THE GROWTH RATE OF SYMPLECTIC FLOER HOMOLOGY

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Abstract. The main theme of this paper is to study for a symplectomorphism of a compact surface, the asymptotic invariant which is defined to be the growth rate of the sequence of the total dimensions of symplectic Floer homologies of the iterates of the symplectomorphism. We prove that the asymptotic invariant coincides with asymptotic Nielsen number and with asymptotic absolute Lefschetz number. We also show that the asymptotic invariant coincides with the largest dilatation of the pseudo-Anosov components of the symplectomorphism and its logarithm coincides with the topological entropy. This implies that symplectic zeta function has a positive radius of convergence.

Contents

1. Introduction

The main theme of this paper is to study for a symplectomorphism $\phi : M \to M$ in given mapping class g of a compact surface M, a asymptotic invariant $F_\infty(g)$, introduced in [9], which is defined to be the growth rate of the sequence $\dim HF_*(\phi^n)$ of the total dimensions of symplectic Floer homologies of the iterates of ϕ. We prove a conjecture from [9] which suggests that the asymptotic invariant coincides with asymptotic Nielsen number and with the largest dilatation of the pseudo-Anosov components of g and its logarithm coincides with topological entropy. The asymptotic invariant also provides the radius of convergence of the symplectic zeta function

$$F_g(t) = F_\phi(t) = \exp\left(\sum_{n=1}^{\infty} \frac{\dim HF_*(\phi^n)}{n} t^n\right)$$

. We show that the symplectic zeta function has a positive radius of convergence which admits exact algebraic estimation via Reidemeister trace formula.

Our main results are the following

Theorem 1.1. *If ϕ is any symplectomorphism with nondegenerate fixed points in given pseudo-Anosov mapping class g with dilatation $\lambda > 1$ of surfase M of genus ≥ 2. Then*

$$F^{\infty}(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = \lambda = \exp(h(\psi)) = L^{\infty}(\psi) = N^{\infty}(\psi)$$

where ψ is a canonical singular pseudo-Anosov representative of g, $h(\psi)$ is the topological entropy and $L^{\infty}(\psi)$ and $N^{\infty}(\psi)$ are asymptotic(absolute) Lefshetz number and asymptotic Nielsen number.

Theorem 1.2. *Let $\bar{\phi}$ be a perturbed standard form map ϕ in a reducible mapping class g of compact surface of genus ≥ 2 and λ is the largest dilatation of the pseudo-Anosov components($\lambda = 1$ if there is no pseudo-Anosov components). Then*

$$F^{\infty}(g) := \mathrm{Growth}(\dim HF_*((\bar{\phi})^n)) = \lambda = \exp(h(\psi)) = L^{\infty}(\psi) = N^{\infty}(\psi)$$

where ψ is canonical representative of mapping class g.

Remark 1.3. The genus one case follows from [9] and Pozniak's thesis[24].

Theorem 1.4. *Let $\bar{\phi}$ be a perturbed standard form map ϕ in a reducible mapping class g of compact surface of genus ≥ 2 and λ is the largest dilatation of the pseudo-Anosov components($\lambda = 1$ if there is no pseudo-Anosov components). Then the symplectic zeta function $F_g(t) = F_{\bar{\phi}(t)}$ has positive radius of convergence $R = \frac{1}{\lambda}$, where λ is the largest dilatation of the pseudo-Anosov components($\lambda = 1$ if there is no pseudo-Anosov components).*

Although the exact evaluation of the asymptotic invariant would be desirable, in general, its estimation is a more realistic goal and as we shall show, one that is sufficient for some applications.

We suggested in [9] that the asymptotic invariant potentially may be important for the applications. Recent paper of Ivan Smith [28] gives application of the asymptotic invariant to the important question of faithfulness of a representation of extended mapping class group via considerations motivated by Homological Mirror Symmetry.

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2. Preliminaries

2.1. **Symplectic Floer homology.**

2.1.1. *Review of monotonicity and weak monotonicity.* In this section we discuss the notion of monotonicity and weak monotonicity as defined in [25, 15, 2]. Monotonicity plays important role for Floer homology in two dimensions. Throughout this article, M denotes a compact connected and oriented 2-manifold of genus ≥ 2. Pick an everywhere positive two-form ω on M.

Let $\phi \in \mathrm{Symp}(M,\omega)$, the group of symplectic automorphisms of the two-dimensional symplectic manifold (M,ω) (when M has boundary we consider the group of orientation-preserving diffeomorphisms of M with no fixed points on the boundary). The mapping torus of ϕ, $T_\phi = \mathbb{R}\times M/(t+1,x) \sim (t,\phi(x))$, is a 3-manifold fibered over $S^1 = \mathbb{R}/\mathbb{Z}$. There are two natural second cohomology classes on T_ϕ, denoted by $[\omega_\phi]$ and c_ϕ. The first one is represented by the closed two-form ω_ϕ which is induced from the pullback of ω to $\mathbb{R}\times M$. The second is the Euler class of the vector bundle $V_\phi = \mathbb{R}\times TM/(t+1,\xi_x) \sim (t, \mathrm{d}\phi_x\xi_x)$, which is of rank 2 and inherits an orientation from TM.

Symplectomorphism $\phi \in \mathrm{Symp}(M,\omega)$ is called **monotone**, if $[\omega_\phi] = (\mathrm{area}_\omega(M)/\chi(M))\cdot c_\phi$ in $H^2(T_\phi;\mathbb{R})$; throughout this article $\mathrm{Symp}^m(M,\omega)$ denotes the set of monotone symplectomorphisms.

Now $H^2(T_\phi;\mathbb{R})$ fits into the following short exact sequence [25, 15]

$$0 \longrightarrow \frac{H^1(M;\mathbb{R})}{\mathrm{im}(\mathrm{id}-\phi^*)} \xrightarrow{d} H^2(T_\phi;\mathbb{R}) \xrightarrow{r^*} H^2(M;\mathbb{R}), \longrightarrow 0. \tag{1}$$

where the map r^* is restriction to the fiber. The map d is defined as follows. Let $\rho : I \to \mathbb{R}$ be a smooth function which vanishes near 0 and 1 and satisfies $\int_0^1 \rho\,\mathrm{d}t = 1$. If θ is a closed 1-form on M, then $\rho\cdot\theta\wedge \mathrm{d}t$ defines a closed 2-form on T_ϕ; indeed $d[\theta] = [\rho\cdot\theta\wedge\mathrm{d}t]$. The map $r : M \hookrightarrow T_\phi$ assigns to each $x \in M$ the equivalence class of $(1/2, x)$. Note, that $r^*\omega_\phi = \omega$ and r^*c_ϕ is the Euler class of TM. Hence, by (1), there exists a unique class $m(\phi) \in H^1(M;\mathbb{R})/\mathrm{im}(\mathrm{id}-\phi^*)$ satisfying $d\,m(\phi) = [\omega_\phi] - (\mathrm{area}_\omega(M)/\chi(M))\cdot c_\phi$, where $\chi(M)$ denotes the Euler characteristic of M. Therefore, ϕ is monotone if and only if $m(\phi) = 0$.

Because c_ϕ controls the index, or expected dimension, of moduli spaces of holomorphic curves under change of homology class and ω_ϕ controls their energy under change of homology class, the monotonicity condition ensures that the energy is constant on the index one components of the moduli space, which implies compactness and, as a corollary, finite count in a differential of the Floer complex.

We recall the fundamental properties of $\mathrm{Symp}^m(M,\omega)$ from [25, 15]. Let $\mathrm{Diff}^+(M)$ denotes the group of orientation preserving diffeomorphisms of M.

(Identity) $\mathrm{id}_M \in \mathrm{Symp}^m(M,\omega)$.

(Naturality) If $\phi \in \mathrm{Symp}^m(M,\omega), \psi \in \mathrm{Diff}^+(M)$, then $\psi^{-1}\phi\psi \in \mathrm{Symp}^m(M,\psi^*\omega)$.

(Isotopy) Let $(\psi_t)_{t\in I}$ be an isotopy in $\mathrm{Symp}(M,\omega)$, i.e. a smooth path with $\psi_0 = \mathrm{id}$. Then $m(\phi\circ\psi_1) = m(\phi) + [\mathrm{Flux}(\psi_t)_{t\in I}]$ in $H^1(M;\mathbb{R})/\mathrm{im}(\mathrm{id}-\phi^*)$; see [25, Lemma 6]. For the definition of the flux homomorphism see [22].

(Inclusion) The inclusion $\mathrm{Symp}^m(M,\omega) \hookrightarrow \mathrm{Diff}^+(M)$ is a homotopy equivalence. In particular $\mathrm{Symp}^m(M,\omega)$ is path connected.

(Floer homology) To every $\phi \in \mathrm{Symp}^m(M,\omega)$ symplectic Floer homology theory assigns a $\mathbb{Z}_2$-graded vector space $HF_*(\phi)$ over $\mathbb{Z}_2$, with an additional multiplicative structure, called the quantum cap product, $H^*(M;\mathbb{Z}_2)\otimes HF_*(\phi) \longrightarrow HF_*(\phi)$. For $\phi = \mathrm{id}_M$ the symplectic Floer homology $HF_*(\mathrm{id}_M)$ are canonically isomorphic to ordinary homology $H_*(M;\mathbb{Z}_2)$ and quantum cap product agrees with the ordinary cap product. Each $\psi \in \mathrm{Diff}^+(M)$ induces an isomorphism $HF_*(\phi) \cong HF_*(\psi^{-1}\phi\psi)$ of $H^*(M;\mathbb{Z}_2)$-modules.

(Invariance) If $\phi, \phi' \in \mathrm{Symp}^m(M,\omega)$ are isotopic, then $HF_*(\phi)$ and $HF_*(\phi')$ are naturally isomorphic as $H^*(M;\mathbb{Z}_2)$-modules. This is proven in [25, Page 7]. Note that every

Hamiltonian perturbation of ϕ (see [4]) is also in $\mathrm{Symp}^m(M,\omega)$.

Now let g be a mapping class of M, i.e. an isotopy class of $\mathrm{Diff}^+(M)$. Pick an area form ω and a representative $\phi \in \mathrm{Symp}^m(M,\omega)$ of g. $HF_*(\phi)$ is an invariant as ϕ is deformed through monotone symplectomorphisms. These imply that we have a symplectic Floer homology invariant $HF_*(g)$ canonically assigned to each mapping class g given by $HF_*(\phi)$ for any monotone symplectomorphism ϕ. Note that $HF_*(g)$ is independent of the choice of an area form ω by Moser's isotopy theorem [21] and naturality of Floer homology.

We give now, following A. Cotton-Clay [2], a notion of *weak monotonicity* such that $HF_*(\phi)$ is well-defined for and invariant among weakly monotone simplectomorphisms. Monotonicity implies weak monotonicity, and so $HF_*(g) = HF_*(\phi)$ for any weakly monotone ϕ in mapping class g. The properties of weak monotone symplectomorphism of surface play a crucial role in the computation of Floer homology for pseudo-Anosov and reducible mapping classes(see [2]).

A symplectomorphism $\phi : M \to M$ is **weakly monotone** if $[\omega_\phi]$ vanishes on $\ker(c_\phi|_{T(T_\phi)})$, where $T(T_\phi) \subset H_2(M_\phi;\mathbb{R})$ is generated by tori T such that $\pi|_T : T \to S^1$ is a fibration with fiber S^1, where the map $\pi : T_\phi \to S^1$ is the projection. Throughout this article $\mathrm{Symp}^{wm}(M,\omega)$ denotes the set of weakly monotone symplectomorphisms.

2.1.2. *Floer homology.* Let $\phi \in \mathrm{Symp}(M,\omega)$.There are two ways of constructing Floer homology detecting its fixed points, $\mathrm{Fix}(\phi)$. Firstly, the graph of ϕ is a Lagrangian submanifold of $M\times M, (-\omega)\times\omega)$ and its fixed points correspond to the intersection points of graph(ϕ) with the diagonal $\Delta = \{(x,x) \in M \times M\}$. Thus we have the Floer homology of the Lagrangian intersection $HF_*(M\times M, \Delta, graph(\phi))$. This intersection is transversal if the fixed points of ϕ are nondegenerate, i.e. if 1 is not an eigenvalue of $d\phi(x)$, for $x \in \mathrm{Fix}(\phi)$. The second approach was mentioned by Floer in [13] and presented with details by Dostoglou and Salamon in [4].We follow here Seidel's approach [25] which, comparable with [4], uses a larger class of perturbations, but such that the perturbed action form is still cohomologous to the unperturbed. As a consequence, the usual invariance of Floer homology under Hamiltonian isotopies is extended to the stronger property stated above. Let now ϕ is monotone or weakly monotone. Firstly, we give the definition of $HF_*(\phi)$ in the special case where all the fixed points of ϕ are non-degenerate, i.e. for all $y \in \mathrm{Fix}(\phi)$, $\det(\mathrm{id} - \mathrm{d}\phi_y) \neq 0$, and then following Seidels approach [25] we consider general case when ϕ has degenerate fixed points. Let $\Omega_\phi = \{y \in C^\infty(\mathbb{R}, M) \,|\, y(t) = \phi(y(t+1))\}$ be the twisted free loop space, which is also the space of sections of $T_\phi \to S^1$. The action form is the closed one-form α_ϕ on Ω_ϕ defined by

$$\alpha_\phi(y)Y = \int_0^1 \omega(dy/dt, Y(t))\, dt.$$

where $y \in \Omega_\phi$ and $Y \in T_y\Omega_\phi$, i.e. $Y(t) \in T_{y(t)}M$ and $Y(t) = \mathrm{d}\phi_{y(t+1)}Y(t+1)$ for all $t \in \mathbb{R}$.

The tangent bundle of any symplectic manifold admits an almost complex structure $J : TM \longrightarrow TM$ which is compatible with ω in sense that $(v,w) = \omega(v, Jw)$ defines a Riemannian metric. Let $J = (J_t)_{t\in\mathbb{R}}$ be a smooth path of ω-compatible almost complex structures on M such that $J_{t+1} = \phi^* J_t$. If $Y, Y' \in T_y\Omega_\phi$, then $\int_0^1 \omega(Y'(t), J_tY(t))\mathrm{d}t$ defines a metric on the loop space Ω_ϕ. So the critical points of α_ω are the constant paths in Ω_ϕ and hence the fixed points of ϕ. The negative gradient lines of α_ω with respect to the metric above are solutions of the partial differential equations with boundary conditions

$$\begin{cases} u(s,t) = \phi(u(s,t+1)), \\ \partial_s u + J_t(u)\partial_t u = 0, \\ \lim_{s\to\pm\infty} u(s,t) \in \mathrm{Fix}(\phi) \end{cases} \tag{2}$$

These are exactly Gromov's pseudoholomorphic curves [16].

For $y^{\pm} \in \mathrm{Fix}(\phi)$, let $\mathcal{M}(y^-, y^+; J, \phi)$ denote the space of smooth maps $u : \mathbb{R}^2 \to M$ which satisfy the equations (2). Now to every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ we associate a Fredholm operator D_u which linearizes (2) in suitable Sobolev spaces. The index of this operator is given by the so called Maslov index $\mu(u)$, which satisfies $\mu(u) = \deg(y^+) - \deg(y^-) \bmod 2$, where $(-1)^{\deg y} = \mathrm{sign}(\det(\mathrm{id} - \mathrm{d}\phi_y))$. We have no bubbling, since for surface $\pi_2(M) = 0$. For a generic J, every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ is regular, meaning that D_u is onto. Hence, by the implicit function theorem, $\mathcal{M}_k(y^-, y^+; J, \phi)$ is a smooth k-dimensional manifold and is the subset of those $u \in \mathcal{M}(y^-, y^+; J, \phi)$ with $\mu(u) = k \in \mathbb{Z}$. Translation of the s-variable defines a free $\mathbb{R}$-action on 1-dimensional manifold $\mathcal{M}_1(y^-, y^+; J, \phi)$ and hence the quotient is a discrete set of points. The energy of a map $u : \mathbb{R}^2 \to M$ is given by $E(u) = \int_{\mathbb{R}} \int_0^1 \omega\big(\partial_t u(s,t), J_t \partial_t u(s,t)\big)\, \mathrm{d}t\mathrm{d}s$ for all $y \in \mathrm{Fix}(\phi)$. P.Seidel and A. Cotton-Clay have proved in [25] and [2] that if ϕ is monotone or weakly monotone, then the energy is constant on each $\mathcal{M}_k(y^-, y^+; J, \phi)$. Since all fixed points of ϕ are nondegenerate the set $\mathrm{Fix}(\phi)$ is a finite set and the $\mathbb{Z}_2$-vector space $CF_*(\phi) := \mathbb{Z}_2^{\#\,\mathrm{Fix}(\phi)}$ admits a $\mathbb{Z}_2$-grading with $(-1)^{\deg y} = \mathrm{sign}(\det(\mathrm{id} - \mathrm{d}\phi_y))$, for all $y \in \mathrm{Fix}(\phi)$. The boundedness of the energy $E(u)$ for monotone or weakly monotone ϕ implies that the 0-dimensional quotients $\mathcal{M}_1(y_-, y_+, J, \phi)/\mathbb{R}$ are actually finite sets. Denoting by $n(y_-, y_+)$ the number of points mod 2 in each of them, one defines a differential $\partial_J : CF_*(\phi) \to CF_{*+1}(\phi)$ by $\partial_J y_- = \sum_{y_+} n(y_-, y_+) y_+$. Due to gluing theorem this Floer boundary operator satisfies $\partial_J \circ \partial_J = 0$. For gluing theorem to hold one needs again the boundedness of the energy $E(u)$. It follows that $(CF_*(\phi), \partial_J)$ is a chain complex and its homology is by definition the Floer homology of ϕ denoted $HF_*(\phi)$. It is independent of J and is an invariant of ϕ.

If ϕ has degenerate fixed points one needs to perturb equations (2) in order to define the Floer homology. Equivalently, one could say that the action form needs to be perturbed. The necessary analysis is given in [25], it is essentially the same as in the slightly different situations considered in [4]. But Seidel's approach also differs from the usual one in [4]. He uses a larger class of perturbations, but such that the perturbed action form is still cohomologous to the unperturbed.

2.2. **Nielsen classes and Reidemeister trace.** Before discussing the results of the paper, we briefly describe the few basic notions of Nielsen fixed point theory which will be used. We assume X to be a connected, compact polyhedron and $f : X \to X$ to be a continuous map. Let $p : \tilde{X} \to X$ be the universal cover of X and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lifting of f, i.e. $p \circ \tilde{f} = f \circ p$. Two liftings $\tilde{f}$ and $\tilde{f}'$ are called *conjugate* if there is a $\gamma \in \Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(Fix(\tilde{f})) \subset Fix(f)$ is called *the fixed point class of f determined by the lifting class* $[\tilde{f}]$.Two fixed points x_0 and x_1 of f belong to the same fixed point class iff there is a path c from x_0 to x_1 such that $c \cong f \circ c$ (homotopy relative endpoints). This fact can be considered as an equivalent definition of a non-empty fixed point class. Every map f has only finitely many non-empty fixed point classes, each a compact subset of X. A fixed point class is called *essential* if its index is nonzero. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$.The Nielsen number is always finite. $R(f)$ and $N(f)$ are homotopy invariants. In the category of compact, connected polyhedra, the Nielsen number of a map is, apart from certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as f.

Let $f : X \to X$ be given, and let a specific lifting $\tilde{f} : \tilde{X} \to \tilde{X}$ be chosen as reference. Let Γ be the group of covering translations of $\tilde{X}$ over X. Then every lifting of f can be written uniquely as $\alpha \circ \tilde{f}$, with $\alpha \in \Gamma$. So elements of Γ serve as coordinates of liftings with respect to the reference $\tilde{f}$. Now for every $\alpha \in \Gamma$ the composition $\tilde{f} \circ \alpha$ is a lifting of f so there is a unique $\alpha' \in \Gamma$ such that $\alpha' \circ \tilde{f} = \tilde{f} \circ \alpha$. This correspondence $\alpha \to \alpha'$ is determined by the reference $\tilde{f}$, and is obviously a homomorphism. The endomorphism $\tilde{f}_* : \Gamma \to \Gamma$ determined by the lifting $\tilde{f}$ of f is defined by $\tilde{f}_*(\alpha) \circ \tilde{f} = \tilde{f} \circ \alpha$. It is well known that $\Gamma \cong \pi_1(X)$. We shall identify $\pi = \pi_1(X, x_0)$ and Γ in the usual way.

We have seen that $\alpha \in \pi$ can be considered as the coordinate of the lifting $\alpha \circ \tilde{f}$. We can tell the conjugacy of two liftings from their coordinates: $[\alpha \circ \tilde{f}] = [\alpha' \circ \tilde{f}]$ iff there is $\gamma \in \pi$ such that $\alpha' = \gamma\alpha\tilde{f}_*(\gamma^{-1})$.

So we have the Reidemeister bijection: Lifting classes of f are in 1-1 correspondence with $\tilde{f}_*$-conjugacy classes in group π, the lifting class $[\alpha\circ\tilde{f}]$ corresponds to the $\tilde{f}_*$-cojugacy class of α.

By an abuse of language, we say that the fixed point class $p(\mathrm{Fix}\,\alpha \circ \tilde{f})$, which is labeled with the lifting class $[\alpha \circ \tilde{f}]$,corresponds to the $\tilde{f}_*$-conjugacy class of α. Thus the $\tilde{f}_*$-conjugacy classes in π serve as coordinates for the fixed point classes of f, once a reference lifting $\tilde{f}$ is chosen.

2.2.1. *Reidemeister trace.* The results of this section are well known(see [19],[8, 12]).We shall use this results later in section 3 to estimate the asymptotic invariant and the radius of convergence of the symplectic zeta function. The fundamental group $\pi = \pi_1(X, x_0)$ splits into $\tilde{f}_*$-conjugacy classes.Let π_f denote the set of $\tilde{f}_*$-conjugacy classes,and $\mathbb{Z}\pi_f$ denote the Abelian group freely generated by π_f .We will use the bracket notation $a \to [a]$ for both projections $\pi \to \pi_f$ and $\mathbb{Z}\pi \to \mathbb{Z}\pi_f$. Let x be a fixed point of f.Take a path c from x_0 to x.The $\tilde{f}_*$-conjugacy class in π of the loop $c \cdot (f \circ c)^{-1}$,which is evidently independent of the choice of c, is called the coordinate of x.Two fixed points are in the same fixed point class F iff they have the same coordinates.This $\tilde{f}_*$-conjugacy class is thus called the coordinate of the fixed point class F and denoted $cd_\pi(F, f)$ (compare with description in section 2). The generalized Lefschetz number or the Reidemeister trace [19] is defined as

$$L_\pi(f) := \sum_F \mathrm{ind}(F, f) \cdot cd_\pi(F, f) \in \mathbb{Z}\pi_f, \tag{3}$$

the summation being over all essential fixed point classes $F$of f.The Nielsen number $N(f)$ is the number of non-zero terms in $L_\pi(f)$,and the indices of the essential fixed point classes appear as the coefficients in $L_\pi(f)$.This invariant used to be called the Reidemeister trace because it can be computed as an alternating sum of traces on the chain level as follows [19] . Assume that X is a finite cell complex and $f : X \to X$ is a cellular map. A cellular decomposition e^d_j of X lifts to a π-invariant cellular structure on the universal covering $\tilde{X}$.Choose an arbitrary lift $\tilde{e}^d_j$ for each e^d_j . They constitute a free $\mathbb{Z}\pi$-basis for the cellular chain complex of $\tilde{X}$. The lift $\tilde{f}$ of f is also a cellular map.In every dimension d, the cellular chain map $\tilde{f}$ gives rise to a $\mathbb{Z}\pi$-matrix $\tilde{F}_d$ with respect to the above basis,i.e $\tilde{F}_d = (a_{ij})$ if $\tilde{f}(\tilde{e}_i^d) = \sum_j a_{ij}\tilde{e}_j^d$,where $a_{ij} \in \mathbb{Z}\pi$.Then we have the Reidemeister trace formula

$$L_\pi(f) = \sum_d (-1)^d[\mathrm{Tr}\, \tilde{F}_d] \in \mathbb{Z}\pi_f. \tag{4}$$

Now we describe alternative approach to the Reidemeister trace formula proposed by Jiang [19]. This approach is useful when we study the periodic points of f,i.e. the fixed points of the iterates of f.

The mapping torus T_f of $f : X \to X$ is the space obtained from $X \times [o, \infty)$ by identifying $(x, s+1)$ with $(f(x), s)$ for all $x \in X, s \in [0, \infty)$.On T_f there is a natural semi-flow $\phi : T_f \times [0, \infty) \to T_f, \phi_t(x, s) = (x, s + t)$ for all $t \geq 0$.Then the map $f : X \to X$ is the return map of the semi-flow ϕ.A point $x \in X$ and a positive number $\tau > 0$ determine the orbit curve $\phi_{(x,\tau)} := \phi_t(x)_{0\leq t\leq \tau}$ in T_f. Take the base point x_0 of X as the base point of T_f.It is known that the fundamental group $H := \pi_1(T_f, x_0)$ is obtained from π by adding a new generator z and adding the relations $z^{-1}gz = \tilde{f}_*(g)$ for all $g \in \pi = \pi_1(X, x_0)$.Let H_c denote the set of conjugacy classes in H. Let $\mathbb{Z}H$ be the integral group ring of H, and let $\mathbb{Z}H_c$ be the free Abelian group with basis H_c.We again use the bracket notation $a \to [a]$ for both projections $H \to H_c$ and $\mathbb{Z}H \to \mathbb{Z}H_c$. If F^n is a fixed point class of f^n, then $f(F^n)$ is also fixed point class of f^n and $\mathrm{ind}(f(F^n), f^n) = \mathrm{ind}(F^n, f^n)$. Thus f acts as an index-preserving permutation among fixed point classes of f^n.By definition, an n-orbit class O^n of f to be the union of elements of an orbit of this action.In other words, two points $x, x' \in \mathrm{Fix}(f^n)$ are said to be in the same n-orbit class of f if and only if some $f^i(x)$ and some $f^j(x')$ are in the same fixed point class of f^n.The set $\mathrm{Fix}(f^n)$ splits into a disjoint union of n-orbits classes.Point x is a fixed point of f^n or a periodic point of period n if and only if orbit curve $\phi_{(x,n)}$ is a closed curve. The free homotopy class of the closed curve $\phi_{(x,n)}$ will be called the H -coordinate of point x,written $cd_H(x, n) = [\phi_{(x,n)}] \in H_c$.It follows that periodic points x of period n and x' of period n' have the same H-coordinate if and only if $n = n'$ and x,x' belong to the same n-orbits class of f. Thus it is possible equivalently define $x, x' \in \mathrm{Fix}(f^n)$ to be in the same n-orbit class if and only if they have the same $H-$coordinate. Jiang [19] has considered generalized Lefschetz number with respect to H

$$L_H(f^n) := \sum_{O^n} \mathrm{ind}(O^n, f^n) \cdot cd_H(O^n) \in \mathbb{Z}H_c, \tag{5}$$

and proved following trace formula:

$$L_H(f^n) = \sum_d (-1)^d[\mathrm{Tr}(z\tilde{F}_d)^n] \in \mathbb{Z}H_c, \tag{6}$$

where $\tilde{F}_d$ be $\mathbb{Z}\pi$-matrices defined in (16) and $z\tilde{F}_d$ is regarded as a $\mathbb{Z}H$-matrix.

2.2.2. *Twisted Lefschetz numbers and twisted Lefschetz zeta function.* Let R be a commutative ring with unity. Let $GL_n(R)$ be the group of invertible $n \times n$ matrices in R, and $M_{n\times n}(R)$ be the algebra of $n\times n$ matrices in R. Suppose a representation $\rho : H \to GL_n(R)$

is given. It extends to a representation $\rho : \mathbb{Z}H \to M_{n\times n}(R)$. Following Jiang [19] we define ρ-twisted Lefschetz number

$$L_\rho(f^n) := \mathrm{Tr}(L_H(f^n))^\rho = \sum_{O^n} \mathrm{ind}(O^n, f^n) \cdot \mathrm{Tr}(cd_H(O^n))^\rho \in R, \tag{7}$$

where h^ρ is ρ-image of $h \in \mathbb{Z}H$. It has the trace formula(see [19])

$$L_\rho(f^n) = \sum_d (-1)^d \,\mathrm{Tr}((z\tilde{F}_d)^\rho)^n \in R, \tag{8}$$

where for a $\mathbb{Z}H$-matrix A, its ρ-image A^ρ means the block matrix obtained from A by replacing each element a_{ij} with $n \times n$ R-matrix a_{ij}^ρ. Twisted Lefschetz zeta function is defined as formal power series

$$L_\rho^f(t) := \exp\left(\sum_{n=1}^{\infty} \frac{L_\rho(f^n)}{n} t^n\right).$$

It is in the multiplicative subgroup $1 + tR[[t]]$ of the formal power series ring $R[[t]]$. The trace formula for the twisted Lefschetz numbers implies that $L_\rho^f(t)$ is a rational function in R given by the formula

$$L_\rho^f(t) = \prod_d \det\left(E - t(z\tilde{F}_d)^\rho\right)^{(-1)^{d+1}} \in R(t), \tag{9}$$

where E stands for suitable identity matrices. Twisted Lefschetz zeta function enjoys the same invariance properties as that of $L_H(f^n)$.

2.3. **Computation of symplectic Floer homology.** In this section we describe known results from [2], [15], [9, 10] about computation of symplectic Floer homology for different mapping classes.

2.3.1. *Thurston classification theorem and standard form maps.* We recall firstly Thurston classification theorem for homeomorphisms of surfase M of genus ≥ 2.

Theorem 2.1. [29] *Every homeomorphism $\phi : M \to M$ is isotopic to a homeomorphism f such that either*
(1) f is a periodic map; or
(2) f is a pseudo-Anosov map, i.e. there is a number $\lambda > 1$, the dilation of f, and a pair of transverse measured foliations (F^s, μ^s) and (F^u, μ^u) such that $f(F^s, \mu^s) = (F^s, \frac{1}{\lambda}\mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda\mu^u)$; or
(3)f is reducible map, i.e. there is a system of disjoint simple closed curves $\gamma = \{\gamma_1, ..., \gamma_k\}$ in $intM$ such that γ is invariant by f (but γ_i may be permuted) and γ has a f-invariant tubular neighborhood U such that each component of $M \setminus U$ has negative Euler characteristic and on each(not necessarily connected) f-component of $M \setminus U$, f satisfies (1) or (2).

The map f above is called a Thurston canonical representative of ϕ. In (3) it can be chosen so that some iterate f^m is a generalised Dehn twist on U. A key observation is that if f is canonical representative, so are all iterates of f.

Thurston classification theorem for homeomorphisms of surfase implies that every mapping class of M is precisely one of the following: periodic, pseudo-Anosov or reducible.

In this section we review standard form maps as discussed in [15] and [2]. These are special representative of mapping classes adopted to the symplectic geometry. For the identity mapping class, a standard form map is a small perturbation of the identity map by the Hamiltonian flow associated to a Morse function for which the boundary components are locally minima and maxima. Every fixed point is in the same Nielsen class. This Nielsen class has index given by the Euler characteristic of the surface. For non-identity periodic mapping classes, a standard form map is an isometry with respect to a hyperbolic structure on the surface with geodesic boundary. Every fixed point is in a separate Nielsen class and each of the Nielsen classes for which there is a fixed point has index one. For a pseudo-Anosov mapping classes, a standard form map is a symplectic smoothing(see [2]) of the singularities and boundary components of the canonical singular representative. Each singularity has a number $p \geq 3$ of prongs and each boundary component has a number $p \geq 1$ of prongs. If a singularity or boundary component is (setwise) fixed, it has some fractional rotation number modulo p(see [2]). There is a separate Nielsen class for every smooth fixed point, which is of index one or minus one; for every fixed singularity, which when symplectically smoothed gives $p-1$ fixed points all of index minus one if the rotation number is zero modulo p or one fixed point of index one otherwise; and for every fixed boundary component with rotation number zero modulo p, which when symplectically smoothed gives p fixed points all of index minus one [2].

From this discussion, we see that for non-identity periodic and pseudo-Anosov mapping classes, the standard form map is such that all fixed points are nondegenerate of index +1 or -1 and, for every Nielsen class F, the number of fixed points in F is $|\operatorname{ind}(F)|$. We now turn to reducible maps and the identity map.

By Thurston's classification (see [29] and [5]; also [15, Definition 8] and [2, Definition 4.6]), in a reducible mapping class g, there is a (not necessarily smooth) map ϕ which satisfies the following:

Definition 2.2. A reducible map ϕ is in *standard form* if there is a ϕ-and-ϕ^{-1}-invariant finite union of disjoint noncontractible (closed) annuli $U \subset M$ such that:

(1) For N a component of $U$and ℓ the smallest positive integer such that ϕ^ℓ maps N to itself, the map $\phi^\ell|_N$ is either a *twist map* or a *flip-twist map*. That is, with respect to coordinates $(q,p) \in [0,1] \times S^1$, we have one of

$$\begin{aligned} (q,p) &\mapsto (q, p - f(q)) \qquad \text{(twist map)} \\ (q,p) &\mapsto (1-q, -p+f(q)) \qquad \text{(flip-twist map)}, \end{aligned}$$

where $f:[0,1] \to \mathbb{R}$ is a strictly monotonic smooth map. We call the (flip-)twist map *positive* or *negative* if f is increasing or decreasing, respectively. Note that these maps are area-preserving.

(2) Let N and ℓ be as in (1). If $\ell = 1$ and $\phi|_U$ is a twist map, then $\operatorname{Im}(f) \subset [0,1]$. That is, $\phi|_{\operatorname{int}(N)}$ has no fixed points. (If we want to twist multiple times, we separate the twisting region into parallel annuli separated by regions on which the map is the identity.) We further require that parallel twisting regions twist in the same direction.

(3) For S a component of $M \backslash N$ and ℓ the smallest integer such that ϕ^ℓ maps S to itself, the map $\phi^\ell|_S$ is area-preserving and is either isotopic to the *identity*, *periodic*,

or *pseudo-Anosov*. In these cases, we require the map to be in standard form as above.

Thurston classification theorem for homeomorphisms of surfase implies that every mapping class of M is precisely one of the following: periodic, pseudo-Anosov or reducible.

2.3.2. *Periodic mapping classes.*

Theorem 2.3. [15], [10] *If ϕ is a non-trivial, orientation preserving, standard form periodic diffeomorphism of a compact connected surface M of Euler characteristic $\chi(M) \leq 0$, then ϕ is monotone symplectomorphism with respect to some ϕ-invariant area form and*

$$\dim HF_*(\phi) = L(\phi) = N(\phi)$$

where $L(\phi), N(\phi)$ denote the Lefschetz and the Nielsen number of ϕ correspondingly.

2.3.3. *Algebraically finite mapping classes.* A mapping class of M is called algebraically finite if it does not have any pseudo-Anosov components in the sense of Thurston's theory of surface diffeomorphism.The term algebraically finite goes back to J. Nielsen
In [15] the diffeomorphisms of finite type were defined . These are reducible map in standard form which are special representatives of algebraically finite mapping classes adopted to the symplectic geometry.

By M_{id} we denote the union of the components of $M \setminus \text{int}(U)$, where ϕ restricts to the identity.

The monotonicity of diffeomorphisms of finite type was investigated in details in [15]. Let ϕ be a diffeomorphism of finite type and ℓ be as in (1). Then ϕ^ℓ is the product of (multiple) Dehn twists along U. Moreover, two parallel Dehn twists have the same sign. We say that ϕ has uniform twists, if ϕ^ℓ is the product of only positive, or only negative Dehn twists.

Furthermore, we denote by ℓ the smallest positive integer such that ϕ^ℓ restricts to the identity on $M \setminus U$.

If ω' is an area form on M which is the standard form $\mathrm{d}q \wedge \mathrm{d}p$ with respect to the (q,p)-coordinates on U, then $\omega := \sum_{i=1}^{\ell} (\phi^i)^* \omega'$ is standard on U and ϕ-invariant, i.e. $\phi \in \text{Symp}(M, \omega)$. To prove that ω can be chosen such that $\phi \in \text{Symp}^m(M, \omega)$, Gautschi distinguishes two cases: uniform and non-uniform twists. In the first case he proves the following stronger statement.

Lemma 2.4. [15] *If ϕ has uniform twists and ω is a ϕ-invariant area form, then $\phi \in \text{Symp}^m(M, \omega)$.*

In the non-uniform case, monotonicity does not hold for arbitrary ϕ-invariant area forms.

Lemma 2.5. [15] *If ϕ does not have uniform twists, there exists a ϕ-invariant area form ω such that $\phi \in \text{Symp}^m(M, \omega)$. Moreover, ω can be chosen such that it is the standard form $\mathrm{d}q \wedge \mathrm{d}p$ on U.*

Theorem 2.6. [15] *Let ϕ be a diffeomorphism of finite type, then ϕ is monotone with respect to some ϕ-invariant area form and*

$$\dim HF_*(\phi) = \dim H_*(M_{\text{id}}, \partial_{M_{\text{id}}}; \mathbb{Z}_2) + L(\phi | M \setminus M_{\text{id}}).$$

Here, L denotes the Lefschetz number.

2.3.4. *Pseudo-Anosov mapping classes.* For a pseudo-Anosov mapping classes, a standard form map is a symplectic smoothing of the singularities and boundary components of the canonical singular representative. Full description of the symplectic smoothing is given by A. Cotton-Clay in [2]. Each singularity has a number $p \geq 3$ of prongs and each boundary component has a number $p \geq 1$ of prongs. If a singularity or boundary component is (setwise) fixed, it has some fractional rotation number modulo p. There is a separate Nielsen class for every smooth fixed point, which is of index one or minus one; for every fixed singularity, which when symplectically smoothed gives $p-1$ fixed points all of index minus one if the rotation number is zero modulo p or one fixed point of index one otherwise; and there is a separate Nielsen class for every fixed boundary component with rotation number zero modulo p, which when symplectically smoothed gives p fixed points all of index minus one.

Theorem 2.7. *(* [2], *see also* [10]*) If ϕ is any symplectomorphism with nondegenerate fixed points in given pseudo-Anosov mapping class g, then ϕ is weakly monotone, $HF_*(\phi)$ is well defined and*

$$\dim HF_*(\phi) = \dim HF_*(g) = \sum_{x \in \mathrm{Fix}(\psi)} |\operatorname{Ind}(x)|,$$

where ψ is the singular canonical pseudo-Anosov representative of g.

2.3.5. *Reducible mapping classes.* Recently, A. Cotton-Clay [2] calculated Seidel's symplectic Floer homology for reducible mapping classes. This result completing all previous computations.

In the case of reducible mapping classes a energy estimate forbids holomorphic discs from crossing reducing curves except when a pseudo-Anosov component meets an identity component (with no twisting). Let us introduce some notation following [2]. Recall the notation of M_{id} for the collection of fixed components as well as the tree types of boundary: 1) $\partial_+ M_{id}$, $\partial_- M_{id}$ denote the collection of components of ∂M_{id} on which we've joined up with a positive(resp. negative) twist; 2) the collection of components of ∂M_{id} which meet a pseudo-Anosov component will be denoted $\partial_p M_{\mathrm{id}}$. Additionally let M_1 be the collection of periodic components and let M_2 be the collection of pseudo-Anosov components with punctures(i.e. before any perturbation) instead of boundary components wherever there is a boundary component that meets a fixed component. We further subdivide M_{id}. Let M_a be the collection of fixed components which don't meet any pseudo-Anosov components. Let $M_{b,p}$ be the collection of fixed components which meet one pseudo-Anosov component at a boundary with p prongs. In this case, we assign the boundary components to $\partial_+ M_{\mathrm{id}}$ (this is an arbitrary choice). Let $M^o_{b,p}$ be the collection of the $M_{b,p}$ with each component punctured once. Let $M_{c,q}$ be the collection of fixed components which meets at least two pseudo-Anosov components such that the total number of prongs over all the boundaries is q. In this case, we assign at least one boundary component to $\partial_+ M_{\mathrm{id}}$ and at least one to $\partial_- M_{\mathrm{id}}$ (and beyond that, it does not matter).

Theorem 2.8. [2] *If $\bar{\phi}$ is a perturbed standard form map ϕ in a reducible mapping class g with choices of the signs of components of $\partial_p M_{\mathrm{id}}$. Then $\bar{\phi}$ is weakly monotone, $HF_*(\bar{\phi})$ is well-defined and*

$$\dim HF_*(g) = \dim HF_*(\bar{\phi}) = \dim H_*(M_a, \partial_+ M_{\mathrm{id}}; \mathbb{Z}_2) +$$

$$+\sum_p (\dim H_*(M^0_{b,p}, \partial_+ M_{b,p}; \mathbb{Z}_2) + (p-1)|\pi_0(M_{b,p})|)+$$
$$+\sum_q (\dim H_*(M_{c,q}, \partial_+ M_{c,q}; \mathbb{Z}_2) + q|\pi_0(M_{c,q)})|)+$$
$$+L(\bar{\phi}|M_1) + \dim HF_*(\bar{\phi}|M_2),$$

where $L(\bar{\phi}|M_1)$ is the Lefschetz number of $\bar{\phi}|M_1$, the $L(\bar{\phi}|M_1)$ summand is all in even degree, the other two summands(with $p-1$ and q are all in odd degree, and $HF_*(\bar{\phi}|M_2)$ denotes the Floer homology for $\bar{\phi}$ on the pseudo-Anosov components M_2

Remark 2.9. The first summand and the $L(\bar{\phi}|M_1)$ are as in R. Gautschi's Theorem 2.6 [15]. The last summand comes from the pseudo-Anosov components and is calculated via the Theorem 2.7. The sums over p and q arise in the same manner as the first summand.

Corollary 2.10. *As an application, A. Cotton-Clay gave recently* [3] *a sharp lower bound on the number of fixed points of area-preserving map in any prescribed mapping class(rel boundary), generalising the Poincare-Birkhoff fixed point theorem.*

3. The growth rate of symplectic Floer homology

3.1. **Topological entropy and Nielsen numbers.** The most widely used measure for the complexity of a dynamical system is the topological entropy. For the convenience of the reader, we include its definition. Let $f : X \to X$ be a self-map of a compact metric space. For given $\epsilon > 0$ and $n \in \mathbb{N}$, a subset $E \subset X$ is said to be (n, ϵ)-separated under f if for each pair $x \neq y$ in E there is $0 \leq i < n$ such that $d(f^i(x), f^i(y)) > \epsilon$. Let $s_n(\epsilon, f)$ denote the largest cardinality of any (n, ϵ)-separated subset E under f. Thus $s_n(\epsilon, f)$ is the greatest number of orbit segments $x, f(x), ..., f^{n-1}(x)$ of length n that can be distinguished one from another provided we can only distinguish between points of X that are at least ϵ apart. Now let

$$h(f, \epsilon) := \limsup_n \frac{1}{n} \cdot \log\ s_n(\epsilon, f)$$
$$h(f) := \limsup_{\epsilon \to 0} h(f, \epsilon).$$

The number $0 \leq h(f) \leq \infty$, which to be independent of the metric d used, is called the topological entropy of f. If $h(f, \epsilon) > 0$ then, up to resolution $\epsilon > 0$, the number $s_n(\epsilon, f)$ of distinguishable orbit segments of length n grows exponentially with n. So $h(f)$ measures the growth rate in n of the number of orbit segments of length n with arbitrarily fine resolution.

A basic relation between topological entropy $h(f)$ and Nielsen numbers was found by N. Ivanov [18]. We present here a very short proof by Boju Jiang of the Ivanov's inequality.

Lemma 3.1. [18]

$$h(f) \geq \limsup_n \frac{1}{n} \cdot \log N(f^n)$$

Proof. Let δ be such that every loop in X of diameter $< 2\delta$ is contractible. Let $\epsilon > 0$ be a smaller number such that $d(f(x), f(y)) < \delta$ whenever $d(x, y) < 2\epsilon$. Let $E_n \subset X$ be a set consisting of one point from each essential fixed point class of f^n. Thus $\mid E_n \mid = N(f^n)$. By the definition of $h(f)$, it suffices to show that E_n is (n, ϵ)-separated. Suppose it is not so. Then there would be two points $x \neq y \in E_n$ such that $d(f^i(x), f^i(y)) \leq \epsilon$ for $o \leq i < n$ hence for all $i \geq 0$. Pick a path c_i from $f^i(x)$ to $f^i(y)$ of diameter $< 2\epsilon$ for $o \leq i < n$ and let $c_n = c_0$. By the choice of δ and ϵ , $f \circ c_i \simeq c_{i+1}$ for all i, so $f^n \circ c_0 \simeq c_n = c_0$. This means x, y in the same fixed point class of f^n, contradicting the construction of E_n.

□

This inequality is remarkable in that it does not require smoothness of the map and provides a common lower bound for the topological entropy of all maps in a homotopy class.

3.2. **Asymptotic invariant.** Let $\Gamma = \pi_0(Diff^+(M))$ be the mapping class group of a closed connected oriented surface M of genus ≥ 2. Pick an everywhere positive two-form ω on M. A isotopy theorem of Moser [21] says that each mapping class of $g \in \Gamma$, i.e. an isotopy class of $Diff^+(M)$, admits representatives which preserve ω. Due to Seidel[25] and Cotton-Clay [2] we can pick a monotone(weakly monotone) representative $\phi \in \mathrm{Symp}^m(M, \omega)$(or $\phi \in \mathrm{Symp}^{wm}(M, \omega)$ of g such that $HF_*(\phi)$ is an invariant as ϕ is deformed through monotone(weakly monotone) symplectomorphisms. These imply that we have a symplectic Floer homology invariant $HF_*(g)$ canonically assigned to each mapping class g given by $HF_*(\phi)$ for any monotone(weakly monotone) symplectomorphism ϕ.

Note that $HF_*(g)$ is independent of the choice of an area form ω by Moser's theorem and naturality of Floer homology.

Taking a dynamical point of view, we consider now the iterates of monotone(weakly monotone) symplectomorphism ϕ. Symplectomorphisms ϕ^n are also monotone(weakly monotone) for all $n > 0$ [15, 2].

The growth rate of a sequence a_n of complex numbers is defined by

$$\mathrm{Growth}(a_n) := max\{1, \limsup_{n\to\infty} |a_n|^{1/n}\}$$

which could be infinity. Note that $\mathrm{Growth}(a_n) \geq 1$ even if all $a_n = 0$. When $\mathrm{Growth}(a_n) > 1$, we say that the sequence a_n grows exponentially.

In [9] we have introduced the asymptotic invariant $F^\infty(g)$ assigned to mapping class $g \in Mod_M = \pi_0(Diff^+(M))$ via the growth rate of the sequence $\{a_n = \dim HF_*(\phi^n)\}$ for a monotone(or weakly monotone) representative $\phi \in \mathrm{Symp}^m(M, \omega)$ of g:

$$F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n))$$

Example 3.2. If ϕ is a non-trivial orientation preserving standard form periodic diffeomorphism of a compact connected surface M of Euler characteristic $\chi(M) < 0$, then the periodicity of the sequence $\dim HF_*(\phi^n)$ implies that for the corresponding mapping class g the asymptotic invariant

$$F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = 1$$

Example 3.3. Let ϕ be a monotone diffeomorphism of finite type of a compact connected surface M of Euler characteristic $\chi(M) < 0$ and g a corresponding algebraically finite mapping class. Then the total dimension of $HF_*(\phi^n)$ grows at most linearly (see [2, 28, 10]. Taking the growth rate in n, we get that the asymptotic invariant

$$F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = 1$$

.

For any set S let $\mathbb{Z}S$ denote the free Abelian group with the specified basis S.The norm in $\mathbb{Z}S$ is defined by

$$\|\sum_i k_i s_i\| := \sum_i \mid k_i \mid \in \mathbb{Z}, \tag{10}$$

when the s_i in S are all different.

For a $\mathbb{Z}H$-matrix $A = (a_{ij})$,define its norm by $\|A\| := \sum_{i,j} \|a_{ij}\|$.Then we have inequalities $\|AB\| \leq \|A \mid \cdot \|B\|$ when A, B can be multiplied, and $\|\operatorname{tr} A\| \leq \|A\|$ when A is a square matrix.For a matrix $A = (a_{ij})$ in $\mathbb{Z}S$, its matrix of norms is defined to be the matrix $A^{norm} := (\|a_{ij}\|)$ which is a matrix of non-negative integers.In what follows, the set S wiill be π, H or H_c.We denote by $s(A)$ the spectral radius of A, $s(A) = \lim_n \sqrt[n]{\|A^n\|}$ which coincide with the largest modul of an eigenvalue of A.

Remark 3.4. The norm $\|L_H(f^n)\|$ is the sum of absolute values of the indices of all the n-orbits classes O^n . It equals $\|L_\pi(f^n)\|$, the sum of absolute values of the indices of all the fixed point classes of f^n, because any two fixed point classes of f^n contained in the same n-orbit class O^n must have the same index. The norm $\|L_\pi(f^n)\|$ is homotopy type invariant.

We define the asymptotic absolute Lefschetz number [19] to be the growth rate

$$L^\infty(f) = \text{Growth}(\|L_\pi(f^n)\|)$$

We also define the asymptotic Nielsen number[18] to be the growth rate

$$N^\infty(f) = \text{Growth}(N(f^n))$$

All these asymptotic numbers are homotopy type invariants.

Lemma 3.5. *If ϕ is any symplectomorphism with nondegenerate fixed points in given pseudo-Anosov mapping class g, then*

$$\dim HF_*(\phi) = \dim HF_*(g) = \|L_\pi(\psi)\|,$$

where ψ is a singular canonical pseudo-Anosov representative of g.

Proof. It is known that for pseudo-Anosov map ψ fixed points are topologically separated, i.e. each essential fixed point class of ψ consists of a single fixed point(see [29, 18, 8]). Then the generalized Lefschetz number or the Reidemeister trace [19] is

$$L_\pi(\psi) := \sum_F \operatorname{ind}(F, \psi) \cdot cd_\pi(F, \psi) = \sum_{x \in \operatorname{Fix}(\psi)} \operatorname{ind}(x) \cdot cd_\pi(x, \psi) \in \mathbb{Z}\pi_\psi, \tag{11}$$

where the summation being over all essential fixed point classes F of ψ i.e over all fixed points of ψ. So, the result follows from the theorem 2.7 and the definition of the norm $\|L_\pi(\psi)\|$.

Remark 3.6. Lemma 3.5 provides via Reidemeister trace formula a new combinatorial formula to compute $\dim HF_*(g)$ comparable to the train-track combinatorial formula of A. Cotton-Clay in [2].

Theorem 3.7. [5, 18, 19] *Let f be a pseudo-Anosov homeomorphism with dilatation $\lambda > 1$ of surfase M of genus ≥ 2. Then*

$$h(f) = log(\lambda) = \log N^{\infty}(f) = \log L^{\infty}(f)$$

Theorem 3.8. [19] *Suppose f is canonical representative of a homeomorphism of surfase M of genus ≥ 2 and λ is the largest dilatation of the pseudo-Anosov components($\lambda = 1$ if there is no pseudo-Anosov components). Then*

$$h(f) = log(\lambda) = \log N^{\infty}(f) = \log L^{\infty}(f)$$

Theorem 3.9. *If ϕ is any symplectomorphism with nondegenerate fixed points in given pseudo-Anosov mapping class g with dilatation $\lambda > 1$ of surfase M of genus ≥ 2. Then*

$$F^{\infty}(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = \lambda = \exp(h(\psi)) = L^{\infty}(\psi) = N^{\infty}(\psi)$$

where ψ is a canonical singular pseudo-Anosov representative of g.

Proof. By lemma 3.5 we have that $\dim HF_*(\phi^n) = \|L_\pi(\psi^n)\|$ for every n. So, the result follows from theorem 3.7.

Theorem 3.10. *Let $\bar{\phi}$ be a perturbed standard form map ϕ in a reducible mapping class g of compact surface of genus ≥ 2 and λ is the largest dilatation of the pseudo-Anosov components($\lambda = 1$ if there is no pseudo-Anosov components). Then*
$F^{\infty}(g) := \mathrm{Growth}(\dim HF_*((\bar{\phi})^n)) = \lambda = \exp(h(\psi)) = L^{\infty}(\psi) = N^{\infty}(\psi)$ *where ψ is canonical representative of mapping class g.*

Proof. It follows from the theorem 2.8 that for every n

$$\begin{aligned}\dim HF_*(g^n) = \dim HF_*((\bar{\phi})^n) = \dim H_*(M_a, \partial_+ M_{\mathrm{id}}; \mathbb{Z}_2)+ \\ + \sum_p (\dim H_*(M^0_{b,p}, \partial_+ M_{b,p}; \mathbb{Z}_2) + (p-1)|\pi_0(M_{b,p})|)+ \\ + \sum_q (\dim H_*(M_{c,q}, \partial_+ M_{c,q}; \mathbb{Z}_2) + q|\pi_0(M_{c,q)})|)+ \\ + L((\bar{\phi})^n|M_1) + \dim HF_*((\bar{\phi})^n|M_2).\end{aligned}$$

We need to investigate only the growth of the last summand in this formula because the rest part in the formula grows at most linearly [2, 28]. We have $\dim HF_*((\bar{\phi})^n|M_2) = \sum_j \dim HF_*((\bar{\phi}_j)^n|M_2)$, where the sum is taken over different pseudo-Anosov components of $\phi^n|M_2$. It follows from the theorem 3.9 that $\dim HF_*((\bar{\phi}_j)^n|M_2)$ grows as λ_j^n, where λ_j is the dilatation of the pseudo-Anosov component $\bar{\phi}_j|M_2$.

Taking growth rate in n, we get $F^{\infty}(g) := \mathrm{Growth}(\dim HF_*((\bar{\phi})^n)) = \max_j \lambda_j = \lambda = \exp(h(\psi))$.

Corollary 3.11. *The asymptotic invariant $F^{\infty}(g) > 1$ if and only if ψ has a pseudo-Anosov component.*

Although the exact evaluation of the asymptotic invariant $F^{\infty}(g)$ would be desirable, its estimation is a more realistic goal. We carry out such estimation using notations and results from sections 2.2.1 and 2.2.2.

Proposition 3.12. *Suppose $\rho : H \to U(n)$ is a unitary representation and ψ is canonical representative of reducible mapping class g of compact surface of genus ≥ 2. Let w be a zero or a pole of the rational function $L_\rho^\psi(t) \in \mathbb{C}(t)$. Then*

$$\frac{1}{\mid w \mid} \leq F^\infty(g) \leq \max_d \|z\tilde{F}_d\|$$

Proof. We know from complex analysis and definition of the twisted Lefschetz zeta function that $Growth(L_\rho(\psi^n))$ is the reciprocal of the radius of convergence of the function $log(L_\rho^\psi(t)$, hence $Growth(L_\rho(\psi^n)) \geq \frac{1}{|w|}$.

On other hand, according to section 2.2.1, H-coordinates of n-orbit classes are in the form $[z^n]g$ with $g \in \pi$. So we can assume $L_H(\psi^n) = \sum_i k_i[z^n g_i]$, where the $[z^n g_i]$ are different conjugacy classes in H. Since the trace of a unitary matrix is bounded by its dimension, we get that $L_\rho(\psi^n) \in \mathbb{C}$ are bounded by

$$\mid L_\rho(\psi^n) \mid = \mid \sum_i k_i \operatorname{tr}(z^n g_i)^\rho \mid \leq \sum_i \mid k_i \mid\mid \operatorname{tr}(z^n g_i)^\rho \mid \leq \sum_i \mid k_i \mid = \|L_H(\psi^n)\|.$$

Hence $Growth(L_\rho(\psi^n)) \leq L^\infty(\psi)$. From theorem 3.10 it follows that $F^\infty(g) = L^\infty(\psi)$. So we get the estimation from below $\frac{1}{|w|} \leq F^\infty(g)$. The initial data of our lower estimation is the knowledge of the $\mathbb{Z}H$-matrices $\tilde{F}_d$ provided by a cellular map, which enables us to compute the twisted Lefschetz zeta function. There is also a way to derive an upper bound from the same data. We have

$$\begin{aligned}\dim HF_*(g^n) &= \|L_\pi(\psi^n)\| = \|L_H(\psi^n)\| = \|\sum_d (-1)^d [\operatorname{tr}(z\tilde{F}_d)^n]\| \leq \sum_d \|[\operatorname{tr}(z\tilde{F}_d)^n]\| \\ &\leq \sum_d ||\operatorname{tr}(z\tilde{F}_d)^n|| \leq \sum_d \operatorname{tr}((z\tilde{F}_d)^n)^{norm} \leq \sum_d \operatorname{tr}((z\tilde{F}_d)^{norm})^n \\ &\leq \sum_d \operatorname{tr}((\tilde{F}_d)^{norm})^n.\end{aligned}$$

Hence

$$\begin{aligned}F^\infty(g) &= Growth(\|L_\pi(\psi^n)\|) = Growth(\|L_H(\psi^n)\|) \leq Growth(\sum_d \operatorname{tr}((\tilde{F}_d)^{norm})^n) \\ &= max_d(Growth(\operatorname{tr}((\tilde{F}_d)^{norm})^n)) = max_d(s(\tilde{F}_d)^{norm}).\end{aligned}$$

Remark 3.13. A practical difficulty in the use of $L_\rho(\psi^n))$ and twisted Lefschetz zeta function $L_\rho^\psi(t)$ for estimation is to find a useful representation ρ. Following approach of Boju Jiang in [19], section 1.7 we can weaken the assumption on ρ in proposition 3.12. There are many examples in [19], chapter 4 which illustrate method of estimation above for surface homeomorphisms.

3.3. **Radius of convergence of the symplectic zeta function.** In [9] we have introduced a symplectic zeta function

$$F_g(t) = F_\phi(t) \quad = \quad \exp\left(\sum_{n=1}^\infty \frac{\dim HF_*(\phi^n)}{n} t^n\right)$$

assigned to mapping class g via zeta function $F_\phi(t)$ of a monotone(or weakly monotone) representative $\phi \in \mathrm{Symp}^m(M,\omega)$ of g. Symplectomorphisms ϕ^n are also monotone(weakly monotone) for all $n>0$ [15, 2] so, symplectic zeta function $F_\phi(t)$ is an invariant as ϕ is deformed through monotone(or weakly monotone) symplectomorphisms in g. These imply that we have a symplectic Floer homology invariant $F_g(t)$ canonically assigned to each mapping class g. A motivation for the definition of this zeta function was a connection [9, 15] between Nielsen numbers and Floer homology and nice analytic properties of Nielsen zeta function [23, 6, 9, 7, 8, 11, 12]

We denote by R the radius of convergence of the symplectic zeta function $F_g(t)=F_\phi(t)$.

In this section we give exact algebraic lower estimation for the radius R using Reidemeister trace formula for generalized Lefschetz numbers from section 2.2.1

Theorem 3.14. *If ϕ is any symplectomorphism with nondegenerate fixed points in given pseudo-Anosov mapping class g with dilatation $\lambda>1$ of compact surface M of genus ≥ 2, then the symplectic zeta function $F_g(t)$ has positive radius of convergence $R=\frac{1}{\lambda}$. Radius of convergence R admits following estimations*

$$R \geq \frac{1}{\max_d \|z\tilde{F}_d\|} > 0 \tag{12}$$

and

$$R \geq \frac{1}{\max_d s(\tilde{F}_d^{norm})} > 0 \tag{13}$$

Proof. It follows from lemma 3.5 that $\dim HF_*(\phi^n)=\|L_\pi(\psi^n)\|$. By the homotopy type invariance of the right hand side we can estimate it. We can suppose that ψ is a cell map of a finite cell complex.The norm $\|L_H(\psi^n)\|$ is the sum of absolute values of the indices of all the n-orbits classes O^n . It equals $\|L_\pi(\psi^n)\|$, the sum of absolute values of the indices of all the fixed point classes of ψ^n, because any two fixed point classes of ψ^n contained in the same n-orbit class O^n must have the same index. From this we have $\dim HF_*(\phi^n)=\|L_\pi(\psi^n)\|=\|L_H(\psi^n)\|=\|\sum_d(-1)^d[\mathrm{tr}(z\tilde{F}_d)^n]\| \leq \sum_d \|[\mathrm{tr}(z\tilde{F}_d)^n]\| \leq \sum_d \|\mathrm{tr}(z\tilde{F}_d)^n\| \leq \sum_d \|(z\tilde{F}_d)^n\| \leq \sum_d \|(z\tilde{F}_d)\|^n$. The radius of convergence R is given by Caushy-Adamar formula:

$$\frac{1}{R}=\limsup_n \sqrt[n]{\frac{\dim HF_*(\phi^n)}{n}}=\limsup_n \sqrt[n]{\dim HF_*(\phi^n)}=\lambda.$$

Therefore we have:

$$R=\frac{1}{\limsup_n \sqrt[n]{\dim HF_*(\phi^n)}} \geq \frac{1}{\max_d \|z\tilde{F}_d\|} > 0.$$

Inequalities:

$$\begin{aligned}\dim HF_*(\phi^n)=\|L_\pi(\psi^n)\|=\|L_H(\psi^n)\|=\|\sum_d(-1)^d[\mathrm{tr}(z\tilde{F}_d)^n]\| \leq \sum_d \|[\mathrm{tr}(z\tilde{F}_d)^n]\| \\ \leq \sum_d \|\mathrm{tr}(z\tilde{F}_d)^n\| \leq \sum_d \mathrm{tr}((z\tilde{F}_d)^n)^{norm} \leq \sum_d \mathrm{tr}((z\tilde{F}_d)^{norm})^n \\ \leq \sum_d \mathrm{tr}((\tilde{F}_d)^{norm})^n\end{aligned}$$

and the definition of spectral radius give estimation:

$$R = \frac{1}{\limsup_n \sqrt[n]{\dim HF_*(\phi^n)}} \geq \frac{1}{\max_d s(\tilde{F}_d^{norm})} > 0.$$

Theorem 3.15. *Let $\bar{\phi}$ be a perturbed standard form map ϕ in a reducible mapping class g of compact surface of genus ≥ 2 and λ is the largest dilatation of the pseudo-Anosov components($\lambda = 1$ if there is no pseudo-Anosov components). Then the symplectic zeta function $F_g(t) = F_{\bar{\phi}}(t)$ has positive radius of convergence $R = \frac{1}{\lambda}$, where λ is the largest dilatation of the pseudo-Anosov components($\lambda = 1$ if there is no pseudo-Anosov components).*

Proof. The radius of convergence R is given by Caushy-Adamar formula:

$$\frac{1}{R} = \limsup_n \sqrt[n]{\frac{\dim HF_*(\phi^n)}{n}} = \limsup_n \sqrt[n]{\dim HF_*(\phi^n)}.$$

By Theorem 3.10 we have

$$\limsup_n \sqrt[n]{\dim HF_*(\phi^n)} = \mathrm{Growth}(\dim HF_*(\phi^n)) = \lambda$$

Example 3.16. Let X be surface with boundary, and $f : X \to X$ be a map.Fadell and Husseini(see [19]) devised a method of computing the matrices of the lifted chain map for surface maps.Suppose $\{a_1, ..., a_r\}$ is a free basis for $\pi_1(X)$. Then X has the homotopy type of a bouquet B of r circles which can be decomposed into one 0-cell and r 1-cells corresponding to the a_i,and f has the homotopy type of a cellular map $g : B \to B$. By the homotopy type invariance of the invariants,we can replace f with g in computations.The homomorphism $\tilde{f}_* : \pi_1(X) \to \pi_1(X)$ induced by f and g is determined by the images $b_i = \tilde{f}_*(a_i), i = 1, .., r$.The fundamental group $\pi_1(T_f)$ has a presentation $\pi_1(T_f) =< a_1, ..., a_r, z | a_i z = z b_i, i = 1, .., r >$.Let

$$D = (\frac{\partial b_i}{\partial a_j})$$

be the Jacobian in Fox calculus(see [19]).Then,as pointed out in [19], the matrices of the lifted chain map $\tilde{g}$ are

$$\tilde{F}_0 = (1), \tilde{F}_1 = D = (\frac{\partial b_i}{\partial a_j}).$$

Now, we can find estimations for the radius R as above.

Let $\mu(d), d \in \mathbb{N}$, be the Möbius function.

Theorem 3.17. [9] *Let ϕ be a non-trivial orientation preserving standard form periodic diffeomorphism of least period m of a compact connected surface M of Euler characteristic $\chi(M) < 0$. Then the symplectic zeta function $F_g(t) = F_\phi(t)$ is a radical of a rational function and*

$$F_g(t) = F_\phi(t) = \prod_{d|m} \sqrt[d]{(1 - t^d)^{-P(d)}},$$

where the product is taken over all divisors d of the period m, and $P(d)$ is the integer $P(d) = \sum_{d_1|d} \mu(d_1) \dim HF_*(\phi^{d/d_1})$.

We denote by $L_\phi(t)$ the Weil zeta function

$$L_\phi(t) := \exp\left(\sum_{n=1}^{\infty} \frac{L(\phi^n)}{n} t^n\right),$$

where $L(\phi^n)$ is the Lefschetz number of of ϕ^n.

Theorem 3.18. [9] *If ϕ is a hyperbolic diffeomorphism of a 2-dimensional torus T^2, then the symplectic zeta function $F_g(t) = F_\phi(t)$ is a rational function and $F_g(t) = F_\phi(t) = (L_\phi(\sigma \cdot t))^{(-1)^r}$, where r is equal to the number of $\lambda_i \in Spec(\tilde{\phi})$ such that $\mid \lambda_i \mid > 1$, p is equal to the number of $\mu_i \in Spec(\tilde{\phi})$ such that $\mu_i < -1$ and $\sigma = (-1)^p$, here $\tilde{\phi}$ is a lifting of ϕ to the universal cover.*

3.4. **Concluding remarks and questions.**

Remark 3.19. For a symplectic manifold X the (conjugation-invariant) *Floer-type entropy* of $g \in Symp(X)/Ham(X)$, a mapping class of ϕ, is defined in [28] as

$$h_F(g) \ = \ \limsup \frac{1}{n} \log rk\, HF(\phi^n) = \log F^\infty(g)$$

This is a kind of robust version of the periodic entropy, robust in the sense that it depends on a symplectic diffeomorphism only through its mapping class; by contrast topological and periodic entropy are typically very sensitive to perturbation. As we proved above, for area-preserving diffeomorphisms of a surface M, the Floer-type entropy coincides with the topological entropy of the canonical representative in corresponding mapping class; moreover, $h_F(g) > 0$ if and only if g has a pseudo-Anosov component.

Question 3.20. *(Entropy conjecture for symplectomorphisms)*
Is it always true that for symplectomorphisms of compact symplectic manifolds

$$h(\phi) \geq \log F^\infty(g) = \log \mathrm{Growth}(\dim HF_*(\phi^n)) = h_F(g)?$$

Question 3.21. *(A weak version of the Entropy conjecture for symplectomorphisms)*
Is it always true that for symplectomorphisms of compact symplectic manifolds

$$h(\phi) \geq \log \mathrm{Growth}(\mid \chi(HF_*(\phi^n)) \mid)?$$

Here $\chi(HF_*(\phi^n))$ is the Euler characteristic of symplectic Floer homology of ϕ^n. If for every n all the fixed points of ϕ^n are non-degenerate, i.e. for all $x \in \mathrm{Fix}(\phi^n)$, $\det(\mathrm{id} - \mathrm{d}\phi^n(x)) \neq 0$, then

$$\chi(HF_*(\phi^n)) = \sum_{x=\phi^n(x)} \mathrm{sign}(\det(\mathrm{id} - \mathrm{d}\phi^n(x))) = L(\phi^n).$$

This implies that the question above is a version of the question of Shub [27].

Question 3.22. *Is it true that for a symplectomorphism ϕ of an aspherical compact symplectic manifold*

$$F^\infty(g) := \mathrm{Growth}(\dim HF_*(\phi^n)) = L^\infty(\phi) = N^\infty(\phi)?$$

Inspired by the Hasse-Weil zeta function of an algebraic variety over a finite field, Artin and Mazur [1] defined the zeta function for an arbitrary map $f : X \to X$ of a topological space X:

$$AM_f(t) := \exp\left(\sum_{n=1}^{\infty} \frac{\#\operatorname{Fix}(f^n)}{n} t^n\right),$$

where $\#\operatorname{Fix}(f^n)$ is the number of isolated fixed points of f^n. Artin and Mazur showed that for a dense set of the space of smooth maps of a compact smooth manifold into itself the number of periodic points $\#\operatorname{Fix}(f^n)$ grows at most exponentially and the Artin-Mazur zeta function $AM_f(t)$ has a positive radius of convergence [1]. Later Manning [20] proved the rationality of the Artin - Mazur zeta function for diffeomorphisms of a smooth compact manifold satisfying Smale axiom A. On the other hand there exist maps for which Artin-Mazur zeta function is transcendental . The symplectic zeta function $F_\phi(t)$ can be considered as some analog of the Artin-Mazur zeta function $AM_f(t)$ because periodic points of ϕ^n provide the generators of symplectic Floer homologies $HF_*(\phi^n)$. This motivate following

Conjecture 3.23. *For any compact symplectic manifold M and symplectomorphism $\phi : M \to M$ with well defined Floer homology groups $HF_*(\phi^n)$, $n \in \mathbb{N}$ the symplectic zeta function $F_g(t) = F_\phi(t)$ has a positive radius of convergence.*

Question 3.24. *Is the symplectic zeta function $F_g(t) = F_\phi(t)$ an algebraic function of z?*

Remark 3.25. Given a symplectomorphism ϕ of surface M, one can form the symplectic mapping torus $M^4_\phi = T^3_\phi \rtimes S^1$, where T^3_ϕ is usual mapping torus . Ionel and Parker [17] have computed the degree zero Gromov invariants [17](these are built from the invariants of Ruan and Tian) of M^4_ϕ and of fiber sums of the M^4_ϕ with other symplectic manifolds. This is done by expressing the Gromov invariants in terms of the Lefschetz zeta function $L_\phi(z)$ [17]. The result is a large set of interesting non-Kahler symplectic manifolds with computational ways of distinguishing them. In dimension four this gives a symplectic construction of the exotic elliptic surfaces of Fintushel and Stern [14]. This construction arises from knots. Associated to each fibered knot K in S^3 is a Riemann surface M and a monodromy diffeomorphism f_K of M. Taking $\phi = f_K$ gives symplectic 4-manifolds $M^4{}_\phi(K)$ with Gromov invariant $Gr(M^4{}_\phi(K)) = A_K(t)/(1-t)^2 = L_\phi(t)$, where $A_K(t)$ is the Alexander polynomial of knot K. Next, let $E^4(n)$ be the simply-connected minimal elliptic surface with fiber F and canonicla divisor $k = (n-2)F$. Forming the fiber sum $E^4(n,K) = E^4(n)\#_{(F=T^2)}M^4{}_\phi(K)$ we obtain a symplectic manifold homeomorphic to $E^4(n)$. Then for $n \geq 2$ the Gromov and Seiberg-Witten invariants of $E^4(K)$ are $Gr(E^4(n,K)) = SW(E^4(n,K)) = A_K(t)(1-t)^{n-2}$ [14, 17]. Thus fibered knots with distinct Alexander polynomials give rise to symplectic manifolds $E^4(n,K)$ which are homeomorphic but not diffeomorphic. In particular, there are infinitely many distinct symplectic 4-manifolds homeomorphic to $E^4(n)$ [14] .

In higher dimensions it gives many examples of manifolds which are diffeomorphic but not equivalent as symplectic manifolds. Theorem 13 in [9] implies that the Gromov invariants of M^4_ϕ are related to symplectic Floer homology of ϕ via Lefschetz zeta function $L_\phi(t)$. We hope that the symplectic zeta function $F_\phi(t)$ give rise to a new invariant of symplectic 4-manifolds.

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