathbb{1}_k \otimes A)(\tau^{-it} \otimes \sigma^{-it}) = \mathbb{1}_k \otimes \mathrm{Ad}_{\sigma^{it}}(A)$$

for every $A \in \mathbb{M}_n(\mathbb{C})$, which proves the first implication.

$(iv \Rightarrow iii)$ Suppose the modular group $\mathbb{R} \ni t \mapsto \mathfrak{m}^t_{(\mathbb{M}_{nk}(\mathbb{C}),\omega)} = \mathrm{Ad}(\rho^{it})$ leaves the subalgebra $F(\mathbb{M}_n(\mathbb{C}))$ invariant. Since $\rho^{it} = e^{it\log(\rho)}$ and ρ is a strictly positive matrix, the functional calculus guarantees ρ^z exists for all $z \in \mathbb{C}$. By assumption,

$$\Big[\mathrm{Ad}_{\rho^{it}}\big(\mathbb{1}_k \otimes \mathbb{M}_n(\mathbb{C})\big), \mathbb{M}_n(\mathbb{C}) \otimes \mathbb{1}_k\Big] = 0 \qquad \forall\, t \in \mathbb{R}. \tag{2.25}$$

To see that this identity extends to all complex t as well, let $v, w \in \mathbb{C}^{nk}$, $A \in \mathbb{M}_n(\mathbb{C})$, and $A' \in \mathbb{M}_k(\mathbb{C})$ and define

$$\mathbb{C} \ni z \mapsto f(z) := \Big\langle v, \big[\rho^z(\mathbb{1}_k \otimes A)\rho^{-z}, A' \otimes \mathbb{1}_n\big] w\Big\rangle.$$

Then f is holomorphic for all $z \in \mathbb{C}$. By assumption (2.25), f equals zero on the imaginary axis and therefore is identically zero by the identity theorem (Theorem A.1). Since this holds for all $v, w \in \mathbb{C}^{nk}$, $A \in \mathbb{M}_n(\mathbb{C})$, and $A' \in \mathbb{M}_k(\mathbb{C})$, this proves

$$\rho^z\big(\mathbb{1}_k \otimes \mathbb{M}_n(\mathbb{C})\big)\rho^{-z} \subseteq \mathbb{1}_k \otimes \mathbb{M}_n(\mathbb{C}) \qquad \forall\, z \in \mathbb{C}.$$

Setting $z = 1$, for each $A \in \mathbb{M}_n(\mathbb{C})$, there exists a $B \in \mathbb{M}_n(\mathbb{C})$ such that

$$\rho(\mathbb{1}_k \otimes A) = (\mathbb{1}_k \otimes B)\rho.$$

If we write ρ as $\rho = \sum_{i,j} E_{ij} \otimes \rho_{ij}$, where E_{ij} are the matrix units in $\mathbb{M}_k(\mathbb{C})$, then this condition is equivalent to

$$\rho_{ij} A = B \rho_{ij} \qquad \forall\, i, j.$$

Since ρ is *strictly* positive, each of the blocks ρ_{jj} are strictly positive because $\langle v, \rho v\rangle > 0$ for all non-zero vectors $v \in \mathbb{C}^{nk}$. Hence, ρ_{jj} is invertible for all j. In particular, we obtain $\rho_{11} A \rho_{11}^{-1} = B$. Upon plugging this into the arbitrary ij equations we obtain

$$\rho_{ij} A = B \rho_{ij} = \rho_{11} A \rho_{11}^{-1} \rho_{ij} \iff \rho_{11}^{-1}\rho_{ij} A = A \rho_{11}^{-1}\rho_{ij} \qquad \forall\, A \in M_n(\mathbb{C}).$$

In other words, $\rho_{11}^{-1}\rho_{ij}$ is in the commutant of $\mathbb{M}_n(\mathbb{C})$, which is just $\mathbb{C}\mathbb{1}_n$. Thus, there exists a $\lambda_{ij} \in \mathbb{C}$ such that $\rho_{11}^{-1}\rho_{ij} = \lambda_{ij}\mathbb{1}_n$, i.e.,

$$\rho_{ij} = \lambda_{ij}\rho_{11}.$$

Therefore, ρ can be expressed as the tensor product

$$\rho = \left(\mathrm{tr}(\rho_{11}) \sum_{i,j} \lambda_{ij} E_{ij}\right) \otimes \left(\frac{\rho_{11}}{\mathrm{tr}(\rho_{11})}\right) =: \tau' \otimes \sigma'$$

of two density matrices. The fact that the left factor τ' is positive is simply because ρ and ρ_{11} are positive while the fact that it is a density matrix follows from the computation

$$1 = \operatorname{tr}(\rho) = \sum_j \operatorname{tr}(\rho_{jj}) = \sum_j \operatorname{tr}(\lambda_{jj}\rho_{11}) = \sum_j \lambda_{jj} \operatorname{tr}(\rho_{11}) = \operatorname{tr}\left(\operatorname{tr}(\rho_{11}) \sum_{i,j} \lambda_{ij} E_{ij}\right).$$

It immediately follows from this that $\sigma' = \sigma$ by the condition $\omega \circ F = \xi$, which holds if and only if $\operatorname{tr}_{\mathbb{M}_k(\mathbb{C})}(\rho) = \sigma$. □

Remark 2.26. The equivalence $(i \Leftrightarrow ii)$, under the present faithfulness assumption on ω, holds more generally and with the same proof for unital injective $*$-homomorphisms F between von Neumann algebras, cf. [35, Lemma 7.2]. Under the faithfulness assumption on ω, the equivalence $(ii \Leftrightarrow iv)$ is Takesaki's theorem [56], which we reproved above for completeness (passing through *iii*) in the finite-dimensional matrix algebra context. We shall further discuss and generalize it in Section 4.

A generalization of Proposition 2.23 to the case of *non-faithful states* (e.g., pure states on matrix algebras) is not entirely trivial. This is mainly because items *iii* and *iv* are problematic when ω is non-faithful—one cannot simply work with the truncated modular group, see Remark 2.29 below. Nevertheless, items *i* and *ii* above are still equivalent, and a modified version of item *iii* holds, as the following proposition shows.

Proposition 2.27. *Given the same assumptions as in Proposition 2.23 with the exception that the state $\omega = \operatorname{tr}(\rho\,\cdot\,)$ need not be faithful, the following conditions are equivalent.*

i. The pair (F, ω) admits a disintegration.

ii. There exists an ω-preserving conditional expectation $E : \mathbb{M}_{nk}(\mathbb{C}) \rightsquigarrow \mathbb{M}_{nk}(\mathbb{C})$ onto the subalgebra $\mathbb{1}_k \otimes \mathbb{M}_n(\mathbb{C})$.

iii. There exists a density matrix $\tau \in \mathbb{M}_k(\mathbb{C})$ such that $\rho = \tau \otimes \sigma$.

Proof. It is easy to see that item *ii* implies *i*. As for the converse, if G is a disintegration, so that it satisfies $G \circ F \underset{\xi}{=} \mathrm{id}_{\mathbb{M}_n(\mathbb{C})}$, it follows that G actually satisfies $G \circ F = \mathrm{id}_{\mathbb{M}_n(\mathbb{C})}$ by [43, Theorem 4.3] (which itself relies on an important result on a.e. equivalence [43, Theorem 2.48] and on the factoriality of $\mathbb{M}_n(\mathbb{C})$), so that *i* implies *ii* as well.

The implication $(iii \Rightarrow ii)$ is also easy to see, while for the explicit implication $(ii \Rightarrow iii)$, with τ a (not necessarily invertible) density matrix, follows from Lemma 2.22. In more detail, for finite type I subfactors, every (normal) faithful conditional expectation $E : \mathbb{M}_m(\mathbb{C}) \cong \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C}) \rightsquigarrow F(\mathbb{M}_n(\mathbb{C})) = \mathbb{1}_k \otimes \mathbb{M}_n(\mathbb{C})$, with $m = nk$, by Lemma 2.22 is represented as a partial trace with respect to a density matrix $\tau \in \mathbb{M}_k(\mathbb{C})$. Thus, the condition of E being ω-preserving, i.e., $\omega = \omega \circ E$, reads

$$\begin{aligned} \operatorname{tr}(\rho(A \otimes B)) &= \operatorname{tr}(\rho(\mathbb{1}_k \otimes \operatorname{tr}(\tau A)B)) \\ &= \operatorname{tr}(\tau A) \operatorname{tr}(\rho(\mathbb{1}_k \otimes B)) \\ &= \operatorname{tr}(\tau A) \operatorname{tr}(\sigma B). \end{aligned}$$

Since the tensor product of the traces on $\mathbb{M}_k(\mathbb{C})$ and $\mathbb{M}_n(\mathbb{C})$ is the trace on $\mathbb{M}_m(\mathbb{C})$, we get $\operatorname{tr}((\rho - \tau \otimes \sigma)(A \otimes B)) = 0$ for all A, B. Therefore, by the faithfulness of the trace, we conclude $\rho = \tau \otimes \sigma$. □

Remark 2.28. One can also directly show the equivalence ($i \Leftrightarrow iii$) holds by different arguments cf. [43, Theorem 5.1].

Remark 2.29. The least trivial condition to generalize to the non-faithful setting is item *iv* in Proposition 2.23. One naive replacement for the non-faithful setting would be to work with the Hilbert space $\mathcal{H}_\omega := P_\omega \mathbb{C}^{nk}$ and the C^*-algebra $P_\omega \mathbb{M}_{nk}(\mathbb{C}) P_\omega$, where P_ω is the support projection of ω. On this subalgebra, ρ defines an invertible element, denoted $\rho_\restriction$, and $\omega_\restriction := \mathrm{tr}(\rho_\restriction \cdot\ )$ is faithful with associated modular group $\mathfrak{m}^t_{(P_\omega \mathbb{M}_{nk}(\mathbb{C}) P_\omega, \omega_\restriction)}$. However, even after one makes sense of a condition analogous to invariance under the modular group, this condition alone does *not* guarantee the existence of a state-preserving conditional expectation. More on this will be discussed in Section 4, where we generalize Takesaki's theorem [56] and we find the extra condition needed to guarantee the existence of a state-preserving conditional expectation.

2.3 Disintegrations on multi-matrix algebras

After being exposed to the simpler matrix algebra case, we extend Proposition 2.23 and Proposition 2.27 to the finite-dimensional C^*-algebra case in this section. In particular, we show the equivalence between disintegrations and state-preserving conditional expectations on finite-dimensional C^*-algebras with not necessarily faithful states. The next general lemma will be used throughout.

Lemma 2.30. *Let $F : (\mathcal{B}, \xi) \rightsquigarrow (\mathcal{A}, \omega)$ be a state-preserving UCP map between quantum probability spaces. Then the nullspaces satisfy $F(N_\xi) \subseteq N_\omega$. Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are finite-dimensional (or more generally, W^*-algebras), then $F(P_\xi^\perp) \leq P_\omega^\perp$ and $F(P_\xi) \geq P_\omega$. In particular, $P_\omega F(P_\xi) P_\omega = P_\omega$.*

Proof. To see $F(N_\xi) \subseteq N_\omega$, let $B \in \mathcal{B}$ satisfy $\xi(B^*B) = 0$. Then

$$0 \leq \omega\big(F(B)^* F(B)\big) \leq \omega\big(F(B^*B)\big) = \xi(B^*B) = 0,$$

where the Kadison–Schwarz inequality for F was used in the second inequality. From this, it immediately follows that $F(P_\xi^\perp) \in N_\omega = \mathcal{A} P_\omega^\perp$. Since F is $*$-preserving (because it is positive), $F(P_\xi^\perp)$ is self-adjoint and therefore $F(P_\xi^\perp) \in P_\omega^\perp \mathcal{A} P_\omega^\perp$. Furthermore, since F is order-preserving and unital, $F(P_\xi^\perp) \leq 1_\mathcal{A}$. But the largest element in $P_\omega^\perp \mathcal{A} P_\omega^\perp$ that is bounded from above by $1_\mathcal{A}$ is precisely $P_\omega^\perp$. Hence, $F(P_\xi^\perp) \leq P_\omega^\perp$. The claim $F(P_\xi) \geq P_\omega$ follows immediately from this and the definition of $^\perp$. Finally, since $P_\omega \leq F(P_\xi) \leq 1_\mathcal{A}$, applying the CP (and hence order-preserving) map Ad_{P_ω} to this pair of inequalities gives $P_\omega \leq P_\omega F(P_\xi) P_\omega \leq P_\omega$, which proves the last claim $P_\omega F(P_\xi) P_\omega = P_\omega$. □

Notation 2.31. The following notation will be used throughout this section. Set $\mathcal{A} = \bigoplus_{i=1}^s \mathbb{M}_{m_i}(\mathbb{C})$, $\mathcal{B} = \bigoplus_{j=1}^t \mathbb{M}_{n_j}(\mathbb{C})$. A $*$-homomorphism $F : \mathcal{B} \to \mathcal{A}$ is determined by its multiplicities $\{c_{ij} \in \mathbb{N} \cup \{0\}\}$ in the following sense. First, $m_i = \sum_{j=1}^t c_{ij} n_j$ for every i. As a result, every element $A_i \in \mathbb{M}_{m_i}(\mathbb{C})$ can be expressed as a $t \times t$ matrix

$$A_i \equiv \begin{bmatrix} A_{i;11} & \cdots & A_{i;1t} \\ \vdots & & \vdots \\ A_{i;t1} & \cdots & A_{i;tt} \end{bmatrix},$$

where the kl-th subblock, $A_{i;kl}$, is a $(c_{ik}n_k) \times (c_{il}n_l)$ matrix. A block diagonal matrix, i.e., $A_{i;kl} = 0$ for all $k \neq l$ will often be denoted by $\operatorname{diag}(A_{i;11}, \ldots, A_{i;tt})$ or $\coprod_{j=1}^{t} A_{i;jj}$. Second, up to unitary conjugation on the codomain, F has the form $F(\bigoplus_j B_j) = \bigoplus_i \coprod_j (\mathbb{1}_{c_{ij}} \otimes B_j) \equiv \bigoplus_i \operatorname{diag}(\mathbb{1}_{c_{i1}} \otimes B_1, \ldots, \mathbb{1}_{c_{it}} \otimes B_t)$. For convenience, set $X := \{1, \ldots, s\}$ and $Y := \{1, \ldots, t\}$. A state ω on $\mathcal{A}$ will often be decomposed as $\omega = \sum_{x \in X} p_x \operatorname{tr}(\rho_x \cdot\ )$, with $\rho_x \in \mathbb{M}_{m_x}(\mathbb{C})$ a density matrix and p a probability distribution on X. Similarly, write $\xi = \sum_{y \in Y} q_y \operatorname{tr}(\sigma_y \cdot\ )$ for a state on $\mathcal{B}$.

Theorem 2.32. *Let $F : \mathcal{B} \to \mathcal{A}$ be a (unital) $*$-homomorphism of finite-dimensional C^*-algebras, let ω be a not necessarily faithful state on $\mathcal{A}$ and let $\xi := \omega \circ F$ be the corresponding state on $\mathcal{B}$. Set $\mathcal{N} := F(\mathcal{B})$. Then, the following conditions are equivalent.*

i. The pair (F, ω) admits a disintegration.

ii. An ω-preserving conditional expectation $E : \mathcal{A} \rightsquigarrow \mathcal{A}$ onto $\mathcal{N}$ exists.

iii. For each $i \in X$ and $j \in Y$, there exist non-negative matrices $\tau_{ij} \in \mathcal{M}_{c_{ij}}(\mathbb{C})$ such that

$$\operatorname{tr}\left(\sum_{i=1}^{s} \tau_{ij}\right) = 1 \qquad \forall\, j \in Y \setminus N_q$$

and

$$p_i \rho_i = \coprod_{j=1}^{t} (q_j \tau_{ij} \otimes \sigma_j) \equiv \operatorname{diag}(q_1 \tau_{i1} \otimes \sigma_1, \ldots, q_t \tau_{it} \otimes \sigma_t) \qquad \forall\, i \in X.$$

If, in addition, ω is faithful, then these conditions are equivalent to

iv. The modular group $\mathfrak{m}^t_{(\mathcal{A},\omega)}$ leaves the subalgebra $\mathcal{N} \subseteq \mathcal{A}$ invariant for every $t \in \mathbb{R}$.

Proof. Let $N_F := \{ j \in Y \,:\, c_{ij} = 0 \;\forall\, i \in X \}$. Note that $N_F \subseteq N_q$, where N_q is the nullspace of the uniquely determined probability measure q associated to $\xi \equiv \sum_{y \in Y} q_y \operatorname{tr}(\sigma_y \cdot\ )$. This last fact follows from Lemma 2.30 since $F(N_\xi) \subseteq N_\omega$ and that if $F(B) \in N_\omega$, then $0 = \omega(F(B)^* F(B)) = \omega(F(B^* B)) = \xi(B^* B)$, which shows that $B \in N_\xi$. In particular, if $F(B) = 0$, then $B \in N_\xi$. Also note that F restricts to a $*$-isomorphism $F_\restriction : \bigoplus_{y \in Y \setminus N_F} \mathbb{M}_{n_j}(\mathbb{C}) \xrightarrow{\cong} \mathcal{N}$.

($i \Leftrightarrow iii$) This equivalence was proved in [43, Theorem 5.108]. In the original proof, one sees that it is still possible to choose τ_{ji} such that $\sum_i \tau_{ji}$ is a density matrix for all $j \in Y \setminus N_F$. Since $N_F \subseteq N_q$, this allows one to obtain a disintegration of the form

$$G_{ji}(A_i) = \begin{cases} \operatorname{tr}_{\mathbb{M}_{c_{ij}}(\mathbb{C})} \left((\tau_{ij} \otimes \mathbb{1}_{n_j}) A_{i;jj} \right) & \text{if } j \in Y \setminus N_F \\ \frac{1}{s m_i} \operatorname{tr}(A_i) \mathbb{1}_{n_j} & \text{if } j \in N_F \end{cases}, \tag{2.33}$$

which is ξ-a.e. equivalent to the formula provided in [43].

($i \Leftarrow ii$) Suppose E is an ω-preserving conditional expectation onto $\mathcal{N}$. Then, since every linear map $G : \mathcal{A} \rightsquigarrow \mathcal{B}$ is determined by the values on different factors, set G to be the map uniquely determined by the two composites

$$\mathcal{A} \overset{E_\restriction}{\rightsquigarrow} \mathcal{N} \xrightarrow{F_\restriction^{-1}} \bigoplus_{j \in Y \setminus N_F} \mathbb{M}_{n_j}(\mathbb{C})$$

and

$$\bigoplus_{j\in N_F} \frac{\mathrm{tr}(\,\cdot\,)}{\sqrt{\dim(\mathcal{A})}}\mathbb{1}_{n_j} : \mathcal{A} \rightsquigarrow \bigoplus_{j\in N_F} \mathbb{M}_{n_j}(\mathbb{C}).$$

Then G is a disintegration of (F,ω).

$(i \Rightarrow ii)$ Suppose (F,ω) admits a disintegration G. Then, Equation (2.33) provides one such disintegration. Using this formula, one sees that $F\circ G$ defines an ω-preserving conditional expectation onto $\mathcal{N}$. Indeed,

$$(F\circ G)(A) = \bigoplus_{i'\in X}\sum_{i\in X}\left(\bigsqcup_{j\in Y\setminus N_F} \mathbb{1}_{c_{i'j}} \otimes \mathrm{tr}_{\mathbb{M}_{c_{ij}}(\mathbb{C})}\left((\tau_{ij}\otimes \mathbb{1}_{n_j})A_{i;jj}\right)\right).$$

Note that the second expression in Equation (2.33) vanishes because $F(B)=0$ for all $B\in\mathbb{M}_{n_j}(\mathbb{C})$ with $j\in N_F$. Therefore, the fact that $F\circ G$ fixes $\mathcal{N}$ follows immediately from this calculation upon taking A to be in $\mathcal{N}$, which means it must be of the form $\bigoplus_{i\in X}\bigsqcup_{j\in Y}(\mathbb{1}_{c_{ij}}\otimes A_j)$, with $A_j\in\mathbb{M}_{n_j}(\mathbb{C})$.

$(iii \Leftrightarrow iv)$ If ω is faithful, this equivalence follows from a proof analogous to the one in Proposition 2.23 when combined with Lemma 2.20. □

Remark 2.34. One can also directly prove $(ii \Leftrightarrow iii)$ in Theorem 2.32 by the classification of *not necessarily faithful state-preserving* conditional expectations on direct sums of matrix algebras. This is different from the proof of $(i \Leftrightarrow iii)$ in [43, Theorem 5.108], which uses Kraus operators and facts about C^*-modules. The proof using conditional expectations provides useful techniques and is given in Appendix B.

Remark 2.35. An analogue of the end of Remark 2.29 applies here as well regarding the modular group for non-faithful states. We will come back to this in Section 5.

3 On a theorem of Bayes, Accardi, Cecchini, and Petz

3.1 Bayesian inverses

Recent work in categorical probability theory has allowed a potentially powerful and completely diagrammatic formulation of a version of Bayes' theorem involving the idea of a Bayesian inverse [20, 12, 40].[6] We begin with a definition followed by a partially historical account of the appearance of this concept, though under different names, in the literature on operator algebras. We will use interchangeably the notation $\mathcal{A} \overset{G}{\rightsquigarrow} \mathcal{B}$ for $G:\mathcal{A}\rightsquigarrow\mathcal{B}$, and $\mathcal{A}\overset{G}{\to}\mathcal{B}$ for $G:\mathcal{A}\to\mathcal{B}$ (cf. Notation 2.1).

Definition 3.1. Let $\mathcal{B}\overset{F}{\rightsquigarrow}\mathcal{A}$ be a UCP map between finite-dimensional C^*-algebras, let $\mathcal{A}\overset{\omega}{\rightsquigarrow}\mathbb{C}$ be a state, and set $\xi := \omega\circ F$. A ***Bayesian inverse*** of (F,ω) is a UCP map $\mathcal{A}\overset{G}{\rightsquigarrow}\mathcal{B}$ such that $\xi\big(G(A)B\big) = \omega\big(AF(B)\big)$ for all $A\in\mathcal{A}$ and $B\in\mathcal{B}$. The notation $\overline{F}$ will also be used to denote a Bayesian inverse of (F,ω).

[6] See also [41, Section 3] for a concise review and a subtle distinction between two versions of Bayes' theorem. The present work only deals with the Bayesian inversion form.

The equation $\xi\big(G(A)B\big) = \omega\big(AF(B)\big)$, in the form presented above, seemed to have first appeared in the work of Accardi and Cecchini in 1982 [1], though they did not explicitly mention any connection to Bayes' theorem. Indeed, they were mainly concerned with faithful states ω and ξ, modular theory, and generalizing Takesaki's theorem. In the case of faithful states, they showed that a Bayesian inverse is a generalization of the notion of a state-preserving conditional expectation by extending a theorem of Takesaki [56], where Takesaki's theorem was the special case where F is a unital injective $*$-homomorphism (which corresponds to a subalgebra of $\mathcal{A}$). In fact, they introduced a more general notion of conditional expectation (called the *φ-conditional expectation* in [1]), which always exists in the not necessarily commutative setting, even when a state-preserving conditional expectation does not.

In follow-up work, Accardi and Cecchini [2] and Frigerio [19] continued investigations with this generalized conditional expectation, providing further examples and properties. Accardi and Cecchini proved that the generalized conditional expectation also specializes to the usual notion of classical conditional expectation for commutative algebras (see also [11], where a lucid exposition is given in the finite-dimensional setting). In 1984, Petz generalized this further to allow for UCP maps (not necessarily subalgebra inclusions), and provided many properties of the generalized conditional expectation [47], which was eventually called the transpose channel [38]. This map is known as the Petz recovery map due to all the work by Petz that followed in subsequent decades [48, 49]. The Petz recovery map has taken precedence in the quantum information community, particularly in recent years due to the intimate connection between the existence of recovery maps and saturation of certain measure distances (like relative entropies, f-divergences, and data-processing inequalities) between quantum states [49, 15, 29, 30, 60, 54, 28].

Both Bayesian inverses and Petz recovery maps agree a.e. in the case of commutative algebras, so that both can technically be viewed as generalizations of Bayesian inversion to the non-commutative setting. However, they are in general different (not even a.e. equivalent) on non-commutative algebras when the corresponding states are not faithful. For some illustrative examples exemplifying the difference in Bayesian inference in quantum systems, see [41]. In this paper, we will mainly focus on the Bayesian inverse, which we feel deserves further study. But before getting there, we will extend some of the results of [44] regarding the existence of Bayesian inverses.

3.2 Bayesian inversion and the modular group

Proposition 3.2. *Let $\mathcal{B} := \mathbb{M}_n(\mathbb{C}) \overset{F}{\rightsquigarrow} \mathbb{M}_m(\mathbb{C}) =: \mathcal{A}$ be a UCP map and let $\mathcal{A} \overset{\omega=\mathrm{tr}(\rho\,\cdot\,)}{\rightsquigarrow} \mathbb{C}$ be a faithful state on $\mathcal{A}$, with pullback $\xi := \omega \circ F =: \mathrm{tr}(\sigma\,\cdot\,)$ that is also faithful. Then the following are equivalent.*

i. A UCP Bayesian inverse of (F, ω) exists (and is necessarily unique).

ii. $F(\sigma B)\rho = \rho F(B\sigma)$ for all $B \in \mathcal{B}$.

iii. F acts as an intertwiner for the modular groups of ω and ξ, i.e., $F \circ \mathfrak{m}^t_{(\mathcal{B},\xi)} = \mathfrak{m}^t_{(\mathcal{A},\omega)} \circ F$ for all $t \in \mathbb{R}$.

We will call the intertwining condition in item *iii* of Proposition 3.2 the ***Accardi–Cecchini (AC) condition***.

Proof. The equivalence between items *i* and *ii* is covered by the results in [44, Section 5]. The equivalence between *i* and *iii* is proved in [1, Proposition 6.1] (see also [4, Lemma 2.5]). Nevertheless, we feel it is instructive to see a direct proof of the equivalence between *ii* and *iii*. We first prove *iii* implies *ii*. Since

$$\mathfrak{m}^t_{(\mathcal{A},\omega)} = \mathrm{Ad}\big(\rho^{it}\big) \quad \text{and} \quad \mathfrak{m}^t_{(\mathcal{B},\xi)} = \mathrm{Ad}\big(\sigma^{it}\big)$$

for all $t \in \mathbb{C}$, item *iii* reads

$$F(\sigma^{it} B \sigma^{-it}) = \rho^{it} F(B) \rho^{-it}$$

for all $t \in \mathbb{R}$ and $B \in \mathcal{B}$. By finite-dimensionality, this equation also holds for all $t \in \mathbb{C}$ by the identity theorem (Theorem A.1). Hence, setting $t = -i$ gives $F(\sigma B \sigma^{-1}) = \rho F(B) \rho^{-1}$ for all $B \in \mathcal{B}$. In particular, choosing B of the form $B\sigma$ gives $F(\sigma B) = \rho F(B\sigma)\rho^{-1}$. Multiplying by ρ on the right gives condition *ii*. The direction *ii* implies *iii* is a bit more involved. First, note that $F(\sigma B)\rho = \rho F(B\sigma)$ for all $B \in \mathcal{B}$ implies

$$F(\sigma^k B)\rho^k = \rho F(\sigma^{k-1} B \sigma)\rho^{k-1} = \cdots = \rho^k F(B\sigma^k)$$

for all $B \in \mathcal{B}$ and for all $k \in \mathbb{N}$. Note that this is also true when $k = 0$. Fixing B and vectors $v, w \in \mathbb{C}^n$, define

$$\mathbb{C} \ni z \mapsto \Phi(z) := \langle w | F(\sigma^z B \sigma^{-z}) - \rho^z F(B) \rho^{-z} | v \rangle.$$

Then Φ satisfies the conditions of Carlson's theorem (Theorem A.4) with constants

$$\gamma := 2\max\{\|\log\sigma\|, \|\log\rho\|\}, \qquad C := 2\|F\|\|B\|\|v\|\|w\|, \qquad \gamma' := 0$$

and noting that the map $\mathcal{B} \ni B \mapsto \sigma^z B \sigma^{-z}$ has norm bounded by $e^{2|z|\|\log\sigma\|}$ (the rest of the argument follows the same ideas those from the proof of Proposition 2.23). Since v, w, and B were arbitrary, this proves *iii*. □

The previous result also generalizes to finite-dimensional C^*-algebras.

Proposition 3.3. *Let $\mathcal{B} := \bigoplus_{y\in Y} \mathbb{M}_{n_y}(\mathbb{C}) \overset{F}{\rightsquigarrow} \bigoplus_{x\in X} \mathbb{M}_{m_x}(\mathbb{C}) =: \mathcal{A}$ be a UCP map of finite-dimensional C^*-algebras and let $\mathcal{A} \overset{\omega=\sum_{x\in X} p_x \operatorname{tr}(\rho_x\,\cdot\,)}{\rightsquigarrow} \mathbb{C}$ be a faithful state on $\mathcal{A}$, with pullback $\xi := \omega \circ F =: \sum_{y\in Y} q_y \operatorname{tr}(\sigma_y\,\cdot\,)$ that is also faithful. Write $F_{xy} : \mathbb{M}_{n_y}(\mathbb{C}) \rightsquigarrow \mathbb{M}_{m_x}(\mathbb{C})$ for the map ${}_{\mathbb{M}_{m_x}(\mathbb{C})\downarrow} F_{\restriction \mathbb{M}_{n_y}(\mathbb{C})}$ (cf. Notation 2.3). Then the following are equivalent.*

i. A UCP Bayesian inverse of (F, ω) exists (and is necessarily unique).

ii. $F_{xy}(\sigma_y B_y)\rho_x = \rho_x F_{xy}(B_y \sigma_y)$ for all $B_y \in \mathcal{M}_{n_y}$ and for all $x \in X$, $y \in Y$.

iii. F acts as an intertwiner for the modular groups of ω and ξ, i.e., $F \circ \mathfrak{m}^t_{(\mathcal{B},\xi)} = \mathfrak{m}^t_{(\mathcal{A},\omega)} \circ F$ for all $t \in \mathbb{R}$.

Proof. The equivalence between *i* and *ii* is covered by [44, Section 6]. Therefore, we prove *ii* is equivalent to *iii*. By Lemma 2.20, the modular groups associated to ω and ξ are given by

$$\mathfrak{m}^t_{(\mathcal{A},\omega)} = \bigoplus_{x\in X} \mathrm{Ad}\big(\rho_x^{it}\big) \qquad \text{and} \qquad \mathfrak{m}^t_{(\mathcal{B},\xi)} = \bigoplus_{y\in Y} \mathrm{Ad}\big(\sigma_y^{it}\big)$$

for all $t \in \mathbb{R}$, though these automorphisms are well-defined even for $t \in \mathbb{C}$, provided that one uses the inverse operator rather than the adjoint (the automorphism is not a $*$-isomorphism in general). A quick calculation shows that *iii* holds if and only if

$$F_{xy}(\sigma_y^{it} B_y \sigma_y^{-it}) = \rho_x^{it} F_{xy}(B_y) \rho_x^{-it}$$

for all $B_y \in \mathcal{M}_{n_y}$, for all $t \in \mathbb{R}$, and for all $x \in X$ and $y \in Y$. Thus, the same techniques from the proof of Proposition 3.2 apply here. □

These two results show that the condition $F(\sigma B)\rho = \rho F(B\sigma)$ for matrix algebras (and the more general equation for direct sums) is equivalent to the Accardi–Cecchini condition for the modular group.

Remark 3.4. Note that the condition $F(\sigma B)\rho = \rho F(B\sigma)$ is computationally easier to check than the modular group condition for two reasons: (1) a single time suffices and (2) there is no need of taking exponentials of density matrices. In fact, for $\mathcal{B} = \mathbb{M}_n(\mathbb{C})$, one needs to only check at most n^2 equations since the condition $F(\sigma B)\rho = \rho F(B\sigma)$ is linear in B, so that one can plug in matrix units $B = E_{ij}^{(n)}$ (or any basis) to check this condition.

More still needs to be said in the non-faithful setting, where an equation such as $F(\sigma B)\rho = \rho F(B\sigma)$ still makes sense, while the modular group condition does not. This will be elaborated upon in the remaining subsections.

3.3 Bayesian inverses and disintegrations

Bayesian inverses are generalizations of disintegrations just as the adjoints of Accardi–Cecchini are generalizations of state-preserving conditional expectations. This section will explain this in more detail as well as provide some of the functorial properties of Bayesian inverses [20, 40].

Proposition 3.5. *Under the same assumptions as in Proposition 2.23, all conditions are equivalent to*

v. The pair (F, ω) admits a Bayesian inverse (or any of the equivalent conditions in Proposition 3.2).

Rather than proving this, we state a much more general result that is valid for not necessarily faithful states.

Theorem 3.6. *Let $(\mathcal{B}, \xi) \overset{F}{\rightsquigarrow} (\mathcal{A}, \omega)$ be a state-preserving UCP map between two finite-dimensional non-commutative probability spaces with $\mathcal{A} = \bigoplus_{i=1}^{s} \mathbb{M}_{m_i}(\mathbb{C})$ and $\mathcal{B} = \bigoplus_{j=1}^{t} \mathbb{M}_{n_j}(\mathbb{C})$ for some finite sets X and Y. Then the following conditions are equivalent.*

i. The pair (F, ω) admits a Bayesian inverse and F is ω-a.e. deterministic in the sense of Definition 2.12.

ii. The pair (F, ω) admits a disintegration.

Proof. This follows from [40, Corollary 8.6]. □

Remark 3.7. If ω and ξ are faithful, the conditions in Theorem 3.6 are equivalent to any of the conditions in Proposition 3.3 and therefore also to the conditions in Theorem 2.32 because the existence of a state-preserving UCP left-inverse between non-degenerate quantum probability spaces *guarantees* that F is an injective $*$-homomorphism, see e.g. [36, Theorem 5].

However, if the states are *not* faithful, then $\mathcal{N} := F(\mathcal{B})$ need not be a subalgebra of $\mathcal{A}$ even if the disintegration condition holds, and so it is not even possible to formulate conditions *ii*, *iii*, and *iv* of Theorem 2.32 as stated. In this way, the notion of disintegration generalizes that of conditional expectation since it requires fewer assumptions and uses less structure explicit in its definition.

It is well-known that conditional expectations obey functoriality/compositionality. The same can be said of disintegrations and Bayesian inverses. Since we will need these statements in some theorems later, we provide them now.

Proposition 3.8 (Compositional properties of Bayesian inverses)**.** *In what follows, let* $(\mathcal{C},\zeta),(\mathcal{B},\xi)$*, and* $(\mathcal{A},\omega)$ *be finite-dimensional quantum probability spaces.*

i. The identity map $\mathrm{id}_{\mathcal{A}}$ *is a Bayesian inverse of* $(\mathrm{id}_{\mathcal{A}},\omega)$*.*

ii. Let $(\mathcal{C},\zeta) \overset{G}{\rightsquigarrow} (\mathcal{B},\xi) \overset{F}{\rightsquigarrow} (\mathcal{A},\omega)$ *be a pair of composable state-preserving UCP maps that admit Bayesian inverses* $(\mathcal{A},\omega) \overset{\overline{F}}{\rightsquigarrow} (\mathcal{B},\xi) \overset{\overline{G}}{\rightsquigarrow} (\mathcal{C},\zeta)$*. Then* $\overline{G}\circ\overline{F}$ *is a Bayesian inverse of* $(G\circ F,\omega)$*.*

iii. Let $(\mathcal{B},\xi) \overset{F}{\rightsquigarrow} (\mathcal{A},\omega)$ *be an invertible UCP map, whose inverse* F^{-1} *is UCP. Then* F^{-1} *is a Bayesian inverse of* (F,ω)*.*

Remark 3.9. We have been careful about the statements of Proposition 3.8 when the states are not faithful. The following comments justify this caution.

i. If $\overline{\mathrm{id}_{\mathcal{A}}}$ is a Bayesian inverse of $(\mathcal{A},\omega)$, then $\overline{\mathrm{id}_{\mathcal{A}}} \underset{\omega}{=} \mathrm{id}_{\mathcal{A}}$. In other words, the two maps need not be equal on the nose when ω is not faithful. However, if $\mathcal{A}$ is a matrix algebra, then $\overline{\mathrm{id}_{\mathcal{A}}} = \mathrm{id}_{\mathcal{A}}$, though this is a non-trivial fact [43, Theorem 2.48].

ii. If $\overline{F\circ G}$ denotes a Bayesian inverse of $(F\circ G,\omega)$, then $\overline{F\circ G} \underset{\omega}{=} \overline{G}\circ\overline{F}$. In other words, the composite of Bayesian inverses need not equal an arbitrary Bayesian inverse of $(F\circ G,\omega)$, but they are a.e. equal.

iii. Note that F and F^{-1} being UCP automatically implies F and F^{-1} are $*$-isomorphisms (for a string-diagrammatic proof, see [40, Corollary 4.15]).

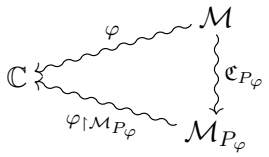
4 Takesaki's theorem for non-faithful states

If ω is a state on a finite-dimensional C^*-algebra $\mathcal{A}$ with support projection P_ω that is strictly less than $1_{\mathcal{A}}$, then the modular automorphism group as in Definition 2.17 does not exist. Instead, one can either define the modular automorphism group on the support algebra $P_\omega\mathcal{A}P_\omega$, where the state ω restricts to a faithful state, or one can define a modular automorphism semigroup on $\mathcal{A}$. If now $\mathcal{B} \overset{F}{\to} \mathcal{A}$ is a unital injective $*$-homomorphism and $\xi := \omega\circ F$ is the induced state, invariance of the subalgebra $F(\mathcal{B})$ under the modular group is not enough to guarantee the existence of a state-preserving conditional expectation. In fact, it need not be the case that the induced map $P_\xi\mathcal{B}P_\xi \rightsquigarrow P_\omega\mathcal{A}P_\omega$ is even a $*$-homomorphism, so asking for the invariance of a subalgebra on the supported algebras under the modular group does not even make sense. The purpose of this section is to address this and generalize Takesaki's theorem [56], which relates the existence of state-preserving conditional expectations to the modular group, to the setting of (not necessarily faithful) states on finite-dimensional C^*-algebras.

4.1 Matrix algebra case

Definition 4.1. Let $\mathcal{M} \overset{\varphi}{\rightsquigarrow} \mathbb{C}$ be a state on a finite-dimensional C^*-algebra $\mathcal{M}$, let P_φ be the projection of φ, and set $\mathcal{M}_{P_\varphi} := P_\varphi \mathcal{M} P_\varphi$ to be the ***support algebra*** associated to $(\mathcal{M}, \varphi)$ (this is also called the ***corner algebra*** in the literature). Also, let $\mathfrak{C}_{P_\varphi} : \mathcal{M} \rightsquigarrow \mathcal{M}_{P_\varphi}$ be the corestriction ${}_{\mathcal{M}_{P_\varphi}\downarrow} \mathrm{Ad}_{P_\varphi}$ map, which is UCP.

It is immediate from this definition that the diagram



(4.2)

commutes, where $\varphi_\uparrow := \varphi \circ \iota_{P_\varphi}$ is the induced faithful state from the (non-unital) inclusion $\mathcal{M}_{P_\varphi} \overset{\iota_{P_\varphi}}{\hookrightarrow} \mathcal{M}$.

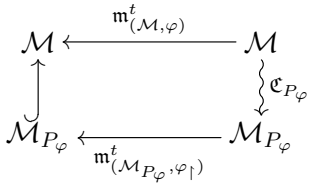
Although the modular group is not defined for φ, one can still define a closely related object with many similar properties since the modular automorphism group associated to $\varphi_\uparrow$ is well-defined. Indeed, set

$$\mathbb{R} \ni t \mapsto \mathfrak{m}^t_{(\mathcal{M},\varphi)} := \iota_{P_\varphi} \circ \mathfrak{m}^t_{(P_\varphi \mathcal{M} P_\varphi, \varphi_\uparrow)} \circ \mathfrak{C}_{P_\varphi}.$$

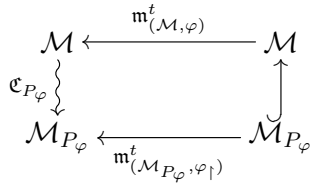
Although not an automorphism group, this provides a family of partial isometries on $\mathcal{M}$ that agrees with the modular group when restricted to $P_\varphi \mathcal{M} P_\varphi$ and sends the remaining vector subspaces $P_\varphi \mathcal{M} P_\varphi^\perp$, $P_\varphi^\perp \mathcal{M} P_\varphi$, and $P_\varphi^\perp \mathcal{M} P_\varphi^\perp$ to zero.

As such, this family of maps will be called the ***modular automorphism semigroup*** associated to the state.

In summary, the modular automorphism group on the support algebra and the modular automorphism semigroup on the original algebra are related by the commutative diagrams



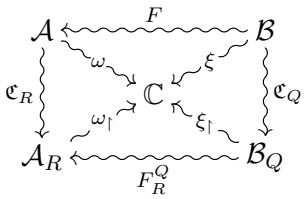
and



for all $t \in \mathbb{R}$.

Notation 4.3. It will be helpful to set up the following notation for the next few subsections. Let $(\mathcal{B}, \xi) \overset{F}{\rightsquigarrow} (\mathcal{A}, \omega)$ be a state-preserving UCP map. Let $R := P_\omega$ and $Q := P_\xi$ be the support projections. Let $\mathcal{B} \overset{\mathfrak{C}_Q}{\rightsquigarrow} \mathcal{B}_Q$ and $\mathcal{A} \overset{\mathfrak{C}_R}{\rightsquigarrow} \mathcal{A}_R$ be the projections onto the support algebras. Finally, let $\mathcal{B}_Q \overset{F_R^Q}{\rightsquigarrow} \mathcal{A}_R$ be the induced map given by $F_R^Q := \mathfrak{C}_R \circ F \circ j_Q$, where $\mathcal{B}_Q \overset{j_Q}{\hookrightarrow} \mathcal{B}$ is the non-unital inclusion.

Lemma 4.4. *In terms of Notation 4.3, the map F_R^Q is UCP and the diagram*



commutes.

Proof. Lemma 2.30 implies $F_R^Q(Q) = RF(Q)R \geq R$. However, since F_R^Q is the composite of a sub-unital map, the non-unital inclusion $\mathcal{B}_Q \hookrightarrow \mathcal{B}$, followed by two unital maps, F and $\mathfrak{C}_R$, it follows that F_R^Q is sub-unital, i.e., $F_R^Q(Q) \leq R$. Hence $F_R^Q(Q) = R$ so that F_R^Q is unital. Since F_R^Q is the composite of CP maps, F_R^Q is UCP. Finally, the diagram commutes because

$$\begin{aligned} RF(B)R &= RF\left(QBQ + QBQ^\perp + Q^\perp BQ + Q^\perp BQ^\perp\right) R \\ &= RF(QBQ)R + R\underbrace{F(QBQ^\perp)}_{\in \mathcal{A}R^\perp} R + R\underbrace{F(Q^\perp BQ)}_{\in R^\perp \mathcal{A}} R + R\underbrace{F(Q^\perp BQ^\perp)}_{\in R^\perp \mathcal{A}R^\perp} R \\ &= RF(QBQ)R \\ &= F_R^Q(QBQ). \end{aligned}$$

□

Lemma 4.5. *In terms Notation 4.3, the Accardi–Cecchini (AC) condition on the support algebra is equivalent to*

$$\mathrm{Ad}_{P_\omega} \circ F \circ \mathfrak{m}_\xi^t = \mathfrak{m}_\omega^t \circ F \circ \mathrm{Ad}_{P_\xi} \qquad \forall\, t \in \mathbb{R}. \tag{4.6}$$

Proof. Indeed, temporarily let $i : \mathcal{A}_{P_\omega} \hookrightarrow \mathcal{A}$ and $j : \mathcal{B}_{P_\xi} \hookrightarrow \mathcal{B}$ denote the non-unital inclusions. If the AC condition holds on the support algebras, then

$$\begin{aligned} \mathrm{Ad}_{P_\omega} \circ F \circ \mathfrak{m}_\xi^t &= j \circ \mathfrak{C}_{P_\omega} \circ F \circ i \circ \mathfrak{m}_{\xi_\uparrow}^t \circ \mathfrak{C}_{P_\xi} \\ &= j \circ F_{P_\omega}^{P_\xi} \circ \mathfrak{m}_{\xi_\uparrow}^t \circ \mathfrak{C}_{P_\xi} \\ &= j \circ \mathfrak{m}_{\omega_\uparrow}^t \circ F_{P_\omega}^{P_\xi} \circ \mathfrak{C}_{P_\xi} \\ &= j \circ \mathfrak{m}_{\omega_\uparrow}^t \circ \mathfrak{C}_{P_\omega} \circ F \circ \mathrm{Ad}_{P_\xi} \\ &= \mathfrak{m}_\omega^t \circ F \circ \mathrm{Ad}_{P_\xi}\,. \end{aligned}$$

Conversely, if (4.6) holds, then

$$\begin{aligned} F_{P_\omega}^{P_\xi} \circ \mathfrak{m}_{\xi_\uparrow}^t &= \mathfrak{C}_{P_\omega} \circ F \circ \underbrace{j \circ \mathfrak{C}_{P_\xi}}_{=\mathrm{Ad}_{P_\xi}} \circ \mathfrak{m}_\xi^t \circ j \\ &= \mathfrak{C}_{P_\omega} \circ \mathrm{Ad}_{P_\omega} \circ F \circ \mathfrak{m}_\xi^t \circ j \\ &= \mathfrak{C}_{P_\omega} \circ \mathfrak{m}_\omega^t \circ F \circ \underbrace{\mathrm{Ad}_{P_\xi} \circ j}_{=j} \\ &= \underbrace{\mathfrak{C}_{P_\omega} \circ i}_{=\mathrm{id}_{\mathcal{A}_{P_\omega}}} \circ \mathfrak{m}_{\omega_\uparrow}^t \circ \mathfrak{C}_{P_\omega} \circ F \circ j \\ &= \mathfrak{m}_{\omega_\uparrow}^t \circ F_{P_\omega}^{P_\xi}. \end{aligned}$$

□

Definition 4.7. Both conditions from Lemma 4.5 will be referred to as the ***Accardi–Cecchini (AC) condition***.

Although the AC condition is a consequence of the Bayes condition, which reads $\omega(AF(B)) = \xi(G(A)B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, it is not equivalent to the Bayes condition. In fact, even if F is an injective $*$-homomorphism (and hence describes a unital subalgebra inclusion), then

the invariance of the subalgebra $F(\mathcal{B})$ under the modular automorphism semigroup (which is equivalent to the AC condition) is *not* sufficient for the existence of a state-preserving conditional expectation, when the states in question are not faithful. We will soon show that demanding F_R^Q to be a $*$-homomorphism is a necessary condition, and combined with the AC condition it is also sufficient.

Before analyzing the general case, we first consider the more explicit case of (unital) inclusions of matrix algebras. In terms of Notation 4.3, let $\mathcal{A} = \mathbb{M}_{kn}(\mathbb{C})$ and $\mathcal{B} = \mathbb{M}_n(\mathbb{C})$, with $k \in \mathbb{N}$, and $F(B) = \mathbb{1}_k \otimes B$ for all $B \in \mathcal{B}$. Represent the states ω and ξ, respectively on $\mathbb{M}_{kn}(\mathbb{C})$ and $\mathbb{M}_n(\mathbb{C})$, by density matrices as $\omega = \mathrm{tr}(\rho\,\cdot\,)$ and $\xi = \mathrm{tr}(\sigma\,\cdot\,)$. In this case, F_R^Q is given by

$$\begin{aligned} F_R^Q : \mathbb{M}_n(\mathbb{C})_Q &\rightsquigarrow \mathbb{M}_{kn}(\mathbb{C})_R \\ QBQ &\mapsto R(\mathbb{1}_k \otimes QBQ)R. \end{aligned} \tag{4.8}$$

Theorem 4.9 (Non-faithful state generalization of Takesaki's theorem on matrix algebras)**.** *Given the data set up in the previous paragraph, the following are equivalent.*

i. There exists a unique ω-preserving conditional expectation $\mathbb{M}_{kn}(\mathbb{C}) \overset{E}{\rightsquigarrow} \mathbb{M}_{kn}(\mathbb{C})$ onto the subalgebra $\mathbb{1}_k \otimes \mathbb{M}_n(\mathbb{C}) \cong \mathbb{M}_n(\mathbb{C})$, i.e., (F, ω) admits a disintegration.

ii. The pair $(F_R^Q, \omega_\restriction)$ has a disintegration.

iii. The map F_R^Q defined in (4.8) is a unital $*$-homomorphism and satisfies the Accardi–Cecchini condition, i.e.,

$$\mathfrak{m}^t_{(\mathbb{M}_{kn}(\mathbb{C})_R, \omega_\restriction)} \circ F_R^Q = F_R^Q \circ \mathfrak{m}^t_{(\mathbb{M}_n(\mathbb{C})_Q, \xi_\restriction)} \qquad \forall\, t \in \mathbb{R}. \tag{4.10}$$

Proof. The proof of item i implies item ii will be provided in much greater generality in Theorem 4.14. The equivalence between items ii and iii proceeds as follows. First, note that the Bayes condition is equivalent to the AC condition by Proposition 3.5. Therefore, Theorem 3.6 shows items ii and iii are equivalent since the state $\omega_\restriction$ is faithful.

The only thing left to prove is therefore the implication ($iii \Rightarrow i$), which proceeds as follows. We will freely use the equivalence between items ii and iii. Since $\mathbb{1}_k \otimes Q \geq R$ by the state-preserving condition on the inclusion F, the non-unital inclusion $\mathbb{M}_{kn}(\mathbb{C})_R \hookrightarrow \mathbb{M}_{kn}(\mathbb{C})$ factors through $\mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})_Q$. Therefore, the composite F_R^Q followed by this non-unital inclusion gives a non-unital $*$-homomorphism $\mathbb{M}_n(\mathbb{C})_Q \to \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})_Q$. Hence, there exists a projection $S \in \mathbb{M}_k(\mathbb{C})$ and a unitary $R(\mathbb{C}^k \otimes \mathbb{C}^n) \xrightarrow{U} (S \otimes Q)(\mathbb{C}^k \otimes \mathbb{C}^n)$ such that

$$\left(\mathrm{Ad}_U \circ F_R^Q\right)(QBQ) = S \otimes QBQ \qquad \forall\, QBQ \in \mathbb{M}_n(\mathbb{C})_Q. \tag{4.11}$$

Since $\mathbb{1}_k \otimes Q \geq R$, the vector spaces $R(\mathbb{C}^k \otimes \mathbb{C}^n)$ and $(S \otimes Q)(\mathbb{C}^k \otimes \mathbb{C}^n)$ are both subspaces of $(\mathbb{1}_k \otimes Q)(\mathbb{C}^k \otimes \mathbb{C}^n)$. Hence, there exists a unitary extension $(\mathbb{1}_k \otimes Q)(\mathbb{C}^k \otimes \mathbb{C}^n) \xrightarrow{V} (\mathbb{1}_k \otimes Q)(\mathbb{C}^k \otimes \mathbb{C}^n)$ acting as U on the subspace $R(\mathbb{C}^k \otimes \mathbb{C}^n)$. Thus, V can be viewed as an element of $\mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})_Q$. Then $L := VR \in \mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})_Q$ is a partial isometry such that ${}_{\downarrow(S\otimes Q)(\mathbb{C}^k \otimes \mathbb{C}^n)} L_{\restriction R(\mathbb{C}^k \otimes \mathbb{C}^n)} = U$ as a linear operator. In particular, $L(\mathbb{1}_k \otimes QBQ)L^* = S \otimes QBQ$ for all $QBQ \in \mathbb{M}_n(\mathbb{C})_Q$ by (4.11).

In the next step of the argument, we will show $\mathrm{Ad}_L = \mathrm{Ad}_{W \otimes Q}$, when restricted to $\mathbb{1}_k \otimes \mathbb{M}_n(\mathbb{C})_Q$, for some partial isometry $W \in \mathbb{M}_k(\mathbb{C})$ such that $WW^* = S$. Write $L =$

$\sum_\alpha W_\alpha \otimes V_\alpha$ as a generic tensor product, where the sum is over a finite index set and where $W_\alpha \in \mathbb{M}_k(\mathbb{C})$ and $V_\alpha \in \mathbb{M}_n(\mathbb{C})_Q$ for all α. By assumption, $\sum_{\alpha,\beta} W_\alpha W_\beta^* \otimes V_\alpha Y V_\beta^* = S \otimes Y$ for all $Y \in \mathbb{M}_n(\mathbb{C})_Q$. Let $\{e_i^{(Q)}\}$ be an orthonormal basis of $Q\mathbb{C}^n$ with corresponding basis $\{E_{ij}^{(Q)} = |e_i^{(Q)}\rangle\langle e_j^{(Q)}|\}$ of $\mathbb{M}_n(\mathbb{C})_Q$ so that $i,j \in \{1,\dots,\mathrm{rank}(Q)\}$. Then the previous equation is equivalent to $\sum_{\alpha,\beta} W_\alpha W_\beta^* \otimes V_\alpha E_{ij}^{(Q)} V_\beta^* = S \otimes E_{ij}^{(Q)}$ for all i,j. Let $(V_\gamma)_{ij}^{(Q)} := \langle e_i^{(Q)}|V_\gamma|e_j^{(Q)}\rangle$ be the corresponding matrix entries of V_γ with $\gamma \in \{\alpha,\beta\}$. Therefore, expanding the left-hand gives

$$S \otimes E_{ij}^{(Q)} = \sum_{\alpha,\beta} W_\alpha W_\beta^* \otimes V_\alpha E_{ij}^{(Q)} V_\beta^* = \sum_{\alpha,\beta} \sum_{i',j'} (V_\alpha)_{i'i}^{(Q)} \overline{(V_\beta)_{j'j}^{(Q)}} W_\alpha W_\beta^* \otimes E_{i'j'}^{(Q)} \qquad \forall\, i,j.$$

By linear independence of the matrix units,

$$\sum_{\alpha,\beta} (V_\alpha)_{i'i}^{(Q)} \overline{(V_\beta)_{j'j}^{(Q)}} W_\alpha W_\beta^* = \sum_{\alpha,\beta} (V_\alpha)_{ii}^{(Q)} \overline{(V_\beta)_{jj}^{(Q)}} \delta_{ii'}\delta_{jj'} W_\alpha W_\beta^* \qquad \forall\, i,j,i',j'.$$

Setting

$$W := \sum_\alpha \mathrm{tr}(V_\alpha) W_\alpha \equiv \sum_\alpha \sum_i (V_\alpha)_{ii}^{(Q)} W_\alpha,$$

we conclude

$$\mathrm{Ad}_{W\otimes Q}(\mathbb{1}_k \otimes Y) = \mathrm{Ad}_L(\mathbb{1}_k \otimes Y) \qquad \forall\, Y \in \mathbb{M}_n(\mathbb{C})_Q.$$

Note that W is a partial isometry since $WW^* = S$. Hence, there exists a unitary extension $\widetilde{W} \in \mathbb{M}_k(\mathbb{C})$ such that[7]

$$\mathrm{Ad}_{\widetilde{W}\otimes Q}(F_R^Q(Y)) = \mathrm{Ad}_{W\otimes Q}(F_R^Q(Y)) = S \otimes Y$$

for all $Y \in \mathbb{M}_n(\mathbb{C})_Q$. Now, since $\widetilde{W}$ is unitary, the inverse of $\mathrm{Ad}_{\widetilde{W}\otimes Q}$ is given by

$$\mathrm{Ad}_{\widetilde{W}\otimes Q}^{-1} = \mathrm{Ad}_{\widetilde{W}^*\otimes Q}$$

as $*$-isomorphisms from the algebra $\mathbb{M}_k(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})_Q$ to itself and satisfying $\mathrm{Ad}_{\widetilde{W}\otimes Q}^{-1}(S\otimes Y) = R(\mathbb{1}_k \otimes Y)R$ for all $Y \in \mathbb{M}_n(\mathbb{C})_Q$, i.e.,

$$F_R^Q(Y) = \widetilde{W}^* S \widetilde{W} \otimes Y$$

for all $Y \in \mathbb{M}_n(\mathbb{C})_Q$. In particular, taking $Y = Q$ gives

$$R = T \otimes Q, \quad \text{where} \quad T := \widetilde{W}^* S \widetilde{W}.$$

Therefore, $\mathbb{M}_{kn}(\mathbb{C})_R = \mathbb{M}_k(\mathbb{C})_T \otimes \mathbb{M}_n(\mathbb{C})_Q$ and $F_R^Q(Y) = T \otimes Y$ for all $Y \in \mathbb{M}_n(\mathbb{C})_Q$.

Now, by *iii*, [1, Proposition 6.1] (see also [4, Lemma 2.5]) implies the existence of a Bayesian inverse $\mathbb{M}_{kn}(\mathbb{C})_R \overset{G}{\rightsquigarrow} \mathbb{M}_n(\mathbb{C})_Q$ of $(F_R^Q, \varphi_\restriction)$. Hence, G is a disintegration by [40, Proposition 7.31] (see also Theorem 3.6). Proposition 2.23 then implies $\rho = \tau \otimes \sigma$ for some invertible density matrix $\tau \in \mathbb{M}_k(\mathbb{C})_T$. Viewing this equation in $\mathbb{M}_{kn}(\mathbb{C})$ shows that $\rho = \tau \otimes \sigma$ and the map $\mathbb{M}_{kn}(\mathbb{C}) \overset{E}{\rightsquigarrow} \mathbb{M}_n(\mathbb{C})$ defined by $E := \mathrm{tr}_{\mathbb{M}_k(\mathbb{C})}(\tau \otimes \mathbb{1}_n\, \cdot\,)$ is a disintegration, which defines a state-preserving conditional expectation. Uniqueness of this expectation follows from [43, Theorem 4.3], for example. □

[7] Warning: It need not be the case that $\mathrm{Ad}_{\widetilde{W}\otimes Q}(\mathbb{1}_k \otimes Y) = \mathrm{Ad}_{W\otimes Q}(\mathbb{1}_k \otimes Y)$ for all $Y \in \mathbb{M}_n(\mathbb{C})_Q$ unless W was already unitary to begin with. It is only true that $\mathrm{Ad}_{\widetilde{W}\otimes Q}$ and $\mathrm{Ad}_{W\otimes Q}$ are equal when acting on the subalgebra $\mathbb{M}_{kn}(\mathbb{C})_R$.

In order to show the necessity of both the Accardi–Cecchini (AC) condition and F_R^Q being a $*$-homomorphism in Theorem 4.9, we provide two counterexamples where only one of the two conditions holds, showing that there does not exist a state-preserving conditional expectation.

Example 4.12 (AC does not hold, but F_R^Q is a $*$-homomorphism). It is enough to take a faithful state on $\mathbb{M}_{kn}(\mathbb{C})$ in such a way that the Takesaki condition is not satisfied (i.e., the density matrix is not a pure tensor) and then F_R^Q is just the injection, which is a $*$-homomorphism of course.

Example 4.13 (AC holds, but F_R^Q is not a $*$-homomorphism). This example is illustrated in [43, Theorem 4.3] and it is about the EPR state. Let $\mathbb{M}_2(\mathbb{C}) \overset{\iota}{\hookrightarrow} \mathbb{M}_4(\mathbb{C})$ be the inclusion and set $\rho := \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which is the projection onto the subspace generated by $\frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1)$. Set $\varphi := \mathrm{tr}(\rho\, \cdot\,)$. The density matrix associated to the state $\psi := \varphi \circ \iota$ is $\sigma = \frac{1}{2}\mathbb{1}_2$. Hence, the support algebra of $\mathbb{M}_4(\mathbb{C})$ becomes isomorphic to $\mathbb{C}$ because the support of ρ is one dimensional, while the support algebra of $\mathbb{M}_2(\mathbb{C})$ is itself, since the state ψ is faithful on it. Moreover since $\mathbb{C}$ is commutative, and since ψ is tracial, we have that the AC condition is satisfied (the modular groups both act as the identity), but F_R^Q cannot be a $*$-homomorphism, since it is a map from a higher dimensional simple algebra to a lower dimensional one. In this case a direct proof of the non disintegrability of the system is given in [43].

4.2 Multi-matrix algebra case

We now generalize Theorem 4.9 to the setting of arbitrary finite-dimensional C^*-algebras. Since the relationships between conditional expectations, disintegrations, Bayesian inverses, and the AC condition have already been established, we state the result in its greatest generality.

Theorem 4.14 (Non-faithful state generalization of Takesaki's theorem on finite-dimensional C^*-algebras)**.** *In terms of Notation 4.3 with $\mathcal{A}, \mathcal{B}$ arbitrary finite-dimensional C^*-algebras and ω, ξ states, as well as assuming F is a unital $*$-homomorphism, the following are equivalent.*

i. The pair (F, ω) admits a disintegration, i.e., there exists a ω-preserving conditional expectation from $\mathcal{A}$ to itself onto the subalgebra $F(\mathcal{B})$.

ii. The pair $(F_R^Q, \omega_\restriction)$ admits a disintegration. In particular, F_R^Q is a unital injective $*$-homomorphism.

Remark 4.15. In contrast to Theorem 4.9, The conditional expectation in item i of Theorem 4.14 need not be unique. Indeed, let $m, n \in \mathbb{N}$, with $n > 1$, and consider the unital inclusion $\mathbb{M}_m(\mathbb{C}) \oplus \mathbb{C} \hookrightarrow \mathbb{M}_m(\mathbb{C}) \oplus \mathbb{M}_n(\mathbb{C})$. Let ω be the state on $\mathbb{M}_m(\mathbb{C}) \oplus \mathbb{M}_n(\mathbb{C})$ uniquely determined by sending $A \oplus B$ to $\omega(A \oplus B) := \mathrm{tr}(\rho A)$ for some density matrix $\rho \in \mathbb{M}_m(\mathbb{C})$. Then $\mathrm{id}_m \oplus \varphi : \mathbb{M}_m(\mathbb{C}) \oplus \mathbb{M}_n(\mathbb{C}) \rightsquigarrow \mathbb{M}_m(\mathbb{C}) \oplus \mathbb{M}_n(\mathbb{C})$ defines an ω-preserving conditional expectation onto the subalgebra $\mathbb{M}_m(\mathbb{C}) \oplus \mathbb{C}$ for *any* state φ on $\mathbb{M}_n(\mathbb{C})$.

Proof.

($i \Rightarrow ii$) Let $\overline{F}$ be a disintegration of (F, ω). Set $G := \mathfrak{C}_Q \circ \overline{F} \circ j_R$. We claim that G is a disintegration of $(F_R^Q, \omega_\restriction)$. We will prove this in three steps by first showing G preserves states, then showing G is unital, and finally showing that G is a left-inverse of F_R^Q.

Step 1. G preserves states because

$$\begin{aligned}
\xi_\restriction\big(G(A)\big) &= \xi_\restriction\big(Q\overline{F}(RAR)Q\big) && \text{by definition of } G\\
&= \xi\big(\overline{F}(RAR)\big) && \text{by (4.2) for } \xi\\
&= \omega(RAR) && \text{since } \overline{F} \text{ is state-preserving}\\
&= \omega_\restriction(A) && \text{by (4.2) for } \omega
\end{aligned}$$

for all $A = RAR \in \mathcal{A}_R$.

Step 2. Unitality of G, meaning $G(R) = Q$, then follows from $G(R) = Q\overline{F}(R)Q = Q$, where the second equality holds by Lemma 2.30 since $\overline{F}$ is UCP and state-preserving.

Step 3. The condition $G \circ F_R^Q = \mathrm{id}_{\mathcal{B}_Q}$ for G to be a disintegration[8]

is equivalent to $Q\overline{F}\big(RF(B)R\big)Q = QBQ$ for all $B = QBQ \in \mathcal{B}_Q$. Since every B can be written as a linear combination of at most four positive elements, it suffices to prove $Q\overline{F}\big(RF(B^*B)R\big)Q = QB^*BQ$ for all *positive* $B^*B \in \mathcal{B}_Q$. For this, we first prove that $Q\overline{F}\big(RF(B^*B)R\big)Q \geq QB^*BQ$ (and afterwards, we will prove the reverse inequality). This follows from

$$\begin{aligned}
Q\overline{F}\big(RF(B^*B)R\big)Q &\geq Q\overline{F}\big(RF(B)^*F(B)R\big)Q && \text{by Kadison–Schwarz for } F\\
&= Q\overline{F}\Big(\big(F(B)R\big)^*\big(F(B)R\big)\Big)Q\\
&\geq Q\overline{F}\big(F(B)R\big)^*\overline{F}\big(F(B)R\big)Q && \text{by Kadison–Schwarz for } \overline{F}\\
&= \Big(\overline{F}\big(F(B)R\big)Q\Big)^*\Big(\overline{F}\big(F(B)R\big)Q\Big)\\
&= \big(B\overline{F}(R)Q\big)^*\big(B\overline{F}(R)Q\big) && \text{by right } \xi\text{-a.e. modularity}^9 \text{ of } \overline{F}\\
&= Q\overline{F}(R)B^*B\overline{F}(R)Q\\
&= Q\overline{F}(R)QB^*BQ\overline{F}(R)Q && \text{since } B^*B = QB^*BQ\\
&= QB^*BQ && \text{since } Q\overline{F}(R)Q = Q \text{ by Lemma 2.30.}
\end{aligned}$$

Using this, we can prove the other inequality as follows:

$$\begin{aligned}
0 &\leq \xi_\restriction\Big(Q\overline{F}\big(RF(B^*B)R\big)Q - QB^*BQ\Big) && \text{since } Q\overline{F}\big(RF(B^*B)R\big)Q \geq QB^*BQ\\
&= \omega_\restriction\big(RF(B^*B)R\big) - \xi_\restriction(QB^*BQ) && \text{since } \xi \circ \overline{F} = \omega\\
&= \omega_\restriction\big(F_R^Q(QB^*BQ)\big) - \xi_\restriction(QB^*BQ) && \text{since } R \leq F(R) \text{ by Lemma 2.30}\\
&= 0 && \text{since } \omega_\restriction \circ F_R^Q = \xi_\restriction.
\end{aligned}$$

Since $\xi_\restriction$ is faithful and the above argument is positive, this proves $Q\overline{F}\big(RF(B^*B)R\big)Q = QB^*BQ$ for all $B \in \mathcal{B}$. As stated above, since every element of $\mathcal{B}_Q$ can be written as a linear combination of positive elements, this proves $G \circ F_R^Q = \mathrm{id}_{\mathcal{B}_Q}$ and completes the proof that G is a disintegration of $(F_R^Q, \omega_\restriction)$.

[8] This is an equality because the states are faithful on the support algebras.

[9] Assuming the states to be faithful and F to be a unital injective $*$-homomorphism, this property (a.e. modularity) is also called *left-inverse property* in [14, Def. 3.2]. Under the same assumptions, it is known to be essentially equivalent to the conditional expectation property, see [35, Lemma 7.2].

$(i \Leftarrow ii)$ We will follow Notation 2.31 closely with some minor differences to avoid clutter. Since $*$-isomorphisms are automatically disintegrable, it suffices to assume

$$\bigoplus_{x\in X} \mathbb{M}_{m_x}(\mathbb{C}) =: \mathcal{A} \xleftarrow{F} \mathcal{B} := \bigoplus_{y\in Y} \mathbb{M}_{n_y}(\mathbb{C})$$

is given in standard form as

$$F\left(\bigoplus_{y\in Y} B_y\right) := \bigoplus_{x\in X} \bigsqcup_{y\in Y} (\mathbb{1}_{c_{xy}} \otimes B_y).$$

The states ω and ξ have support projections $R = \bigoplus_{x\in X} R_x$ and $Q = \bigoplus_{y\in Y} Q_y$, respectively. By Lemma 2.30, $R_x \leq \bigsqcup_{y\in Y}(\mathbb{1}_{c_{xy}} \otimes Q_y)$ for all $x \in X$. In a similar way, if

$$\mathcal{A} \xleftarrow{G} \mathcal{C} := \bigoplus_{x,y} \mathbb{M}_{c_{xy}}(\mathbb{C})$$

is the map

$$G\left(\bigoplus_{x,y} C_{xy}\right) := \bigoplus_{x\in X} \bigsqcup_{y\in Y} (C_{xy} \otimes \mathbb{1}_{n_y})$$

associated to the commutant of $F(\mathcal{B})$, then we also conclude $R_x \leq \bigsqcup_{y\in Y}(S'_{xy} \otimes \mathbb{1}_{n_y})$, where S'_{xy} is the support projection of the induced state $\omega \circ G$ restricted to $\mathbb{M}_{c_{xy}}(\mathbb{C})$. Since these commute with each other, we arrive at

$$R_x \leq \bigsqcup_{y\in Y} (S'_{xy} \otimes Q_y)$$

for every $x \in X$. In particular, there exist projections $R_{xy} \in \mathbb{M}_{c_{yx}}(\mathbb{C}) \otimes \mathbb{M}_{n_y}(\mathbb{C})$ such that $R_x = \bigsqcup_{y\in Y} R_{xy}$ for all $x \in X$. Because of this decomposition,

$$R_x \mathbb{C}^{m_x} = \bigoplus_{y\in Y} R_{xy}(\mathbb{C}^{c_{xy}} \otimes \mathbb{C}^{n_y})$$

for all $x \in X$.

Now, using the assumption that F_R^Q is a unital $*$-homomorphism, there exists a collection of projections $S_{xy} \in \mathbb{M}_{c_{xy}}(\mathbb{C})$ and a collection of unitaries

$$R_x \mathbb{C}^{m_x} \xrightarrow{U_x} \bigoplus_{y\in Y} (S_{xy} \otimes Q_y)(\mathbb{C}^{c_{xy}} \otimes \mathbb{C}^{n_y})$$

such that

$$\mathcal{B}_Q \ni B \mapsto \left(\mathrm{Ad}_U \circ F_R^Q\right)\left(\bigoplus_{y\in Y} B_y\right) = \bigoplus_{x\in X} \bigsqcup_{y\in Y} (S_{xy} \otimes B_y),$$

where $U := \bigoplus_{x\in X} U_x$. Hence, by the decomposition of R_x into $\bigsqcup_{y\in Y} R_{xy}$ and the fact that $U_x R_{xy} U_x^* = S_{xy} \otimes Q_y$ for all x, y, we conclude that U_x itself breaks up as a direct sum of unitaries[10] $U_x = \bigoplus_{y\in Y} U_{xy}$ with each unitary acting as

$$R_{xy}(\mathbb{C}^{c_{xy}} \otimes \mathbb{C}^{n_y}) \xrightarrow{U_{xy}} (S_{xy} \otimes Q_y)(\mathbb{C}^{c_{xy}} \otimes \mathbb{C}^{n_y}).$$

[10]Recall, if $\mathcal{V} \subseteq \mathcal{W}$ is a subspace of a finite-dimensional Hilbert space with corresponding orthogonal projection $P_{\mathcal{V}}$, and if $U : \mathcal{W} \to \mathcal{H}$ is a unitary map, then $U(\mathcal{V})$ has orthogonal projection $UP_{\mathcal{V}}U^*$.

Hence, there exist unitary extensions

$$(\mathbb{1}_{c_{xy}} \otimes Q_y)(\mathbb{C}^{c_{xy}} \otimes \mathbb{C}^{n_y}) \xrightarrow{V_{xy}} (\mathbb{1}_{c_{xy}} \otimes Q_y)(\mathbb{C}^{c_{xy}} \otimes \mathbb{C}^{n_y})$$

of the U_{xy} and which can be viewed as elements of $\mathbb{M}_{c_{xy}}(\mathbb{C}) \otimes \mathbb{M}_{n_y}(\mathbb{C})_{Q_y}$. Thus, $V_x := \bigoplus_{y \in Y} V_{xy}$ is a unitary extension of U_x for each $x \in X$. Set $L_{xy} := V_{xy} R_{xy}$ and $L_x := V_x R_x$. At this point, we can follow a similar analysis that was done in the case of factors to show the existence of a partial isometry $W_{xy} \in \mathbb{M}_{c_{xy}}(\mathbb{C})$ such that $W_{xy} W_{xy}^* = S_{xy}$ and $\mathrm{Ad}_{L_{xy}} = \mathrm{Ad}_{W_{xy} \otimes Q_y}$ as CP maps on $\mathbb{M}_{c_{xy}}(\mathbb{C}) \otimes \mathbb{M}_{n_y}(\mathbb{C})$. Hence, there exist unitary extensions $\widetilde{W}_{xy} \in \mathbb{M}_{c_{xy}}(\mathbb{C})$ of the W_{xy} such that

$$\mathrm{Ad}_{\widetilde{W}_{xy} \otimes Q_y}\left(R_{xy}(\mathbb{1}_{c_{xy}} \otimes B_y) R_{xy}\right) = S_{xy} \otimes B_y$$

for all $B_y \in \mathbb{M}_{n_y}(\mathbb{C})_{Q_y}$. Setting $T_{xy} := \widetilde{W}_{xy}^* S_{xy} \widetilde{W}_{xy}$, we conclude that $R_{xy} := T_{xy} \otimes Q_y$. This automatically implies that

$$F_R^Q\left(\bigoplus_{y \in Y} B_y\right) = \bigoplus_{x \in X} \bigboxplus_{y \in Y} (T_{xy} \otimes B_y)$$

for all $B = \bigoplus_y B_y \in \bigoplus_y \mathbb{M}_{n_y}(\mathbb{C})_{Q_y}$.

Finally, since F_R^Q is disintegrable by assumption, there exist invertible positive matrices $\tau_{xy} \in \mathbb{M}_{c_{xy}}(\mathbb{C})_{T_{xy}}$ such that

$$\rho_x = \bigboxplus_{y \in Y} (\tau_{xy} \otimes \sigma_y)$$

for all $x \in X$, which is a relation that also therefore holds in $\mathcal{A}_x = \mathbb{M}_{m_x}(\mathbb{C})$ for each $x \in X$. Note that in this expression, the probabilities were included inside the definitions of ρ_x and σ_y to avoid clutter. Hence, ρ_x and σ_y are not necessarily density matrices but are the associated positive operators on their respective components. By Theorem 2.32, a disintegration $\overline{F} : \mathcal{A} \to \mathcal{B}$ of (F, ω) exists. □

Remark 4.16. Note that the proof of $(i \Rightarrow ii)$ in Theorem 4.14 did not use the fact that F is a $*$-*homomorphism* nor did it use the *finite-dimensionality* of the algebras $\mathcal{A}$ and $\mathcal{B}$. All that was needed was that F is a state-preserving UCP map between von Neumann algebras equipped with normal states.

5 Non-commutative Bayesian inversion on matrix algebras

Similar to the case of disintegrations and state-preserving conditional expectations, consider now the more general case of a state-preserving UCP map $(\mathcal{B}, \xi) \overset{F}{\rightsquigarrow} (\mathcal{A}, \omega)$ between C^*-algebras equipped with states. The following theorem is an enhancement of the quantum Bayes' theorem for matrix algebras from [44] combined with the results of the present paper. If A is a matrix, we use the notation $\widehat{A}$ to indicate its Moore–Penrose inverse (pseudoinverse) [46], i.e., the unique matrix such that

$$A\widehat{A}A = A, \qquad \widehat{A}A\widehat{A} = \widehat{A}, \qquad (A\widehat{A})^* = A\widehat{A}, \quad \text{and} \quad (\widehat{A}A)^* = \widehat{A}A.$$

It follows from this definition that $A\widehat{A}$ and $\widehat{A}A$ are orthogonal projections onto the range of A and A^*, respectively.

Theorem 5.1 (Non-commutative Bayesian inversion on matrix algebras)**.** *Let $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$ be a UCP map between matrix algebras $\mathcal{A} := \mathbb{M}_m(\mathbb{C})$ and $\mathcal{B} := \mathbb{M}_n(\mathbb{C})$, let $\mathcal{A} \overset{\omega=\mathrm{tr}(\rho\,\cdot\,)}{\rightsquigarrow} \mathbb{C}$ be a state, and set $\xi := \omega \circ F \equiv \mathrm{tr}(\sigma\,\cdot\,)$. Let P_ω and P_ξ be the support projections of ω and ξ, respectively. Let $G^L, G^R : \mathcal{A} \rightsquigarrow \mathcal{B}$ be any two unital linear maps satisfying $P_\xi G^L(A) = \widehat{\sigma} F^*(\rho A)$ and $G^R(A) P_\xi = F^*(A\rho)\widehat{\sigma}$ for all $A \in \mathcal{A}$ (such unital linear maps are called* **left** *and* **right Bayes maps***, respectively[11]). Finally, set*[12]

$$\mathfrak{A} := \sum_{i,j=1}^{\sqrt{dim(\mathcal{A})}} E_{ij}^{(m)} \otimes \widehat{\sigma} F^*(\rho E_{ij}^{(m)}) P_\xi \qquad \text{and} \qquad \mathfrak{B} := \sum_{i,j=1}^{\sqrt{dim(\mathcal{A})}} E_{ij}^{(m)} \otimes \widehat{\sigma} F^*(\rho E_{ij}^{(m)}) P_\xi^\perp$$

Then the following conditions are equivalent.

i. *The map $\mathrm{Ad}_{P_\xi} \circ G^R$ (or $\mathrm{Ad}_{P_\xi} \circ G^L$) is $*$-preserving.*

ii. $\mathrm{Ad}_{P_\xi} \circ G^L = \mathrm{Ad}_{P_\xi} \circ G^R$.

iii. $\mathfrak{A}^* = \mathfrak{A}$.

iv. *$P_\xi F^*(\rho A)\sigma = \sigma F^*(A\rho) P_\xi$ for all $A \in \mathbb{M}_m(\mathbb{C})$.*

v. *$F(\sigma B)\rho = \rho F(B\sigma)$ for all $B \in P_\xi \mathcal{B} P_\xi$.*

vi. *$\widehat{\sigma} F^*(\rho E_{ij}^{(m)} P_\omega^\perp) P_\xi = 0$ for all i, j and the map F_R^Q is an intertwiner for the modular groups associated to the states on the support algebras, i.e., $F_R^Q \circ \mathfrak{m}_{\xi_\uparrow}^t = \mathfrak{m}_{\omega_\uparrow}^t \circ F_R^Q$ for all $t \in \mathbb{R}$.*

vii. *The map $\mathrm{Ad}_{P_\xi} \circ G^R$ (or $\mathrm{Ad}_{P_\xi} \circ G^L$) is UCP.*

When one, and hence all, of these conditions hold, a formula for $G := \mathrm{Ad}_{P_\xi} \circ G^L \equiv \mathrm{Ad}_{P_\xi} \circ G^R$ is given by

$$G = \mathrm{Ad}_{\sqrt{\widehat{\sigma}}} \circ F^* \circ \mathrm{Ad}_{\sqrt{\rho}}.$$

Moreover, if any (and hence all) of the above conditions hold, then the following additional conditions are equivalent.

(a) *A Bayesian inverse of (F, ω) exists.*

(b) $\mathrm{tr}_{\mathcal{A}}\left(\mathfrak{B}^* \hat{\mathfrak{A}} \mathfrak{B}\right) \leq P_\xi^\perp$.

In other words, the AC condition on the support algebras is not enough to guarantee the existence of a Bayesian inverse when the states are not faithful. Two additional constraints are needed, namely

$$\widehat{\sigma} F^*(\rho E_{ij}^{(m)} P_\omega^\perp) P_\xi = 0 \quad \forall\, i, j \qquad \text{and} \qquad \mathrm{tr}_{\mathcal{A}}\left(\mathfrak{B}^* \hat{\mathfrak{A}} \mathfrak{B}\right) \leq P_\xi^\perp.$$

This is to be contrasted with the previous theorems on disintegrations.

[11] The left Bayes map satisfies the Bayes condition $\omega(AF(B)) = \xi(G^L(A)B)$ for all inputs, while the right Bayes map satisfies the reversed Bayes condition $\omega(F(B)A) = \xi(BG^R(A))$ for all inputs [41].

[12] Notice that $\mathfrak{A}$ is the the Choi matrix associated to $\mathrm{Ad}_{P_\xi} \circ G^L$ and $\mathfrak{A}^*$ is the Choi matrix associated to $\mathrm{Ad}_{P_\xi} \circ G^R$.

Proof. The theorem contains two sets of claims. The equivalence between items *i*, *iv*, *v*, and *vii* together with the resulting formula for G was proved in [44, Proposition 5.12].[13] Furthermore, the equivalence between these items and items *ii* and *iii* can be easily deduced from this. Hence, the first set of equivalent conditions will be established by proving that item *vi* is equivalent to any of the other conditions. We prove item *vi* is equivalent to item *iv*. We first note that

$$\rho E_{ij}^{(m)} = \rho(P_\omega + P_\omega^\perp)E_{ij}^{(m)} = \rho P_\omega E_{ij}^{(m)} = \rho P_\omega E_{ij}^{(m)} P_\omega + \rho P_\omega E_{ij}^{(m)} P_\omega^\perp.$$

Hence, if we take $A = P_\omega E_{ij}^{(m)} P_\omega^\perp$, item *iv* implies $\widehat{\sigma} F^*(\rho E_{ij}^{(m)} P_\omega^\perp) P_\xi = 0$. Moreover, the reverse implication holds if matrices in item *iv* are of the form $AP_\omega^\perp$. Since F is $*$-preserving, the same is true for matrices of the form $P_\omega^\perp A$. In the remaining case, for matrices of the form $A = P_\omega A P_\omega$, Proposition 3.2 completes the proof, since on the support algebra $P_\omega \mathbb{M}_m(\mathbb{C}) P_\omega$ the state ω is faithful and Bayesian invertibility there (which is equivalent to item *iv*) is equivalent to the Accardi–Cecchini condition.

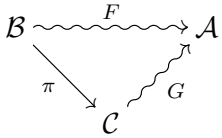
Finally, the equivalence between conditions (a) and (b), provided that the stated assumptions hold, were established in [44, Theorem 5.62]. □

Remark 5.2. The theorem suggests that the AC condition does not suffice to guarantee the existence of a Bayesian inverse $\overline{F}$ to (F, ω). This turns out to indeed be the case. Even if items *i* through *vii* hold, it is not automatic that the condition $\mathrm{tr}_{\mathcal{A}}\left(\mathfrak{B}^* \hat{\mathfrak{A}} \mathfrak{B}\right) \leq P_\xi^\perp$ holds. A simple explicit counter-example is provided in [44, Example 5.85]. In particular, we cannot simply extend G arbitrarily to a UCP map of the form

$$\widetilde{G}(A) := G(A) + P_\xi^\perp \zeta(A)$$

for some state $\mathcal{B} \overset{\zeta}{\rightsquigarrow} \mathbb{C}$ as is often done for Petz recovery maps in the literature [29, 30]. The reason is because such a $\widetilde{G}$ need not satisfy the Bayes condition (which is stronger than the AC condition). This should be compared with Theorems 4.9 and 4.14, where disintegrability on the support algebras sufficed for disintegrability on the original algebras.

Remark 5.3. Although every UCP map $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{A}$ between finite-dimensional C^*-algebras has a Stinespring representation of the form



with π a unital $*$-homomorphism and G a pure[14] UCP map, it is not necessarily the case that

[13] Technically, G^L was used in the statement and proof of [44, Proposition 5.12], but the proof is completely analogous with G^R. We have used G^R in this paper to more easily connect to other works in the literature, such as Carlen and Vershynina's recent theorem [11], which we will explain in Remark 5.4 is a special case of our Bayes' theorem.

[14] Pure maps/processes as defined in [51, Definition 2.32] between multi-matrix algebras are characterized in [51, Proposition B.2]. If $\mathcal{A} = \bigoplus_{x \in X} \mathbb{M}_{m_x}(\mathbb{C})$ and $\mathcal{B} = \bigoplus_{y \in Y} \mathbb{M}_{n_y}(\mathbb{C})$, then Stinespring's construction applied to the x component $F_x : \mathcal{B} \rightsquigarrow \mathcal{A} \to \mathbb{M}_{m_x}(\mathbb{C})$ provides a Hilbert space $\mathcal{H}_x$, a representation $\mathcal{B} \xrightarrow{\pi_x} \mathcal{B}(\mathcal{H}_x)$, and an isometry $\mathbb{C}^{m_x} \xrightarrow{V_x} \mathcal{H}_x$ such that $F_x = \mathrm{Ad}_{V_x} \circ \pi_x$ (see [39, Section 5] for details). Each Ad_{V_x} is a pure map. Hence, $\mathcal{C}$ can be taken to be the direct sum $\mathcal{C} = \bigoplus_{x \in X} \mathcal{B}(\mathcal{H}_x)$, the map $\mathcal{B} \xrightarrow{\pi} \mathcal{C}$ sends B to $\bigoplus_{x \in X} \pi_x(B)$, and the pure map $\mathcal{C} \overset{G}{\rightsquigarrow} \mathcal{A}$ can be taken to be the direct sum $G := \bigoplus_{x \in X} \mathrm{Ad}_{V_x}$.

such a decomposition can be used to determine if a state-preserving UCP map $(\mathcal{B},\xi) \overset{F}{\rightsquigarrow} (\mathcal{A},\omega)$ has a Bayesian inverse.

More precisely, given such a Stinespring representation, let $\zeta := \omega \circ G$ be the induced state on $\mathcal{C}$. Then all maps in the Stinespring representation are state-preserving. Furthermore, $(\mathcal{C},\zeta) \overset{G}{\rightsquigarrow} (\mathcal{A},\omega)$ is always Bayesian invertible (without any conditions) assuming the Stinespring construction from footnote 14 is used (cf., [44, Proposition 5.38]). Therefore, one might guess that a Bayesian inverse of (F,ω) exists if and only if a Bayesian inverse (disintegration) of $(\mathcal{B},\xi) \overset{\pi}{\to} (\mathcal{C},\zeta)$ exists. Although it is true that if π has a Bayesian inverse, then F has a Bayesian inverse, which we know can be taken as the composite $\overline{F} := \overline{\pi} \circ \overline{G}$ of Bayesian inverses (by Proposition 3.8), there exist situations where (F,ω) has a Bayesian inverse without (π,ζ) having one.

To illustrate this, if we take $\mathcal{A},\mathcal{B},\mathcal{C}$ to be matrix algebras, π to be the usual $*$-homomorphism in standard form, and $G = \mathrm{Ad}_V$ for some coisometry V, the claim is: *If ξ is faithful and a Bayesian inverse $\overline{\pi}$ of (π,ζ) exists, then (F,ω) admits a disintegration (in particular, F is a $*$-homomorphism).*[15] Indeed, set $Q := V^*V$ and let $\overline{G}$ be any Bayesian inverse, such as

$$\mathcal{A} \ni A \mapsto \overline{G}(A) := V^*AV + \nu(A)Q,$$

where ν is any state on $\mathcal{A}$. Then

$$(\overline{\pi} \circ \overline{G} \circ F)(B) = (\overline{\pi} \circ \overline{G} \circ G \circ \pi)(B) = \overline{\pi}\left(Q\pi(B)Q + \nu(V\pi(B)V^*)Q^\perp\right)$$

for all $B \in \mathcal{B}$. By [44, Lemma 5.4], $Q^\perp \leq P_\zeta^\perp$, where P_ζ is the support projection of ζ. Hence, by Lemma 2.30,

$$(\overline{\pi} \circ \overline{G} \circ F)(B) = \overline{\pi}\left(Q\pi(B)Q\right)$$

since $N_\xi = 0$. Similarly, since $\overline{\pi}$ is $*$-preserving,

$$\overline{\pi}\left(\pi(B)\right) = \overline{\pi}\left(Q\pi(B)Q\right)$$

for all $B \in \mathcal{B}$. But since $\overline{\pi}$ is a disintegration of (π,ζ), this proves $\overline{\pi} \circ \overline{G}$ is a disintegration of (F,ω).

Remark 5.4. The implication $(i \Rightarrow vii)$ from Theorem 5.1 holds in full generality for finite-dimensional C^*-algebras, as it is shown in [44, Lemma 6.19]. It generalizes a recent result of Carlen and Vershynina [11, Theorem 3.1], which restricted itself to the case of faithful states and injective $*$-homomorphisms.[16] Let us explain this result and its generalization in some detail.

First, let $\langle\,\cdot\,,\,\cdot\,\rangle_\omega$ denote the GNS bilinear form on $\mathcal{A}$ with respect to a state ω on $\mathcal{A}$, and similarly for $\langle\,\cdot\,,\,\cdot\,\rangle_\xi$ on $\mathcal{B}$ with ξ a state on $\mathcal{B}$. Assume ξ is faithful so that $\langle\,\cdot\,,\,\cdot\,\rangle_\xi$ is an inner product. Let $(\mathcal{B},\xi) \overset{F}{\to} (\mathcal{A},\omega)$ be a state-preserving $*$-homomorphism. Let $\mathcal{A} \overset{G}{\rightsquigarrow} \mathcal{B}$ be the right Bayes map[17] of (F,ω). Then G automatically satisfies Equation (1.17) in [11] since that equation reads

$$\langle F(B), A\rangle_\omega = \langle B, G(A)\rangle_\xi \qquad \forall\ A \in \mathcal{A},\ B \in \mathcal{B},$$

[15] And since we know there are examples of (F,ω) that are Bayesian invertible but not disintegrable, this tells us that Bayesian inverses cannot just be computed using Stinespring dilations and the disintegration theorem.

[16] Generalizing their result to the case of UCP maps is not just a straightforward application of Stinespring's theorem, as explained in Remark 5.3.

[17] If $\mathcal{A}$ and $\mathcal{B}$ are matrix algebras, the formula for $G := G^R$ was given in Theorem 5.1. More generally, if $\mathcal{A} = \bigoplus_{x\in X} \mathbb{M}_{m_x}(\mathbb{C})$, $\mathcal{B} = \bigoplus_{y\in Y} \mathbb{M}_{n_y}(\mathbb{C})$, $\omega = \sum_x p_x \operatorname{tr}(\rho_x\,\cdot\,)$, and $\xi = \sum_y q_y \operatorname{tr}(\sigma_y\,\cdot\,)$ as in the statement

which in terms of the definition of the GNS bilinear forms becomes

$$\omega\big(F(B)^*A\big) = \xi\big(B^*G(A)\big) \qquad \forall\, A \in \mathcal{A},\ B \in \mathcal{B}.$$

The agrees exactly with the Bayes condition written in reverse order because F is $*$-preserving and $*$ is an involution. In other words, it is equivalent to

$$\omega\big(F(B)A\big) = \xi\big(BG(A)\big) \qquad \forall\, A \in \mathcal{A},\ B \in \mathcal{B}. \tag{5.5}$$

Note that G is uniquely determined in this case, even though ω is not necessarily faithful. Furthermore, it follows from this formula, and the fact that F is a $*$-homomorphism, that $E := F \circ G$ is *automatically* a projection (meaning $E^2 = E$) onto $F(\mathcal{B})$ (the fact that it is orthogonal in the sense of [11] is precisely the GNS inner product condition), but is not necessarily $*$-preserving, nor CP. In fact, $G \circ F = \mathrm{id}_{\mathcal{B}}$. To see this, first note that

$$\begin{aligned}
\xi\Big(BG\big(F(B')\big)\Big) &= \omega\Big(F(B)F(B')\Big) && \text{by (5.5)} \\
&= (\omega \circ F)\big(BB'\big) && \text{since } F \text{ is deterministic} \\
&= \xi\big(BB'\big) && \text{since } F \text{ is state-preserving}
\end{aligned}$$

for all $B, B' \in \mathcal{B}$. In other words, $G \circ F \underset{\xi}{=} \mathrm{id}_{\mathcal{B}}$. But since ξ is faithful, this means $G \circ F = \mathrm{id}_{\mathcal{B}}$.

When F is therefore replaced with a UCP map, as in Theorem 5.1, it no longer makes sense to ask for a projection onto some subalgebra of $\mathcal{A}$. For one, $F(\mathcal{B})$ is only an *operator system* inside $\mathcal{A}$. Secondly, if we replaced the projection condition with some left-inverse condition, such as $G \circ F \underset{\xi}{=} \mathrm{id}_{\mathcal{B}}$, then we know that this necessarily implies that F is ω-a.e. deterministic. Nevertheless, one always has the right Bayes map (which reduces to the orthogonal projection of [11] when F is an injective $*$-homomorphism). Thus, Theorem 5.1 item *i* (G is $*$-preserving) implies item *vii* (G is CP) is a generalization of [11, Theorem 3.1].

6 Discussion and outlook

In this article, we showed how the Tomita–Takesaki modular automorphism group (or semigroup) is related to disintegrations and Bayesian inverses, concepts that arise naturally in the setting of synthetic probability [12, 20, 40]. This brings the categorical approach towards probability closer to the algebraic approach pioneered by Segal [50], Umegaki [59], and others. We reviewed how the Accardi–Cecchini (AC) condition generalizes the modular group invariance of a subalgebra to the case of UCP maps, rather than injective $*$-homomorphisms. We then demonstrated how the Bayes condition generalizes the AC condition to allow for non-faithful states. Indeed, in the case of non-faithful states, we saw that the AC condition is not enough to guarantee the existence of a state-preserving conditional expectation, or more generally a Bayesian inverse. The remaining condition for the existence of Bayesian inverses was discovered in [44] and enhanced in the present paper (Theorem 5.1). Furthermore, a simplified condition, in terms of disintegrations, was presented for the first time in this paper for the existence of state-preserving conditional expectations (Theorem 4.14).

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of Proposition 3.3, then a formula for the yx component $\mathbb{M}_{m_x}(\mathbb{C}) \overset{G_{yx}}{\rightsquigarrow} \mathbb{M}_{n_y}(\mathbb{C})$ is given by $G_{yx}(A_x) = \frac{p_x}{q_y} F_{xy}^*(A_x\rho_x)\sigma_y^{-1}$ for all $A_x \in \mathbb{M}_{m_x}(\mathbb{C})$ (cf., [44, Lemma 6.19]). Here, $\mathbb{M}_{n_y}(\mathbb{C}) \overset{F_{xy}}{\rightsquigarrow} \mathbb{M}_{m_x}(\mathbb{C})$ is the xy component of F.

In the quantum information theory literature, the Petz recovery map and its swiveled/rotated variant have played an important role in terms of information recovery [29, 30]. However, for non-faithful states, the Petz recovery map does not specify the action off the support algebra. On the other hand, the Bayesian inverse does not always exist, unlike the Petz recovery map. As such, it is important to study approximate versions of Bayesian inverses. From this, one might suspect the existence of some interpolation between these two approaches towards quantum Bayesian inference. Furthermore, just like perfect error-correction is related to disintegrations [40], which has its approximate versions [33], one might guess that approximate versions of Bayesian inverses could be used in an alternative approach towards approximate error-correction.

Finally, the Petz recovery map and its swiveled/rotated variants do not generally work in generalizing the strengthened data-processing inequality to the quantum setting. Classically, this inequality states that if p is a probability measure on a finite set X, and $X \overset{f}{\rightsquigarrow} Y$ is a stochastic map to a finite set Y, then there exists a *recovery map*, i.e., a probability-preserving stochastic map $(Y, f \circ p) \overset{g}{\rightsquigarrow} (X, p)$ such that

$$S(q \parallel p) - S(f \circ q \parallel f \circ p) \geq S(q \parallel g \circ f \circ q)$$

for *all* probability measures q on X. Here, $S(q \parallel p)$ denotes the relative entropy of q given p and $f \circ p$ denotes the push-forward of the probability p along f. It is known that in full generality, no such recovery map exists (see the discussion at the beginning of Section 5 in [17] and the end of Section 5 in [34]). Therefore, it would be convenient to find sufficient and/or necessary conditions for a quantum analogue of this inequality to hold.

A Carlson's theorem

We recall some facts from complex analysis [3].

Theorem A.1 (The identity theorem)**.** *Let $f : D \to \mathbb{C}$ be a holomorphic function on a domain (an open and connected subset) $D \subseteq \mathbb{C}$ and let $S \subseteq D$ be a subset with an accumulation point in S. If $f(z) = 0$ for all $z \in S$, then $f \equiv 0$ on all of D.*

Proof. See [3, Chapter 4 Section 3.2 page 127]. □

Definition A.2. An ***entire function*** is a $\mathbb{C}$-valued holomorphic function whose domain is all of $\mathbb{C}$.

Theorem A.3 (Liouville's theorem)**.** *If f is a bounded entire function, then f is a constant.*

Proof. See [3, Chapter 4 Section 2.3 page 122]. □

As a corollary to Liouville's theorem, the set of bounded entire functions is the one-dimensional vector subspace of constant functions inside the infinite-dimensional vector space of all entire functions. Therefore, one often distinguishes the different classes of non-bounded entire functions by their asymptotic growth rates. Such a situation occurs in the following theorem, which is used multiple times in this work.

Theorem A.4 (Carlson's theorem)**.** *Let f be an entire function satisfying the following conditions:*

i. there exist constants $C, \gamma, \gamma' \in \mathbb{R}$ with $\gamma' < \pi$ such that

$$|f(z)| \leq Ce^{\gamma|z|} \quad \forall\, z \in \mathbb{C} \qquad \textit{and} \qquad |f(it)| \leq Ce^{\gamma'|t|} \quad \forall\, t \in \mathbb{R}$$

ii. $f(n) = 0$ for all $n \in \mathbb{N}$.

Then $f \equiv 0$.

Proof. This follows from [9, Theorem 9.2.1]. We include the argument for completeness. First, define

$$h(\theta) := \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{|r|}.$$

By the assumptions in [9, Theorem 9.2.1], f is regular and of exponential type, which implies the constants C, γ, γ' exist. Then

$$h(\pm\pi/2) = \limsup_{r\to\infty} \frac{\log|f(\pm ir)|}{|r|} \leq \limsup_{r\to\infty} \frac{\log|Ce^{\gamma'|r|}|}{|r|} = \limsup_{r\to\infty} \left(\frac{\log|C|}{|r|} + \gamma'\right) = \gamma'.$$

Hence, $h(\pi/2) + h(-\pi/2) \leq 2\gamma'$, which shows that $\gamma' < \pi$. □

B State-preserving conditional expectations

As in Section 2.3, let $\mathcal{A} \xleftarrow{F} \mathcal{B}$ be a unital $*$-homomorphism of finite-dimensional C^*-algebras (multi-matrix algebras). Set $\mathcal{N} := F(\mathcal{B})$ and $\mathcal{M} := \mathcal{A}$ so that $\mathcal{N} \subseteq \mathcal{M}$.

In this section, we characterize explicitly the states ω on $\mathcal{M}$ (not necessarily faithful) that admit a state-preserving conditional expectation $E : \mathcal{M} \rightsquigarrow \mathcal{M}$ onto $\mathcal{N}$, namely such that

$$\omega = \omega \circ E.$$

Before proving the proposition, we need to recall some notation. Let P_i, $i = 1, \ldots, s$, and Q_j, $j = 1, \ldots, t$, be the minimal projects in the center of $\mathcal{M}$ and $\mathcal{N}$ respectively, in some ordering. As in Section 2.3, set $X := \{1, \ldots, s\}$ and $Y := \{1, \ldots, t\}$. Let $\mathcal{M}_i := P_i\mathcal{M}P_i = P_i\mathcal{M}$ and $\mathcal{N}_i := Q_j\mathcal{N}Q_j = Q_j\mathcal{N}$. Then $\mathcal{M}_i \cong \mathbb{M}_{m_i}(\mathbb{C})$ and $\mathcal{N}_j \cong \mathbb{M}_{n_j}(\mathbb{C})$ for some $m_i, n_j \in \mathbb{N}$, and

$$\mathcal{M} = \bigoplus_{i\in X} \mathcal{M}_i \cong \bigoplus_{i\in X} \mathbb{M}_{m_i}(\mathbb{C}), \quad \mathcal{N} = \bigoplus_{j\in Y} \mathcal{N}_j \cong \bigoplus_{j\in Y} \mathbb{M}_{n_j}(\mathbb{C}).$$

Let also $\mathcal{M}_{ij} := P_iQ_j\mathcal{M}P_iQ_j = P_iQ_j\mathcal{M}Q_j$ and $\mathcal{N}_{ij} := P_iQ_j\mathcal{N}P_iQ_j = P_iQ_j\mathcal{N}$. Observe that $P_iQ_j = Q_jP_i$ and assume $P_iQ_j \neq 0$. Then $\mathcal{N}_{ij} \cong \mathcal{N}_j$, as $P_i \in \mathcal{N}_j{}'$ and $\mathcal{N}_j$ is a factor, thus $x \in \mathcal{N}_j \mapsto P_ix \in \mathcal{N}_{ij}$ is a $*$-isomorphism. Moreover, $\mathcal{N}_{ij} \subseteq \mathcal{M}_{ij}$ is a type I_n subfactor. Explicitly,

$$\mathcal{N}_{ij} \cong \mathbb{M}_{n_j}(\mathbb{C}) \otimes \mathbb{1}_{c_{ij}} \subseteq \mathbb{M}_{n_j}(\mathbb{C}) \otimes \mathbb{M}_{c_{ij}}(\mathbb{C}) \cong \mathcal{M}_{ij}$$

where $c_{ij} \in \mathbb{N}$ are some multiplicities describing the type I_n subfactors, i.e., $(c_{ij})_{i,j}$ is the Bratteli inclusion matrix of $\mathcal{N} \subseteq \mathcal{M}$, extended to all pairs (i, j) by setting $c_{ij} := 0$ if $P_iQ_j = 0$. Note that $m_i = \sum_j c_{ij}n_j$ because $\mathcal{N} \subseteq \mathcal{M}$ is unital.

Lastly, as $\sum_i P_i = \mathbb{1} = \sum_j Q_j$, every $A \in \mathcal{M}$ can be written as $A = \sum_{i,u,v} Q_uP_iAP_iQ_v$ where i runs in X and u, v run in Y. We write for short $A_i := P_iAP_i = P_iA \in \mathcal{M}_i$ and $A_{i;uv} := Q_uP_iAQ_v$. Thus $A = \sum_i A_i$, but also $A = \sum_{i,u,v} A_{i;uv}$, and $A_{i;uv} \in \mathcal{M}_{ij}$ if $u = v = j$. Note that $\mathcal{M}_{ij} = 0 = \mathcal{N}_{ij}$ whenever $P_iQ_j = 0$.

The following two lemmas are a consequence of the condition $\sum_i P_i = \mathbb{1} = \sum_j Q_j$ and they hold for arbitrary von Neumann algebras with finite-dimensional centers. The proof of the first is immediate, for the second we refer to [27, Sec. 2], [24, Sec. 2], [23].

Lemma B.1. *Every state ω on $\mathcal{M}$, not necessarily faithful, can be written as follows. Let $A \in \mathcal{M}$, then*

$$\omega(A) = \sum_i p_i \omega_i(A_i)$$

where $p_i \geq 0$, $\sum_i p_i = 1$ are defined by $p_i := \omega(P_i)$, and ω_i is the state on $\mathcal{M}_i$ defined by $\omega_i(A_i) := p_i^{-1}\omega(A_i)$ if $p_i \neq 0$, or the zero functional on $\mathcal{M}_i$ otherwise.

Lemma B.2. *Every conditional expectation $E : \mathcal{M} \rightsquigarrow \mathcal{M}$ onto $\mathcal{N}$, not necessarily faithful, nor state-preserving, can be written as follows. Let $A \in \mathcal{M}$, then*

$$E(A) = \sum_{i,j} \lambda_{ij} \mathcal{Q}_{ij}(E_{ij}(A_{i;jj}))$$

where $\lambda_{ij} \geq 0$ are defined by $\lambda_{ij} Q_j := E(P_i Q_j)$, $E_{ij} : \mathcal{M}_{ij} \rightsquigarrow \mathcal{M}_{ij}$ are the conditional expectations onto $\mathcal{N}_{ij}$ defined by $E_{ij}(A_{i;jj}) := \lambda_{ij}^{-1} P_i Q_j E(A_{i;jj}) = \lambda_{ij}^{-1} P_i Q_j E(P_i A)$ if $\lambda_{ij} \neq 0$, or the zero map on $\mathcal{M}_{ij}$ otherwise, and $\mathcal{Q}_{ij} : \mathcal{N}_{ij} \to \mathcal{N}_j$ is the inverse of the $*$-isomorphism $A \mapsto P_i A$ if $P_i Q_j \neq 0$, or the zero map on $\mathcal{N}_{ij}$ otherwise.

Now we state and prove the main result of this section.

Proposition B.3. *Let $\mathcal{N} \subseteq \mathcal{M}$ be a unital inclusion of multi-matrix algebras. A state ω on $\mathcal{M} \cong \bigoplus_{i \in X} \mathbb{M}_{m_i}(\mathbb{C})$, not necessarily faithful, admits a conditional expectation $E : \mathcal{M} \rightsquigarrow \mathcal{M}$ onto $\mathcal{N} \cong \bigoplus_{j \in Y} \mathbb{M}_{n_j}(\mathbb{C})$ such that $\omega = \omega \circ E$ if and only if it is of the form*

$$\omega(\,\cdot\,) = \sum_i p_i \operatorname{tr}(\rho_i \,\cdot\,)$$

where ρ_i is the density matrix associated with the restriction of ω to $\mathcal{M}_i \cong \mathbb{M}_{m_i}(\mathbb{C})$, $i \in X$, or zero, and it holds for every $j, u, v \in Y$ with $u \neq v$

$$P_{iu} \rho_i P_{iv} = 0, \quad P_{ij} \rho_i P_{ij} = \mu_{ij} \sigma_j \otimes \tau_{ij}$$

where P_{ij} is the projection in $\mathbb{M}_{m_i}(\mathbb{C})$ corresponding to $P_i Q_j \in \mathcal{M}_i$, $\mu_{ij} \geq 0$ are some proportionality coefficients, σ_j is the density matrix in $\mathbb{M}_{n_j}(\mathbb{C})$ associated with the restriction of ω to $\mathcal{N}_j \cong \mathbb{M}_{n_j}(\mathbb{C})$ or zero, and τ_{ij} is the density matrix in $\mathbb{M}_{c_{ij}}(\mathbb{C})$ associated with the partial trace $E_{ij} : \mathcal{M}_{ij} \cong \mathbb{M}_{n_j}(\mathbb{C}) \otimes \mathbb{M}_{c_{ij}}(\mathbb{C}) \to \mathcal{N}_{ij} \cong \mathbb{M}_{n_j}(\mathbb{C}) \otimes \mathbb{1}_{c_{ij}}$ or zero.

Proof. By Lemma B.1, for every state ω on $\mathcal{M}$ we have

$$\omega(A) = \sum_i p_i \omega_i(A_i).$$

Similarly, for the restriction $\xi := \omega_{|\mathcal{N}}$ we have

$$\xi(B) = \sum_j q_j \xi_j(B_j)$$

$B_j := Q_j B Q_j = Q_j B \in \mathcal{N}_j$, $q_j := \xi(Q_j)$ fulfill $q_j \geq 0$, $\sum_j q_j = 1$, and $\xi_j(B_j) := q_j^{-1}\xi(B_j)$ if $q_j \neq 0$, or the zero functional on $\mathcal{N}_j$ otherwise.

By Lemma B.2, for every conditional expectation $E : \mathcal{M} \rightsquigarrow \mathcal{M}$ onto $\mathcal{N}$ we have

$$E(A) = \sum_{i,j} \lambda_{ij} \mathcal{Q}_{ij}(E_{ij}(A_{i;jj})).$$

Assume $\omega = \omega \circ E$, then the two sides of the equality

$$\omega(A) = \omega(E(A))$$

read

$$\omega(A) = \sum_{i,u,v} p_i \omega_i(A_{i;uv})$$

and

$$\begin{aligned}\omega(E(A)) &= \sum_{i,j} \lambda_{ij} \omega(\mathcal{Q}_{ij}(E_{ij}(A_{i;jj}))) \\ &= \sum_{i,j} \lambda_{ij} q_j \xi_j(\mathcal{Q}_{ij}(E_{ij}(A_{i;jj}))).\end{aligned}$$

In our case at hand, $\mathcal{M}_{ij} \cong \mathbb{M}_{n_j}(\mathbb{C}) \otimes \mathbb{M}_{c_{ij}}(\mathbb{C})$, $\mathcal{N}_{ij} \cong \mathbb{M}_{n_j}(\mathbb{C}) \otimes \mathbb{1}_{c_{ij}}$ and $\mathcal{N}_j \cong \mathbb{M}_{n_j}(\mathbb{C})$. So by Lemma 2.22, E_{ij} can be viewed as the partial trace defined on simple tensors $B_j \otimes C_{ij}$ in $\mathbb{M}_{n_j}(\mathbb{C}) \otimes \mathbb{M}_{c_{ij}}(\mathbb{C})$ by

$$E_{ij}(B_j \otimes C_{ij}) = \operatorname{tr}(\tau_{ij} C_{ij}) B_j \otimes \mathbb{1}_{c_{ij}}$$

for some density matrix $\tau_{ij} \in \mathbb{M}_{c_{ij}}(\mathbb{C})$, or the zero map, and $\mathcal{Q}_{ij}$ can be viewed as the $*$-isomorphism $B_j \otimes \mathbb{1}_{c_{ij}} \mapsto B_j$. Note that $P_i Q_j = 0$ if and only if $c_{ij} = 0$. In view of the identifications $\mathcal{M}_i \cong \mathbb{M}_{m_i}(\mathbb{C})$ and $\mathcal{N}_j \cong \mathbb{M}_{n_j}(\mathbb{C})$, if ω_j and ξ_j are not zero, we have $\omega_i(A_i) = \operatorname{tr}(\rho_i A_i)$, $A_i \in \mathcal{M}_i$, and $\xi_j(B_j) = \operatorname{tr}(\sigma_j A_j)$, $B_j \in \mathcal{N}_j$, for some density matrices $\rho_i \in \mathbb{M}_{m_i}(\mathbb{C})$ and $\sigma_j \in \mathbb{M}_{n_j}(\mathbb{C})$.

Choose $A = A_{i;jj} \in \mathcal{M}_{ij} \subseteq \mathcal{M}_i$, for i, j fixed. Then $Q_u A_{i;jj} P_i Q_v = A_{i;jj}$ if $u = v = j$, zero otherwise. More particularly, choose $A = A_{i;jj}$ to be identified with a simple tensor $B_j \otimes C_{ij}$ in $\mathbb{M}_{n_j}(\mathbb{C}) \otimes \mathbb{M}_{c_{ij}}(\mathbb{C})$. Then $\omega(A) = \omega(E(A))$ implies

$$p_i \operatorname{tr}_{\mathbb{M}_{m_i}(\mathbb{C})}((P_{ij} \rho_i P_{ij}) B_j \otimes C_{ij}) = \lambda_{ij} q_j \operatorname{tr}(\sigma_j B_j) \operatorname{tr}(\tau_{ij} C_{ij})$$

where P_{ij} is the projection in $\mathbb{M}_{m_i}(\mathbb{C})$ corresponding to $P_i Q_j \in \mathcal{M}_i$. In particular, $\sum_j P_{ij} = \mathbb{1}_{m_i}$. By taking linear combinations of simple tensors, we get

$$P_{ij} \rho_i P_{ij} = \frac{\lambda_{ij} q_j}{p_i} \sigma_j \otimes \tau_{ij}$$

if $p_i \neq 0$ and $P_{ij} \neq 0$. In order to determine ρ_i completely, we need to determine also $P_{iu} \rho_i P_{iv}$ with $u \neq v$ and we can assume $P_{iu} \neq 0$, $P_{iv} \neq 0$. Choose $A = A_{i;uv} \in \mathcal{M}_i$ with $u \neq v$, and denote again by $A_{i;uv}$ its corresponding element in $\mathbb{M}_{m_i}(\mathbb{C})$. Then $\omega(A) = \omega(E(A))$ implies $p_i \omega_i(A_{i;uv}) = 0$, because $Q_j A_{i;uv} P_i Q_j = 0$ if $u \neq v$, hence

$$0 = p_i \operatorname{tr}(\rho_i A_{i;uv}) = p_i \operatorname{tr}(\rho_i (P_{iu} A_{i;uv} P_{iv})) = p_i \operatorname{tr}((P_{iv} \rho_i P_{iu}) A_{i;uv})$$

which, if $p_i \neq 0$, yields $P_{iv} \rho_i P_{iu} = 0$, concluding the proof. □

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