

# SOME ASPECTS OF DYNAMICAL TOPOLOGY: DYNAMICAL COMPACTNESS AND SLOVAK SPACES

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**ABSTRACT.** The area of dynamical systems where one investigates dynamical properties that can be described in topological terms is "Topological Dynamics". Investigating the topological properties of spaces and maps that can be described in dynamical terms is in a sense the opposite idea. This area is recently called as "Dynamical Topology". For (discrete) dynamical systems given by compact metric spaces and continuous (surjective) self-maps we (mostly) survey some results on two new notions: "Slovak Space" and "Dynamical Compactness". Slovak space is a dynamical analogue of the rigid space: a nontrivial compact metric space whose homeomorphism group is cyclic and generated by a minimal homeomorphism. Dynamical compactness is a new concept of chaotic dynamics. The omega-limit set of a point is a basic notion in theory of dynamical systems and means the collection of states which "attract" this point while going forward in time. It is always nonempty when the phase space is compact. By changing the time we introduced the notion of the omega-limit set of a point with respect to a Furstenberg family. A dynamical system is called *dynamically compact* (with respect to a Furstenberg family) if for any point of the phase space this omega-limit set is nonempty. A nice property of dynamical compactness: all dynamical systems are dynamically compact with respect to a Furstenberg family if and only if this family has the finite intersection property.

By a (*topological*) *dynamical system*  $(X, T)$  we mean a compact metric space  $X$  with a metric  $d$  and a continuous self-surjection  $T$  of  $X$ . We say it *trivial* if the space is a singleton. *Throughout this paper, we are only interested in a nontrivial (topologically transitive) dynamical system, where the state space is a compact metric space without isolated points.*

When mathematicians are considering various classes of functions with a natural topology, often one of the first questions that comes to their mind is about the topological properties of those spaces. While those questions have been answered long ago in Mathematical Analysis and Topology, and also in Ergodic Theory there is a series of papers where topological properties of the group of measure preserving bijections or homeomorphisms are investigated, see e.g. [24], [30], [13], [10], [40], [41], it seems that they even have not been asked before in Dynamical Systems. There are very few papers on topological properties of spaces of maps that have some specific dynamical properties. Perhaps, the only exception is the paper of Farrell

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and Gogolev [14] about the spaces of Anosov diffeomorphisms, that was being written at the same time as [31] (and completely independently of it).

The area of Dynamical Systems where one investigates dynamical properties that can be described in topological terms is *Topological Dynamics*. Investigating topological properties of spaces of maps that can be described in dynamical terms is in a sense the opposite idea. Therefore in [31] was proposed to call this area **Dynamical Topology**.

Let  $X$  be a compact metric space and let  $T : X \rightarrow X$  be continuous. The dynamical system  $(X, T)$  is called *topologically transitive* (or just *transitive*) if for every pair of nonempty open sets  $U$  and  $V$  in  $X$  there is a nonnegative integer  $n$  such that  $T^n(U) \cap V \neq \emptyset$ . If the space  $X$  has no isolated points, this is equivalent to the existence of a point  $x \in X$  whose orbit  $\{x, T(x), \dots, T^n(x), \dots\}$  is dense in  $X$ . Consequently, a topologically transitive dynamical system cannot be decomposed into two disjoint sets with nonempty interiors which do not interact under the transformation. In particular, transitivity is an ingredient of several definitions of chaos. For more information on topological transitivity see, e.g., [1], [36], [38] and references there.

In fact we survey some recent results in the following areas of Dynamical Topology:

- (1) Topological properties of the space of all transitive maps of a compact interval to itself and its subspaces (very briefly).
- (2) Dynamical compactness, especially transitive and sensitive compactnesses as new concepts of chaoticity of a dynamical system.
- (3) Slovak spaces - compact metric spaces whose homeomorphism group is cyclic and generated by a minimal homeomorphism (very briefly).

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## 1. SPACES OF TOPOLOGICALLY TRANSITIVE INTERVAL MAPS

This section is based on the papers [31], [32]. The space  $\mathcal{T}$  of all transitive maps on the interval  $I \rightarrow I$ , with uniform metric has the following nice properties: *is separable; is locally infinite dimensional; is not complete; is not locally compact; is nowhere dense in the space of all continuous maps  $I \rightarrow I$ ; is a Baire space; and is arcwise connected and is locally arcwise connected.* Our main aim is to investigate loops in subspaces of  $\mathcal{T}$ .

By a *map* we mean a continuous map. A *lap* of an interval map is a maximal interval on which this map is monotone. The *modality* of a piecewise monotone map is the number of laps minus 1. A *turning point* is a point that belongs to two distinct laps. When we say “piecewise”, we mean that there are finitely many pieces. By “slope” we mean the absolute value of the derivative. A *full  $n$ -horseshoe* is a piecewise monotone map with constant slope and  $n$  laps, each of which is mapped to the whole domain of the map.

We will use the following notation.

- (1)  $I = [0, 1]$ ;
- (2)  $\mathcal{T}$  – (continuous !) transitive maps  $I \rightarrow I$ ;
- (3)  $\mathcal{T}_{\text{PM}}$  – piecewise monotone transitive maps  $I \rightarrow I$ ;
- (4)  $\mathcal{T}_{\text{PL}}$  – piecewise linear transitive maps  $I \rightarrow I$ ;
- (5)  $\mathcal{T}_n$  – elements of  $\mathcal{T}_{\text{PM}}$  of modality  $n$ ;
- (6)  $\mathcal{T}_n^+$  – elements of  $\mathcal{T}_n$  increasing on the first lap;
- (7)  $\mathcal{T}_n^-$  – elements of  $\mathcal{T}_n$  decreasing on the first lap;
- (8)  $\mathcal{CS}_n$  – piecewise linear maps  $I \rightarrow I$ , with constant slope and of modality  $n$ ;
- (9)  $\mathcal{TCS}_n$  – transitive maps from  $\mathcal{CS}_n$ .

All those spaces are considered with the  $C^0$ -metric  $d$ :

$$d(f, g) = \sup_{x \in I} |f(x) - g(x)|.$$

By an interval we mean a nondegenerate interval. If not stated otherwise, it is assumed to be closed. For an interval  $J$  we will denote its length by  $|J|$ . When we speak about symmetry, we mean conjugacy via the symmetry map of  $I$ , that is  $x \mapsto 1 - x$ .

### 1.1. Connectedness properties of spaces of transitive maps.

**Theorem 1.1.** *The spaces  $\mathcal{T}$ ,  $\mathcal{T}_{\text{PM}}$  and  $\mathcal{T}_{\text{PL}}$  are contractible (hence arcwise connected) and locally arcwise connected.*

*Idea of the proof of contractibility of  $\mathcal{T}$*

Given a closed interval  $K \subseteq I$ , we consider *box map*:

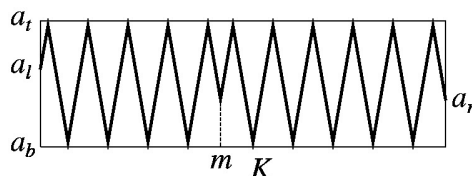


Figure 1. Box map.

It depends continuously (jointly) on 5 parameters:

- (1)  $a_l$  and  $a_r$  – the values at the left and right endpoint of  $K$ ,
- (2)  $a_b$  and  $a_t$  – the bottom and the top of the box,
- (3)  $a_s$  – the slope multiplier  
(slope is  $a_s \cdot \frac{a_t - a_b}{|K|}$ , here  $a_s = 20$ )

Given  $f \in \mathcal{T}$ , we put

- (1)  $g_{f,0} = f$ ;
- (2) to define  $g_{f,t}$  for  $0 < t \leq 1$ , we partition  $[0, 1]$  into intervals  $I_0, I_1, \dots, I_s$  of lengths  $t$  (the rightmost one can be shorter). Over each  $I_i$  we construct a box:

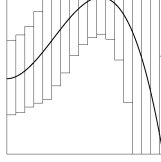


Figure 2. Boxes  $I_i \times I_j$ .

In each box we choose a box map which coincides with  $f$  at the endpoints of  $I_i$  and has  $a_s = 20$ . The obtained map  $I \rightarrow I$  is  $g_{f,t}$ .

Vertical sides of the boxes can be chosen such that:

- (1)  $g_{f,t}$  is transitive;
- (2)  $g_{f,t}$  depends continuously on  $f$  and  $t$  (jointly).

If  $t = 1$  then we have just one box and so the maps  $g_{f,1}$  are box maps with  $K = [0, 1]$ ,  $a_b = 0$ ,  $a_t = 1$  and  $a_s = 20$ . They depend only on  $a_l = f(0) \in [0, 1]$  and  $a_r = f(1) \in [0, 1]$ . Hence the set  $Z = \{g_{f,1} : f \in \mathcal{T}\}$ , being homeomorphic to the square, is contractible.

Our family  $g_{f,t}$  can be treated as a homotopy joining (1) the identity  $\text{id}_{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}$ ,  $f \mapsto f = g_{f,0}$  and (2) the map  $\mathcal{T} \rightarrow Z \subseteq \mathcal{T}$ ,  $f \mapsto g_{f,1}$ . Since  $Z$  is contractible, also  $\mathcal{T}$  is contractible.

**Theorem 1.2.** *Each space  $\mathcal{T}_n$  has two connected components,  $\mathcal{T}_n^+$  and  $\mathcal{T}_n^-$ , and they are arcwise connected. The distance between  $\mathcal{T}_1^+$  and  $\mathcal{T}_1^-$  is positive. While the distance between  $\mathcal{T}_n^+$  and  $\mathcal{T}_n^-$  is zero for all  $n \geq 2$ .*

**1.2. How to recognize that a piecewise linear map with constant slope is transitive.** We use a coding of maps  $f \in \mathcal{CS}_n$ :

- (1) code of  $f = (f(0), f(\text{1st turning pt}), \dots, f(\text{nth turning pt}), f(1))$ .
- (2) When we consider a union of spaces  $\mathcal{CS}_n$  with different  $n$ 's, we use the common (largest) length of codes, say

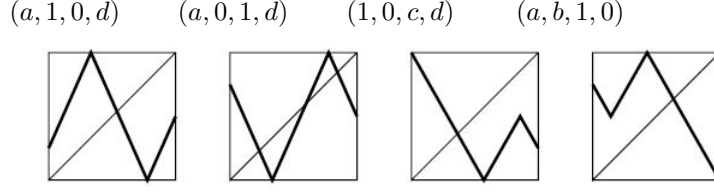
$$\text{code of tent map} = (0, 1, 0) = (0, 0, 1, 0) = (0, 1, 0, 0) = \dots,$$

- (3) code  $(a_0, \dots, a_{n+1}) \implies \text{slope} = \sum_{j=1}^{n+1} |a_j - a_{j-1}|$
- (4)  $f \in \bigcup_{i=1}^n \mathcal{CS}_i$  depends continuously on the parameters  $a_0, a_1, \dots, a_{n+1}$  (jointly).

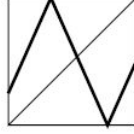
**Lemma 1.2.1.** *A map  $f \in \mathcal{CS}_1$  is transitive if and only if it has the code*

- (1)  $(a, 1, 0)$  where  $a \in [0, 2 - \sqrt{2}]$ , or
- (2)  $(1, 0, c)$  where  $c \in [\sqrt{2} - 1, 1]$ .

**Lemma 1.2.2.** *Let  $f \in \mathcal{CS}_2$  be transitive. Then it has one of the four codes:*



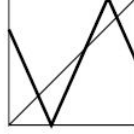
In the case  $(a, 1, 0, d)$ , if  $x$  is the fixed point in the second lap, transitivity is equivalent to  $a \leq x$  or  $d \geq x$ :



**Lemma 1.2.3.**  $f \in \mathcal{CS}_2$  with a code  $(a, 1, 0, d)$  is transitive  $\iff$

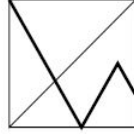
$$d \leq a - 4 + \frac{2}{a} \quad \text{or} \quad 1 - a \leq (1 - d) - 4 + \frac{2}{1 - d}.$$

In the case  $(a, 0, 1, d)$ , transitivity is equivalent to the slope being larger than 2:



**Lemma 1.2.4.**  $f \in \mathcal{CS}_2$  with a code  $(a, 0, 1, d)$  is transitive  $\iff a > d$ .

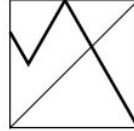
In the case  $(1, 0, c, d)$ , if the fixed point in the first lap is  $x$ , then transitivity is equivalent to  $c \geq x$ :



**Lemma 1.2.5.**  $f \in \mathcal{CS}_2$  with a code  $(1, 0, c, d)$  is transitive  $\iff$

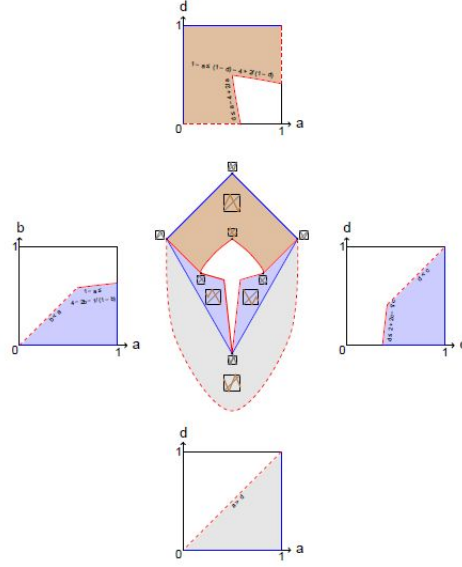
$$d \leq 2 + 2c - \frac{1}{c}.$$

In the case  $(a, b, 1, 0)$ , if the fixed point in the third lap is  $x$ , then transitivity is equivalent to  $b \leq x$ :



**Lemma 1.2.6.**  $f \in \mathcal{CS}_2$  with a code  $(a, b, 1, 0)$  is transitive  $\iff$

$$1 - a \leq 2 + 2(1 - b) - \frac{1}{1 - b}.$$

Figure 3. The space  $\mathcal{TCS}_1 \cup \mathcal{TCS}_2$ .

We have described the spaces  $\mathcal{TCS}_1$  and  $\mathcal{TCS}_2$ . What about  $\mathcal{TCS}_n$  for  $n \geq 3$ ?

**Lemma 1.2.7.**  *$f \in \mathcal{CS}_n$ , slope  $\lambda > 2$ , image of every lap (except perhaps the leftmost and the rightmost ones) is the whole  $I \implies f$  is transitive.*

**Lemma 1.2.8.**  *$f \in \mathcal{CS}_n$ , slope  $\lambda > 3$ , image of every lap (except perhaps one or two leftmost or one or two rightmost ones) is the whole  $I \implies f$  is transitive.*

Put  $E(f) := \{\text{turning points of } f\} \cup \{\text{endpoints of the interval } I\}$ .

**Lemma 1.2.9.**  *$f \in \mathcal{CS}_n$ , slope  $\lambda > 3$ ,  $f(\{0, 1\}) \subseteq \{0, 1\}$ , out of any four consecutive points of  $E(f)$  at least one is mapped to 0 and at least one is mapped to 1  $\implies f$  is transitive.*

**1.3. Loops of transitive interval maps.** In what follows: loops of transitive maps with constant slopes (transitivity always follows from previous lemmas).

**Theorem 1.3.** *For every  $n \geq 1$  there is a loop in  $\mathcal{T}_n \cup \mathcal{T}_{n+1}$ , which is not contractible in  $\mathcal{T}_n \cup \mathcal{T}_{n+1}$ .*

In fact, we have found such a loop in  $\mathcal{TCS}_n \cup \mathcal{TCS}_{n+1}$ . We call it the basic loop of order  $n$  and denote it  $L_n$ .

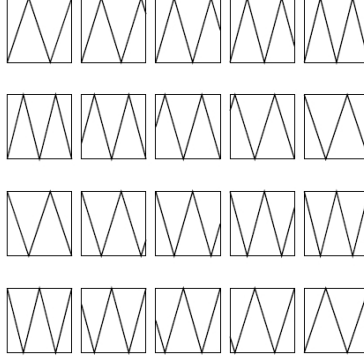


Figure 4. Basic loop  $L_2$  (it consists of four arcs).

In particular:  $L_2 \in \mathcal{TCS}_2 \cup \mathcal{TCS}_3 \dots$  not contractible in  $\mathcal{TCS}_2 \cup \mathcal{TCS}_3$  (even not in  $\mathcal{T}_2 \cup \mathcal{T}_3$ , by Theorem 3.1)

We show that it is contractible in  $\mathcal{TCS}_2 \cup \mathcal{TCS}_3 \cup \mathcal{TCS}_4$ . First we find the auxiliary loop of order 2 homotopic to  $L_2$  in the space  $\mathcal{TCS}_2 \cup \mathcal{TCS}_3 \cup \mathcal{TCS}_4$ . And then one can show similarly that the auxiliary loop is contractible in  $\mathcal{TCS}_2 \cup \mathcal{TCS}_4$ .

The basic loop  $L_2$  consists of four arcs, so we need to show how to deform these four arcs, to obtain the auxiliary loop.

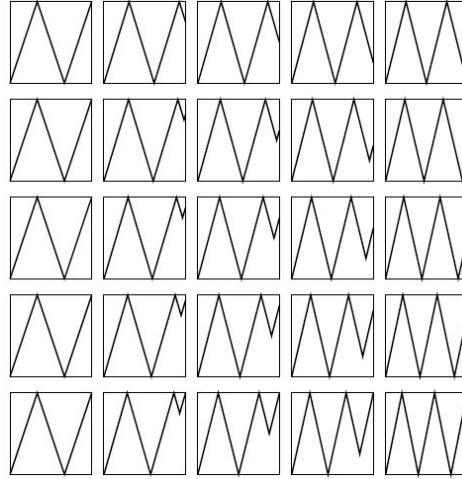


Figure 5. Deformation of the 1st arc of the basic loop  $L_2$ .

In codes:  $(0, 1, 0, 1, 1 - s, 1 - s + st)$ ,  $s$  varies from 0 (left end) to 1 (right end),  $t$  varies from 0 (for basic loop) to 1 (for auxiliary loop).

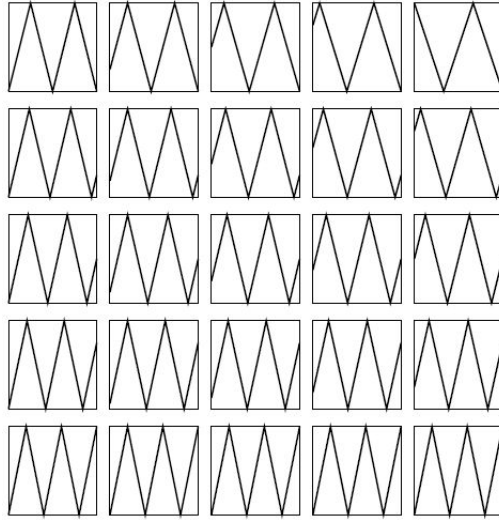


Figure 6. Deformation of the 2nd arc of the basic loop  $L_2$ .

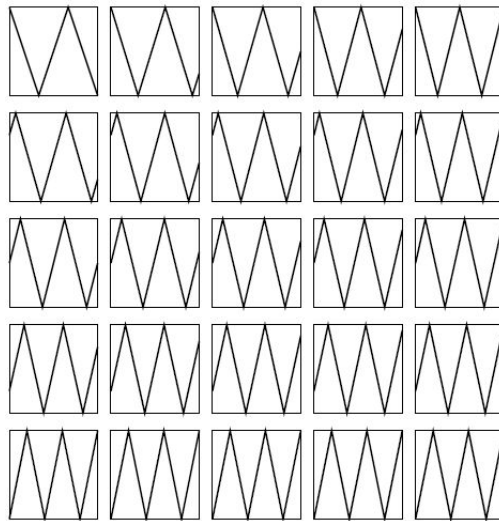


Figure 7. Deformation of the 3rd arc of the basic loop  $L_2$ .



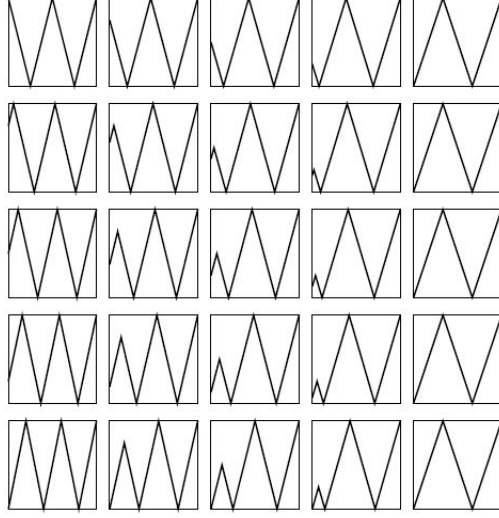
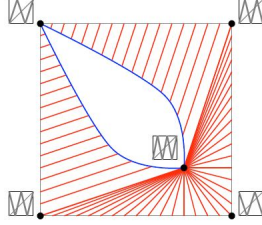


Figure 8. Deformation of the 4th arc of the basic loop  $L_2$ .

The basic loop  $L_2$  consisting of four arcs has been deformed to the auxiliary loop consisting of two arcs (four arcs, but the second and the third are constant, so we can ignore them):



The situation seems to be similar as for the following model, although we do not know how far we can go with this analogy. Think about the sequence of spaces  $\mathbb{R}^n$ ,  $n = 0, 1, 2, \dots$ , where each space is a subset of the next one. Set  $R_n = \mathbb{R}^n \setminus \mathbb{R}^{n-1}$  for  $n = 1, 2, 3, \dots$ . Then the fundamental group of the space

$$R_n \cup R_{n+1} = \mathbb{R}^{n+1} \setminus \mathbb{R}^{n-1} = (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^{n-1}$$

is nontrivial, while the fundamental group of the space

$$R_n \cup R_{n+1} \cup R_{n+2} = \mathbb{R}^{n+2} \setminus \mathbb{R}^{n-1} = (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^{n-1}$$

is trivial.

Additionally, we described the topology (and, in a sense, geometry) of the space  $\mathcal{TCS}_1 \cup \mathcal{TCS}_2$ .

## 2. DYNAMICAL COMPACTNESS

This section is based on the papers [27] and [28]. Let  $\mathbb{Z}_+$  be the set of all nonnegative integers and  $\mathbb{N}$  the set of all positive integers. Before going on, let us recall the notion of a Furstenberg family from [1]. Denote by  $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$  the set of all subsets of  $\mathbb{Z}_+$ . A subset  $\mathcal{F} \subset \mathcal{P}$  is a (*Furstenberg*) *family*, if it is *hereditary upward*, that is,  $F_1 \subset F_2$  and  $F_1 \in \mathcal{F}$  imply  $F_2 \in \mathcal{F}$ . Any subset  $\mathcal{A}$  of  $\mathcal{P}$  clearly generates a family  $\{F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A}\}$ . Denote by  $\mathcal{B}$  the family of all infinite subsets of  $\mathbb{Z}_+$ , and by  $\mathcal{P}_+$  the family of all nonempty subsets of  $\mathbb{Z}_+$ . For a family  $\mathcal{F}$ , the *dual family* of  $\mathcal{F}$ , denoted by  $k\mathcal{F}$ , is defined as

$$\{F \in \mathcal{P} : F \cap F' \neq \emptyset \text{ for any } F' \in \mathcal{F}\}.$$

A family  $\mathcal{F}$  is *proper* if it is a proper subset of  $\mathcal{P}$ , that is,  $\mathbb{Z}_+ \in \mathcal{F}$  and  $\emptyset \notin \mathcal{F}$ . By a *filter*  $\mathcal{F}$  we mean a proper family closed under intersection, that is,  $F_1, F_2 \in \mathcal{F}$  implies  $F_1 \cap F_2 \in \mathcal{F}$ . A filter is *free* if the intersection of all its elements is empty. We extend this concept, a family  $\mathcal{F}$  is called *free* if the intersection of all elements of  $\mathcal{F}$  is empty.

For any  $F \in \mathcal{P}$ , every point  $x \in X$  and each subset  $G \subset X$ , we define the orbit  $T^F x = \{T^i x : i \in F\}$ , and the visiting times  $n_T(x, G) = \{n \in \mathbb{Z}_+ : T^n x \in G\}$ . The  $\omega$ -*limit set of  $x$  with respect to  $\mathcal{F}$*  (see [1]), or shortly the  $\omega_{\mathcal{F}}$ -*limit set of  $x$* , denoted by  $\omega_{\mathcal{F}}(x)$ , is defined as

$$\bigcap_{F \in \mathcal{F}} \overline{T^F x} = \{z \in X : n_T(x, G) \in k\mathcal{F} \text{ for every neighborhood } G \text{ of } z\}.$$

Let us note that not always  $\omega_{\mathcal{F}}(x)$  is a subset of the  $\omega$ -*limit set*  $\omega_T(x)$ , which is defined as

$$\bigcap_{n=1}^{\infty} \overline{\{T^k x : k \geq n\}} = \{z \in X : n_T(x, G) \in \mathcal{B} \text{ for every neighborhood } G \text{ of } z\}.$$

For instance, if each element of  $\mathcal{F}$  contains 0 then any point  $x \in \omega_{\mathcal{F}}(x)$ . Nevertheless, if a family  $\mathcal{F}$  is free, then  $\omega_{\mathcal{F}}(x) \subset \omega_T(x)$  for any point  $x \in X$  and if  $(X, T)$  has a nonrecurrent point<sup>1</sup>, then the converse is true.

A dynamical system  $(X, T)$  is called *compact with respect to  $\mathcal{F}$* , or shortly *dynamically compact*, if the  $\omega_{\mathcal{F}}$ -limit set  $\omega_{\mathcal{F}}(x)$  is nonempty for all  $x \in X$ .

H. Furstenberg started a systematic study of transitive systems in his paper on disjointness in topological dynamics and ergodic theory [16], and the theory was further developed in [18] and [17]. Recall that the system  $(X, T)$  is (*topologically*) *transitive* if  $N_T(U_1, U_2) = \{n \in \mathbb{Z}_+ : U_1 \cap T^{-n}U_2 \neq \emptyset\} (= \{n \in \mathbb{Z}_+ : T^n U_1 \cap U_2 \neq \emptyset\}) \in \mathcal{P}_+$  for any *opene*<sup>2</sup> subsets  $U_1, U_2 \subset X$ , equivalently,  $N_T(U_1, U_2) \in \mathcal{B}$  for any opene subsets  $U_1, U_2 \subset X$ .

**Transitive compactness** — one of possible dynamical compactnesses. Let  $\mathcal{N}_T$  be the set of all subsets of  $\mathbb{Z}_+$  containing some  $N_T(U, V)$ , where  $U, V$  are opene subsets of  $X$ . A dynamical system  $(X, T)$  is called *transitive compact*, if for any point  $x \in X$  the  $\omega_{\mathcal{N}_T}$ -limit set  $\omega_{\mathcal{N}_T}(x)$  is nonempty, in other words, for any point  $x \in X$  there exists a point  $z \in X$  such that

$$n_T(x, G) \cap N_T(U, V) \neq \emptyset$$

<sup>1</sup>A point  $x \in X$  is called *recurrent* if  $x \in \omega_T(x)$ .

<sup>2</sup>Because we so often have to refer to open, nonempty subsets, we will call such subsets *opene*.

for any neighborhood  $G$  of  $z$  and any open subsets  $U, V$  of  $X$ .

Let  $(X, T)$  and  $(Y, S)$  be two dynamical systems and  $k \in \mathbb{N}$ . The product system  $(X \times Y, T \times S)$  is defined naturally, and denote by  $(X^k, T^{(k)})$  the product system of  $k$  copies of the system  $(X, T)$ . Recall that the system  $(X, T)$  is *minimal* if it does not admit a nonempty, closed, proper subset  $K$  of  $X$  with  $TK \subset K$ , and is *weakly mixing* if the product system  $(X^2, T^{(2)})$  is transitive. Any transitive compact system is obviously topologically transitive, and observe that each weakly mixing system is transitive compact ([3]).

**Sensitive compactness** — another possible dynamical compactness. The notion of sensitivity was first used by Ruelle [43]. It captures the idea that in a chaotic system a small change in the initial condition can cause a big change in the trajectory. According to the works by Guckenheimer [23], Auslander and Yorke [7] a dynamical system  $(X, T)$  is called *sensitive* if there exists  $\delta > 0$  such that for every  $x \in X$  and every neighborhood  $U_x$  of  $x$ , there exist  $y \in U_x$  and  $n \in \mathbb{N}$  with  $d(T^n x, T^n y) > \delta$ . Such a  $\delta$  is called a *sensitive constant* of  $(X, T)$ . Recently in [39] Moothathu initiated a way to measure the sensitivity of a dynamical system, by checking how large is the set of nonnegative integers for which the sensitivity occurs. For a positive  $\delta$  and a subset  $U \subset X$  define

$$S_T(U, \delta) = \{n \in \mathbb{Z}_+ : \text{there are } x_1, x_2 \in U \text{ such that } d(T^n x_1, T^n x_2) > \delta\}.$$

Let  $\mathcal{S}_T := \mathcal{S}_T(\delta)$  be the set of all subsets of  $\mathbb{Z}_+$  containing some  $S_T(U, \delta)$ , where  $U$  is an open subset of  $X$ . A dynamical system  $(X, T)$  is called *sensitive compact*, if for any point  $x \in X$  the  $\omega_{\mathcal{S}_T}$ -limit set  $\omega_{\mathcal{S}_T}(x)$  is nonempty, in other words, for any point  $x \in X$  there exists a point  $z \in X$  such that

$$n_T(x, G) \cap \mathcal{S}_T(U, \delta) \neq \emptyset$$

for any neighborhood  $G$  of  $z$  and any open subset  $U$  in  $X$ .

A dynamical system  $(X, T)$  is called *multi-sensitive* if there exists  $\delta > 0$  such that  $\bigcap_{i=1}^k S_T(U_i, \delta) \neq \emptyset$  for any finite collection of open  $U_1, \dots, U_k \subset X$ . Such a  $\delta$  is called a *constant of multi-sensitivity* of  $(X, T)$ .

Recall that a collection  $A$  of subsets of a set  $Y$  has the *finite intersection property* (FIP) if the intersection of all sets in any finite subcollection of  $A$  is nonempty. It is well known that the FIP is useful in formulating an alternative definition of compactness of a topological space: *a topological space is compact if and only if every collection of closed subsets satisfying the FIP has a nonempty intersection itself*.

We recall that if  $(X, T)$  is weakly mixing then it is well known that the family  $\mathcal{N}_T$  is a filter (see [16], [1]), and hence has FIP. If  $(X, T)$  is a multi-sensitive system with a constant of multi-sensitivity  $\delta > 0$  then obviously the family  $\mathcal{S}_T(\delta)$  has FIP. Since all of these families are also free, actually they have the *strong finite intersection property* (SFIP).

In fact we can say more — the FIP is useful in characterizing the dynamical compactness.

**Theorem FIP** ([27]). *All dynamical systems are dynamically compact with respect to  $\mathcal{F}$  if and only if the family  $\mathcal{F}$  has the finite intersection property.*

*Proof. Sufficiency.* Suppose that  $\mathcal{F}$  has FIP. Take arbitrary dynamical system  $(X, T)$  and let  $x \in X$ . Obviously the family  $\{\overline{T^F x} : F \in \mathcal{F}\}$  also has FIP, and then

by compactness of  $X$  the family  $\{\overline{T^F x} : F \in \mathcal{F}\}$  has a nonempty intersection itself, i.e.,  $\omega_{\mathcal{F}}(x) \neq \emptyset$ . Thus  $(X, T)$  is dynamically compact with respect to  $\mathcal{F}$ .

*Necessity.* Suppose that the family  $\mathcal{F}$  has no FIP, and then there is a collection  $\{F_1, \dots, F_k\} \subset \mathcal{F}$  with  $\bigcap_{i=1}^k F_i = \emptyset$ . Let  $A = \{a_1, \dots, a_k\}$  be an alphabet and let  $(X, T) := (\Sigma, \sigma)$  be the full (one-sided)  $A$ -shift. We are going to define a point  $x \in X$  with  $\omega_{\mathcal{F}}(x) = \emptyset$ . Let  $x_0 = a_1$ . For any  $n \geq 1$  there is  $i$  with  $n \notin F_i$ , else the intersection of  $F_1, \dots, F_k$  would be nonempty. Then define  $x_n := a_i$ . Finally, let  $x = x_0 x_1 x_2 x_3 \dots$  and the construction is finished.

Assume the contrary that we can take  $z \in \omega_{\mathcal{F}}(x)$ , and that  $z$  begins with  $a_i \in A$ . Take  $G_z = C[a_i]$ . As  $z \in \omega_{\mathcal{F}}(x)$  we have  $n_T(x, G_z) \cap F_i \neq \emptyset$ . But if  $n \in n_T(x, G_z)$ , then  $x_n = a_i$  and so  $n \notin F_i$  by the construction, a contradiction.  $\square$

**2.1. Dynamical compactness with respect to an arbitrary family.** As we have mentioned, any filter has FIP; if  $(X, T)$  is weakly mixing then the family  $\mathcal{N}_T$  is a filter; if  $A$  a weakly mixing subset of  $(X, T)$  then the family  $\mathcal{N}_T(A)$  has FIP; and if  $(X, T)$  is a multi-sensitive system with a constant of multi-sensitivity  $\delta > 0$  then the family  $\mathcal{S}_T(\delta)$  also has FIP.

A collection  $\mathcal{H} \subset \mathcal{F}$  will be called a *base* for  $\mathcal{F}$  if for any  $F \in \mathcal{F}$  there is  $H \in \mathcal{H}$  with  $H \subset F$ . We are interested in those families which have a *countable base*, that is, there exists a base  $\mathcal{H}$  which is countable.

Note that not every Furstenberg family  $\mathcal{F}$  has a countable base, for example, the family  $\mathcal{B}$ . Assume the contrary that  $\mathcal{B}$  admits a countable base  $\{F_n : n \in \mathbb{N}\}$ . We take  $k_1 \in F_1$ , and once  $k_m \in F_m, m \in \mathbb{N}$  is defined we choose  $k_{m+1} \in F_{m+1}$  with  $k_{m+1} > k_m + m + 1$ . Set  $E = \{k_n : n \in \mathbb{N}\}$  and  $F = \mathbb{Z}_+ \setminus E$ . Then  $E \cap F_n \neq \emptyset$  for all  $n \in \mathbb{N}$ , and  $F \supset \{k_m + m : m \in \mathbb{N}\}$  and hence  $F \in \mathcal{B}$ , in particular, there exists no  $n \in \mathbb{N}$  with  $F_n \subset F$ , a contradiction.

It is not hard to show even the existence of a family with FIP, but without a countable base. Nevertheless the families  $\mathcal{N}_T$  and  $\mathcal{S}_T(\delta)$  have countable bases. Indeed, we can consider a countable base  $\mathcal{U}$  of open sets for the space  $X$ . Note that  $U_1 \subset U, V_1 \subset V$  implies  $N_T(U_1, V_1) \subset N_T(U, V)$  and  $S_T(U_1, \delta) \subset S_T(U, \delta)$ . Then  $\{N_T(U, V) : U, V \in \mathcal{U}\}$  and  $\{S_T(U, \delta) : U \in \mathcal{U}\}$  are countable bases for  $\mathcal{N}_T$  and  $\mathcal{S}_T(\delta)$ , respectively.

The following is a general result that will be especially useful for families with countable bases.

**Proposition 2.1.** *Let  $(X, T)$  be a dynamical system and let  $\mathcal{F}$  be a family such that there exists  $x \in \text{Tran}_{\mathcal{F}}(X, T)$ . Then  $\text{orb}_T(x) \subset \text{Tran}_{\mathcal{F}}(X, T)$ .*

**Proposition 2.2.** *Assume that  $\mathcal{F}$  admits a countable base  $\mathcal{H}$ . Then  $\text{Tran}_{k\mathcal{F}}(X, T)$  is a  $G_{\delta}$  subset of  $X$ . Moreover, the following are equivalent:*

- (1) *The system  $(X, T)$  is  $k\mathcal{F}$ -transitive,*
- (2)  *$\text{Tran}_{k\mathcal{F}}(X, T)$  is a dense  $G_{\delta}$  subset of  $X$ ,*
- (3)  *$\text{Tran}_{k\mathcal{F}}(X, T) \neq \emptyset$ .*

**2.2. Transitive sensitivity and sensitive compactness.** A dynamical system  $(X, T)$  is *transitively sensitive* if there exists  $\delta > 0$  such that  $S_T(W, \delta) \cap N_T(U, V) \neq \emptyset$  for any open subsets  $U, V, W$  of  $X$ , and recall is sensitive compact if there exists  $\delta > 0$  such that for any point  $x \in X$  the set  $\omega_{\mathcal{S}_T(\delta)}(x)$  is nonempty. Sometimes in that cases we will say also  $(X, T)$  is transitively sensitive with a sensitive constant  $\delta$  and  $(X, T)$  is sensitive compact with a sensitive constant  $\delta$ . Then

**Theorem 2.3.** *Let  $(X, T)$  be a minimal system. Then the following conditions are equivalent:*

- (1)  $(X, T)$  is multi-sensitive.
- (2)  $(X, T)$  is sensitive compact.
- (3) There exists  $\delta > 0$  such that  $\omega_{\mathcal{S}_T(\delta)}(x) = X$  for each  $x \in X$ .
- (4) There exist  $\delta > 0$  and  $x \in X$  with  $\omega_{\mathcal{S}_T(\delta)}(x) = X$ .
- (5)  $(X, T)$  is transitively sensitive.

Before proceeding, we need:

**Lemma 2.2.1.** *Let  $\delta > 0$  and  $x \in X$ . If  $T : X \rightarrow X$  is almost open, then the family  $\mathcal{S}_T(\delta)$  is negatively invariant, and the subset  $\omega_{\mathcal{S}_T(\delta)}(x)$  is positively  $T$ -invariant.*

The following result gives a characterization of transitive sensitivity for a general dynamical system in terms of dynamical compactness.

**Proposition 2.4.** *Let  $(X, T)$  be a dynamical system. Then the family  $\mathcal{S}_T(\delta)$  is positively invariant for any  $\delta > 0$ . Furthermore, the following conditions are equivalent:*

- (1)  $(X, T)$  is transitively sensitive.
- (2) There exist a  $\delta > 0$  and a dense  $G_\delta$  subset  $X_0 \subset X$  such that  $\omega_{\mathcal{S}_T(\delta)}(x) = X$  for each  $x \in X_0$ .
- (3) There exist a  $\delta > 0$  and a point  $x \in X$  with  $\omega_{\mathcal{S}_T(\delta)}(x) = X$ .

Recall that by [20, Corollary 1.7] the sensitivity of a dynamical system can be lifted up from a factor to an extension by an almost open factor map between transitive systems. The following result shows that the transitive sensitivity can be lifted up to an extension from a factor by an almost one-to-one factor map and that the transitive sensitivity is projected from an extension to the sensitivity of a factor by a weakly almost one-to-one factor map.

**Lemma 2.2.2.** *Let  $\pi : (X, T) \rightarrow (Z, R)$  be a factor map between dynamical systems.*

- (1) Assume that  $\pi$  is almost one-to-one. If  $(Z, R)$  is transitively sensitive with a sensitive constant  $\delta > 0$  then  $(X, T)$  is also transitively sensitive.
- (2) Assume that there exists  $z \in Z$  whose fiber is a singleton. If  $(X, T)$  is transitively sensitive then  $(Z, R)$  is sensitive, in particular,  $Eq(Z, R) = \emptyset$ .

Now let us give the proof of Theorem 2.3 from [28].

*Proof of Theorem 2.3.* (1)  $\Rightarrow$  (2) follows directly from the definitions. As the system  $(X, T)$  is minimal, the map  $T : X \rightarrow X$  is almost open. Observing that  $\omega_{\mathcal{S}_T(\delta)}(x)$  is a closed subset of  $X$  for each  $x \in X$ , the implication of (2)  $\Rightarrow$  (3) follows from Lemma 2.2.1 and the minimality of  $(X, T)$ . The implication of (3)  $\Rightarrow$  (4)  $\Rightarrow$  (5) follows from Proposition 2.4. Since a minimal system is either multi-sensitive or a weakly almost one-to-one extension of its maximal equicontinuous factor by [29], then (5)  $\Rightarrow$  (1) follows from Lemma 2.2.2. This finishes the proof.  $\square$

Clearly each multi-sensitive system is sensitive compact. Observe that from each non-proximal, transitive compact system is multi-sensitive by [27, Theorem 4.7] we

**Proposition 2.5.** *Each non-proximal, transitive compact system  $(X, T)$  is multi-sensitive.*

In particular, each minimal transitive compact system is multi-sensitive, as each minimal proximal system is trivial by [3] and all dynamical systems considered are assumed to be nontrivial. Nevertheless, there are many minimal, non transitive compact, multi-sensitive systems. For example, consider the classical dynamical system  $(X, T)$  given by  $X = \mathbb{R}^2/\mathbb{Z}^2$  and  $T : (x, y) \mapsto (x + \alpha, x + y)$  with  $\alpha \notin \mathbb{Q}$  (see [17, Chapter 1]). As commented in [27, Page 1816],  $(X, T)$  is an invertible minimal multi-sensitive system; note that  $(X, T)$  is not weakly mixing, since  $(X, T)$  admits an irrational rotation as its nontrivial equicontinuous factor and any equicontinuous factor of a weakly mixing system is trivial. Recall that by [27, Corollary 3.10] for a minimal system the system is transitive compact if and only if it is weakly mixing, and then the constructed system  $(X, T)$  is not transitive compact.

**Proposition 2.6.** *Each nontrivial weakly mixing system  $(X, T)$  is transitively sensitive.*

We give a sufficient condition for a dynamical system being transitively sensitive (by Proposition 2.4) as the last result of this section.

**Lemma 2.2.3.** *Assume  $\omega_{S_T(\varepsilon)}(x) = X$  for some  $x \in X$  and  $\varepsilon > 0$ . Then there is  $\delta > 0$  such that for any opene subset  $U$  of  $X$  and each neighbourhood  $U_x$  of  $x$  there are  $y \in U_x$  and  $n \in n_T(x, U)$  with  $d(T^n x, T^n y) > \delta$ . If in addition, the map  $T : X \rightarrow X$  is almost one-to-one, then the converse holds.*

**2.3. Transitive compact (non weakly mixing) systems.** Recall that the system  $(X, T)$  is *totally transitive* if  $(X, T^k)$  is transitive for each  $k \in \mathbb{N}$ ; and is *topologically mixing* if  $N_T(U, V) \in \mathcal{F}_{\text{cof}}$  for any opene subsets  $U, V$  in  $X$ . Note that  $(X, T)$  is weakly mixing if and only if  $N_T(U, V) \in \mathcal{F}_{\text{thick}}$  for any opene sets  $U, V$  in  $X$  by [16, 42], and so any weakly mixing system is totally transitive. It is direct to check that each weakly mixing system is transitive compact. We extend it as follows:

**Theorem 2.7.** *There are non-totally transitive, transitive compact systems and totally transitive, transitive compact systems which are not weakly mixing.*

The following result is proved independently in [9] and [45].

**Lemma 2.3.1.** *Any  $\omega$ -limit set  $\omega_T(x)$  can not be decomposed into  $\alpha$  disjoint closed, nonempty, positively  $T$ -invariant subsets, where  $2 \leq \alpha \leq \aleph_0$ .*

Before proceeding, we need the following example, for which we fail to find a reference and hence provide a detailed construction, as it is crucial in our arguments.

**Proposition 2.8.** *For any given compact metric space  $Z$ , there exists a topologically mixing system  $(X, T)$  such that,  $Z$  can be realized as the set of all of its minimal points, furthermore, its each minimal point is a fixed point.*

The following result shows that in general there is no topological structure similar to Lemma 2.3.1 for the  $\omega_{N_T}$ -limit sets.

**Theorem 2.9.** *For any given compact metric space  $Z$ , there exists a non totally transitive, transitive compact system  $(X, T)$  such that,  $Z$  can be realized as the set of all its minimal points with its each minimal point being a fixed point, furthermore,  $Z$  is realized as  $\omega_{N_T}(x)$  for some  $x \in X$ .*

Note that a dynamical system is proximal if and only if it contains the unique fixed point, which is the only minimal point of the system [3]. Thus, as a direct corollary of Lemma 2.3.1 and Theorem 2.9, we have:

**Corollary 1.** *There exists a non-proximal, non totally transitive, transitive compact system  $(X, T)$  and a point  $x_0 \in X$  such that  $\omega_{N_T}(x_0) \neq \omega_T(x)$  for all  $x \in X$ .*

Nevertheless is still open the following

**Question.** *Let  $(X, T)$  be a weakly mixing system. Is there a point  $x \in X$  and  $2 \leq \alpha \leq \aleph_0$  such that  $\omega_{N_T}(x)$  can be decomposed into  $\alpha$  disjoint closed, nonempty, positively  $T$ -invariant subsets?*

At the end of this section let us mention one more chaotic property of transitive compact systems in addition to already known.

A pair of points  $x, y \in X$  is *proximal* (*asymptotic*) if  $\liminf_{n \rightarrow \infty} d(T^n x, T^n y) = 0$  ( $\liminf_{n \rightarrow \infty} d(T^n x, T^n y) = 0$ , respectively). Denote by  $Prox_T(X)$  and by  $Asym_T(X)$  the set of all proximal pairs and asymptotic pairs of points, respectively. Any pair  $(x, y) \in Prox_T(X) \setminus Asym_T(X)$  is called a *Li-Yorke pair*. Recall that a dynamical system  $(X, T)$  is *Li-Yorke chaotic* if there exists an uncountable set  $S \subset X$  with  $(S \times S) \setminus \Delta_2(X) \subset Prox_T(X) \setminus Asym_T(X)$ , where  $\Delta_2(X) = \{(x, x) : x \in X\}$ .

**Proposition 2.10.** *Each transitive compact system  $(X, T)$  is Li-Yorke chaotic.*

Observe that in [33] we initiated another way to measure the sensitivity of a system, that is, gave quantitative measures of the sensitivity of a dynamical system by introducing the **Lyapunov numbers**:

$$\begin{aligned} \mathbb{L}_r &= \sup\{\delta : \text{for every } x \in X \text{ and every open neighborhood } U_x \text{ of } x \text{ there} \\ &\quad \text{exist } y \in U_x \text{ and a nonnegative integer } n \text{ with} \\ &\quad d(T^n x, T^n y) > \delta\}; \end{aligned}$$

$$\begin{aligned} \bar{\mathbb{L}}_r &= \sup\{\delta : \text{for every } x \in X \text{ and every open neighborhood } U_x \text{ of } x \text{ there} \\ &\quad \text{exists } y \in U_x \text{ with } \limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \delta\}; \end{aligned}$$

$$\mathbb{L}_d = \sup\{\delta : \text{in any opene } U \subset X \text{ there exist } x, y \in U \text{ and a nonnegative} \\ \text{integer } n \text{ with } d(T^n x, T^n y) > \delta\};$$

$$\begin{aligned} \bar{\mathbb{L}}_d &= \sup\{\delta : \text{in any opene } U \subset X \text{ there exist } x, y \in U \text{ with} \\ &\quad \limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \delta\}. \end{aligned}$$

Here we set  $\sup \emptyset = 0$  by convention. Various definitions of sensitivity, formally give us different Lyapunov numbers. Nevertheless, as was shown in [33], for minimal topologically weakly mixing systems all these Lyapunov numbers are the same.

The motivation of [33] comes from the following proposition according to [3]:

**Proposition 2.11.** *The following conditions are equivalent:*

1.  $(X, T)$  is sensitive.
2. There exists  $\delta > 0$  such that for every  $x \in X$  and every neighborhood  $U_x$  of  $x$ , there exists  $y \in U_x$  with  $\limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \delta$ .
3. There exists  $\delta > 0$  such that in any opene  $U$  in  $X$  there are  $x, y \in U$  and a nonnegative integer  $n$  with  $d(T^n x, T^n y) > \delta$ .

4. There exists  $\delta > 0$  such that in any opene  $U \subset X$  there are  $x, y \in U$  with  $\limsup_{n \rightarrow \infty} d(T^n x, T^n y) > \delta$ .

The following Figure 9 is a concluding remark for results presented in this Section 2 and presents a comparison between stronger forms of sensitivity for transitive systems. It includes also some new elements on the diagram, which did not present in [27] and [28].

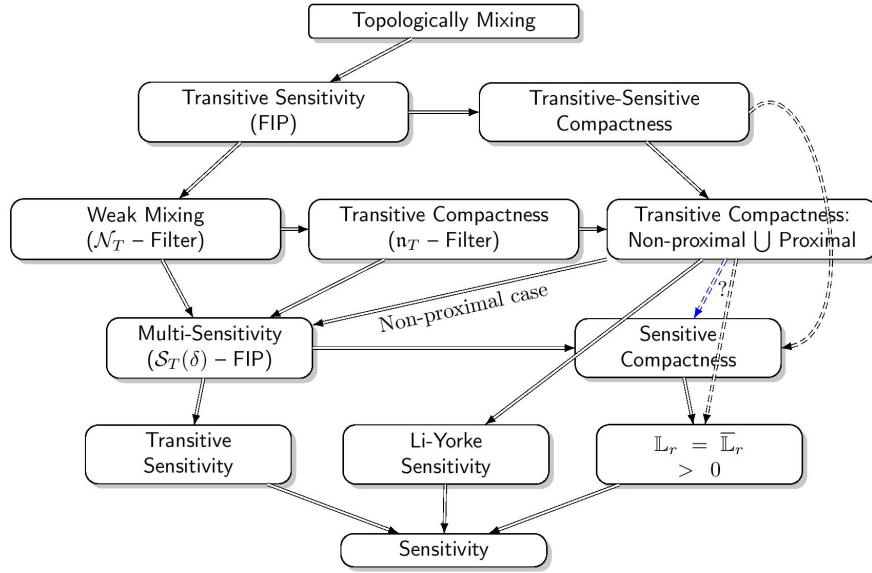


Figure 9. “Topologically transitive systems”.

**Some remarks.** First of all recall some definitions which we did not mention before. Recall that a pair of points  $(x, y)$  is *proximal* if  $\liminf_{n \rightarrow \infty} d(T^n x, T^n y) = 0$ . A dynamical system  $(X, T)$  is called proximal if for any  $x, y \in X$  the pair  $(x, y)$  is proximal. Note, it is known that  $(X, T)$  is proximal iff it has a unique minimal point which is fixed (see [3]).

$(X, T)$  is called *Li-Yorke sensitive* if there exists  $\delta > 0$  such that for every  $x \in X$  and every neighborhood  $U_x$  of  $x$ , there exist  $y \in U_x$  such that  $(x, y)$  is a proximal pair while  $\limsup_{n \rightarrow +\infty} d(T^n x, T^n y) > \delta$ .

Let  $(X, T)$  be a dynamical system which is compact regarding to a (dynamical) Furstenberg family  $\mathcal{F}$ . Say,  $\mathcal{F}$  is  $\mathcal{N}_T$  or  $\mathcal{S}_T$ . Then not so hard to show that in that case for any point  $x \in X$  there exist a minimal subset  $M_x$  of  $X$  and a point  $z \in M_x$  such that

$$n_T(x, G_z) \cap F \neq \emptyset$$

for any neighborhood  $G$  of  $z$  in  $X$  and any  $F \in \mathcal{F}$ . If  $A$  is a set in  $X$ , by  $B_\varepsilon[A]$  we will denote the union of all open balls of radius  $\varepsilon > 0$  whose centers run over  $A$ . Similarly as in the proof of [27, Lemma 3.12] one can show that for any  $x \in X$  and any  $B_\varepsilon[M_x]$  the set  $n_T(x, B_\varepsilon[M_x])$  is thickly syndetic. Let  $\mathbf{n}_T$  be the set of all subsets of  $\mathbb{Z}_+$  containing some  $n_T(x, B_\varepsilon[M_x])$ , where  $x$  is a point of  $X$ ,  $\varepsilon > 0$  and  $M_x \subset \omega_T(x)$  is a minimal subset of  $X$ .



Figure 9 also presents an open question. Are all transitive compact systems sensitive compact? More precisely - does there exist a proximal, transitive compact system which is not sensitive compact?

### 3. SLOVAK SPACES

This section is based on the papers [34], and [12]. One of the main topics of our research is to study relations between some properties (structure) of the topological semigroup  $S(X)$ , group  $H(X)$  and possible values of the topological entropy (and/or some other chaotic properties) of its elements (continuous maps and homeomorphisms, respectively).

In [34] mostly we have considered the following two questions: 1) when does a compact metric space admit a continuous map (homeomorphism) with positive topological entropy? 2) when does the existence of a positive-entropy continuous map on a compact metric space imply the existence of a  $+\infty$ -entropy continuous map? Also there we have proved the following

**Theorem 3.1.** *Let  $X$  be a compact metric space. If  $S(X)$  is compact, then for any  $f \in S(X)$  topological entropy of  $(X, f)$  and topological entropy the functional envelope  $(S(X), F_f)$  is zero. If  $H(X)$  is compact, then for any  $f \in H(X)$  topological entropy of  $(X, f)$  is also zero.*

Now it is not so much known about the topological structure of compact semigroup  $S(X)$  and group  $H(X)$ . The compactness of the  $S(X)$  and  $H(X)$  is not a very strict condition and takes place sometimes, because both of them may be “very small”. Recall that a topological space  $X$  is said to be *rigid*, if the full topological homeomorphism group  $H(X)$  is the identity. De Groot and Wille (in [21] showed the existence of rigid spaces even as locally connected one-dimensional continua (*Peano curves*) of the plane. The main idea of the construction of such a space is similar to the *Sierpinski carpet* — a square with interiors of a dense family of subsquares removed.

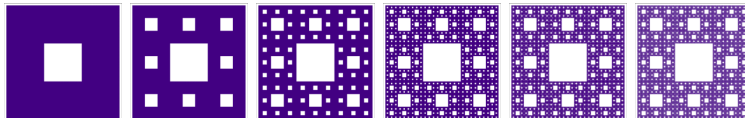
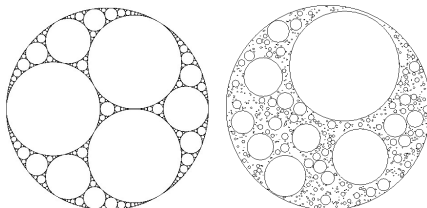


Figure 10. The first 6 steps in the construction of the Sierpinski carpet

The Sierpinski carpet, also known as the *Sierpinski universal curve*, is a one-dimensional planar Peano continuum.



*De Groot - Wille rigid plane locally connected one-dimensional continua* is a disc with interiors of a dense family of propellers (with different numbers of blades) removed.

Consider a disc  $D$  in the plane. Let  $\{a_i\}$  be a countable dense subset of the interior of  $D$ . We define a sequence of "propellers" in  $D$ . The first is bounded by a two-bladed curve having  $a_1$  as its centre, which avoids the boundary of  $D$ . Moreover, we take care that the diameters of the propellers tend to zero. The space  $P$  is the disc  $D$  with the interiors of all the propellers removed. Then  $P$  is a continuum as the intersection of a countable, decreasing sequence of continua. Routine procedure shows the local connectedness of  $P$ .  $P$  has dimension one, since it does not contain a subset, open in the plane. Since the total area of the interiors of the propellers can be chosen as small as we want it to be,  $P$  can also have positive measure.

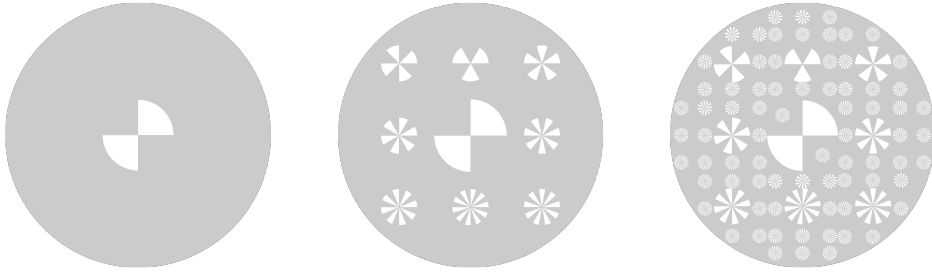


Figure 11. The first 3 steps in the construction of the De Groot - Wille rigid plane continua.

In fact, it must look like the following Julia set (a picture of Volodymyr Nekrashevych)

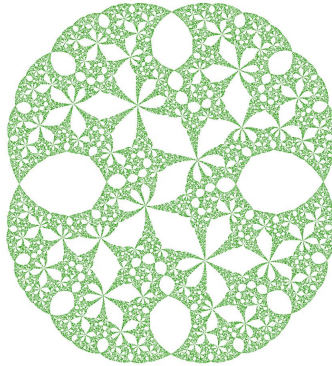


Figure 12. A Julia set.

In fact, by using such kind of ideas, de Groot [22] proved the following

**Theorem 3.2.** *Let  $G$  be an arbitrary group. Then there exists a connected, locally connected, complete metric space  $X$  (or alternatively compact, Hausdorff) for which the group of all autohomeomorphisms  $H(X)$  is isomorphic to  $G$ .*

However, such a space need not exist in the class of compact metric spaces, because a compact metric space has cardinality at most  $c$ , while there are groups of arbitrary cardinalities. Nevertheless, as De Groot and Wille proved, if  $G$  is countable then  $X$  can be chosen to be a Peano continuum of any positive dimension.

Howard Cook (in [8]) constructed two metric continua:

- (1) the space  $S(X)$  consists only of the constant maps (sending all points from  $X$  to a fixed point of  $X$ ) and the identity on  $X$ ;
- (2) the space  $H(X)$  is topologically equivalent to the Cantor set.

But it is still an open problem what can we say about the topological structure of the compact full topological homeomorphism group  $H(X)$  and of the compact full topological homeomorphism group  $S(X)$  (see the conjecture below).

Recall that a topological group is called *profinite group* if it is Hausdorff, compact, and totally disconnected. Gartside and Glyn ([19]) have established that every metric profinite group is the full homeomorphism group of a continuum. Recently Hofmann and Morris in [25] proved that a compact full homeomorphism group of a Tychonoff space is a profinite topological group. But the following conjecture is still open (see [26] for details):

**Conjecture.** *Let  $G$  be a compact group. Then the following conditions are equivalent:*

- (1) *There is a compact connected space  $X$  such that  $G \cong H(X)$ .*
- (2) *There is a compact space  $X$  such that  $G \cong H(X)$ .*
- (3)  *$G$  is profinite.*

A dynamical system  $(X, T)$  gives rise to several “hyper-systems”. For example,  $T$  acts naturally on  $2^X$ , the space of all compact subsets of  $X$  equipped with the Hausdorff metric. The induced map here is the continuous map  $\hat{T} : 2^X \mapsto 2^X$  given by  $\hat{T}(A) = T(A)$  for each  $A \in 2^X$ .

In the same spirit (although now the domains are not necessarily compact), jointly with J. Auslander and L. Snoha in [6], we introduced and started to study the “hyper-system”  $(S(X), F_T)$ , where the transformation

$$F_T : S(X) \mapsto S(X) \text{ is defined by } F_T\varphi = T \circ \varphi \text{ for any } \varphi \in S(X).$$

We called the system as the “*Functional Envelope*” of  $(X, T)$ . Why “functional envelope”? Because the system  $(S(X), F)$  contains (properly, if  $\text{card } X \geq 2$ ) an isomorphic copy of  $(X, f)$ . The map  $\iota : X \rightarrow S(X)$  sending a point  $a \in X$  to the constant map  $\text{con}_a$ , is an isometry (regardless of whether the uniform or the Hausdorff metric is used in  $S(X)$ ) and also a topological conjugacy, i.e.,  $\iota \circ f = F \circ \iota$ .

The functional envelopes may differ in some dynamical properties which depend on the metric (for example the topological entropy in non-compact systems does). On the other hand, the essential dynamical properties of the functional envelopes do not depend on the metric applied on  $X$ .

Now suppose additionally that the map  $T : X \mapsto X$  is a homeomorphism. Then the space  $H(X)$  is invariant under  $F_T$  and  $(H(X), F_T)$  becomes another kind of “functional envelope”

$$F_T : H(X) \mapsto H(X) \text{ is defined by } F_T\varphi = T \circ \varphi \text{ for any } \varphi \in H(X).$$

**3.1. Topological entropy: Bowen and Dinaburg definition.** *Topological entropy* measures the evolution of distinct (distinguishable) orbits over time, thereby providing an idea of how complex the orbit structure of a system is. Recall the

Bowen-Dinaburg definition of topological entropy (see [47]). Let  $(Z, \varrho)$  be a metric space and  $f : Z \rightarrow Z$  be uniformly continuous. For any integer  $n \geq 1$  the function

$$\varrho_n(x, y) := \max_{0 \leq j \leq n-1} \varrho(f^j x, f^j y)$$

defines a metric on  $Z$  equivalent with  $\varrho$ .

Fix an integer  $n \geq 0$  and  $\varepsilon > 0$  and let  $K$  be a compact set in  $Z$ . A subset  $E \subset K$  is called  $(n, f, \varepsilon)$ -separated, if for any two distinct points  $x, y \in E$ ,  $\varrho_n(x, y) > \varepsilon$ . We say that a subset  $F \subset Z$   $(n, f, \varepsilon)$ -spans the set  $K$ , if for every point  $x \in K$  there exists a point  $y \in F$  such that  $\varrho_n(x, y) \leq \varepsilon$ . Note that since  $K$  is compact,  $E$  is finite and  $F$  may be infinite, but a finite subset that still spans  $K$  exists.

Denote by  $\text{sep}(n, f, \varepsilon; K)$  the maximal cardinality of a  $(n, f, \varepsilon)$ -separated set in  $K$ , and by  $\text{span}(n, f, \varepsilon; K)$  the minimal cardinality of a set which  $(n, f, \varepsilon)$ -spans  $K$ .

For every  $\varepsilon > 0$  and  $n \geq 0$  it holds

$$\text{span}(n, f, \varepsilon; K) \leq \text{sep}(n, f, \varepsilon; K) \leq \text{span}(n, f, \varepsilon/2; K) .$$

The *topological entropy*  $h(f, K)$  on a compact set  $K \subset Z$ , is defined by

$$h(f, K) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{sep}(n, f, \varepsilon; K) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{span}(n, f, \varepsilon; K).$$

Then the *topological entropy*  $h(f)$  of a map  $f : Z \rightarrow Z$  is defined by

$$h_\varrho(f) = h(f) := \sup \{ h(f, K) : K \subset Z \text{ and } K \text{ is compact} \} .$$

If  $\text{span}^K(n, f, \varepsilon; K)$  denotes the minimal cardinality of subsets of the compact set  $K$  which  $(n, f, \varepsilon)$ -span  $K$  then  $\text{span}^K(n, f, \varepsilon; K) \leq \text{sep}(n, f, \varepsilon; K) \leq \text{span}^K(n, f, \varepsilon/2; K)$  and  $h(f, K) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{span}^K(n, f, \varepsilon; K)$  (see [47]).

For uniformly equivalent metrics  $\varrho_1$  and  $\varrho_2$  we have  $h_{\varrho_1}(X, f) = h_{\varrho_2}(X, f)$ , and if  $\varrho_1$  is stronger than  $\varrho_2$ , then  $h_{\varrho_1}(X, f) \geq h_{\varrho_2}(X, f)$ .

We know that if  $X$  is compact, then the map  $F_f : \varphi \mapsto f \circ \varphi$  is uniformly continuous on both the spaces  $S_U(X)$  and  $S_H(X)$ , and so we can study its topological entropy. Recall that  $d_U(\varphi_1, \varphi_2) \geq d_H(\varphi_1, \varphi_2)$  for all  $\varphi_1, \varphi_2 \in S(X)$  and that these two metrics are equivalent on  $S(X)$ , hence uniformly equivalent on compact subsets of  $S(X)$ .

Some basic properties of topological entropy:

- (1)  $h(T) \geq h(S)$  if  $(Y, S)$  is a "topological factor" of  $(X, T)$ , i.e.,  $\varphi \circ T = S \circ \varphi$ , where  $\varphi : X \rightarrow Y$  is a continuous surjection.
- (2)  $h(T) \geq h(T|_Y)$  if  $Y$  is a nonempty closed invariant subset of  $X$ .
- (3)  $h(T^n) = n \cdot h(T)$  for  $n \geq 0$ .
- (4)  $h(T \times S) = h(T) + h(S)$ .
- (5)  $h(T \circ S) = h(S \circ T)$ .

Let us gather some basic known facts and open problems concerning the entropy of the functional envelopes:

(1) The entropy of  $(S(X), F_T)$  is always not smaller than that of  $(X, T)$ ; to see this it suffices to consider the subsemigroup of constant self-maps, which is conjugate to  $(X, T)$ .

(2) On the other hand, it was not known whether the same inequality holds for  $(H(X), F_T)$ .

(3) Our with Semikina conjecture says that the functional envelope  $(S(X), F_T)$  has entropy either zero or infinity. In [34] we proved that the conjecture holds true for all Peano continua and for all compact spaces with continuum many connected components. Otherwise the conjecture was open. The same problem can be posed for the functional envelope  $(H(X), F_T)$  in case  $T$  is a homeomorphism.

Recently, Downarowicz, Snoha and Tywoniuk [12] gave a positive answer for the question (2) for homogeneous spaces and extended the knowledge on the question (3). Namely,

**Theorem 3.3.** *Let  $T : X \mapsto X$  be a homeomorphism of a homogeneous compact space. Then the entropy of the functional envelope  $(H(X), F_T)$  is at least as large as that of  $(X, T)$ .*

**Theorem 3.4.** *Let  $T : X \mapsto X$  be a self-homeomorphism of a compact zero-dimensional space. Then the entropies of  $(S(X), F_T)$  and  $(H(X), F_T)$  are either both zero or both infinite. They are equal to zero if and only if  $T$  is equicontinuous.*

Nevertheless, they also showed that there exists a positive (even infinite) entropy homeomorphism  $T$  of a compact space  $X$  such that  $(H(X), F_T)$  has entropy zero (see [12] and below). This proves that the general question (2) (the inequality between topological entropies of  $(X, T)$  and its functional envelope  $(H(X), F_T)$ ) has a negative answer.

It was solved using a new class of spaces, which they have called **Slovak spaces**, defined by the combination of two properties: the existence of a (minimal) homeomorphism, say  $T$ , and nonexistence of homeomorphisms other than the powers of  $T$ .

For such spaces obviously that the functional envelope  $(H(X), F_T)$  always has entropy zero.

**3.2. Existence of uniquely minimal spaces and applications.** Recall that a map  $T : X \mapsto X$  is called **minimal** if the orbit  $\{x, Tx, T^2x, \dots, T^n x, \dots\}$  of any point  $x \in X$  is dense in  $X$ .

For a compact metric space  $X$  there are two possibilities:

- (1)  $X$  does not admit any minimal homeomorphism (e.g., interval, disk, ... , any space with the fixed point property);
- (2)  $X$  admits a minimal homeomorphism (Cantor set, circle, torus, ... ). In this case, if  $X$  is infinite then in known examples usually  $X$  admits uncountably many homeomorphisms and even uncountably many of them are minimal).

Is there a third possibility? That is, does there exist an infinite compact metric space  $X$  such that it admits, but only “a few”, minimal homeomorphisms?

An infinite compact metric space  $X$  is called *Slovak* if it is *uniquely minimal* in the following sense:  $X$  admits a minimal homeomorphism  $T$  and  $H(X) = \{T^n : n \in \mathbb{Z}\}$ .

The assumption that  $X$  is infinite eliminates two trivial examples: the one-point space and the two-point space. If  $X$  is Slovak then  $\text{card } X = c$ ,  $X$  has no isolated point and all iterates  $T^n$ ,  $n \in \mathbb{Z}$  are different, i.e.  $\mathbf{H}(X) \approx \mathbb{Z}$ . Moreover, *all iterates  $T^n$ , except identity, are minimal*.

**Theorem 3.5.** *There exist Slovak spaces in the class of metric continua. Moreover, the topological entropies of generating homeomorphisms  $T$  exhaust the interval  $[0, \infty]$ .*

**3.3. Idea of a construction of uniquely minimal spaces.** Every Slovak space is a non-degenerate continuum.

**Step 1.** Let  $h : C \rightarrow C$  be a minimal homeomorphism on the Cantor set  $C$  (with arbitrary entropy).

**Step 2.** Define a generalized solenoid induced by  $(C, h)$  as following

$$SOL := C \times [0, 1] / \sim,$$

where  $(x, 1) \sim (hx, 0)$ .

Recall that solenoids are among the simplest examples of indecomposable homogeneous continua. They are neither arcwise connected nor locally connected.

On the pictures below: a solid torus  $\mathbb{S}^1 \times D$  wrapped twice around inside another solid torus in  $\mathbb{R}^3$ . And each solenoid may be constructed as the intersection of a nested system of embedded solid tori in  $\mathbb{R}^3$ .



Figure 13. The first 5 steps in the construction of the Smale-Williams attractor.

Take the suspension flow over  $h$  (with ceiling function  $\equiv 1$ )  $\Phi_t, t \in \mathbb{R}$  defined on  $SOL$  by the formula

$$\Phi_t(y, s) := (h^{[t+s]}y, \{t+s\}),$$

where  $[\cdot]$  and  $\{\cdot\}$  denote the integer and fractional parts of a real number, respectively.

**Step 3.** It is well known that there exists  $t_0 \in \mathbb{R}$  such that the map

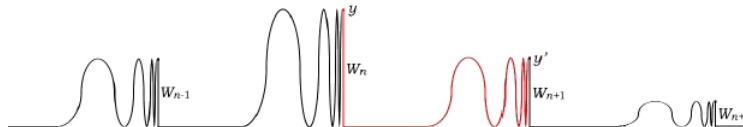
$$T := \text{time } t_0\text{-map of flow } \Phi$$

is a minimal homeomorphism on  $SOL$  (see also [15]).

**Step 4.** It is the main technical step. The Slovak space (which were constructed in [12]) will be a subset of the cylinder  $SOL \times [0, 1]$ . The main element of this construction is a topologist's sine curve. For instance, a topologist's sine curve is a subset of the plane that is the union of the graph of the function  $f(x) = \sin(1/x)$ ,  $0 < x \leq 1$  with the segment  $-1 \leq y \leq 1$  of the  $y$ -axis.

1. The continuum  $SOL$  has uncountably many composants (orbits of the flow  $\Phi$ ); choose a component  $\gamma$  and a point  $x_0 \in \gamma$ . The component of a point  $p$  in a continuum  $A$  is the union of all proper subcontinua of  $A$  that contain  $p$ . If a continuum is indecomposable, then its composants are pairwise disjoint. The composants of a continuum are dense in that continuum.
2. On a closed arc around  $x_0$  (minus the point  $x_0$  itself) and lying in  $\gamma$ , we define a function which looks like a one-sided topologist's sine curve (values in  $[0, 1]$ , wiggles of height 1 in any left neighbourhood of  $x_0$ , constant value 0 to the right of  $x_0$ ). It is continuous and not defined at  $x_0$ .
3. Extend it to a continuous function  $f : SOL \setminus \{x_0\} \rightarrow [0, 1]$ .
4. Let  $F = \sum_{n \in \mathbb{Z}} a_n f \circ T^n$ , where the coefficients  $a_n$  are all strictly positive,  $\sum_{n \in \mathbb{Z}} a_n = 1$  and satisfying some technical assumptions ( $F$  is defined on  $SOL$  minus the  $T$ -orbit of  $x_0$ ).

5. Then one can show that both, the mapping  $(x, F(x)) \mapsto (Tx, F(Tx))$  and its inverse are uniformly continuous homeomorphisms of the graph of  $F$ . Therefore, the map  $(x, F(x)) \mapsto (Tx, F(Tx))$  extends to a homeomorphism  $\bar{T}$  of  $\bar{F}$  (=the closure of the graph of  $F$ ).
6.  $\bar{F} \subseteq SOL \times [0, 1]$  is our Slovak space, looks as follows:  
The compositant  $\bar{\gamma}$  of  $\bar{F}$  “above”  $\gamma$  has basically this shape:



The other compositants of  $\bar{F}$  are continuous bijective images of the real line.

**3.4. Some generalizations.** The question when a system  $(X, f)$  can be embedded as a subsystem of some  $(Y, g)$  so that  $X = \omega_g(y)$  for some  $y \in Y$  was answered by Dowker and Friedlander [9] for homeomorphisms and by Sharkovsky [44] in general (see also [46]). Recall that a system  $(X, f)$  is *f-connected* if, for any proper, nonempty, closed subset  $U \subsetneq X$ , the intersection  $f(U) \cap \overline{X \setminus U}$  is nonempty. They show that  $(X, f)$  can be embedded as the omega limit set in some larger system if and only if it is *f-connected*.

In [4] Akin and Rautio have considered the related problem of when a space  $X$  admits a homeomorphism  $f$  so that  $(X, f)$  is the omega limit set in a larger system. As we will see, their results are somewhat different from the map case. They reinterpreted the problem by using some other notion. Given  $\varepsilon \geq 0$ , a finite or infinite sequence  $\{x_n \in X\}$  with at least two terms is an  $\varepsilon$ -chain for  $(X, f)$  if  $d(fx_k, x_{k+1}) \leq \varepsilon$  for all terms  $x_k$  of the sequence (except the last one). The system  $(X, f)$  is called *chain transitive* when every pair of points of  $X$  can be connected by some finite  $\varepsilon$ -chain for every positive  $\varepsilon$ . A subset  $A \subset X$  is called a *chain transitive* subset when it is closed and  $f$ -invariant (i.e.,  $f(A) = A$ ) and the subsystem  $(A, f)$  is chain transitive.

It is well known that chain transitivity and *f-connectedness* are equivalent concepts (see also [4] for details). Akin and Rautio generalized several known results (in particular, regarding rigid spaces). They proved the existence of the following compact metric spaces:

- (1) Suppose that  $G$  is a finitely generated (and hence countable) group. Then there exist a space  $X$  such that the homeomorphism group  $H(X)$  is isomorphic to  $G$  and every  $f \in H(X)$  is chain transitive.
- (2) There exist a space  $X$  such that the homeomorphism group  $H(X)$  contains a nontrivial path-connected subgroup and every  $f \in H(X)$  is chain transitive.
- (3) There exist a space  $X$  such that the homeomorphism group  $H(X)$  is isomorphic to the homeomorphism group of the Cantor set and every  $f \in H(X)$  is chain transitive.

Akin and Rautio in [4] also have extended the result of Downarowicz, Snoha and Tywoniuk on rigid spaces by showing the existence of spaces (*Slovakian spaces* in their terminology) which admit just a topologically transitive homeomorphism and its iterations.

## 4. APPENDIX

**4.1. Basic concepts in topological dynamics.** Recall that  $x \in X$  is a *fixed point* if  $Tx = x$ , and an  *$\mathcal{F}$ -transitive point* of  $(X, T)$  [37] if  $N_T(x, U) \in \mathcal{F}$  for any opene subset  $U$  of  $X$ . It is a trivial observation that if a family  $\mathcal{F}$  admits an  $\mathcal{F}$ -transitive dynamical system  $(X, T)$  without isolated points, then  $\mathcal{F}$  is free. Since  $k(k\mathcal{F}) = \mathcal{F}$ , it is easy to see that  $x \in X$  is an  $\mathcal{F}$ -transitive point of  $(X, T)$  if and only if  $\omega_{k\mathcal{F}}(x) = X$ . Denote by  $\text{Tran}_{\mathcal{F}}(X, T)$  the set of all  $\mathcal{F}$ -transitive points of  $(X, T)$ . The system  $(X, T)$  is  *$\mathcal{F}$ -point transitive* if  $\text{Tran}_{\mathcal{F}}(X, T) \neq \emptyset$ , and is  *$\mathcal{F}$ -transitive* if  $N_T(U, V) \in \mathcal{F}$  for any opene subsets  $U, V$  of  $X$ . Write  $\text{Tran}(X, T) = \text{Tran}_{\mathcal{P}_+}(X, T)$  for short, and we also call the point  $x$  *transitive* if  $x \in \text{Tran}(X, T)$ , equivalently, its *orbit*  $\text{orb}_T(x) = \{T^n x : x = 0, 1, 2, \dots\}$  is dense in  $X$ . Since  $T$  is surjective, the system  $(X, T)$  is transitive if and only if  $\text{Tran}(X, T)$  is a dense  $G_\delta$  subset of  $X$ .

In general, a subset  $A$  of  $X$  is  *$T$ -invariant* if  $TA = A$ , and *positively  $T$ -invariant* if  $TA \subset A$ . If  $A$  is a closed, nonempty,  $T$ -invariant subset then  $(A, T|_A)$  is called the associated *subsystem*. A *minimal subset* of  $X$  is a closed, nonempty,  $T$ -invariant subset such that the associated subsystem is minimal. Clearly,  $(X, T)$  is minimal if and only if  $\text{Tran}(X, T) = X$ , if and only if it admits no a proper, closed, nonempty, positively  $T$ -invariant subset. A point  $x \in X$  is called *minimal* if it lies in some minimal subset. In this case, in order to emphasize the underlying system  $(X, T)$  we also say that  $x \in X$  is a *minimal point of  $(X, T)$* . Zorn's Lemma implies that every closed, nonempty, positively  $T$ -invariant set contains a minimal set.

A pair of points  $x, y \in X$  is called *proximal* if  $\liminf_{n \rightarrow \infty} d(T^n x, T^n y) = 0$ . In this case each of points from the pair is said to be *proximal* to another. Denote by  $\text{Prox}_T(X)$  the set of all proximal pairs of points. For each  $x \in X$ , denote by  $\text{Prox}_T(x)$ , called the *proximal cell of  $x$* , the set of all points which are proximal to  $x$ . Recall that a dynamical system  $(X, T)$  is called *proximal* if  $\text{Prox}_T(X) = X \times X$ . The system  $(X, T)$  is proximal if and only if  $(X, T)$  has the unique fixed point, which is the only minimal point of  $(X, T)$  (e.g. see [3]).

The opposition to the notion of sensitivity is the concept of equicontinuity. Recall that  $x \in X$  is an *equicontinuity point* of  $(X, T)$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that  $d(x, x') < \delta$  implies  $d(T^n x, T^n x') < \varepsilon$  for any  $n \in \mathbb{Z}_+$ . Denote by  $\text{Eq}(X, T)$  the set of all equicontinuity points of  $(X, T)$ . The system  $(X, T)$  is called *equicontinuous* if  $\text{Eq}(X, T) = X$ . Each dynamical system admits a maximal equicontinuous factor. Recall that by a *factor map*  $\pi : (X, T) \rightarrow (Y, S)$  between dynamical systems  $(X, T)$  and  $(Y, S)$ , we mean that  $\pi : X \rightarrow Y$  is a continuous surjection with  $\pi \circ T = S \circ \pi$ . In this case, we call  $\pi : (X, T) \rightarrow (Y, S)$  an *extension*; and  $(X, T)$  an *extension* of  $(Y, S)$ ,  $(Y, S)$  a *factor* of  $(X, T)$ .

**4.2. Basic concepts of Furstenberg families.** In this subsection we recall from [1] basic concepts about Furstenberg families.

Let  $F \in \mathcal{P}$ . Recall that a subset  $F$  is *thick* if it contains arbitrarily long runs of positive integers. Denote by  $\mathcal{F}_{\text{thick}}$  the set of all thick subsets of  $\mathbb{Z}_+$ , and define  $\mathcal{F}_{\text{syn}} = k\mathcal{F}_{\text{thick}}$ . Each element of  $\mathcal{F}_{\text{syn}}$  is said to be *syndetic*, equivalently,  $F$  is syndetic if and only if there is  $N \in \mathbb{N}$  such that  $\{i, i+1, \dots, i+N\} \cap F \neq \emptyset$  for every  $i \in \mathbb{Z}_+$ . We say that  $F$  is *thickly syndetic* if for every  $N \in \mathbb{N}$  the positions where length  $N$  runs begin form a syndetic set. Denote by  $\mathcal{F}_{\text{cof}}$  the set of all cofinite subsets of  $\mathbb{Z}_+$ . Note that by the classic result of Gottschalk a point  $x \in X$  is minimal if and only if  $n_T(x, U) = \{n \in \mathbb{Z}_+ : T^n x \in U\}$  is syndetic for any



neighborhood  $U$  of  $x$ . Hence, for any minimal system  $(X, T)$ , the subset  $N_T(U, V)$  is syndetic for any open subsets  $U, V$  of  $X$ .

Recall that a family  $\mathcal{F}$  is proper if it is a proper subset of  $\mathcal{P}$ , that is,  $\mathbb{Z}_+ \in \mathcal{F}$  and  $\emptyset \notin \mathcal{F}$ . By a filter  $\mathcal{F}$  we mean a proper family closed under intersection, that is,  $F_1, F_2 \in \mathcal{F}$  implies  $F_1 \cap F_2 \in \mathcal{F}$ . For families  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , we define the family  $\mathcal{F}_1 \cdot \mathcal{F}_2 := \{F_1 \cap F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$  and call it the *interaction of  $\mathcal{F}_1$  and  $\mathcal{F}_2$* . Thus we have  $\mathcal{F}_1 \cup \mathcal{F}_2 \subset \mathcal{F}_1 \cdot \mathcal{F}_2$ ; and it is easy to check that  $\mathcal{F}$  is a filter if and only if  $\mathcal{F} = \mathcal{F} \cdot \mathcal{F}$ , and  $\mathcal{F}_1 \cdot \mathcal{F}_2$  is proper if and only if  $\mathcal{F}_2 \subset k\mathcal{F}_1$ .

For each  $i \in \mathbb{Z}_+$ , we define  $g^i : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+, j \mapsto i + j$ . Let  $\mathcal{F}$  be a family. Recall that  $\mathcal{F}$  is *positively invariant* if for every  $i \in \mathbb{Z}_+$ ,  $F \in \mathcal{F}$  implies  $g^i(F) \in \mathcal{F}$ ; *negatively invariant* if for every  $i \in \mathbb{Z}_+$ ,  $F \in \mathcal{F}$  implies  $g^{-i}(F) \in \mathcal{F}$ , where  $g^{-i}(F) = (g^i)^{-1}(F) = \{j - i : j \in F, j \geq i\}$ ; and *translation invariant* if it is both positively and negatively invariant, equivalently, for every  $i \in \mathbb{Z}_+$ ,  $F \in \mathcal{F}$  if and only if  $g^{-i}(F) \in \mathcal{F}$ .

As  $g^{-i}(g^i A) = A$  and  $g^i(g^{-i} A) \subset A$  for any  $i \in \mathbb{Z}_+$ , it is easy to obtain that the family  $\mathcal{F}$  is positively (negatively, translation, respectively) invariant if and only if  $k\mathcal{F}$  is negatively (positively, translation, respectively) invariant (see for example [1, Proposition 2.5.b]). And then we have:

**Proposition 4.1.** *Let  $x \in X$ . Then  $T\omega_{\mathcal{F}}(x) \subset \omega_{\mathcal{F}}(Tx)$ . Additionally, if  $\mathcal{F}$  is negatively (positively, translation, respectively) invariant then  $\omega_{\mathcal{F}}(Tx) \subset (\supset, =, \text{respectively}) \omega_{\mathcal{F}}(x)$ .*

**Proposition 4.2.** *Let  $(X, T)$  be a dynamical system and let  $\mathcal{F}$  be a family.*

- (i) *If  $\mathcal{F}$  is free, then  $\omega_{\mathcal{F}}(x) \subset \omega_T(x)$  for any  $x \in X$ . Moreover, if  $(X, T)$  has a nonrecurrent point, then the converse implication is true.*
- (ii) *If  $\mathcal{F}$  is free and has FIP then it has SFIP.*

**4.3. The concept of an almost one-to-one map.** Let  $\varphi : X \rightarrow Y$  be a continuous surjective map from a compact metric space  $X$  onto a compact Hausdorff space  $Y$ . Recall that  $\varphi$  is *almost open* if  $\varphi(U)$  has a nonempty interior in  $Y$  for any open  $U \subset X$ . Note that each factor map between minimal systems is almost open [5, Theorem 1.15], in particular, for a minimal system  $(X, T)$  the map  $T : X \rightarrow X$  is almost open [35]. Denote by  $Y_0 \subset Y$  the set of all points  $y \in Y$  whose fiber is a singleton. Then  $Y_0$  is a  $G_\delta$  subset of  $Y$ , because

$$Y_0 = \{y \in Y : \varphi^{-1}(y) \text{ is a singleton}\} = \bigcap_{n \in \mathbb{N}} \left\{ y \in Y : \text{diam}(\varphi^{-1}(y)) < \frac{1}{n} \right\}$$

and the map  $y \mapsto \text{diam}(\varphi^{-1}(y))$  is upper semi-continuous. Here, we denote by  $\text{diam}(A)$  the diameter of a subset  $A \subset X$ . Recall that the function  $f : Y \rightarrow \mathbb{R}_+$  is *upper semi-continuous* if  $\limsup_{y \rightarrow y_0} f(y) \leq f(y_0)$  for each  $y_0 \in Y$ . Denote by  $X_0 \subset X$  the set of all points  $x \in X$  such that the pre-image of  $\varphi(x)$  is a singleton. Then  $X_0 = \pi^{-1}(Y_0)$  is a  $G_\delta$  subset of  $X$ .

We call  $\varphi$  *weakly almost one-to-one* if  $Y_0$  is dense in  $Y$ , and *almost one-to-one*<sup>3</sup> if  $X_0$  is dense in  $X$ . It is not hard to show that: if  $\varphi$  is weakly almost one-to-one, then for any  $\delta > 0$  and any open subset  $U$  of  $Y$  there exists open  $V \subset U$  with  $\text{diam}(\varphi^{-1}V) < \delta$ ; and if  $\varphi$  is almost one-to-one, then for any open subset  $U^*$  of  $X$

<sup>3</sup>Here we use the concept of *almost one-to-one* following [2], and the concept of *almost one-to-one* used in [11, 29, 35] is in fact our weakly almost one-to-one.

there exists an open subset  $V^*$  of  $Y$  with  $\varphi^{-1}V^* \subset U^*$ . Clearly almost one-to-one is much stronger than weakly almost one-to-one. For example, let  $X$  be the closed unit interval, define  $Tx = 2x$  for  $x \in [0, \frac{1}{2}]$  and  $Tx = 1$  for  $x \in [\frac{1}{2}, 1]$ , and then  $T : X \rightarrow X$  is clearly not almost one-to-one but weakly almost one-to-one.

For each minimal system  $(X, T)$ , the map  $T : X \rightarrow X$  is weakly almost one-to-one [35, Theorem 2.7], and in fact almost one-to-one [29, Proposition 2.3]. The following result characterizes the relationship between weakly almost one-to-one and almost one-to-one, which extends [29, Proposition 2.3].

**Proposition 4.3.** *Let  $\varphi : X \rightarrow Y$  be a continuous surjective map from a compact metric space  $X$  onto a compact Hausdorff space  $Y$ . Then  $\varphi$  is almost one-to-one if and only if it is almost open and weakly almost one-to-one.*

As a direct corollary, we have:

**Corollary 2.** *Let  $\varphi : X \rightarrow Y$  and  $\pi : Y \rightarrow Z$  be continuous surjective maps between compact metric spaces. Then the composition map  $\pi \circ \varphi : X \rightarrow Z$  is almost one-to-one if and only if both  $\varphi$  and  $\pi$  are almost one-to-one.*

Let  $\pi : (X, T) \rightarrow (Y, S)$  be a factor map between dynamical systems. If the map  $\pi : X \rightarrow Y$  is almost one-to-one (weakly almost one-to-one, respectively), then we also call  $(X, T)$  an *almost one-to-one extension* (a *weakly almost one-to-one extension*, respectively) of  $(Y, S)$ . The main result of [29] states that a minimal system is either multi-sensitive or a weakly almost one-to-one extension of its maximal equicontinuous factor. This is an analog of the well-known Auslander-Yorke dichotomy theorem: a minimal system is either sensitive or equicontinuous.

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