A LIE THEORETIC APPROACH TO RENORMALIZATION

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Abstract. Motivated by recent work of Connes and Marcolli, based on the Connes–Kreimer approach to renormalization, we augment the latter by a combinatorial, Lie algebraic point of view. Our results rely both on the properties of the Dynkin idempotent, one of the fundamental Lie idempotents in the theory of free Lie algebras, and on the fine properties of Hopf algebras and their associated descent algebras. Besides leading very directly to proofs of the main combinatorial properties of the renormalization procedures, the new techniques do not depend on the geometry underlying the particular case of dimensional regularization and the Riemann–Hilbert correspondence. This is illustrated with a discussion of the BPHZ renormalization scheme.

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Contents

1. Introduction

From its inception, renormalization theory in perturbative quantum field theory (pQFT) had a combinatorial flavour, as well as an analytic one. The former manifests itself in the self-similarity of Feynman graphs, the building blocks of pQFT. The intricate combinatorics of extracting and combining subgraphs, required in the renormalization process, is encoded in the Bogoliubov recursion, respectively its solution via Zimmermann's forest formula [11, 13, 38, 49, 50].

Kreimer's discovery of a Hopf algebra structure underlying Bogoliubov and Zimmermann's formulae and illuminating their internal structure [33] was the starting point of a new approach in the field. Then Connes–Kreimer's decomposition à la Birkhoff–Wiener–Hopf (BWH) of Feynman rules captured abstractly the process of renormalization in pQFT [16]. Further work by Connes, Kreimer and others has since then established various links between the BWH decomposition of characters in renormalizable quantum field theories and

relevant mathematical topics, culminating recently in work by Connes and Marcolli on motivic Galois theory [18, 20] and by Bloch, Esnault and Kreimer on the motivic nature of primitive Feynman graphs [4].

In the present work, largely motivated by [18], we return to the origin of Connes–Kreimer's theory and concentrate on algebraic features of renormalization relevant to pQFT, trying to unravel further fundamental properties of renormalization schemes by methods inspired on the classical theory of free Lie algebras (FLAs).

It has been known since the mid-nineties that many properties of FLAs, as exposed e.g. in [7, 44], can be lifted to general graded Lie algebras and their enveloping algebras. In other terms —particularly emphasized in reference [41]— Lie theory is relevant to the study of arbitrary graded connected cocommutative or commutative Hopf algebras. The observation applies to the Hopf algebras of renormalization, yet it has not received the attention it deserves. Here we develop it systematically, and show that some of the deepest combinatorial properties of renormalization schemes (dimensional regularization, but also other schemes) are codified in the composition with the Dynkin idempotent of Lie theory, and in its inverse map. We derive in particular from their study the properties of characters under locality assumptions for counterterms in renormalization and, in the process, establish that the data relevant to their computation are contained in the 'beta function'. The phenomenon is well known in pQFT; the Lie theoretic approach, however, provides a remarkably efficient way to deduce it from the locality assumptions. Our proofs should be compared both with the classical approaches and with Connes–Kreimer's more geometrically-oriented study of characters.

The article is organized as follows. We consider abstract renormalization Hopf algebras H and commutative target algebras A of quantum amplitudes. After settling some notation in the next section, we ponder in Section 3 the convolution algebra of linear maps $\mathrm{Lin}(H, A)$. It cannot be made into a Hopf algebra in general; but a suitable algebra of *characteristic functions* can. This is our playground; it encodes, at the Hopf algebra level, the properties of the group of A-valued characters of H. Section 4 is the heart of the paper: starting from a short survey on the Dynkin idempotent for cocommutative Hopf algebras, we establish the formal properties of its sibling in the commutative case, then introduce and exhaustively study the inverse Dynkin map. In particular, we show that the latter bijectively sends infinitesimal characters into characters of commutative connected Hopf algebras —applying in particular to the Hopf algebras of Feynman diagrams and rooted trees of renormalization theory and the corresponding Feynman rules. In Section 5 we recall the BWH decomposition of characters, choosing once again to obtain it algebraically from Rota–Baxter operator theory and the 'Baker–Campbell–Hausdorff (BCH) recursion'. After that, our Lie theoretic machine is fully operational.

In the rest of the paper, we show the relevance of that machine to pQFT. In Section 6 we briefly review some standard lore of renormalization and remind the reader of the dictionary between it and the Connes–Kreimer paradigm. Next we study in Section 7 the locality properties for dimensional regularization (DR) schemes by exploiting the properties of the Dynkin pair of maps, together with the BWH decomposition. The main results concerning Connes–Kreimer's beta function and the renormalization group (RG) in DR are rederived from direct computations in the algebra of characteristic functions; hopefully, the role of that beta function is thereby illuminated. Sections 8 and 9 are essays on the same question in other renormalization frameworks; in the second we invoke the BPHZ scheme of renormalization and exemplify with the (Ginzburg–Landau–Wilson) φ_4^4 scalar model in Euclidean field theory.

To finish, in Section 10 we go back to the mathematical setting, trying to place our results in the 'great scheme of things' of combinatorial Hopf algebra theory. We show there how

the Connes–Marcolli "motivic Galois group" of renormalization relates with FLAs as well as the theory of descent algebras and noncommutative symmetric functions. Together with the links between the same group and Connes–Moscovici's index theorem in noncommutative geometry, these new connections give further evidence for Connes' and Kreimer's —already much documented— claim that the divergences of pQFT do reveal the presence of deep mathematical structures.

2. Notational conventions

Let $H = \bigoplus_{n=0}^{\infty} H_n$ be a graded connected commutative Hopf algebra (of finite type) over a field k of characteristic zero; this is necessarily free as a commutative algebra [41]. We write ϵ for the augmentation from H to $H_0 = k \subset H$ and H^+ for the augmentation ideal $\bigoplus_{n=1}^{\infty} H_n$ of H. The identity map of H is denoted I. The product in H is written π or simply by concatenation. The coproduct is written δ; we use Sweedler's notation and write $h^{(1)} \otimes h^{(2)}$ for $\delta(h)$, $h \in H_n$; or $\sum_{i=0}^{n} h_i^{(1)} \otimes h_{n-i}^{(2)}$ when the grading has to be taken into account. The usual restrictions apply, that is, $h^{(1)} \otimes h^{(2)}$ stands for a sum $\sum_{j\in J} h_j^{(1)} \otimes h_j^{(2)}$ and should not be interpreted as the tensor product of two elements of H. The same convention applies in forthcoming notation such as $h^{(1)} \otimes g^{(1)} \otimes h^{(2)} \otimes g^{(2)}$, that should be understood as $\sum_{j\in J}\sum_{k\in K} h_j^{(1)} \otimes g_k^{(1)} \otimes h_j^{(2)} \otimes g_k^{(2)}$, where $\delta(g) = \sum_{k\in K} g_k^{(1)} \otimes g_k^{(2)}$.

Graduation phenomena are essential for all our forthcoming computations, since in the examples of physical interest they incorporate information such as the number of loops (or vertices) in Feynman graphs, and the subdivergence structure, relevant for the RG. They are expressed by the action of the grading operator $Y : H \to H$, given by:

$$Y(h) = \sum_{n\in\mathbb{N}} n h_n \quad \text{for} \quad h = \sum_{n\in\mathbb{N}} h_n \in \bigoplus_{n\in\mathbb{N}} H_n.$$

We write $|h_n| := n$. Notice that Y is usually denoted by D in the FLA literature. In the present article we stick to the notation most often used in the context of the Hopf algebra approach to renormalization [15, 16, 17, 18, 23, 28] and reserve D for the Dynkin operator.

3. The Hopf algebra of characteristic functions

A *character* is a linear map γ of unital algebras from H to the base field k:

$$\gamma(hh') = \gamma(h)\gamma(h').$$

It should be clear that the product on the right hand side is the one in k. We write γ_n for the restriction of γ to a map from H_n to k.

Let A be a commutative k-algebra, with unit $1_A = \eta_A(1)$, $\eta_A : k \to A$ and with product π_A, which we sometimes denote by a dot: $\pi_A(u\otimes v) =: u\cdot v$. The main examples we have in mind are $A = \mathbb{C}$, $A = \mathbb{C}[[\varepsilon, \varepsilon^{-1}]$ and $A = H$. We extend now the definition of characters and call an (A-valued) character of H any algebra map from H to A. In particular H-valued characters are simply algebra endomorphisms of H.

An *infinitesimal character* is a linear map α from H to k such that:

$$\alpha(hh') = \alpha(h)\epsilon(h') + \epsilon(h)\alpha(h').$$

As for characters, we write $\alpha(h) = \sum_{n\in\mathbb{N}} \alpha_n(h_n)$. The same notational convention will be used in the sequel without further notice: f_n stands for the restriction of an arbitrary map f on H to H_n. We remark that by the very definition of characters and infinitesimal characters $\gamma_0(1_H) = 1$, that is $\gamma_0 = \epsilon$, whereas $\alpha_0(1_H) = 0$.

We extend as well in the obvious way the notion of infinitesimal characters to maps from H to a commutative k-algebra A. We have now:

$$\alpha(hh') = \alpha(h)\cdot e(h') + e(h)\cdot\alpha(h'),$$

where $e := \eta_A \circ \epsilon$. They can be alternatively defined as k-linear maps from H to A with $\alpha_0 = 0$ that vanish on the square of the augmentation ideal of H. In particular, if α is an infinitesimal character, the linear map $\alpha^{|n}$, the restriction of which is 0 on all the graded components of H excepted H_n, and $\alpha^{|n}_n = \alpha_n$, is also an infinitesimal character. Thus α decomposes as a sum of infinitesimal characters $\alpha = \sum_{n>0} \alpha^{|n}$. The vector space of infinitesimal characters, written $\Xi_H(A)$, or just $\Xi(A)$, decomposes accordingly as the direct product of its graded components: $\Xi(A) = \prod_{n\in\mathbb{N}^*} \Xi_n(A)$, where $\Xi_n(A)$ is the linear span of the $\alpha^{|n}$. Thus we regard $\Xi(A)$ as the natural 'topological' completion of the graded vector space $\oplus_{n\in\mathbb{N}^*}\Xi_n(A)$. In more detail, we consider the subspaces $\oplus_{i\geq n\in\mathbb{N}^*}\Xi_i(A)$ and the associated quotients and onto homomorphisms, and we take the inverse limit, whose elements are infinite series. This property we agree to abbreviate from now on to "$\Xi(A)$ is a graded vector space"; the sundry objects defined below are graded in that sense —that is, they are actually completions of graded vector spaces, completions of graded Lie algebras, and so on.

The space $\mathrm{Lin}(H, A)$ of k-linear maps from H to A, $\mathrm{Lin}(H, A) := \prod_{n\in\mathbb{N}} \mathrm{Lin}(H_n, A)$, is naturally endowed with an algebra structure by the *convolution product*:

$$f * g := \pi_A \circ (f \otimes g) \circ \delta : \qquad H \xrightarrow{\delta} H \otimes H \xrightarrow{f\otimes g} A \otimes A \xrightarrow{\pi_A} A.$$

The unit for the convolution product is precisely $e : H \to A$. Especially when $A = H$, the convolution product provides various tools to deal with properties of characters, such as the logarithm of the identity, which is a projector on H with kernel the square of the augmentation ideal. As a corollary, A-valued infinitesimal characters can be characterized as those maps α from H to A such that $\alpha_n \circ I^{*k}(h_n) = k\,\alpha_n(h_n)$, for any $k, n \in \mathbb{N}$. We refer the reader interested in a systematic study of these phenomena and of their connections to the classical structure theorems for Hopf algebras such as the Cartier-Milnor-Moore theorem to [40, 41].

The set $G_H(A)$ of characters, or just $G(A)$, is a group for the convolution product. The inverse is given explicitly by the formula:

$$\gamma^{-1} = \Big(e + \sum_{n\in\mathbb{N}^*} \gamma_n\Big)^{-1} = e + \sum_{k\in\mathbb{N}^*} (-1)^k \Big(\sum_{n\in\mathbb{N}^*} \gamma_n\Big)^{*k}.$$

The last sum is well-defined as a power series, since only a finite number of terms appear in each degree. We denote as usual by S the convolution inverse of the identity map I in $\mathrm{End}(H) := \mathrm{Lin}(H, H)$. Then $\gamma^{-1} = \gamma \circ S \in G(A)$; the reader unfamiliar with this identity can deduce it easily from the next lemma and other notions introduced soon.

Now, $\mathrm{Lin}(H, A)$ is *not* naturally provided with a Hopf algebra structure over the ground field k, except under particular assumptions on the target space A. For example, it is (up to the completion phenomena) a Hopf algebra if $A = k$. This follows from the usual argument to construct a Hopf algebra structure on the graded dual of a graded connected Hopf algebra of finite type. It is not when $A = k[[\varepsilon, \varepsilon^{-1}]$, that is when the coefficient algebra A is the field of Laurent series —an example relevant to renormalization. However, as will be shown below, a Hopf algebra structure can always be defined on certain remarkable spaces naturally related to $\mathrm{Lin}(H, A)$ and, most importantly in view of applications to pQFT, to the group of characters $G(A)$.

Lemma 3.1. *Assume that, for given $\phi, \psi \in \mathrm{Lin}(H, A)$, there exist elements $\phi^{(1)} \otimes \phi^{(2)}$, respectively $\psi^{(1)} \otimes \psi^{(2)}$, in $\mathrm{Lin}(H, A) \otimes \mathrm{Lin}(H, A)$ such that, for any $h, h' \in H$:*

$$\phi^{(1)}(h) \cdot \phi^{(2)}(h') = \phi(hh') \quad \textit{and} \quad \psi^{(1)}(h) \cdot \psi^{(2)}(h') = \psi(hh');$$

then

$$\phi * \psi(hh') = \big(\phi^{(1)} * \psi^{(1)}(h)\big) \cdot \big(\phi^{(2)} * \psi^{(2)}(h')\big).$$

Moreover, when $\psi \in \mathrm{End}(H)$ *and* $\phi \in \mathrm{Lin}(H, A)$*, with the same hypothesis and* $\psi^{(1)}, \psi^{(2)}$ *now in* $\mathrm{End}(H)$*:*

$$\phi \circ \psi(hh') = \big(\phi^{(1)} \circ \psi^{(1)}(h)\big) \cdot \big(\phi^{(2)} \circ \psi^{(2)}(h')\big).$$

The last identity in particular holds when $A = H$*, that is, in* $\mathrm{End}(H)$*.*

Proof. Indeed, we have:

$$\begin{aligned}
\phi * \psi(hh') &= \phi(h^{(1)}h'^{(1)}) \cdot \psi(h^{(2)}h'^{(2)}) \\
&= \phi^{(1)}(h^{(1)}) \cdot \phi^{(2)}(h'^{(1)}) \cdot \psi^{(1)}(h^{(2)}) \cdot \psi^{(2)}(h'^{(2)}) \\
&= \phi^{(1)}(h^{(1)}) \cdot \psi^{(1)}(h^{(2)}) \cdot \phi^{(2)}(h'^{(1)}) \cdot \psi^{(2)}(h'^{(2)}) \\
&= \big(\phi^{(1)} * \psi^{(1)}(h)\big) \cdot \big(\phi^{(2)} * \psi^{(2)}(h')\big),
\end{aligned}$$

an identity that we also write, for later use,

$$(\phi * \psi)^{(1)} \otimes (\phi * \psi)^{(2)} = (\phi^{(1)} \otimes \phi^{(2)}) * (\psi^{(1)} \otimes \psi^{(2)}).$$

We also clearly have:

$$\phi \circ \psi(hh') = \phi(\psi^{(1)}(h)\ \psi^{(2)}(h')) = \big(\phi^{(1)} \circ \psi^{(1)}(h)\big) \cdot \big(\phi^{(2)} \circ \psi^{(2)}(h')\big). \qquad \square$$

Corollary 3.1. *The graded vector space of infinitesimal characters* $\Xi(A)$ *is a graded Lie subalgebra of* $\mathrm{Lin}(H, A)$ *for the Lie bracket induced on the latter by the convolution product.*

Proof. Indeed, infinitesimal characters are precisely the elements α of $\mathrm{Lin}(H, A)$ such that:

$$\alpha^{(1)} \otimes \alpha^{(2)} = \alpha \otimes e + e \otimes \alpha \quad \text{satisfy, for any } h, h' \in H\text{:} \quad \alpha^{(1)}(h) \cdot \alpha^{(2)}(h') = \alpha(hh').$$

According to the foregoing lemma, for α and β two graded infinitesimal characters, we obtain:

$$\begin{aligned}
[\alpha, \beta](hh') &:= (\alpha * \beta - \beta * \alpha)(hh') \\
&= \pi_A[(\alpha \otimes e + e \otimes \alpha) * (\beta \otimes e + e \otimes \beta) \\
&\quad -(\beta \otimes e + e \otimes \beta) * (\alpha \otimes e + e \otimes \alpha)](h \otimes h') \\
&= [\alpha, \beta](h) \cdot e(h') + e(h) \cdot [\alpha, \beta](h'),
\end{aligned}$$

hence the corollary follows. $\square$

Proposition 3.1. *The enveloping algebra* $U(\Xi(A))$ *of the Lie algebra* $\Xi(A)$ *maps naturally to the convolution subalgebra of* $\mathrm{Lin}(H, A)$ *generated by* $\Xi(A)$*.*

The existence of that natural algebra map from $U(\Xi(A))$ to $\mathrm{Lin}(H, A)$ follows from the previous lemma and from the universal property of enveloping algebras.

Notice that $U(\Xi(A))$ is also, as the enveloping algebra of a graded Lie algebra, a graded connected cocommutative Hopf algebra, which we call the Hopf algebra $\mathrm{Char}_H(A)$, or just $\mathrm{Char}(A)$, of characteristic functions on H (with values in A). We write $\overline{*}$ for the product on $\mathrm{Char}(A)$ and use Δ for its coproduct. Thus by definition of $\mathrm{Char}(A)$ the primitive elements are the infinitesimal A-valued characters of H. Besides providing Hopf algebra tools for the study of Feynman rules, the Hopf algebra of characteristic functions —and the associated pro-unipotent group— will play a crucial role in Section 10, when relating the FLA approach to renormalization to the Connes–Marcolli motivic Galois group.

Notice that Δ is not defined in general on $\mathrm{Lin}(H, A)$, and neither on the image of $\mathrm{Char}(A)$ in $\mathrm{Lin}(H, A)$, see [31] and [42] for a discussion on the subject in the particular case $A = H$.

Proposition 3.2. *We have, for any $\phi \in \mathrm{Char}(A)$ and any $h, h' \in H$, the reciprocity law:*

$$\phi(hh') = \phi^{(1)}(h) \cdot \phi^{(2)}(h'),$$

where we use the Sweedler notation for $\Delta(\phi)$, and where the action of ϕ on H is induced by the natural map from $\mathrm{Char}(A)$ *to* $\mathrm{Lin}(H, A)$.

Proof. This is true when ϕ is an infinitesimal character. According to the previous proposition, for ϕ, ϕ' in $\mathrm{Char}(A)$, we have: $\phi \,\overline{*}\, \phi'(hh') = \phi * \phi'(hh')$. Due to the Lemma 3.1, it follows that the identity holds for $\phi \,\overline{*}\, \phi'$ if it holds for ϕ and ϕ'. Since $\mathrm{Char}(A)$ is generated as an associative algebra by infinitesimal characters, the proposition follows. □

We remark that the reciprocity law can be rewritten:

$$\phi \circ \pi = \pi_A \circ \Delta(\phi).$$

In the cocommutative case, the identity playing a similar role (mutatis mutandis) is [42]:

$$\delta \circ \phi = \Delta(\phi) \circ \delta.$$

As a consequence of Proposition 3.2, the set $G'(A)$ of group-like elements in $\mathrm{Char}(A)$ maps to characters, that is elements of $G(A)$ —since the identity $\Delta(\phi) = \phi \otimes \phi$ in $\mathrm{Char}(A)$ translates into the identity $\phi(hh') = \phi(h)\phi(h')$ in H. We show now that, as usual, the convolution exponential and logarithm maps are inverse bijections from $\Xi(A)$ onto $G(A)$ and from $\Xi(A)$ onto $G'(A)$. Indeed, for any $\alpha \in \Xi(A)$, we have in $\mathrm{Char}(A)$:

$$\begin{aligned} \Delta\big(\exp(\alpha)\big) &= \exp\big(\Delta(\alpha)\big) = \exp(\alpha \otimes e + e \otimes \alpha) = \exp(\alpha \otimes e) \,\overline{*}\, \exp(e \otimes \alpha) \\ &= \big(\exp(\alpha) \otimes e\big) \,\overline{*}\, \big(e \otimes \exp(\alpha)\big) = \exp(\alpha) \otimes \exp(\alpha), \end{aligned}$$

which also implies that we have $\exp(\alpha) \in G(A)$ in $\mathrm{Lin}(H, A)$. We have used first that α is a graded infinitesimal character, then that $\alpha \otimes e$ and $e \otimes \alpha$ commute. The other way round, if γ is a character:

$$\begin{aligned} \log(\gamma)(hh') &= \pi_A\big(\log(\gamma \otimes \gamma)(h \otimes h')\big) \\ &= \pi_A\big((\log(\gamma \otimes e) + \log(e \otimes \gamma))(h \otimes h')\big) \\ &= \pi_A\big((\log(\gamma) \otimes e + e \otimes \log(\gamma))(h \otimes h')\big) \\ &= \log(\gamma)(h) \cdot e(h') + e(h) \cdot \log(\gamma)(h'), \end{aligned}$$

whereas if $\gamma \in G'(A)$:

$$\begin{aligned} \Delta(\log \gamma) &= \log(\Delta\gamma) = \log(\gamma \otimes \gamma) \\ &= \log((\gamma \otimes e) \,\overline{*}\, (e \otimes \gamma)) \\ &= \log \gamma \otimes e + e \otimes \log \gamma. \end{aligned}$$

Corollary 3.2. *The natural algebra map from* $\mathrm{Char}(A)$ *to* $\mathrm{Lin}(H, A)$ *induces a bijection between the group $G'(A)$ of group-like elements in* $\mathrm{Char}(A)$ *and $G(A)$, the group of A-valued characters on H.*

We identify $G(A)$ with $G'(A)$ henceforth. In particular, both the identity map I and the antipode S can be viewed as elements of $\mathrm{Char}(H)$, and we won't distinguish between I, S and their respective preimages in $\mathrm{Char}(H)$.

4. Logarithmic derivatives and the Dynkin operator

Although the logarithm is the simplest bijection from group-like elements to primitive elements in a graded connected cocommutative Hopf algebra, the most relevant bijection in view of applications to renormalization is a bit subtler. It is a kind of logarithmic derivative, closely related to a Lie idempotent known as the *Dynkin idempotent*. Presently we survey the properties of the Dynkin operator (the composition of the Dynkin idempotent with the grading map Y) pertinent to our forthcoming computations, and also obtain new results on the operator, such as the existence of the advertised bijection between $G(A)$ and $\Xi(A)$. The results generalize the fine properties of Hopf algebras encapsulated in the notion of descent algebra of a Hopf algebra [41]. They rely as well on the Hopf algebraic treatment of the Dynkin operator given in [42], and on more classical Lie theoretic properties of the operator. We give in particular closed formulas for the inverse map to Dynkin's operator, i.e., from $\Xi(A)$ to $G(A)$.

The classical Dynkin operator is defined as follows. Let $X = \{x_1, \dots, x_n, \dots\}$ be an alphabet. The tensor algebra $T(X) := \bigoplus_{n\geq 0} T_n(X)$ over X is a cocommutative graded Hopf algebra with the set of words $x_{i_1} \dots x_{i_l}$ as a linear basis in degree l. It is also, canonically, the enveloping algebra of the FLA $\mathrm{Lie}(X)$ over X. The product in $T(X)$ is induced by concatenation of words, whereas the coproduct is fully characterized by the property that elements of X are primitive in $T(X)$. The Dynkin operator $D : T(X) \to \mathrm{Lie}(X)$ is given by:

$$D(x_{i_1} \dots x_{i_n}) = [\dots[[x_{i_1}, x_{i_2}], x_{i_3}], \dots, x_{i_n}];$$

with $D_{T_0(X)} = 0$ and $D_{T_1(X)} = \mathrm{id}_X$. According to an idea essentially due to von Waldenfels, this iterated bracketing operator can be represented in a more abstract way as the convolution product of the antipode S of $T(X)$ with the grading operator Y, acting as the multiplication by n on $T_n(X)$:

$$D = S * Y; \quad \text{equivalently} \quad I * D = Y.$$

The most famous property of D is the Dynkin–Specht–Wever theorem, stating that an element v in $T_n(X)$ —a linear combination of words of length n— is in $\mathrm{Lie}(X)$ if and only if $D(v) = nv$. In effect, such a v is in the primitive part of the tensor algebra, and:

$$nv = Yv = \pi(I \otimes D)(1 \otimes v + v \otimes 1) = D(v).$$

The converse is also well known. These definitions and properties have been generalized to bialgebras in [42], that we follow mainly here. However, for our purposes we have to give a somewhat detailed account. Indeed, that reference as well as the classical theory of the Dynkin operator do focus on the study of graded connected cocommutative Hopf algebras, whereas we are mainly interested here in *commutative* Hopf algebras. The interested reader will find further historical and technical information about the Dynkin operator and its relevance to Lie computations in references [29] and [44].

So let H be graded, connected and commutative. Since $I \in \mathrm{Char}(H)$, so its graded components $I_n \in \mathrm{Char}(H)$. Notice, for further use, that the subalgebra of $\mathrm{Char}(H)$ generated by the I_n maps to the descent algebra of H —the convolution subalgebra of $\mathrm{End}(H)$ generated by the I_n, see [41]. Moreover, the grading operator $Y := \sum_{n\in\mathbb{N}} nI_n$ belongs to $\mathrm{Char}(H)$. Its coproduct is given by:

$$\Delta(Y) = Y \otimes I + I \otimes Y,$$

another way of expressing that Y is a derivation of H:

$$Y(hh') = Y(h)h' + hY(h').$$

Let us adopt the notation Yf for $f \circ Y$, according to the custom in distribution theory. Under this guise the operator Y extends naturally to a derivation on $\mathrm{Lin}(H, A)$. We find,

with $f, g \in \mathrm{Lin}(H, A)$ and $h \in H$:

$$\begin{aligned} Y(f*g)(h) &:= f*g\,(Y(h)) = |h|(f*g)(h) \\ &= |h| f(h^{(1)}) g(h^{(2)}) \\ &= |h^{(1)}| f(h^{(1)}) g(h^{(2)}) + |h^{(2)}| f(h^{(1)}) g(h^{(2)}) \\ &= Yf*g\,(h) + f*Yg\,(h), \end{aligned}$$

where we used that $\Delta(Y(h)) = |h|\Delta(h) = \big(|h^{(1)}| + |h^{(2)}|\big)\, h^{(1)} \otimes h^{(2)}$.

Proposition 4.1. *Convolution of the H-valued character S with any derivation d of H yields an H-valued infinitesimal character.*

Proof. Indeed, since d is a derivation, we have $d(hh') = d(h)h' + hd(h')$. Since, furthermore, $\Delta(S) = S \otimes S$, we get:

$$\begin{aligned} (S*d)(hh') &= \pi \circ [(S\otimes S) * (d \otimes I + I \otimes d)](h \otimes h') \\ &= \pi \circ [(S*d) \otimes (S*I) + (S*I) \otimes (S*d)](h \otimes h') \\ &= S*d(h)\cdot e(h') + e(h)\cdot S*d(h'), \end{aligned}$$

hence the proposition follows. □

Corollary 4.1. *The Dynkin operator $D := S*Y$ is an H-valued infinitesimal character of H.*

Notice also that D satisfies $D \circ D = D \circ Y$ —in other terms, D is an idempotent up to a scalar factor depending on the grading, that is, D is a quasi-idempotent. Equivalently, $D \circ Y^{-1}$ is an idempotent on H^+ (also known when H is the cotensor algebra $T^*(X)$ —see below— as the Dynkin idempotent). Indeed, for any $h \in H$,

$$D \circ D(h) = D \circ (S*Y)(h) = D\big(S(h^{(1)})Y(h^{(2)})\big).$$

However, since D is an infinitesimal character, $D(hh') = 0$ if $h, h' \in H^+$ and therefore,

$$D \circ D(h) = D\big(S(h)Y(1_H) + S(1_H)Y(h)\big) = D \circ Y(h),$$

since $Y(1_H) = 0$.

Proposition 4.2. *Right composition with an infinitesimal character $\alpha \in \Xi(H)$ induces a map from $G(H)$ to $\Xi(H)$. This also holds for $G(A)$ and $\Xi(A)$, where A is an arbitrary commutative unital algebra.*

Proof. Indeed, let $\gamma \in G(H)$ or $G(A)$, we have:

$$\gamma \circ \alpha(hh') = \gamma \circ \alpha(h)\, e(h') + e(h)\, \gamma \circ \alpha(h'),$$

by virtue of Lemma 3.1, since $\gamma \circ e = e$ for any character γ. □

Corollary 4.2. *Right composition with the Dynkin operator D induces a map from $G(A)$ to $\Xi(A)$.*

In general, for γ belonging to $G(H)$ or $G(A)$ and any $f_1, \ldots, f_k \in \mathrm{End}(H)$, we have the distributive property:

$$\gamma \circ (f_1 * \cdots * f_k) = (\gamma \circ f_1) * \cdots * (\gamma \circ f_k).$$

Particularly,

$$\gamma \circ D = \gamma \circ (S*Y) = \gamma^{-1} * Y\gamma.$$

Theorem 4.1. *Right composition with D is a bijective map from $G(H)$ to $\Xi(H)$. The inverse map is given by:*

$$\Gamma : \alpha \in \Xi(H) \longmapsto \sum_n \sum_{k_1,\dots,k_n \in \mathbb{N}^*} \frac{\alpha_{k_1} * \cdots * \alpha_{k_n}}{k_1(k_1+k_2)\dots(k_1+\cdots+k_n)} \in G(H). \tag{1}$$

The theorem also holds if $G(H)$ and $\Xi(H)$ are replaced by $G(A)$, respectively $\Xi(A)$.

We show first that Γ is a left inverse to right composition with D. The following lemma has been obtained in [29] in the setting of noncommutative symmetric functions and quasi-determinants. We include the proof, which is elementary.

Lemma 4.1. *For $n \geq 1$ we have:*

$$I_n = \sum_{\substack{k_1,\dots,k_l \in \mathbb{N}^* \\ k_1+\cdots+k_l=n}} \frac{D_{k_1} * \cdots * D_{k_l}}{k_1(k_1+k_2)\dots(k_1+\cdots+k_l)}.$$

Proof. As already remarked, the definition of D implies $I * D = I * S * Y = Y$. In particular, since $D_0 = 0$:

$$Y_n = nI_n = (I * D)_n = \sum_{i=1}^{n} I_{n-i} * D_i.$$

Inserting recursively the value of I_i in the right member of the identity, we obtain:

$$\begin{aligned} I_n &= \frac{D_n}{n} + \sum_{i=1}^{n-1} \sum_{1 \leq j \leq n-i} \frac{I_{n-i-j} * D_j * D_i}{(n-i)n} \\ &= \frac{D_n}{n} + \sum_{\substack{j+i=n \\ i,j \neq 0}} \frac{D_j * D_i}{j \cdot n} + \sum_{i=1}^{n-1} \sum_{1 \leq j \leq n-i-1} \frac{I_{n-i-j} * D_j * D_i}{(n-i)n} \\ &= \sum_{\substack{k_1,\dots,k_l \in \mathbb{N}^* \\ k_1+\cdots+k_l=n}} \frac{D_{k_1} * \cdots * D_{k_l}}{k_1(k_1+k_2)\dots(k_1+\cdots+k_l)}. \end{aligned}$$

□

Now we compute $\gamma = \gamma \circ I$, where I is expanded according to the previous lemma, yielding:

$$\begin{aligned} \gamma &= e + \gamma \circ \left\{ \sum_{n \in \mathbb{N}^*} \sum_{\substack{k_1,\dots,k_l \in \mathbb{N}^* \\ k_1+\cdots+k_l=n}} \frac{D_{k_1} * \cdots * D_{k_l}}{k_1(k_1+k_2)\dots(k_1+\cdots+k_l)} \right\} \\ &= e + \sum_{n \in \mathbb{N}^*} \sum_{\substack{k_1,\dots,k_l \in \mathbb{N}^* \\ k_1+\cdots+k_l=n}} \frac{\gamma \circ D_{k_1} * \cdots * \gamma \circ D_{k_l}}{k_1(k_1+k_2)\dots(k_1+\cdots+k_l)}. \end{aligned}$$

As D preserves the grading, it follows that Γ is a left inverse to the right composition with D.

Similar calculations help to prove that Γ is character-valued, that is, is actually a map from $\Xi(H)$ to $G(H)$. Indeed, let α be any infinitesimal character in $\Xi(H)$. Then we have in $\mathrm{Char}(H)$:

$$\Delta(\Gamma(\alpha)) = e \otimes e + \sum_{n>0} \sum_{k_1,\dots,k_n \in \mathbb{N}^*} \frac{\Delta(\alpha_{k_1} \,\overline{*}\, \cdots \,\overline{*}\, \alpha_{k_n})}{k_1(k_1+k_2)\dots(k_1+\cdots+k_n)}$$

$$= \quad e\otimes e+\sum_{n>0}\sum_{k_1,\dots,k_n\in\mathbb{N}^*}\sum_{\substack{I\sqcup J=\{k_1,\dots,k_n\}\\ |I|=l,|J|=p}}\frac{(\alpha_{i_1}\,\overline{*}\cdots\overline{*}\,\alpha_{i_l})\otimes(\alpha_{j_1}\,\overline{*}\cdots\overline{*}\,\alpha_{j_p})}{k_1(k_1+k_2)\dots(k_1+\cdots+k_n)},$$

where $I=\{i_1,\dots,i_l\}$, $J=\{j_1,\dots,j_p\}$ and we have used that the α_{i_m} are all infinitesimal characters. Particularly, the assertion we intend to prove, that is $\Delta(\Gamma(\alpha))=\Gamma(\alpha)\otimes\Gamma(\alpha)$, amounts to the equality:

$$\left(e+\sum_{n>0}\sum_{k_1,\dots,k_n\in\mathbb{N}^*}\frac{\alpha_{k_1}\,\overline{*}\cdots\overline{*}\,\alpha_{k_n}}{k_1(k_1+k_2)\dots(k_1+\cdots+k_n)}\right)^{\otimes 2}$$
$$= \quad e\otimes e+\sum_{n>0}\sum_{k_1,\dots,k_n\in\mathbb{N}^*}\sum_{\substack{I\sqcup J=\{k_1,\dots,k_n\}\\ |I|=l,|J|=p}}\frac{(\alpha_{i_1}\,\overline{*}\cdots\overline{*}\alpha_{i_l})\otimes(\alpha_{j_1}\,\overline{*}\cdots\overline{*}\,\alpha_{j_p})}{k_1(k_1+k_2)\dots(k_1+\cdots+k_n)}.$$

This follows from the identity:

$$\sum_{I\sqcup J=K}\frac{1}{k_1(k_1+k_2)\dots(k_1+\cdots+k_{p+l})}$$
$$= \sum_{\substack{i_1,\dots,i_l\in\mathbb{N}^*\\ j_1,\dots,j_p\in\mathbb{N}^*}}\frac{1}{i_1(i_1+i_2)\dots(i_1+\cdots+i_l)}\cdot\frac{1}{j_1(j_1+j_2)\dots(j_1+\cdots+j_p)}, \tag{2}$$

where K runs over all sequences $(k_1,...,k_{p+l})$ obtained by shuffling the sequences I and J. In turn, the identity follows if the equation $\Delta(\Gamma(\alpha))=\Gamma(\alpha)\otimes\Gamma(\alpha)$ (that is, $\Gamma(\alpha)\in G(H)$) holds for a particular choice of H and α such that the α_i form a family of algebraically independent generators of a subalgebra of the convolution algebra $\mathrm{End}(H)$ —which are therefore also algebraically independent as elements of $\mathrm{Char}(H)$.

So, let us consider the particular case $H=T^*(X)$ where H is the graded dual of the enveloping algebra of the FLA over an infinite alphabet X and $\alpha=D$ is the Dynkin operator. Then, we already know that $\Gamma(D)=I$, due to the previous lemma, so that $\Gamma(D)$ is group-like in $\mathrm{Char}(T^*(X))$. On the other hand, as is well known, the graded components of the Dynkin operator are algebraically independent in the convolution algebra $\mathrm{End}(T^*(X))$, and the identity follows. (For further details on the algebraic independence of the graded components of the Dynkin operator and, more generally, of the classical Lie idempotents, consult [29, 44].)

Although we have preferred to give a conceptual proof based on FLAs, let us mention that identity (2) is elementary, and well known in boson Fock space theory. It also follows e.g. from the shuffle identity for the product of two iterated integrals [45]:

$$\sum_{I\sqcup J=K}\int_0^1 x_{p+l}^{k_{p+l}-1}\dots\int_0^{x_2}x_1^{k_1-1}dx_1\dots dx_{p+l}$$
$$= \quad\int_0^1 x_l^{i_l-1}\dots\int_0^{x_2}x_1^{i_1-1}dx_1\dots dx_l\cdot\int_0^1 y_p^{j_p-1}\dots\int_0^{y_2}y_1^{j_1-1}dy_1\dots dy_p,$$

which generalizes the integration by parts rule.

To conclude the proof of Theorem 4.1 we show that Γ is also a right inverse to the composition with D. We contend that, for any h in the augmentation ideal of H and arbitrary $\alpha\in\Xi(H)$, the following relation holds:

$$\alpha(h)=\Gamma(\alpha)^{-1}*Y\Gamma(\alpha)\,(h)\quad\text{or, equivalently,}\quad Y\Gamma(\alpha)(h)=\Gamma(\alpha)*\alpha\,(h).$$

Indeed, we have:

$$\begin{aligned} Y\Gamma(\alpha)(h) &:= |h| \sum_{\substack{k_1,\dots,k_l\in\mathbb{N}^* \\ k_1+\cdots+k_l=|h|}} \frac{\alpha_{k_1} * \cdots * \alpha_{k_l}}{k_1(k_1+k_2)\dots(k_1+\cdots+k_l)}(h) \\ &= \sum_{\substack{k_1,\dots,k_l\in\mathbb{N}^* \\ k_1+\cdots+k_l=|h|}} \frac{\alpha_{k_1} * \cdots * \alpha_{k_{l-1}}}{k_1(k_1+k_2)\dots(k_1+\cdots+k_{l-1})} * \alpha_{k_l}(h) \\ &= \Gamma(\alpha) * \alpha(h). \end{aligned}$$

This together with the fact that $\Gamma(\alpha) \in G(H)$ for $\alpha \in \Xi(H)$ implies:

$$\Gamma(\alpha) \circ D = \Gamma(\alpha)^{-1} * Y\Gamma(\alpha) = \alpha.$$

Our task is over. □

When H is both commutative and cocommutative the convolution product is commutative and $\gamma \circ D = Y \log \gamma := \log(\gamma) \circ Y$. In particular in this case $D = Y \log I$. This was put to good use in [5]. Thus clearly D, in the general case, is a noncommutative logarithmic derivative; and the inverse Dynkin operator Γ a extremely powerful tool. We finally remark that $Y * S$, corresponding, in the free cocommutative case and as an operator from the tensor algebra to the FLA, to the right-to-left iteration of bracketings, is another possible form for the noncommutative logarithmic derivative, leading in particular to $\gamma \circ (Y * S) = Y\gamma * \gamma^{-1}$. More generally, any interpolation of the form $S^a * Y * S^b$, with $a + b = 1$, yields a notion of noncommutative logarithmic derivative.

5. Algebraic BWH decomposition of characters

In this section we summarize previous research on Rota–Baxter operators, relevant for our purpose. Let H be still graded, connected and commutative and let again A stand for a commutative unital algebra. Assume the algebra A to split directly, $A = A_+ \oplus A_-$, into the subalgebras A_+, A_- with $1_A \in A_+$. We denote the projectors to $A_\pm$ by $R_\pm$, respectively. The pair (A, R_-) is a special case of a (weight one) Rota–Baxter algebra [21] since R_-, and similarly R_+, satisfies the relation:

$$R_-(x) \cdot R_-(y) + R_-(x \cdot y) = R_-\big(R_-(x) \cdot y + x \cdot R_-(y)\big), \qquad x, y \in A. \tag{3}$$

The reader may verify that integration by parts rule is just the weight zero Rota–Baxter identity, that is, the second term on left hand side of (3) is absent. One easily shows that $\mathrm{Lin}(H, A)$ with the idempotent operator $\mathcal{R}_-$ defined by $\mathcal{R}_-(f) = R_- \circ f$, for $f \in \mathrm{Lin}(H, A)$, is a (in general not commutative) unital Rota–Baxter algebra [21].

The subspace $L^{(1)}$ of $\mathrm{Lin}(H, A)$ made of linear maps that send the Hopf algebra unit to zero forms a Lie algebra with $\Xi(A) \subset L^{(1)}$ as a Lie subalgebra. To $L^{(1)}$ does correspond the group $G_0 = e + L^{(1)} = \exp(L^{(1)})$ of linear maps sending the Hopf algebra unit to the algebra unit. It contains the group of characters $G(A)$ as the subgroup $\exp(\Xi(A))$. Due to the characterization of infinitesimal characters as maps that vanish on the square of the augmentation ideal of H, both $\mathcal{R}_+(\Xi(A))$ and $\mathcal{R}_-(\Xi(A))$ embed into $\Xi(A)$. In particular, both are Lie subalgebras of $\Xi(A)$.

The Lie algebra decomposition, $\Xi(A) = \mathcal{R}_+(\Xi(A)) \oplus \mathcal{R}_-(\Xi(A))$ lifts to the group of characters $G(A)$ as follows. Recall the Baker–Campbell–Hausdorff (BCH) formula [44] for the product of two exponentials, that we write:

$$\exp(x)\exp(y) = \exp\big(x + y + \mathrm{BCH}(x, y)\big).$$

In [21, 22] the following non-linear map was defined, whose properties where further explored in [25]. See also [39]. For $f \in L^{(1)}$, define $\chi^{\mathcal{R}_-}(f) = \lim_{n\to\infty} \chi_n^{\mathcal{R}_-}(f)$ where $\chi_n^{\mathcal{R}_-}(f)$ is given

by what we call the BCH recursion:

$$\begin{aligned}\chi_0^{\mathcal{R}_-}(f) &= f, \qquad \chi_1^{\mathcal{R}_-}(f) = f - \mathrm{BCH}\big(\mathcal{R}_-(f), \mathcal{R}_+(f)\big), \ \dots\ , \\ \chi_{n+1}^{\mathcal{R}_-}(f) &= f - \mathrm{BCH}\Big(\mathcal{R}_-\big(\chi_n^{\mathcal{R}_-}(f)\big), \mathcal{R}_+\big(\chi_n^{\mathcal{R}_-}(f)\big)\Big).\end{aligned}$$

Then the fixed-point map $\chi^{\mathcal{R}_-} : L^{(1)} \to L^{(1)}$ satisfies:

$$\chi^{\mathcal{R}_-}(f) = f - \mathrm{BCH}\Big(\mathcal{R}_-\big(\chi^{\mathcal{R}_-}(f)\big), \mathcal{R}_+\big(\chi^{\mathcal{R}_-}(f)\big)\Big). \tag{4}$$

The superscript $\mathcal{R}_-$ is justified by the dependency of the limit on the Rota–Baxter operator, and the following result.

Lemma 5.1. *The map $\chi^{\mathcal{R}_-}$ in (4) solves the simpler recursion:*

$$\chi^{\mathcal{R}_-}(f) = f + \mathrm{BCH}\Big(-\mathcal{R}_-\big(\chi^{\mathcal{R}_-}(f)\big), f\Big), \quad f \in L^{(1)}.$$

Following [25], the following factorization theorem holds.

Theorem 5.1. *For any $f \in L^{(1)}$, we have the unique factorization:*

$$\exp(f) = \exp\Big(\mathcal{R}_-\big(\chi^{\mathcal{R}_-}(f)\big)\Big) * \exp\Big(\mathcal{R}_+\big(\chi^{\mathcal{R}_-}(f)\big)\Big).$$

Uniqueness of the factorization follows from R_- being idempotent. In the particular case that $f \in \Xi(A)$, the BCH recursion (4) takes place inside the Lie algebra $\Xi(A)$, and the decomposition of $\exp(f)$ holds therefore inside the group of characters $G(A)$. In particular, it follows ultimately from Theorem 5.1 that $G(A)$ decomposes as a product

$$G(A) = G_-(A) * G_+(A), \quad \text{where} \quad G_-(A) = \exp(\mathcal{R}_-(\Xi(A))),\ G_+(A) = \exp(\mathcal{R}_+(\Xi(A))).$$

Corollary 5.1. *For any $\gamma = \exp(\alpha) \in G(A)$, with $\alpha \in \Xi(A)$, we have unique $\alpha_\pm := \mathcal{R}_\pm(\chi^{\mathcal{R}_-}(\alpha)) \in \mathcal{R}_\pm(\Xi(A))$, and unique characters $\gamma_\pm := \exp(\pm\alpha_\pm) \in G_\pm(A)$ such that:*

$$\gamma = \gamma_-^{-1} * \gamma_+. \tag{5}$$

Using again the Rota–Baxter relation (3) the BWH decomposition of Connes and Kreimer, originally found by a more geometrical method [16], is recovered [21, 24, 25]:

Lemma 5.2. *For any $\gamma = \exp(\alpha)$ the unique characters $\gamma_\pm := \exp(\pm\alpha_\pm) \in G_\pm(A)$ in the previous corollary solve the equations:*

$$\gamma_\pm = e \pm \mathcal{R}_\pm(\gamma_- * (\gamma - e)). \tag{6}$$

Proof. There is a sharpened version [22] of Atkinson's theorem [1], stating that the characters $\gamma, \gamma_\pm$ of (5) verify $\gamma_- = e - \mathcal{R}_-(\gamma_- * (\gamma - e))$ and $\gamma_+ = e - \mathcal{R}_+(\gamma_+ * (\gamma^{-1} - e))$. Now:

$$\gamma_+ * (\gamma^{-1} - e) = \gamma_- * \gamma * (\gamma^{-1} - e) = \gamma_- * (e - \gamma)$$

gives (6). □

6. On renormalization procedures

Prima facie in pQFT, to a Feynman graph F does correspond by the Feynman rules a multiple $\mathcal{D}$-dimensional momentum space integral. Each independent loop in a diagram yields one integration:

$$F \mapsto J_F(p) = \Big[\int \prod_{l=1}^{|F|} d^{\mathcal{D}}k_l\Big] I_F(p,k). \tag{7}$$

Here $|F|$ is the number of loops, $k = (k_1, \dots, k_{|F|})$ are the $|F|$ independent internal (loop) momenta and $p = (p_1, \dots, p_N)$, with $\sum_{k=1}^N p_k = 0$, denote the N external momenta. In

the most relevant kind ($\mathcal{D} = 4$, renormalizable with dimensionless couplings) of field theories, these integrals suffer from ultraviolet (UV) divergencies. Concretely, under a scale transformation, the integrand behaves as

$$\Big[\prod_{l=1}^{|F|} d^{\mathcal{D}}(\lambda k_l)\Big] I_F(\lambda p, \lambda k) \sim \lambda^{d(F)},$$

with $d(F)$ the superficial UV degree of divergence of the graph F. Power-counting renormalizable theories are such that all interaction terms in the Lagrangian are of dimension $\leq \mathcal{D}$; then $d(F)$ is bounded by a number independent of the order of the graph. For instance in the φ^4_4 model the superficial UV degree of divergence of a graph with N external legs is:

$$d(F) = 4 - N.$$

The Weinberg–Zimmermann theorem says: "provided all free propagators have nonzero masses, the integral associated to the Feynman graph F is absolutely convergent if its superficial UV degree of divergence and that of each of its one-particle irreducible (1PI) subgraphs is strictly negative". The BPHZ momentum space subtraction method is rooted in this assertion: the idea is to redefine the integrand $I_F(p, k)$ of a divergent integral by subtracting from it the first terms of its Taylor expansion in the external momenta p, after these subtractions have been performed on the integrands corresponding to the 1PI subgraphs of F that are renormalization parts; in such a way the UV degrees of the integral and its subintegrals are lowered until they become all negative. The combinatorics of those subgraph subtractions leads to Bogoliubov's recursive $\bar{R}$-operation and Zimmermann's forest formula; we assume the reader acquainted with the former at least [11, 13, 46, 49].

Less straightforward conceptually, but way more practical, is the DR method. This belongs to the class of regularization prescriptions, which parameterize the divergencies appearing in J_F upon introducing non-physical parameters, thereby rendering them formally finite. In DR [48] one introduces a complex parameter $\varepsilon \in \mathbb{C}$ by changing the integral measure, that is, the space-time dimension, to $\mathrm{D} \in \mathbb{C}$:

$$d^{\mathcal{D}}k \xrightarrow{\dim\,\mathrm{reg}} \mu^{\varepsilon}\, d^{\mathrm{D}}k,$$

where $\varepsilon = (\mathcal{D} - \mathrm{D})$. Henceforth always $\mathcal{D} = 4$. The arbitrary parameter $\mu \neq 0$ ('t Hooft's 'unit-mass' parameter) is introduced for dimensional reasons. Take the φ^4_4 model: in DR, if we wrote the interaction term simply in the form $g\varphi^4/4!$, then the (naive) dimension of g would be $[g] = \mu^{\varepsilon}$. The redefinition $\tilde{g}\mu^{\varepsilon}\varphi^4/4!$ of the original vertex in the Lagrangian includes the mass parameter μ, introduced to make $\tilde{g}$ dimensionless. Now, in any given theory there is a rigid relation between the numbers of loops and vertices, for each given N-point function. For instance in the φ^4_4 model, for graphs associated to the 2-point function the number of vertices is just equal to the number of loops. For graphs associated to the 4-point function the number of vertices is equal to the number of loops plus one, and so an extra $\tilde{g}\mu^{\varepsilon}$ factor comes in; but, because we are correcting the vertex, this extra factor becomes the coupling constant, and is not expanded in powers of the regularization parameter ε; only the expression multiplying it contributes to the renormalization constant Z_g —with a pole term independent of the mass scale μ. The outcome is that in practice one computes as many μ^{ε} factors as loops:

$$F \longrightarrow J_F^{(\varepsilon,\mu)}(p) = \mu^{|F|\varepsilon}\Big[\int \prod_{l=1}^{|F|} d^{\mathrm{D}}k_l\Big]\; I_F(p, k). \tag{8}$$

See the discussion in [16, 17] and [32, Sections 7 and 8] as well. The point is important from the algebraic viewpoint, since it makes the grading of Feynman graphs by the number of loops immediately relevant to the renormalization process.

The next step in DR consists of a specific subtraction rule of those ε-parameterized expressions which allows to take the limit $\varepsilon \downarrow 0$. Now, Connes and Kreimer's BWH decomposition of Feynman rules [16] is extraordinarily well adapted to DR in pQFT. In the Connes–Kreimer paradigm, any renormalizable quantum field theory gives rise to a Hopf algebra H of Feynman graphs, polynomially generated by 1PI Feynman graphs and graded by graph loop number. The coproduct operation of H mirrors the combinatorics of the subgraphs. Looking back to (7), the unrenormalized Feynman integral does define a character because:

$$I_{F_1 \cup F_2}(p_1, p_2, k_1, k_2) = I_{F_1}(p_1, k_1) I_{F_2}(p_2, k_2)$$

for disjoint graphs F_1, F_2. On the Hopf algebra H the Feynman rules single out a distinguished character γ with values in a suitable target algebra of regularized amplitudes. Precisely, Connes and Kreimer establish the above decomposition $G(A) = G_-(A) * G_+(A)$, for A the algebra of amplitude-valued Laurent series in the dimension defect ε, using the minimal subtraction (MS) scheme in DR on momentum space. The characters $\gamma_\pm$ in the decomposition (5) solve Bogoliubov's renormalization formulae —see Corollary 6.1 below— and may be called the renormalized and counterterm character, respectively. The sought after result is given by the 'positive' part in (5). To wit, $\gamma_+^{(\varepsilon,\mu)} = \gamma_-^{(\varepsilon,\mu)} * \gamma^{(\varepsilon,\mu)}$, and the limit $\gamma_+^{(\varepsilon\downarrow 0,\mu)}$ exists, giving the renormalized Feynman amplitude. In waht follows, when dealing with dimensionally regularized characters we drop the superscript ε. We also forget about the other involved variables, that do not enter our considerations, and just write $\mathbb{C}[[\varepsilon, \varepsilon^{-1}]$ for A. Thus R_- will be the projection onto the subalgebra $A_- := \varepsilon^{-1}\mathbb{C}[\varepsilon^{-1}]$. In summary, A encodes the Feynman rules within the chosen regularization procedure, whereas the splitting of A, hence the projector R_-, reflects a renormalization scheme within that choice.

Corollary 6.1. *The map* $\bar{\gamma} := \gamma_- * (\gamma - e) = \gamma_+ - \gamma_-$ *in* (6) *gives Bogoliubov's preparation map* $\bar{R}$.

Indeed with Connes–Kreimer's definition of the Hopf algebra of Feynman graphs, equations (6) coincide with Bogoliubov's recursive formulas for the counterterm and renormalized parts.

7. Locality and Connes–Kreimer's beta function

The results in Sections 3 to 5 apply to any graded connected commutative bialgebra H and any commutative unital algebra A with a direct splitting. In this section we restrict our consideration to the class of Hopf algebra characters possessing a *locality* property, with H as before. This will correspond to the example given by the Feynman rules for a renormalizable pQFT in DR, using the framework of the MS and the $\overline{\text{MS}}$ [13, Section 5.11.2] schemes. There locality comes from the dependency of DR on the arbitrary 'mass parameter' μ. It is handy to provisionally fix the value of μ at some convenient reference point μ_0. The difference between both schemes boils down to:

$$\mu_0 = \overline{\mu}_0 \frac{e^{\gamma_E/2}}{2\sqrt{\pi}}, \tag{9}$$

with $\mu_0, \overline{\mu}_0$ respectively the MS, $\overline{\text{MS}}$ values and γ_E the Euler–Mascheroni constant. Our aim is to recover by our methods the results in [17, 18]; the latter constitute a stronger, algebraic version of a famous theorem by 't Hooft [48]. Place of pride corresponds to Connes–Kreimer's abstract beta function. This is a conceptually very powerful beast, giving rise to

the ordinary beta function through the (tangent morphism to) the morphism (of unipotent groups) from $G(\mathbb{C})$ to the group of transformations of the coupling constants [17].

Any $f \in \mathrm{Lin}(H, A)$ is now of the form:

$$f(h) = \sum_{k=-U}^{\infty} f_{:k}(h)\varepsilon^k =: f(h;\varepsilon)$$

for $h \in H$. Here every $f_{:k} \in \mathrm{Lin}(H, \mathbb{C})$, the dual of H; and the last notation is used when we want to emphasize the dependency on the DR parameter ε. If h is a $|h|$-loop 1PI Feynman graph, a general theorem [46] in DR says that $U = |h|$ at non-exceptional momenta.

We define on the group of A-valued characters $G(A)$ a one-parameter action of $\mathbb{C}^* \ni t$ given for h homogeneous in H by:

$$\psi^t(h;\varepsilon) := t^{\varepsilon|h|}\psi(h;\varepsilon). \tag{10}$$

Physically this amounts to neatly replacing the $\mu_0^{\varepsilon|h|}$ factor present in each dimensionally regularized Feynman diagram (8) by $(\mu_0 t)^{\varepsilon|h|}$; that is, the mass scale is changed from μ_0 to $t\mu_0$ —or from $\overline{\mu}_0$ to $t\overline{\mu}_0$, as the case might be. As ε is a complex variable, there is no harm in taking t complex, too. When dealing with dimensionally regularized characters we often omit the ε in our notation.

It is clear that ψ^t in (10) is still a character, and that $(\psi_1 * \psi_2)^t = \psi_1^t * \psi_2^t$. This last property also holds if ψ_1, ψ_2 in (10) belong more generally to $\mathrm{Lin}(H, A)$. Besides, for future use we store :

$$t\frac{\partial}{\partial t}\psi^t = \varepsilon|h|\psi^t(h;\varepsilon) = \varepsilon\, Y\psi^t \quad \text{ such that } \quad t\frac{\partial}{\partial t}\Big|_{t=1}\psi^t = \varepsilon\, Y\psi. \tag{11}$$

For any t and any homogeneous $h \in H$ we have $t^{\varepsilon|h|} \in \mathcal{R}_+(A) = A_+ := \mathbb{C}[[\varepsilon]]$, so that the one-parameter action on $G(A)$ restricts to a one-parameter action on the group $G_+(A)$:

$$\psi \in G_+(A) \mapsto \psi^t \in G_+(A).$$

We now have for the regularized, but unrenormalized character $\gamma^t \in G(A)$ a BWH decomposition:

$$\gamma^t = (\gamma^t)_-^{-1} * (\gamma^t)_+.$$

Notice that we write instead $(\gamma_-)^t$ and $(\gamma_+)^t$ for the image of γ_- and γ_+ under the one-parameter group action. The locality property simply states that for the counterterm the following holds.

Theorem 7.1. *Let γ be a dimensionally regularized Feynman rule character. The counterterm character in the BWH decomposition $\gamma^t = (\gamma^t)_-^{-1} * (\gamma^t)_+$ satisfies:*

$$t\,\frac{\partial(\gamma^t)_-}{\partial t} = 0 \quad \textit{or } (\gamma^t)_- \textit{ is equal to } \gamma_-\textit{, i.e. independent of } t. \tag{12}$$

We say the A-valued characters $\gamma \in G(A)$ with this property are local characters.

The physical reason for this is that the counterterms can be taken mass-independent; this goes back at least to [12]. For this fact, and more details on the physical significance of the locality property in pQFT, we refer the reader to [2, 13, 16, 18].

In the sequel, albeit mustering Dynkin technology, we partly follow the plan in reference [39]. Denote by $G^{\mathrm{loc}}(A)$ the subset of local elements of $G(A)$ and $G_-^{\mathrm{loc}}(A)$ the subset of elements in $G^{\mathrm{loc}}(A)$ for which $\psi(h) \in A_-$ when h has no scalar part.

Proposition 7.1. *The set $G^{\mathrm{loc}}(A)$ decomposes into the product $G_-^{\mathrm{loc}}(A) * G_+(A)$.*

Proof. Notice first that $G_+(A)$ embeds in $G^{\rm loc}(A)$, since $\psi^t \in G_+(A)$ for any $\psi \in G_+(A)$. Next, if ϕ is local and $\rho \in G_+(A)$, then $\phi * \rho$ is local. Indeed, we have:

$$\phi^t * \rho^t = (\phi^t)_-^{-1} * (\phi^t)_+ * \rho^t,$$

with polar part the one of ϕ^t, which is constant since ϕ is local. Let ϕ still be local. Then $\phi * \phi_+^{-1} = \phi_-^{-1}$, and the proposition follows if we can show $\phi_-^{-1} \in G_-^{\rm loc}(A)$. Now, if ϕ is local, we have:

$$(\phi_-^{-1})^t * (\phi_+)^t = \phi^t = (\phi^t)_-^{-1} * (\phi^t)_+;$$

so that the BWH decomposition of $(\phi_-^{-1})^t$ is given by:

$$(\phi_-^{-1})^t = (\phi^t)_-^{-1} * (\phi^t)_+ * ((\phi_+)^t)^{-1}, \tag{13}$$

with polar part $(\phi^t)_-$, the one of ϕ, constant and equal to ϕ_-. □

Now we wish to complete the study of locality by understanding how the decomposition of $G^{\rm loc}(A)$, which is a refinement of the decomposition $G(A) = G_-(A) * G_+(A)$, is reflected at the Lie algebra level. More precisely, we would like to know if there is a subspace of $\mathcal{R}_-(\Xi(A))$ naturally associated to $G_-^{\rm loc}(A)$. The answer (quite surprising at first sight) is simple and well known to physicists: beta functions are enough. An excellent tool to understand this is the Dynkin operator pair —as shown below. Let now $\beta \in \Xi(\mathbb{C})$ be a *scalar*-valued infinitesimal character. Notice that β/ε can be regarded as an element of $\mathcal{R}_-(\Xi(A))$.

Proposition 7.2. *With* Γ *as defined in Eq. (1) of Theorem 4.1, we find:*

$$\psi_\beta \;:=\; \Gamma(\beta/\varepsilon) \in G_-^{\rm loc}(A).$$

Proof. From Eq. (1) it is clear that :

$$\psi_\beta = \Gamma(\beta/\varepsilon) = \sum_n \left(\sum_{k_1,\ldots,k_n \in \mathbb{N}^*} \frac{\beta_{k_1} * \cdots * \beta_{k_n}}{k_1(k_1+k_2)\ldots(k_1+\cdots+k_n)} \right) \frac{1}{\varepsilon^n} \tag{14}$$

implying $\psi_\beta \in G_-(A)$. Next we observe $\psi_\beta^t = \Gamma(\beta^t/\varepsilon)$, which follows simply from (14) and, on use of (11):

$$(\psi_\beta^t)^{-1} * t\frac{\partial}{\partial t}\psi_\beta^t = \varepsilon(\psi_\beta^t)^{-1} * Y\psi_\beta^t = \varepsilon\psi_\beta^t \circ D = \varepsilon\Gamma(\beta^t/\varepsilon) \circ D = \beta^t.$$

Now, the BWH decomposition $\psi_\beta^t = (\psi_\beta^t)_-^{-1} * (\psi_\beta^t)_+$ is such that :

$$\begin{aligned}(\psi_\beta^t)^{-1} * t\frac{\partial}{\partial t}\psi_\beta^t &= (\psi_\beta^t)^{-1} * t\frac{\partial}{\partial t}(\psi_\beta^t)_-^{-1} * (\psi_\beta^t)_+ + (\psi_\beta^t)^{-1} * (\psi_\beta^t)_-^{-1} * t\frac{\partial}{\partial t}(\psi_\beta^t)_+ \\ &= (\psi_\beta^t)_+^{-1} * (\psi_\beta^t)_- * t\frac{\partial}{\partial t}(\psi_\beta^t)_-^{-1} * (\psi_\beta^t)_+ + (\psi_\beta^t)_+^{-1} * t\frac{\partial}{\partial t}(\psi_\beta^t)_+;\end{aligned}$$

hence we find:

$$(\psi_\beta^t)_+ * \beta^t * (\psi_\beta^t)_+^{-1} \;=\; (\psi_\beta^t)_- * t\frac{\partial}{\partial t}(\psi_\beta^t)_-^{-1} + t\frac{\partial}{\partial t}(\psi_\beta^t)_+ * (\psi_\beta^t)_+^{-1}.$$

Using $\psi_\beta \in G_-(A)$ and that β^t takes values in A_+, we find by applying the projector $\mathcal{R}_-$ on both sides of the last equation :

$$\mathcal{R}_-\Big((\psi_\beta^t)_+ * \beta^t * (\psi_\beta^t)_+^{-1}\Big) = 0 = (\psi_\beta^t)_- * t\frac{\partial}{\partial t}(\psi_\beta^t)_-^{-1} = -t\frac{\partial}{\partial t}(\psi_\beta^t)_- * (\psi_\beta^t)_-^{-1}$$

implying that $(\psi_\beta^t)_-$ is independent of t; thus $\Gamma(\beta/\varepsilon)$ is a local character. □

The last proposition is suggestive of the fact that local A-valued characters are essentially determined by *ordinary* (scalar-valued) infinitesimal characters (beta functions). This is indeed the case. Before proving the main result of this section, we remark that, for any $\phi \in G^{\mathrm{loc}}(A)$, we can write :

$$\phi^t = \phi_-^{-1} * (\phi^t)_+ = \phi * h_\phi^t, \tag{15}$$

where $h_\phi^t := (\phi_+)^{-1} * (\phi^t)_+ \in G_+(A)$. Also, for $\phi \in G_-^{\mathrm{loc}}(A)$ we denote $\phi_{:-1}$ by $\operatorname{Res}\phi$.

Theorem 7.2. *The map $\phi \mapsto \varepsilon(\phi \circ D)$, with D the Dynkin operator, sends $G^{\mathrm{loc}}(A)$ to $\Xi(A_+)$ and $G_-^{\mathrm{loc}}(A)$ to $\Xi(\mathbb{C})$; explicitly, in the second case:*

$$G_-^{\mathrm{loc}}(A) \ni \phi \mapsto \varepsilon(\phi \circ D) = Y \operatorname{Res}\phi \in \Xi(\mathbb{C}).$$

Proof. First, recall from Corollary 4.2 that D sends characters to infinitesimal characters and that by Proposition 7.1 any local $\phi \in G^{\mathrm{loc}}(A)$ decomposes as $\phi^a * \phi^b$ with $\phi^a \in G_-^{\mathrm{loc}}(A), \phi^b \in G_+(A)$. Therefore, we see that:

$$\phi \circ D = \phi^{-1} * Y\phi = (\phi^b)^{-1} * (\phi^a)^{-1} * Y\phi^a * \phi^b + (\phi^b)^{-1} * Y\phi^b;$$

since $(\phi^b)^{-1}$ and $Y\phi^b$ belong to $\mathrm{Lin}(H, A_+)$, the theorem follows if we can prove that:

$$\epsilon(\phi \circ D) = \epsilon(\phi^{-1} * Y\phi) \in \Xi(\mathbb{C})$$

when $\phi \in G_-^{\mathrm{loc}}(A)$. Assume the latter is the case and recall the decomposition $\phi^t = \phi * h_\phi^t$ in (15), with now simply $h_\phi^t = (\phi^t)_+$. So that on the one hand (11) implies :

$$t\frac{\partial}{\partial t}\Big|_{t=1} h_\phi^t = \phi^{-1} * t\frac{\partial}{\partial t}\Big|_{t=1} \phi^t = \phi^{-1} * \varepsilon\, Y\phi = \varepsilon(\phi \circ D).$$

On the other hand, observe that by using the Bogoliubov formula (6) one finds:

$$\begin{aligned} t\frac{\partial}{\partial t}\Big|_{t=1} h_\phi^t &= t\frac{\partial}{\partial t}\Big|_{t=1} (\phi^t)_+ = t\frac{\partial}{\partial t}\Big|_{t=1} \mathcal{R}_+\big(\phi_- * (\phi^t - e)\big) \\ &= t\frac{\partial}{\partial t}\Big|_{t=1} \mathcal{R}_+\big(\phi^{-1} * \phi^t - \phi^{-1}\big) = t\frac{\partial}{\partial t}\Big|_{t=1} \mathcal{R}_+\big(\phi^{-1} * t^{\varepsilon|\cdot|}\phi\big) \qquad (16) \\ &= \mathcal{R}_+\big(\varepsilon\, Y\phi\big) = Y \operatorname{Res}\phi. \qquad (17) \end{aligned}$$

In the step before the last we took into account that $Y(1_H) = 0$ and that $\phi_- \in G_-^{\mathrm{loc}}(A)$ which implies for $h \in H^+$:

$$\phi^{-1} * t^{\varepsilon|\cdot|}\phi(h) = \phi^{-1}(h) + t^{\varepsilon|h|}\phi(h) + \phi^{-1}(\overline{h}^{(1)}) t^{\varepsilon|\overline{h}^{(2)}|}\phi(\overline{h}^{(2)}),$$

where $h^{(1)} \otimes h^{(2)} = h \otimes 1 + 1 \otimes h + \overline{h}^{(1)} \otimes \overline{h}^{(2)}$. Here $\phi^{-1}(h) \in A_-$ and $\phi^{-1}(\overline{h}^{(1)}) t^{\varepsilon|\overline{h}^{(2)}|}\phi(\overline{h}^{(2)})$ are both mapped to zero by $t\frac{\partial}{\partial t}\Big|_{t=1} \mathcal{R}_+$. □

A glance back to Theorem 4.1 and Proposition 7.2 allows us to conclude that for $\phi \in G_-^{\mathrm{loc}}(A)$ one has:

$$\Gamma\left(\frac{Y \operatorname{Res}\phi}{\varepsilon}\right) = \phi, \tag{18}$$

so indeed the polar part of a local character ϕ can be retrieved from its beta function, $\beta(\phi) := Y \operatorname{Res}\phi \in \Xi(\mathbb{C})$, by the universal formula (1).

Equation (18) is equivalent to the 'scattering type formula' of [17]; we can refer to [39] for this. Now it is an easy task to recover the other results of [17, 18]. We ought to remark that, for γ a general local character, one defines $\operatorname{Res}\gamma = -\operatorname{Res}\gamma_-$ —see in this respect formulae [17, Equation (11)] or [18, Equation (2.111)].

Theorem 7.3. *For the renormalized character* $\gamma_{\rm ren}(t) := (\gamma^t)_+(\varepsilon = 0)$ *it holds:*

$$t\frac{\partial}{\partial t}\gamma_{\rm ren}(t) = (Y\,\mathrm{Res}\,\gamma) * \gamma_{\rm ren}(t), \tag{19}$$

the abstract RG equation.

Proof. First, in the proof of Theorem 7.2 we saw already that D verifies a cocycle condition [26]: for $\phi, \psi \in G(A)$:

$$(\phi * \psi) \circ D = \psi^{-1} * (\phi \circ D) * \psi + \psi \circ D.$$

This together with Theorem 7.2 implies for $\phi \in G_-^{\rm loc}(A)$ that $\mathrm{Res}\,\phi = -\,\mathrm{Res}\,\phi^{-1}$. Indeed, this follows by taking the residue Res on both sides of the equation:

$$0 = (\phi^{-1} * \phi) \circ D = \phi^{-1} * (\phi^{-1} \circ D) * \phi + \phi \circ D = \phi^{-1} * \frac{\mathrm{Res}\,\phi^{-1}}{\varepsilon} * \phi + \frac{\mathrm{Res}\,\phi}{\varepsilon}.$$

Now, let $\gamma \in G^{\rm loc}(A)$ with BWH decomposition $\gamma^t = \gamma_-^{-1} * (\gamma^t)_+$. Recall that $(\gamma^t)_+ = \mathcal{R}_+\big(\gamma_- * t^{\varepsilon|\cdot|}\gamma\big)$ maps H^+ into $A_+ \otimes \mathbb{C}[[\log(t)]]$ such that:

$$t\frac{\partial(\gamma^t)_+}{\partial t}(0) = t\frac{\partial}{\partial t}\gamma_{\rm ren}.$$

As $\gamma_- \in G_-^{\rm loc}(A)$, we then find:

$$\begin{aligned}
t\frac{\partial(\gamma^t)_+}{\partial t} &= \gamma_- * t\frac{\partial}{\partial t}\gamma^t = \gamma_- * \varepsilon Y\gamma^t = \gamma_- * \varepsilon Y(\gamma_-^{-1} * (\gamma^t)_+) \\
&= \gamma_- * \varepsilon Y(\gamma_-^{-1}) * (\gamma^t)_+ + \varepsilon Y(\gamma^t)_+ = \varepsilon(\gamma_-^{-1} \circ D) * (\gamma^t)_+ + \varepsilon Y(\gamma^t)_+ \\
&= (Y\,\mathrm{Res}\,\gamma_-^{-1}) * (\gamma^t)_+ + \varepsilon Y(\gamma^t)_+ = -(Y\,\mathrm{Res}\,\gamma_-) * (\gamma^t)_+ + \varepsilon Y(\gamma^t)_+ \\
&= (Y\,\mathrm{Res}\,\gamma) * (\gamma^t)_+ + \varepsilon Y(\gamma^t)_+.
\end{aligned}$$

Therefore both sides have a limit as $\varepsilon \downarrow 0$, yielding the sought after RG equation (19). □

Equation (19) is solved using the beta function $\beta(\gamma) := Y\,\mathrm{Res}\,\gamma \in \Xi(\mathbb{C})$:

$$\gamma_{\rm ren}(t) = \exp(\ln(t)\beta(\gamma)) * \gamma_{\rm ren}(1).$$

The last statement and equation (13) tell us that:

$$\lim_{\varepsilon\to 0} \gamma_- * (\gamma_-^{-1})^t = \lim_{\varepsilon\to 0} (\gamma^t)_+ * ((\gamma_+)^t)^{-1} = \gamma_{\rm ren}(t) * \gamma_{\rm ren}^{-1}(1) = \exp(\ln(t)\beta(\gamma)).$$

The scalar-valued characters

$$\Omega_t(\gamma) := \exp(\ln(t)\beta(\gamma)) \in G(\mathbb{C})$$

obviously form a one-parameter subgroup in $G(A)$: $\Omega_{t_1}(\gamma) * \Omega_{t_2}(\gamma) = \Omega_{t_1 t_2}(\gamma)$, generated by the beta function and controlling the flow of the renormalized Feynman rule character with respect to the mass scale.

8. Through the prism of other renormalization schemes I

We plan now to prospect the usefulness of our approach in other schemes of renormalization. Doubtless DR provides the cleanest formulation of locality in the BWH decomposition for renormalization. However, it is physically clear that in any scheme one has still to parameterize the arbitrariness in separating divergent from finite parts; and that the physical irrelevance of the choices made still gives rise to the RG equation. On the mathematical side, it is worth to recall that the algebraic BWH decomposition of Section 5 is not necessarily linked to loops on the Riemann sphere. It is thus legitimate to ponder the question in schemes other than those based on DR. We plan to exemplify with the BPHZ scheme in the next section, but, before dwelling on that, let us briefly indicate other pieces of evidence.

A first one on concerns old research. In the early seventies, Callan set out to prove that broken scale invariance [8] is all that renormalization was about. He was eventually able to give a treatment of the RG, and proofs of renormalizability of field theories based on the former, by relying entirely in the BPHZ formalism. To derive the beta function, he just set up RG equations by studying the dependency of the N-point functions on the renormalized mass. See in this respect [3, 9]. In a renormalization method without regularization, information about the RG must be stored somehow in the *renormalized* quantities. Concretely, as hinted at by our last theorem, one finds it in the scaling properties of the renormalized integral. This was noted in the context of Epstein–Glaser renormalization in [30]. In DR this shows in the RG equation (19).

A second piece of evidence is furnished by more recent work by Kreimer and collaborators [2, 6, 35, 37]. Indeed, Kreimer has long argued that locality (and renormalizability) is determined by the Hochschild cohomology of renormalization Hopf algebras. This cohomology is trivial in degree greater than one. The coproduct on H can be written recursively in terms of closed Hochschild 1-cochains. Those are grafting maps indexed by primitive 1PI diagrams, that govern the structure of Feynman graphs and give rise through the Feynman rules to integral Dyson–Schwinger equations. Here is not the place for details and we refer the reader to [2, 6, 34, 35, 37], and especially Kreimer's recent review [36].

In the indicated references the Dynkin operator D (and its close cousins $S * Y^n$) appears, again defining the residue, in renormalization schemes without regularization. There Green's functions, $\Sigma = \Sigma(g, p)$, are defined in terms of their (combinatorial) Dyson–Schwinger equations using the Hochschild 1-cocycles; g, p denote the coupling constant and external momenta, respectively. Those Green's functions are expanded as power series in g:

$$\Sigma = 1 + \sum_{k>0} \phi(c_k) g^k,$$

for Feynman rules $\phi \in G(\mathbb{C})$ and with order by order divergent Feynman amplitudes $\phi(c_k)$ as coefficients. Here the c_k's are particular linear combinations of graphs of loop order k in H [34]. Renormalization of Σ is achieved by using a renormalized character $\phi_{\text{ren}} \in G(\mathbb{C})$ defined by subtraction at a specific point $p^2 = \lambda^2$ —corresponding to Taylor expansion up to zeroth order. Here λ plays the role of the mass-parameter μ. Locality is automatically fulfilled in this approach. The renormalized Green's functions $\Sigma_{\text{ren}} = \Sigma_{\text{ren}}(g, p, \lambda)$ can be developed in terms of the parameter $L := \log(p^2/\lambda^2)$, hence $\Sigma_{\text{ren}} = 1 + \sum_{k>0} \alpha_k(g) L^k$, with $\alpha_1(g) \in \Xi(\mathbb{C})$ [37]. Following the above references and adapting partially to our notation, the residue is found to be:

$$\Xi(\mathbb{C}) \ni \sigma_1 := \frac{\partial}{\partial L}(\phi_{\text{ren}} \circ D)\Big|_{L=0} = \alpha_1(g).$$

In [37] Kreimer and Yeats outline how to derive $\alpha_k(g)$, $k > 1$ recursively from $\alpha_1(g)$. This confirms that, in a deeper sense, the beta function *is* composition with the Dynkin operator. The interaction between the Hopf algebra of characteristic functions of H of this paper and the Hochschild 1-cocycles on H is a promising field of interest.

9. Through the prism of other renormalization schemes II

Let us now explore the classical BPHZ scheme in the context of the former sections. With I_F the integrand of (7) corresponding to the graph F, let us write for the Taylor subtraction employed in BPHZ renormalization:

$$I_F(p,k) \mapsto I_F(p,k) - t_p^{d(F)} I_F(p,k) := I_F(p,k) - \sum_{|\alpha| \leq d(F)} \frac{p^\alpha}{\alpha!} \partial_\alpha I_F(0,k).$$

We borrowed here the standard multi-index notation

$$\alpha = \{\alpha_1, \dots, \alpha_n\} \in \mathbb{N}^n, \quad |\alpha| := \sum_{i=1}^n \alpha_i, \quad \alpha! = \prod_{i=1}^n \alpha_i!\,;$$

each α_i takes values between 0 and 3, say. We are assuming that only massive particles are present, otherwise the subtraction at zero momentum gives rise to infrared divergences; the expression of $t_p^{d(F)} I_F$ in practice simplifies because of Lorentz invariance.

Notice that the integral $J_F(p)$ in (7) *does* originally have a meaning: it is a well defined functional on the linear subspace $\mathcal{S}_{d(F)}(\mathbb{R}^{4N})$ of Schwartz functions ϕ on the external momenta, whose first moments $\int p^\alpha \phi(p)\, d^{4N}p$ up to order $|\alpha| \leq d(F)$ happen to vanish. The "divergent" loop integrals inside $J_F(p)$ become harmless when coupled exclusively with Schwartz functions of this type. The Taylor 'jet' projector map t_p^l subtracts polynomials, that are functionals of the same type, in such a way that the result (eventually) becomes moreover a tempered distribution.

The first question is whether we have a Rota–Baxter algebra in the BPHZ framework. Actually, we do have the Rota–Baxter property for t_p^l. Indeed, the following is obtained by a simple calculation from the definitions.

Proposition 9.1. *Let I_{F_i}, $i = 1, 2$ have associated degrees of divergence l_i, $i = 1, 2$. Then*

$$t_{p_1}^{l_1}\big(I_{F_1}\big)\, t_{p_2}^{l_2}\big(I_{F_2}\big) = t_{p_1,p_2}^{l_1+l_2}\big(I_{F_1}\, t_{p_2}^{l_2}(I_{F_2})\big) + t_{p_1,p_2}^{l_1+l_2}\big(t_{p_1}^{l_1}(I_{F_1})\, I_{F_2}\big) - t_{p_1,p_2}^{l_1+l_2}\big(I_{F_1} I_{F_2}\big).$$

We leave the verification of this to the care of the reader. In general, if U is a multiplicative semigroup, a family of linear operators R_u, $u \in U$ on the algebra A is called a Rota–Baxter family if for any $u, v \in U$ and $a, b \in A$, we have

$$R_u(a)R_v(b) = R_{uv}(aR_v(b)) + R_{uv}(R_u(a)b) - R_{uv}(ab), \qquad \text{for all} \quad a, b \in A.$$

Thus the l-jets define a Rota–Baxter family. Now, a Rota–Baxter family is almost the same thing as a Rota–Baxter operator.

Proposition 9.2. *Let $\mathcal{A} = A[U]$ be the semigroup algebra associated to A. Let $R_u : A \to A$, $u \in U$ be a Rota–Baxter family. Define*

$$R : \mathcal{A} \to \mathcal{A}, \quad \text{by} \quad R\Big(\sum_u a_u u\Big) := \sum_u R_u(a_u)u.$$

Then R is a Rota–Baxter operator on $\mathcal{A}$ such that $R(au) = a'u$ with a' in A. Conversely, if $R : \mathcal{A} \to \mathcal{A}$ is a Rota–Baxter operator such that $R(au) = a'u$ with a' in A, then we obtain a Rota–Baxter family $R_u, u \in U$, by defining $R_u(a) = a'$ where $R(a\,u) = a'\,u$.

The proof is immediate[1]. On the strength of the previous result, we may refer to a Rota–Baxter family as a Rota–Baxter operator. Now, pQFT in practice furnishes an even more radical answer to the question of whether one has here the Rota–Baxter framework. For this is obviously the case when one deals only with logarithmic divergences; and indeed most often only the latter is required. In general, differentiation of an amplitude with respect to an external momentum lowers the overall degree of divergence of a diagram. In DR, the Caswell–Kennedy theorem [11] states that the pole part of any diagram, besides being independent of the scale, is a polynomial in the external momentum. This follows easily from that derivation and the projector R_- commute in DR. But even in the BPHZ scheme $\partial_p t^l = t^{l-1}\partial_p$, and this is enough for the differentiation trick to work.

[1]We thank L. Guo for suggesting the notion of Rota–Baxter family.

Let us then consider the $J_F(p) \in \mathcal{S}'_{d(F)}(\mathbb{R}^{4N})$ of (7). Suppose moreover the multi-loop divergent graph F has all its 1PI subgraphs γ made convergent by application of Bogoliubov's preparation map $\bar{R}$. Then the renormalized integral $J_F^{\mathrm{ren,BPHZ}}(p)$ can be defined as

$$J_F^{\mathrm{ren,BPHZ}}(p) = \Big[\int \prod_{l=1}^{|F|} d^4k_l\Big]\big(I_F(p,k) - t_p^{d(F)}\bar{R}I_F(p,k)\big) =: \Big[\int \prod_{l=1}^{|F|} d^4k_l\Big]R_F(p,k). \tag{20}$$

This recipe is however not unique. We can write as well

$$J_F^{\mathrm{ren,BPHZ}}(p) = P^{d(F)}(p) + \Big[\int \prod_{l=1}^{|F|} d^4k_l\Big]R_F(p,k), \tag{21}$$

with $P^{d(F)}$ a polynomial of order $d(F)$ in the external momenta. This effects a 'finite renormalization', in principle undetermined, that might be put to use to fulfil renormalization prescriptions (again, the form of that polynomial is severely restricted by Lorentz invariance).

We now come to the key point. The coefficients of $P^{d(F)}$ in (21) exhibit the ambiguity of renormalization in the context of the BPHZ scheme. On the face of it, the 'pole terms' $t_p^{d(F)}I_F(p,k)$ do not depend at all on the mentioned coefficients, and thus locality of the BWH decomposition is guaranteed, in a trivial sense. On the other hand, the Galois group approach to renormalization theory [18, 19] stems originally from the idea that ambiguities should be, insofar as possible, handled from a group-theoretic point of view, much as classical Galois theory handles the multiple solutions of polynomial equations. Here however the mentioned form of the ambiguity does not apparently lend itself to RG analysis. We contend, however, that the ambiguity is expressed essentially in the same form as before. The Caswell–Kennedy theorem is suggestive of a direct link between the DR and BPHZ formalisms, and next we endeavour to prove the pertinence of the RG to BPHZ renormalization by the most direct possible route: introducing a mass scale in the latter formalism in direct analogy with the former. Although a direct and full generalization to the other schemes of the FLA methods developed to handle DR seems still out of reach, the following results, together with the previous section and the existence of a Rota–Baxter structure in the BPHZ framework, pave the way for the development of the general algebraic approach to renormalization schemes.

It would be useful to have the ambiguity implicit in the $P^{d(F)}$ in (21) expressed in terms of a mass scale, like in DR. This can be done, in a simple-minded but powerful way, by means of the modified BPHZ scheme proposed in [27]. For instance, it is well known that the famous graph  ('fish' graph) giving the first nontrivial contribution to the vertex correction in the φ_4^4 model in the Euclidean yields the amplitude

$$J_{\mathrm{fish}}^{\mathrm{DR}}(p) = \tilde{g}^2\mu^{2\varepsilon}\int \frac{d^{\mathrm{D}}k}{(2\pi)^4}\,\frac{1}{k^2+m^2}\,\frac{1}{(p+k)^2+m^2},$$

where $p = p_1 + p_2$, say, and that, by use of Feynman's parametrization (see below) and relation (9) one obtains

$$J_{\mathrm{fish}}^{\mathrm{DR}}(p) = g\frac{\tilde{g}}{(4\pi)^2}\left[\frac{2}{\varepsilon} + \int_0^1 dz\,\log\frac{\overline{\mu}^2}{p^2z(1-z)+m^2} + O(\varepsilon)\right].$$

Now, the natural 'zero point' for the mass scale in this problem is clearly m, and we note

$$R_+\big(J_{\mathrm{fish}}^{\mathrm{DR}}(p=0;\overline{\mu}=m)\big) = 0,$$

as $\varepsilon \downarrow 0$. This, together with the mentioned Caswell–Kennedy theorem, feeds the suspicion that the last expression is just the $J_{\mathrm{fish}}^{\mathrm{ren,BPHZ}}(p)$ of (20). The suspicion is correct. The

computation required for the renormalized fish graph in the BPHZ scheme is

$$g^2 \int \frac{d^4k}{(2\pi)^4}(1-t_p^0)\bigg(\frac{1}{k^2+m^2}\,\frac{1}{(p+k)^2+m^2}\bigg). \tag{22}$$

Introduce the Feynman trick, prior to the Taylor subtraction,

$$\begin{aligned} &g^2\int_0^1 dz \int \frac{d^4k}{(2\pi)^4}(1-t_p^0)\,\frac{1}{\big[((p+k)^2+m^2)z+(1-z)(k^2+m^2)\big]^2} \\ &= g^2\int_0^1 dz \int \frac{d^4k}{(2\pi)^4}(1-t_p^0)\frac{1}{[k^2+p^2z(1-z)+m^2]^2}. \end{aligned} \tag{23}$$

The translation $k \to k - zp$, depending on the Feynman parameter, has been made in order to obtain here the same denominator as in DR calculations. With Ω_4 the area of the unit sphere in $\mathbb{R}^4$, the integral (23) now becomes

$$\begin{aligned} &\frac{\Omega_4\, g^2}{(2\pi)^4}\int_0^1 dz \int_0^\infty dk \left[\frac{k^3}{[k^2+p^2z(1-z)+m^2]^2} - \frac{k^3}{[k^2+m^2]^2}\right] \\ &= \frac{g^2}{16\pi^2}\int_0^1 dz\, \log\frac{m^2}{m^2+p^2z(1-z)}. \end{aligned}$$

The last step is to convert the p-independent part in the argument of the logarithm into a mass scale: $m \to \overline{\mu}$. With this, we recover on the nose the DR result, in the $\overline{\text{MS}}$ scheme as it turns out.

Incidentally, as remarked in [43], this procedure allows us to give the exact value of the BPHZ integral (22): the expression $\int_0^1 dz\, \log\big(1+\frac{p^2}{m^2}z(1-z)\big)$ is actually well known in statistical physics, and leads by elementary manipulations involving the golden ratio to

$$J_{\text{fish}}^{\text{ren,BPHZ}}(p) = -\frac{g^2}{16\pi^2}\bigg(\sqrt{1+\frac{4m^2}{p^2}}\,\log\frac{\sqrt{4m^2/p^2}+1}{\sqrt{4m^2/p^2}-1} - 2\bigg).$$

Thus, what we have done above amounts to *identify the constant* term —recall $P^{d(F)}(p)$ in (21). We have the right to add to the previous expression the term $g^2/16\pi^2$ times $\log\big(\overline{\mu}^2/m^2\big)$. We note also that one can recover the residue $g^2/8\pi^2$ here from $J_{\text{fish}}^{\text{ren,BPHZ}}$, as the coefficient of the term logarithmic in the scaling factor,

$$J_{\text{fish}}^{\text{ren,BPHZ}}(\lambda p) \sim J_{\text{fish}}^{\text{ren,BPHZ}}(p) - \frac{g^2}{8\pi^2}\log\lambda,$$

as $\lambda \uparrow \infty$.

Recapitulating, the steps of the modified BPHZ procedure are: (i) Introduction of the Feynman parametrization in $J_F^{\text{BPHZ}}(k,p)$. (ii) Exchange of the integrations. (iii) Translation of the integration variables by λp, with λ dependent on the Feynman parameter. (iv) Taylor subtraction. (v) Integration over loops and replacement of the mass m in the p-constant part of the resulting logarithm by a mass scale. There is nothing to forbid the same operations to be performed on any primitive logarithmically divergent graph of any field theory and then we are optimistic that, by use of skeletal expansions and the integral equations, we would be led to a procedure largely parallel to DR, and so to a brute-force proof that the coefficients of the higher powers of the scaling logarithms in BPHZ renormalization are determined by the residues. To verify this with full particulars, however, would take us too far afield.

10. On Connes–Marcolli's motivic Galois theory

In the Connes-Kreimer picture of renormalization, group theory and the scheme-theoretic equivalent theory of commutative Hopf algebras have become a fundamental tool of pQFT. Connes and Marcolli identified recently [18] a new level at which Hopf algebra structures

enter pQFT. Namely, they constructed an affine group scheme U^*, universal with respect to physical theories, and pointed out its connections with number theory, e.g. to the motivic Galois group of the scheme of 4-cyclotomic integers $\mathbb{Z}[i][\frac{1}{2}]$.

In their work the initial physical problem attacked through the Connes–Kreimer paradigm translates into the classification of equisingular G-valued flat connections on the total space of a principal bundle over an infinitesimal punctured disk (with G the group scheme represented by H). From the representation theoretic point of view, the classification is provided by representations $U^* \longrightarrow G^*$, where U^* is the semi-direct product with the grading of the pro-unipotent group U, the Lie algebra of which is the free graded Lie algebra with one generator e_n in each degree $n > 0$, and similarly for G^*. Returning to the geometrical side of the correspondence and featuring the DR setting that leads to the Riemann–Hilbert picture of renormalization, Connes and Marcolli construct a universal singular frame on principal U-bundles over B. A formal expression for it is given by:

$$\gamma(\varepsilon, v) = \sum_{n\geq 0} \sum_{k_j} \frac{e(k_1)\dots e(k_n)}{k_1(k_1+k_2)\dots(k_1+\dots+k_n)} v^{\sum k_j} \varepsilon^{-n}.$$

It is interesting that the coefficients of the frame are essentially those appearing in the index formula of Connes–Moscovici [14], hinting at the rooting of noncommutative geometry in quantum field theory, which has been Connes' contention for a long while.

As we have already shown, other Hopf algebra structures (or, from the scheme-theoretic point of view, pro-unipotent groups) do appear naturally in pQFT, namely the Hopf algebras $\mathrm{Char}(A)$ of characteristic functions associated to commutative target algebras, e.g., although not exclusively, of quantum amplitudes. These Hopf algebra structures arise naturally from algebraic-combinatorial constructions on Hopf algebras, and therefore do not immediately seem related to the geometrical-arithmetical constructions underlying the definition of the motivic Galois group in [18]. Nevertheless, the formula rendering the universal singular frame in the motivic understanding of renormalization also essentially coincides with our map Γ —the inverse of the Dynkin map. This indicates that the practical consequences for renormalization of the Riemann-Hilbert and/or motivic point of view can be lifted to the setting of FLA theory —which, besides being more familiar to many, is independent of the geometry embedded in the DR scheme. In the remainder of the present section, we would like to explain how both viewpoints explicitly connect —although the reasons behind this connection certainly have to be deepened.

As it turns out, the pro-unipotent group/Hopf algebra of characteristic functions $\mathrm{Char}(A)$ is related naturally to the group U. We recall a few facts from the general theory of Hopf algebras (see [41] for details). The projections associated to the grading $I_n : H \longrightarrow H_n$, with H as usual graded, connected and commutative, generate a convolution subalgebra of $\mathrm{End}(H)$, the convolution algebra of linear endomorphisms of H, that we may call the descent algebra of H. Since I is group-like, the I_n form a sequence of divided powers ($\Delta(I_n) = \sum_{i+j=n} I_i \otimes I_j$) in $\mathrm{Char}(H)$, and the descent algebra of H lifts to a Hopf subalgebra of $\mathrm{Char}(H)$. A universal Hopf algebra $\mathfrak{D}^{op}$ underlies these constructions and maps surjectively to this Hopf subalgebra as a Hopf algebra, and to the descent algebra of H as an algebra. The opposite algebra $\mathfrak{D}$ is, for historical reasons, more familiar, and is the one classically known as the descent algebra. In the current understanding, $\mathfrak{D}$ is the fundamental tool of FLA theory, as emphasized in the reference book by Reutenauer on the subject [44]. It is isomorphic to the Hopf algebra of noncommutative symmetric functions and dual to the Hopf algebra of quasi-symmetric functions, the two most natural generalizations, respectively cocommutative and commutative, of the algebra of symmetric functions, see e.g. [10, 29].

We recall the construction of $\mathfrak{D}$ —consult [29, 41, 42, 44] for details on the Hopf algebra structure. Let X be a countably infinite alphabet and $T(X)$, as in Section 4, the tensor algebra over X. Define $\mathfrak{D}$ to be the convolution subalgebra of $\mathrm{End}(T(X))$ generated by the graded projections $I_n : T(X) \to T_n(X)$. The same process as the one used to construct a Hopf algebra structure on $\mathrm{Char}(H)$ allows to define a Hopf algebra structure on $\mathfrak{D}$; this amounts, once again, to require the I_n to form a sequence of divided powers. One should be aware, however, that $\mathfrak{D}$ incorporates a lot of deep algebraic and combinatorial structures that go much beyond the pro-unipotent group/Hopf algebraic features. From this latter point of view $\mathfrak{D}$ is a free associative graded algebra generated by the I_n, which are algebraically independent in $\mathrm{End}(T(X))$. Equivalently, $\mathfrak{D}$ is freely generated by a sequence of primitive elements $e_1, \ldots, e_n, \ldots$, where e_i is of degree i —for which one usually chooses the graded components of the logarithm of the identity $\log(I)$, but the graded components of other Lie idempotents such as Dynkin's would do the job as well. This latter property, already alluded to in the article, was put to use in the proof of the Theorem 4.1. As the enveloping algebra of the free Lie algebra over the e_i, the descent algebra $\mathfrak{D}$ is also cofree (the dual of a free graded commutative algebra). In other terms, the graded dual of $\mathfrak{D}$ (and of the opposite algebra $\mathfrak{D}^{op}$, which is also a free associative algebra) identifies with the ring of coordinates of the group U. Let us mention that the graded components of $\mathfrak{D}$ are provided by the composition product in $\mathrm{End}(T(X))$ with another algebra structure, first introduced by Solomon in order to understand the geometry and representation properties of the classical groups and the associated Coxeter groups. Consult [47].

We hope that the appearance of the descent algebra (or equivalently of the Hopf algebras of noncommutative symmetric functions and quasi-symmetric functions), beyond broadening the scope of the mathematical theory of renormalization, will result into new developments in the field, possibly complementary with the arithmetic ones.

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