Field Rotation Parameters and Limit Cycle Bifurcations

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Abstract

In this paper, the global qualitative analysis of planar polynomial dynamical systems is established and a new geometric approach to solving *Hilbert's Sixteenth Problem* on the maximum number and relative position of their limit cycles in two special cases of such systems is suggested. First, using geometric properties of four field rotation parameters of a new canonical system which is constructed in this paper, we present a proof of our earlier conjecture that the maximum number of limit cycles in a quadratic system is equal to four and the only possible their distribution is $(3:1)$ [15]. Then, by means of the same geometric approach, we solve the *Problem* for Liénard's polynomial system (in this special case, it is considered as *Smale's Thirteenth Problem*). Besides, generalizing the obtained results, we present a solution of *Hilbert's Sixteenth Problem* on the maximum number of limit cycles surrounding a singular point for an arbitrary polynomial system and, applying the Wintner–Perko termination principle for multiple limit cycles, we develop an alternative approach to solving the *Problem.* By means of this approach, for example, we give another proof of the main theorem for a quadratic system and complete the global qualitative analysis of a generalized Liénard's cubic system with three finite singularities. We discuss also some different approaches to the *Problem.*

Keywords: planar polynomial dynamical system; Liénard's polynomial system; generalized Liénard's cubic system; quadratic system; Hilbert's sixteenth problem; Smale's thirteenth problem; field rotation parameter; bifurcation; limit cycle

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1 Introduction

We consider planar dynamical systems

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \tag{1.1}$$

where P_n and Q_n are polynomials with real coefficients in the real variables x, y. The main problem of qualitative theory of such systems is *Hilbert's Sixteenth Problem* on the maximum number and relative position of their limit cycles, i. e., closed isolated trajectories of (1.1) [15], [21]. This *Problem* was formulated as one of the fundamental problems for mathematicians of the XX century, however it has not been solved even in the simplest cases of the polynomial systems: for quadratic systems (when $n = 2$) and for Liénard's polynomial system of the form

$$\dot{x} = y, \quad \dot{y} = -x + \mu_1 \, y + \mu_2 \, y^2 + \mu_3 \, y^3 + \ldots + \mu_{2k} \, y^{2k} + \mu_{2k+1} \, y^{2k+1}. \tag{1.2}$$

In the case of system (1.2), it is considered as *Smale's Thirteenth Problem* becoming one of the main problems for mathematicians of the XXI century [21], [30].

In this paper, we suggest a new geometric approach [17] to studying limit cycle bifurcations of (1.1) and to solving the *Problem* in these two special case of polynomial systems. In particular, in Section 2, we construct two canonical quadratic systems with field rotation parameters, one of which contains four such parameters. Using the canonical systems and geometric properties of the spirals filling the interior and exterior domains of limit cycles, we present a solution of the *Problem* on the maximum number and relative position of limit cycles in the case of quadratic systems. In Section 3, by means of the same geometric approach, we solve *Smale's Thirteenth Problem* for Liénard's polynomial system (1.2). In Section 4, generalizing the obtained results, we present a solution of *Hilbert's Sixteenth Problem* on the maximum number of limit cycles surrounding a singular point for an arbitrary polynomial system. In Section 5, applying the Wintner–Perko termination principle for multiple limit cycles, we develop an alternative approach to solving the *Problem.* By means of this approach, for example, we give another proof of the main theorem for a quadratic system and complete the global qualitative analysis of a generalized Liénard's cubic system with three finite singularities. Finally, in Conclusion, we discuss some different approaches to *Hilbert's Sixteenth Problem.*

2 A quadratic system

In [4], [15], we constructed a canonical quadratic system with two field rotation parameters for studying limit cycle bifurcations:

$$\dot{x} = P(x,y) + \alpha\, Q(x,y), \qquad \dot{y} = Q(x,y) - \alpha P(x,y), \tag{2.1}$$

where

$$P(x,y) = -y + mxy + (n-\gamma)y^2, \quad Q(x,y) = x - x^2 + \gamma\, xy + cy^2.$$

In [4], [15], we show also by which linear transformations of the phase variables x, y arbitrary quadratic system (1.1), where $n=2$, is reduced to form (2.1) and how the parameters of (2.1) are expressed via the parameters of (1.1). System (2.1) is especially convenient for the investigation of quadratic systems in the case two finite singularities when the parameters α, γ rotate the field of (2.1) in the whole phase plane x, y.

Later, we constructed a canonical system with three field rotation parameters, $\alpha,\ \beta,\ \lambda,$

$$\dot{x} = -(1+x)\, y + \alpha\, Q(x,y), \quad \dot{y} = Q(x,y), \tag{2.2}$$

where

$$Q(x,y) = x + \lambda y + ax^2 + \beta(1+x)y + cy^2,$$

which, together with the system

$$\dot{x} = -y + \nu y^2, \quad \dot{y} = Q(x,y), \quad \nu = 0; 1, \tag{2.3}$$

can be used in an arbitrary case of finite singularities [15].

Applying a similar approach, we can construct a canonical system with the maximum number of field rotation parameters, namely: with four such parameters. It is valid the following theorem.

Theorem 2.1. *A quadratic system with limit cycles can be reduced to the canonical form*

$$\begin{aligned} \dot{x} &= -y\,(1+x+\alpha\, y) \equiv P, \\ \dot{y} &= x + (\lambda+\beta+\gamma)y + a\,x^2 + (\alpha+\beta+\gamma)xy + c\,\gamma\, y^2 \equiv Q \end{aligned} \tag{2.4}$$

or

$$\begin{aligned} \dot{x} &= -y\,(1+\nu\, y), \quad \nu = 0; 1, \\ \dot{y} &= x + (\lambda+\beta+\gamma)y + a\,x^2 + (\beta+\gamma)xy + c\,\gamma\, y^2. \end{aligned} \tag{2.5}$$

Proof. In [15] is shown that an arbitrary quadratic system with limit cycles, by means of Erugun's two-isocline method [10], can be reduced to the form

$$\begin{aligned} \dot{x} &= -y + mxy + ny^2, \\ \dot{y} &= x + \lambda y + ax^2 + bxy + cy^2, \end{aligned} \tag{2.6}$$

where $m = -1$ or $m = 0$.

Input the field rotation parameters into this system so that (2.4) corresponds to the case of $m = -1$ and (2.5) corresponds to the case of $m = 0$.

Compare (2.4) with (2.6) when $m = -1$. Firstly, we have changed several parameters: n by $-\alpha$; b by β; c by $c\,\gamma$. Secondly, we have input additional terms into the expression for $\dot{y}:$ $(\beta+\gamma)\,y$ and $(\alpha+\gamma)\,xy$. Similar transformations have been made in system (2.6) when $m = 0$; but in this case, we have denoted n by ν assigning two principal values to this parameter: 0 and 1. It is obvious that all these transformations do not restrict generality of systems (2.4) and (2.5) in comparison with system (2.6), what proves the theorem.

System (2.4) will be a basic system for studying limit cycle bifurcations. It contains four field rotation parameters: λ, α, β, γ. The following lemma is valid for this system (a similar lemma is valid for system (2.5), with respect to the parameters λ, β, γ).

Lemma 2.1. *Each of the parameters λ, β, γ, and α rotates the vector field of* (2.4) *in the domains of existence of its limit cycles, under the fixed other parameters of this system, namely*: *when the parameter λ, β, γ, or α increases* (*decreases*), *the field is rotated in positive* (*negative*) *direction, i. e., counterclockwise* (*clockwise*), *in the domains, respectively*:

$$\begin{gathered} 1 + x + \alpha\, y < 0 \ (> 0); \\ (1 + x)(1 + x + \alpha\, y) < 0 \ (> 0); \\ (1 + x + c\, y)(1 + x + \alpha\, y) < 0 \ (> 0); \\ (\lambda + \beta + \gamma)\, y + (a - 1)\, x^2 + (\beta + \gamma)\, xy + c\,\gamma\, y^2 < 0 \ (> 0). \end{gathered}$$

Proof. Using the definition of a field rotation parameter [6], [15] we can calculate the following determinants:

$$\begin{gathered} \Delta_\lambda = PQ'_\lambda - QP'_\lambda = -y^2(1 + x + \alpha\, y); \\ \Delta_\beta = PQ'_\beta - QP'_\beta = -y^2(1 + x)(1 + x + \alpha\, y); \\ \Delta_\gamma = PQ'_\gamma - QP'_\gamma = -y^2(1 + x + c\, y)(1 + x + \alpha\, y); \\ \Delta_\alpha = PQ'_\alpha - QP'_\alpha = \ y^2((\lambda + \beta + \gamma)\, y + (a - 1)\, x^2 + (\beta + \gamma)\, xy + c\,\gamma\, y^2). \end{gathered}$$

Since, by definition, the vector field is rotated in positive direction (counterclockwise) when the determinant is positive and in negative direction (clockwise) when the determinant is negative [6], [15] and since the obtained domains correspond to the domains of existence of limit cycles of (2.4), the lemma is proved.

By means of canonical systems (2.4) and (2.5), we will study global limit cycle bifurcations of (1.1), where $n = 2$. First of all, let us give a new proof of the following theorem.

Theorem 2.2. *A quadratic system can have at least four limit cycles in the* $(3:1)$*-distribution.*

Proof. To prove the theorem, consider the case of two finite anti-saddles and the only saddle at infinity when, for example, $a = 1/2$ and $c = -1$ in (2.4):

$$\begin{aligned} \dot{x} &= -y\,(1 + x + \alpha\, y), \\ \dot{y} &= x + (\lambda + \beta + \gamma)\, y + (1/2)\, x^2 + (\alpha + \beta + \gamma)\, xy - \gamma\, y^2. \end{aligned} \tag{2.7}$$

Vanish all field rotation parameters: $\alpha = \beta = \gamma = \lambda = 0$. Then we have got a system with two centers which is symmetric with respect to the x-axis.

Under increasing the parameter γ $(0 < \gamma \ll 1)$, the vector field of (2.7) is rotated in negative direction (clockwise) and the centers turn into foci: $(0, 0)$ becomes an unstable focus and $(-2, 0)$ becomes a stable one.

Fix γ and take λ satisfying the condition: $-1 \ll \lambda < -\gamma < 0$ $(-1 \ll \gamma + \lambda < 0)$. Then, in the half-plane $x > -1$, the vector field of (2.7) is rotated in positive direction and the focus $(0, 0)$ changes the character of its stability generating an unstable limit cycle. In the half-plane $x < -1$, the field is rotated in negative direction again and the focus $(-2, 0)$ remains stable.

Fix the parameters γ, λ and take α satisfying the condition: $\gamma + \lambda \ll \alpha < 0$. After rotation of the vector field of system (2.7) in positive direction, the straight line $x = 1$ is destroyed and two limit cycles are generated by the separatrix cycles formed by this line and two Poincaré hemi-circles: a stable limit cycle surrounding the focus $(0, 0)$ and an unstable one surrounding the focus $(-2, 0)$.

Finally, fix the parameters γ, λ, α and take β satisfying the condition: $0 < -\gamma - \lambda < \beta \ll 1$ $(0 < \beta + \gamma + \lambda \ll -\alpha)$. Then, after rotation of the vector field in negative direction in the whole phase plane, the focus $(0, 0)$ changes the character of its stability again generating a stable limit cycle, since the parameter α is non-rough and negative when $\beta = -\gamma - \lambda$. Thus, we have obtained at least three limit cycles surrounding the focus $(0, 0)$, under the co-existence of a limit cycle surrounding the focus $(-2, 0)$, what proves the theorem.

It is valid a much stronger theorem.

Theorem 2.3. *A quadratic system has at most four limit cycles and only in the* $(3:1)$*-distribution.*

Proof. Consider again the most interesting case of quadratic systems: with two finite anti-saddles and the only saddle at infinity when $a = 1/2$ and $c = -1$ in (2.4). All other cases of singular points can be considered in a similar way.

Vanish all field rotation parameters of system (2.7), $\alpha = \beta = \gamma = \lambda = 0$:

$$\begin{aligned} \dot{x} &= -y\,(1 + x), \\ \dot{y} &= x + (1/2)\,x^2. \end{aligned} \tag{2.8}$$

We have got a system with two centers which is symmetric with respect to the x-axis. Let us input successively the field rotation parameters into (2.8).

Begin, for example, with the parameter γ supposing that $\gamma > 0$:

$$\begin{aligned} \dot{x} &= -y\,(1 + x), \\ \dot{y} &= x + \gamma\,y + (1/2)\,x^2 + \gamma\,xy - \gamma\,y^2. \end{aligned} \tag{2.9}$$

Under increasing γ, the vector field of (2.9) is rotated in negative direction (clockwise) and the centers turn into foci: $(0, 0)$ becomes an unstable focus and $(-2, 0)$ becomes a stable one.

Fix γ and input a new parameter, for example, $\lambda < 0$ into (2.9):

$$\begin{aligned} \dot{x} &= -y\,(1 + x), \\ \dot{y} &= x + (\lambda + \gamma)\,y + (1/2)\,x^2 + \gamma\,xy - \gamma\,y^2. \end{aligned} \tag{2.10}$$

Then, in the half-plane $x > -1$, the vector field of (2.10) is rotated in positive direction (counterclockwise) and the focus $(0, 0)$ changes the character of its stability (when $\lambda = -\gamma$) generating an unstable limit cycle. Under decreasing λ, this limit cycle will expand until it disappears in a Poincaré hemi-cycle with a saddle-node lying on the invariant straight line $x = -1$ [15]. In the half-plane $x < -1$, the field is rotated in negative direction again and the focus $(-2, 0)$ remains stable.

Denote the limit cycle by Γ_1, the domain inside the cycle by D_1, the domain outside the cycle by D_2 and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a "trajectory concentration" surrounding the focus $(0, 0)$. It is clear that under decreasing λ, a semi-stable limit cycle cannot appear in the domain D_1, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation in positive direction.

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain D_2. Suppose it appears in this domain for some values of the parameters $\gamma^* > 0$ and $\lambda^* < 0$. Return to initial system (2.8) and change the order of inputting the field rotation parameters. Input first the parameter $\lambda < 0$:

$$\begin{aligned} \dot{x} &= -y\,(1+x), \\ \dot{y} &= x + \lambda\, y + (1/2)\, x^2. \end{aligned} \tag{2.11}$$

Fix it under $\lambda = \lambda^*$. In the half-plane $x > -1$, the vector field of (2.11) is rotated in negative direction and $(0,0)$ becomes a stable focus. Inputting the parameter $\gamma > 0$ into (2.11), we have got again system (2.10), the vector field of which is rotated in positive direction in the half-plane $x > -1$. Under this rotation, an unstable limit cycle Γ_1 will appear from a Poincaré hemi-cycle with a saddle-node on the invariant straight line $x = -1$. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing the parameter γ to the value $\gamma = \gamma^*$. It follows that there are no values of $\gamma = \gamma^*$ and $\lambda = \lambda^*$, for which a semi-stable limit cycle could appear in the domain D_2.

This contradiction proves the uniqueness of a limit cycle surrounding the focus $(0,0)$ in system (2.10) for any values of the parameters γ and λ of different signs. Obviously, if these parameters have the same sign, system (2.10) has no limit cycles surrounding $(0,0)$ at all, like there are no limit cycles surrounding the focus $(-2,0)$ for the parameters γ and λ of different signs.

Let system (2.10) have the unique limit cycle Γ_1. Fix the parameters $\gamma > 0$, $\lambda < 0$ and input the third parameter, $\alpha < 0$, into this system:

$$\begin{aligned} \dot{x} &= -y\,(1+x+\alpha\, y), \\ \dot{y} &= x + (\lambda+\gamma)\, y + (1/2)\, x^2 + (\alpha+\gamma)\, xy - \gamma\, y^2. \end{aligned} \tag{2.12}$$

The vector field of (2.12) is rotated in positive direction again, the invariant straight line $x = -1$ is immediately destroyed and two limit cycles appear from the corresponding Poincaré hemi-cycles containing this straight line: a stable cycle, denoted by Γ_2, surrounding the focus $(0,0)$ and an unstable limit cycle, denoted by Γ_3, surrounding the focus $(-2,0)$. Under further decreasing α, the limit cycle Γ_2 will join with Γ_1 forming a semi-stable limit cycle, Γ_{12}, which will disappear in a “trajectory concentration” surrounding the origin $(0,0)$. Can another semi-stable limit cycle appear around the origin in addition to Γ_{12}? It is clear that such a limit cycle cannot appear neither in the domain D_1 bounded by the origin and Γ_1 nor in the domain D_3 bounded on the inside by Γ_2 because of increasing the distance between the spiral coils filling these domains under decreasing α.

To prove impossibility of the appearance of a semi-stable limit cycle in the domain D_2 bounded by the cycles Γ_1 and Γ_2 (before their joining), suppose the contrary, i. e., for some set of values of the parameters $\gamma^* > 0$, $\lambda^* < 0$, and $\alpha^* < 0$, such a semi-stable cycle exists. Return to system (2.8) again and input the parameters $\alpha < 0$ and $\lambda < 0$:

$$\begin{aligned} \dot{x} &= -y\,(1 + x + \alpha\, y), \\ \dot{y} &= x + \lambda\, y + (1/2)\, x^2 + \alpha\, xy. \end{aligned} \tag{2.13}$$

In the half-plane $x > -1$, both parameters act in a similar way: they rotate the vector field of (2.13) in positive direction turning the origin $(0, 0)$ into a stable focus. In the half-plane $x < -1$, they rotate the field in opposite directions generating an unstable limit cycle from the focus $(-2, 0)$.

Fix these parameters under $\alpha = \alpha^*$, $\lambda = \lambda^*$ and input the parameter $\gamma > 0$ into (2.13) getting again system (2.12). Since, on our assumption, this system has two limit cycles for $\gamma < \gamma^*$, there exists some value of the parameter, γ_{12} $(0 < \gamma_{12} < \gamma^*)$, for which a semi-stable limit cycle, Γ_{12}, appears in system (2.12) and then splits into an unstable cycle, Γ_1, and a stable cycle, Γ_2, under further increasing γ. The formed domain D_2 bounded by the limit cycles Γ_1, Γ_2 and filled by the spirals will enlarge, since, on the properties of a field rotation parameter, the interior unstable limit cycle Γ_1 will contract and the exterior stable limit cycle Γ_2 will expand under increasing γ. The distance between the spirals of the domain D_2 will naturally increase, what will prohibit from the appearance of a semi-stable limit cycle in this domain for $\gamma > \gamma_{12}$. Thus, there are no such values of the parameters, $\gamma^* > 0, \lambda^* < 0, \alpha^* < 0$, for which system (2.12) would have an additional semi-stable limit cycle.

Obviously, there are no other values of the parameters γ, λ, α, for which system (2.12) would have more than two limit cycles surrounding the origin $(0, 0)$ and simultaneously more than one limit cycle surrounding the point $(-2, 0)$ (on the same reasons). It follows that system (2.12) can have at most three limit cycles and only in the $(2 : 1)$-distribution.

Suppose that system (2.12) has two limit cycle, Γ_1 and Γ_2, around the origin $(0, 0)$ and the only limit cycle, Γ_3, around the point $(-2, 0)$. Fix the parameters $\gamma > 0$, $\lambda < 0$, $\alpha < 0$ and input the fourth parameter, $\beta > 0$, into (2.12) getting system (2.7). Under increasing β, the vector field of (2.7) is rotated in negative direction, the focus $(0, 0)$ changes the character of its stability (when $\beta = -\gamma - \lambda$) and a stable limit cycle, Γ_0, appears from the origin. Suppose it happens before the cycle Γ_1 disappears in $(0, 0)$ (this is possible by Theorem 2.2). Under further increasing β, the cycle Γ_0 will join with Γ_1 forming a semi-stable limit cycle, Γ_{01}, which will disappear in a “trajectory concentration” surrounding the origin $(0, 0)$; the other cycles, Γ_2 and Γ_3, will expand tending to Poincaré hemi-cycles with the straight line $x = -1$.

Let system (2.7) have four limit cycles: Γ_0, Γ_1, Γ_2, and Γ_3. Can an additional semi-stable limit cycle appear around the origin under increasing the parameter β? It is clear that such a limit cycle cannot appear neither in the domain D_0 bounded by the origin and Γ_0 nor in the domain D_2 bounded by Γ_1 and Γ_2 because of increasing the distance between the spiral coils filling these domains under increasing β. Consider two other domains: D_1 bounded by the cycles Γ_0, Γ_1 and D_3 bounded on the inside by the cycle Γ_2. As before, we will prove impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters, $\gamma^* > 0$, $\lambda^* < 0$, $\alpha^* < 0$, and $\beta^* > 0$, such a semi-stable cycle exists. Return to system (2.8) again and input first the parameters $\beta > 0$, $\gamma > 0$ and then the parameter $\alpha < 0$:

$$\begin{aligned} \dot{x} &= -y\,(1 + x + \alpha\, y), \\ \dot{y} &= x + (\beta + \gamma)\, y + (1/2)\, x^2 + (\alpha + \beta + \gamma)\, xy - \gamma\, y^2. \end{aligned} \tag{2.14}$$

Fix the parameters β, γ under the values β^*, γ^*, respectively. Under decreasing the parameter α, two limit cycles immediately appear from Poincaré hemicycles with the straight line $x = -1$: a stable cycle, Γ_2, around $(0, 0)$ and an unstable one, Γ_3, around $(-2, 0)$. Fix α under the value α^* and input the parameter $\lambda < 0$ into (2.14) getting system (2.7).

Since, on our assumption, system (2.7) has three limit cycles around the origin $(0, 0)$ for $\lambda > \lambda^*$, there exists some value of the parameter, λ_{01} ($\lambda^* < \lambda_{01} < 0$), for which a semi-stable limit cycle, Γ_{01}, appears in this system and then splits into a stable cycle, Γ_0, and an unstable cycle, Γ_1, under further decreasing λ. The formed domain D_1 bounded by the limit cycles Γ_0, Γ_1 and also the domain D_3 bounded on the inside by the limit cycle Γ_2 will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there, i. e., at most three limit cycles can exist around the origin $(0, 0)$. On the same reasons, a semi-stable limit cannot appear around the point $(-2, 0)$ under decreasing the parameter λ, i. e., at most one limit cycle can exist around this point simultaneously with three limit cycles surrounding $(0, 0)$.

All other combinations of the parameters λ, α, β, γ are considered in a similar way. It follows that system (2.7) has at most four limit cycles and only in the $(3 : 1)$-distribution. Applying the same approach to canonical system (2.5), we can complete the proof of the theorem.

3 Liénard's polynomial system

System (1.2) and more general Liénard's systems have been studying in numerous works (see, for example, [1], [2], [5], [11], [17], [19], [22]–[26], [29]). It is easy to see that (1.2) has the only finite singularity: an anti-saddle at the origin. At infinity, system (1.2) for $k \geqslant 1$ has two singular points: a node at the "ends" of the y-axis and a saddle at the "ends" of the x-axis. For studying the infinite singularities, the methods applied in [1] for Rayleigh's and van der Pol's equations and also Erugin's two-isocline method developed in [15] can be used. Following [15], we will study limit cycle bifurcations of (1.2) by means of a canonical system containing only the field rotation parameters of (1.2). It is valid the following theorem.

Theorem 3.1. *Liénard's polynomial system* (1.2) *with limit cycles can be reduced to the canonical form*

$$\dot{x} = y \equiv P, \quad \dot{y} = -x + \mu_1\, y + y^2 + \mu_3\, y^3 + \ldots + y^{2k} + \mu_{2k+1}\, y^{2k+1} \equiv Q, \quad (3.1)$$

where $\mu_1,\ \mu_3, \ldots,\ \mu_{2k+1}$ *are field rotation parameters of* (3.1).

Proof. Vanish all odd parameters of (1.2),

$$\dot{x} = y, \quad \dot{y} = -x + \mu_2\, y^2 + \mu_4\, y^4 + \ldots + \mu_{2k}\, y^{2k}, \qquad (3.2)$$

and consider the corresponding equation

$$\frac{dy}{dx} = \frac{-x + \mu_2\, y^2 + \mu_4\, y^4 + \ldots + \mu_{2k}\, y^{2k}}{y} \equiv F(x, y). \qquad (3.3)$$

Since $F(x, -y) = -F(x, y)$, the direction field of (3.3) (and the vector field of (3.2) as well) is symmetric with respect to the x-axis. It follows that for arbitrary values of the parameters $\mu_2,\ \mu_4, \ldots,\ \mu_{2k}$ system (3.2) has a center at the origin and cannot have a limit cycle surrounding this point. Therefore, without loss of generality, all even parameters of system (1.2) can be supposed to be equal, for example, to one: $\mu_2 = \mu_4 = \ldots = \mu_{2k} = 1$ (they could be also supposed to be equal to zero).

To prove that the rest (odd) parameters rotate the vector field of (3.1), let us calculate the following determinants:

$$\begin{gathered}
\Delta_{\mu_1} = PQ'_{\mu_1} - QP'_{\mu_1} = y^2 \geq 0, \\
\Delta_{\mu_3} = PQ'_{\mu_3} - QP'_{\mu_3} = y^2 \geq 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\Delta_{\mu_{2k+1}} = PQ'_{\mu_{2k+1}} - QP'_{\mu_{2k+1}} = y^2 \geq 0.
\end{gathered}$$

By definition of a field rotation parameter [6], for increasing each of the parameters $\mu_1, \mu_3, \ldots, \mu_{2k+1}$, under the fixed others, the vector field of system (3.1) is rotated in positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (3.1) is rotated in negative direction (clockwise).

Thus, for studying limit cycle bifurcations of (1.2), it is sufficient to consider canonical system (3.1) containing only its odd parameters, $\mu_1, \mu_3, \ldots, \mu_{2k+1}$, which rotate the vector field of (3.1). The theorem is proved.

By means of canonical system (3.1), let us study global limit cycle bifurcations of (1.2) and prove the following theorem.

Theorem 3.2. *Liénard's polynomial system* (1.2) *has at most k limit cycles.*

Proof. According to Theorem 3.1, for the study of limit cycle bifurcations of system (1.2), it is sufficient to consider canonical system (3.1) containing only the field rotation parameters of (1.2): $\mu_1, \mu_3, \ldots, \mu_{2k+1}$.

Vanish all these parameters:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + y^{2k}. \tag{3.4}$$

System (3.4) is symmetric with respect to the x-axis and has a center at the origin. Let us input successively the field rotation parameters into this system beginning with the parameters at the highest degrees of y and alternating with their signs. So, begin with the parameter μ_{2k+1} and let, for definiteness, $\mu_{2k+1} > 0$:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + y^{2k} + \mu_{2k+1}\, y^{2k+1}. \tag{3.5}$$

In this case, the vector field of (3.5) is rotated in positive direction (counterclockwise) turning the origin into a nonrough unstable focus.

Fix μ_{2k+1} and input the parameter $\mu_{2k-1} < 0$ into (3.5):

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + \mu_{2k-1}\, y^{2k-1} + y^{2k} + \mu_{2k+1}\, y^{2k+1}. \tag{3.6}$$

Then the vector field of (3.6) is rotated in opposite direction (clockwise) and the focus immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of y changes) generating a stable limit cycle. Under further decreasing μ_{2k-1}, this limit cycle will expand infinitely, not disappearing at infinity (because of the parameter μ_{2k+1} at the higher degree of y).

Denote the limit cycle by Γ_1, the domain outside the cycle by D_1, the domain inside the cycle by D_2 and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a “trajectory concentration” surrounding the origin. It is clear that, under decreasing the parameter μ_{2k-1}, a semi-stable limit cycle cannot appear in the domain D_2, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation.

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain D_1. Suppose it appears in this domain for some values of the parameters $\mu^*_{2k+1} > 0$ and $\mu^*_{2k-1} < 0$. Return to initial system (3.4) and change the inputting order for the field rotation parameters. Input first the parameter $\mu_{2k-1} < 0$:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + y^4 + \ldots + \mu_{2k-1}\, y^{2k-1} + y^{2k}. \tag{3.7}$$

Fix it under $\mu_{2k-1} = \mu^*_{2k-1}$. The vector field of (3.7) is rotated clockwise and the origin turns into a nonrough stable focus. Inputting the parameter $\mu_{2k+1} > 0$ into (3.7), we get again system (3.6), the vector field of which is rotated counterclockwise. Under this rotation, a stable limit cycle Γ_1 will immediately appear from infinity, more precisely, from a separatrix cycle of the Poincaré circle form containing infinite singularities of the saddle and node types [1]. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing μ_{2k+1} to the value μ^*_{2k+1}. It follows that there are no values of $\mu^*_{2k-1} < 0$ and $\mu^*_{2k+1} > 0$, for which a semi-stable limit cycle could appear in the domain D_1.

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (3.6) for any values of the parameters μ_{2k-1} and μ_{2k+1} of different signs. Obviously, if these parameters have the same sign, system (3.6) has no limit cycles surrounding the origin at all.

Let system (3.6) have the unique limit cycle Γ_1. Fix the parameters $\mu_{2k+1} > 0$, $\mu_{2k-1} < 0$ and input the third parameter, $\mu_{2k-3} > 0$, into this system:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + \ldots + \mu_{2k-3}\, y^{2k-3} + y^{2k-2} + \ldots + \mu_{2k+1}\, y^{2k+1}. \tag{3.8}$$

The vector field of (3.8) is rotated counterclockwise, the focus at the origin changes the character of its stability and the second (unstable) limit cycle, Γ_2, immediately appears from this point. Under further increasing μ_{2k-3}, the limit cycle Γ_2 will join with Γ_1 forming a semi-stable limit cycle, Γ_{12}, which will disappear in a “trajectory concentration” surrounding the origin. Can another semi-stable limit cycle appear around the origin in addition to Γ_{12}? It is clear that such a limit cycle cannot appear neither in the domain D_1 bounded on the inside by the cycle Γ_1 nor in the domain D_3 bounded by the origin and Γ_2 because of increasing the distance between the spiral coils filling these domains under increasing the parameter μ_{2k-3}.

To prove impossibility of the appearance of a semi-stable limit cycle in the domain D_2 bounded by the cycles Γ_1 and Γ_2 (before their joining), suppose the contrary, i. e., for some set of values of the parameters, $\mu^*_{2k+1} > 0$, $\mu^*_{2k-1} < 0$, and $\mu^*_{2k-3} > 0$, such a semi-stable cycle exists. Return to system (3.4) again and input first the parameters $\mu_{2k-3} > 0$ and $\mu_{2k+1} > 0$:

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + \ldots + \mu_{2k-3}\, y^{2k-3} + y^{2k-2} + y^{2k} + \mu_{2k+1}\, y^{2k+1}. \quad (3.9)$$

Both parameters act in a similar way: they rotate the vector field of (3.9) counterclockwise turning the origin into a nonrough unstable focus.

Fix these parameters under $\mu_{2k-3} = \mu^*_{2k-3}$, $\mu_{2k+1} = \mu^*_{2k+1}$ and input the parameter $\mu_{2k-1} < 0$ into (3.9) getting again system (3.8). Since, on our assumption, this system has two limit cycles for $\mu_{2k-1} > \mu^*_{2k-1}$, there exists some value of the parameter, μ^{12}_{2k-1} ($\mu^*_{2k-1} < \mu^{12}_{2k-1} < 0$), for which a semi-stable limit cycle, Γ_{12}, appears in system (3.8) and then splits into a stable cycle, Γ_1, and an unstable cycle, Γ_2, under further decreasing μ_{2k-1}. The formed domain D_2 bounded by the limit cycles Γ_1, Γ_2 and filled by the spirals will enlarge since, on the properties of a field rotation parameter, the interior unstable limit cycle Γ_2 will contract and the exterior stable limit cycle Γ_1 will expand under decreasing μ_{2k-1}. The distance between the spirals of the domain D_2 will naturally increase, what will prohibit from the appearance of a semi-stable limit cycle in this domain for $\mu_{2k-1} < \mu^{12}_{2k-1}$.

Thus, there are no such values of the parameters, $\mu^*_{2k+1} > 0$, $\mu^*_{2k-1} < 0$, and $\mu^*_{2k-3} > 0$, for which system (3.8) would have an additional semi-stable limit cycle. Obviously, there are no other values of the parameters μ_{2k+1}, μ_{2k-1}, and μ_{2k-3} for which system (3.8) would have more than two limit cycles surrounding the origin. Therefore, two is the maximum number of limit cycles for system (3.8). This result agrees with [29], where it was proved for the first time that the maximum number of limit cycles for Liénard's system of the form

$$\dot{x} = y, \quad \dot{y} = -x + \mu_1\, y + \mu_3\, y^3 + \mu_5\, y^5 \quad (3.10)$$

was equal to two.

Suppose that system (3.8) has two limit cycles, Γ_1 and Γ_2 (this is always possible if $\mu_{2k+1} \gg -\mu_{2k-1} \gg \mu_{2k-3} > 0$), fix the parameters μ_{2k+1}, μ_{2k-1}, μ_{2k-3} and consider a more general system than (3.8) (and (3.10)) inputting the fourth parameter, $\mu_{2k-5} < 0$, into (3.8):

$$\dot{x} = y, \quad \dot{y} = -x + y^2 + \ldots + \mu_{2k-5}\, y^{2k-5} + y^{2k-4} + \ldots + \mu_{2k+1}\, y^{2k+1}. \quad (3.11)$$

Under decreasing μ_{2k-5}, the vector field of (3.11) will be rotated clockwise and the focus at the origin will immediately change the character of its stability generating the third (stable) limit cycle, Γ_3. Under further decreasing μ_{2k-5}, Γ_3 will join with Γ_2 forming a semi-stable limit cycle, Γ_{23}, which will disappear in a "trajectory concentration" surrounding the origin; the cycle Γ_1 will expand infinitely tending to the Poincaré circle at infinity.

Let system (3.11) have three limit cycles: Γ_1, Γ_2, Γ_3. Could an additional semi-stable limit cycle appear under decreasing μ_{2k-5}, after splitting of which system (3.11) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear neither in the domain D_2 bounded by the cycles Γ_1 and Γ_2 nor in the domain D_4 bounded by the origin and Γ_3 because of increasing the distance between the spiral coils filling these domains under decreasing μ_{2k-5}. Consider two other domains: D_1 bounded on the inside by the cycle Γ_1 and D_3 bounded by the cycles Γ_2 and Γ_3. As before, we will prove impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters $\mu^*_{2k+1} > 0$, $\mu^*_{2k-1} < 0$, $\mu^*_{2k-3} > 0$, and $\mu^*_{2k-5} < 0$, such a semi-stable cycle exists. Return to system (3.4) again, input first the parameters $\mu_{2k-5} < 0$, $\mu_{2k-1} < 0$ and then the parameter $\mu_{2k+1} > 0$:

$$\dot{x} = y, \quad \dot{y} = -x+y^2+\ldots+\mu_{2k-5}y^{2k-5}+\ldots+\mu_{2k-1}y^{2k-1}+y^{2k}+\mu_{2k+1}y^{2k+1}. \quad (3.12)$$

Fix the parameters μ_{2k-5}, μ_{2k-1} under the values μ^*_{2k-5}, μ^*_{2k-1}, respectively. Under increasing μ_{2k+1}, the node at infinity will change the character of its stability, the separatrix behaviour of the infinite saddle will be also changed and a stable limit cycle, Γ_1, will immediately appear from the Poincaré circle at infinity [1]. Fix μ_{2k+1} under the value μ^*_{2k+1} and input the parameter $\mu_{2k-3} > 0$ into (3.12) getting system (3.11).

Since, on our assumption, (3.11) has three limit cycles for $\mu_{2k-3} < \mu^*_{2k-3}$, there exists some value of the parameter μ^{23}_{2k-3} $(0 < \mu^{23}_{2k-3} < \mu^*_{2k-3})$ for which a semi-stable limit cycle, Γ_{23}, appears in this system and then splits into an unstable cycle, Γ_2, and a stable cycle, Γ_3, under further increasing μ_{2k-3}. The formed domain D_3 bounded by the limit cycles Γ_2, Γ_3 and also the domain D_1 bounded on the inside by the limit cycle Γ_1 will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there.

All other combinations of the parameters μ_{2k+1}, μ_{2k-1}, μ_{2k-3}, and μ_{2k-5} are considered in a similar way. It follows that system (3.11) has at most three limit cycles. If we continue the procedure of successive inputting the odd parameters, $\mu_{2k-7}, \ldots, \mu_3, \mu_1$, into system (3.4), it is possible first to obtain k limit cycles ($\mu_{2k+1} \gg -\mu_{2k-1} \gg \mu_{2k-3} \gg -\mu_{2k-5} \gg \mu_{2k-7} \gg \ldots$) and then to conclude that canonical system (3.1) (i. e., Liénard's polynomial system (1.2) as well) has at most k limit cycles. The theorem is proved.

4 An arbitrary polynomial system

Let us consider an arbitrary polynomial system

$$\dot{x} = P_n(x, y, \mu_1, \ldots, \mu_k), \quad \dot{y} = Q_n(x, y, \mu_1, \ldots, \mu_k) \tag{4.1}$$

containing k field rotation parameters, $\mu_1, \ldots, \mu_k$, and having an anti-saddle at the origin. Generalizing the main result of the previous section on the maximum number of limit cycles surrounding a singular point in Liénard's polynomial system (1.2), we prove the following theorem.

Theorem 4.1. *Polynomial system* (4.1) *containing k field rotation parameters and having a singular point of the center type at the origin for the zero values of these parameters can have at most $k-1$ limit cycles surrounding the origin.*

Proof. Vanish all parameters of (4.1) and suppose that the obtained system

$$\dot{x} = P_n(x, y, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, 0, \ldots, 0) \tag{4.2}$$

has a singular point of the center type at the origin. Let us input successively the field rotation parameters, $\mu_1, \ldots, \mu_k$, into this system.

Suppose, for example, that $\mu_1 > 0$ and that the vector field of the system

$$\dot{x} = P_n(x, y, \mu_1, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, 0, \ldots, 0) \tag{4.3}$$

is rotated counterclockwise turning the origin into a stable focus under increasing μ_1.

Fix μ_1 and input the parameter μ_2 into (4.3) changing it so that the field of the system

$$\dot{x} = P_n(x, y, \mu_1, \mu_2, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_2, 0, \ldots, 0) \tag{4.4}$$

would be rotated in opposite direction (clockwise). Let be so for $\mu_2 < 0$. Then, for some value of this parameter, a limit cycle will appear in system (4.4). There are three logical possibilities for such a bifurcation: 1) the limit cycle appears from the focus at the origin; 2) it can also appear from some separatrix cycle surrounding the origin; 3) the limit cycle appears from a so-called "trajectory concentration". In the last case, the limit cycle is semi-stable and, under further decreasing μ_2, it splits into two limit cycles (stable and unstable), one of which then disappears at (or tends to) the origin and the other disappears on (or tends to) some separatrix cycle surrounding this point. But since the stability character of both a singular point and a separatrix cycle is quite easily controlled [15], this logical possibility can be excluded. Let us choose one of the two other possibilities: for example, the first one, the so-called Andronov–Hopf bifurcation. Suppose that, for some value of μ_2, the focus at the origin becomes non-rough, changes the character of its stability and generates a stable limit cycle, Γ_1.

Under further decreasing μ_2, three new logical possibilities can arise: 1) the limit cycle Γ_1 disappears on some separatrix cycle surrounding the origin; 2) a separatrix cycle can be formed earlier than Γ_1 disappears on it, then it generates one more (unstable) limit cycle, Γ_2, which joins with Γ_1 forming a semi-stable limit cycle, Γ_{12}, disappearing in a "trajectory concentration" under further decreasing μ_2; 3) in the domain D_1 outside the cycle Γ_1 or in the domain D_2 inside Γ_1, a semi-stable limit cycle appears from a "trajectory concentration" and then splits into two limit cycles (logically, the appearance of such semi-stable limit cycles can be repeated).

Let us consider the third case. It is clear that, under decreasing μ_2, a semi-stable limit cycle cannot appear in the domain D_2, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation. By contradiction, we can prove that a semi-stable limit cycle cannot appear in the domain D_1. Suppose it appears in this domain for some values of the parameters $\mu_1^* > 0$ and $\mu_2^* < 0$. Return to initial system (4.2) and change the inputting order for the field rotation parameters. Input first the parameter $\mu_2 < 0$:

$$\dot{x} = P_n(x, y, \mu_2, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_2, 0, \ldots, 0). \tag{4.5}$$

Fix it under $\mu_2 = \mu_2^*$. The vector field of (4.5) is rotated clockwise and the origin turns into a unstable focus. Inputting the parameter $\mu_1 > 0$ into (4.5), we get again system (4.4), the vector field of which is rotated counterclockwise. Under this rotation, a stable limit cycle, Γ_1, will appear from some separatrix cycle. The limit cycle Γ_1 will contract, the outside spirals winding onto this cycle will untwist and the distance between their coils will increase under increasing μ_1 to the value μ_1^*. It follows that there are no values of $\mu_2^* < 0$ and $\mu_1^* > 0$, for which a semi-stable limit cycle could appear in the domain D_1.

The second logical possibility can be excluded by controlling the stability character of the separatrix cycle [15]. Thus, only the first possibility is valid, i. e., system (4.4) has at most one limit cycle.

Let system (4.4) have the unique limit cycle Γ_1. Fix the parameters $\mu_1 > 0$, $\mu_2 < 0$ and input the third parameter, $\mu_3 > 0$, into this system supposing that μ_3 rotates its vector field counterclockwise:

$$\dot{x} = P_n(x, y, \mu_1, \mu_2, \mu_3, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_2, \mu_3, 0, \ldots, 0). \tag{4.6}$$

Here we can have two basic possibilities: 1) the limit cycle Γ_1 disappears at the origin; 2) the second (unstable) limit cycle, Γ_2, appears from the origin and, under further increasing the parameter μ_3, the cycle Γ_2 joins with Γ_1 forming a semi-stable limit cycle, Γ_{12}, which disappears in a "trajectory concentration" surrounding the origin. Besides, we can also suggest that: 3) in the domain D_2 bounded by the origin and Γ_1, a semi-stable limit cycle, Γ_{23}, appears from a "trajectory concentration", splits into an unstable cycle, Γ_2, and a stable cycle, Γ_3, and then the cycles Γ_1, Γ_2 disappear through a semi-stable limit cycle, Γ_{12}, and the cycle Γ_3 disappears through the Andronov–Hopf bifurcation; 4) a semi-stable limit cycle, Γ_{34}, appears in the domain D_2 bounded by the cycles Γ_1, Γ_2 and, for some set of values of the parameters, μ_1^*, μ_2^*, μ_3^*, system (4.6) has at least four limit cycles.

Let us consider the last, fourth, case. It is clear that a semi-stable limit cycle cannot appear neither in the domain D_1 bounded on the inside by the cycle Γ_1 nor in the domain D_3 bounded by the origin and Γ_2 because of increasing the distance between the spiral coils filling these domains under increasing the parameter μ_3. To prove impossibility of the appearance of a semi-stable limit cycle in the domain D_2, suppose the contrary, i. e., for some set of values of the parameters, $\mu_1^* > 0$, $\mu_2^* < 0$, and $\mu_3^* > 0$, such a semi-stable cycle exists. Return to system (4.2) again and input first the parameters $\mu_3 > 0$, $\mu_1 > 0$:

$$\dot{x} = P_n(x, y, \mu_1, \mu_3, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_3, 0, \ldots, 0). \tag{4.7}$$

Fix these parameters under $\mu_3 = \mu_3^*$, $\mu_1 = \mu_1^*$ and input the parameter $\mu_2 < 0$ into (4.7) getting again system (4.6). Since, on our assumption, this system has two limit cycles for $\mu_2 > \mu_2^*$, there exists some value of the parameter, μ_2^{12} ($\mu_2^* < \mu_2^{12} < 0$), for which a semi-stable limit cycle, Γ_{12}, appears in system (4.6) and then splits into a stable cycle, Γ_1, and an unstable cycle, Γ_2, under further decreasing μ_2. The formed domain D_2 bounded by the limit cycles Γ_1, Γ_2 and filled by the spirals will enlarge, since, on the properties of a field rotation parameter, the interior unstable limit cycle Γ_2 will contract and the exterior stable limit cycle Γ_1 will expand under decreasing μ_2. The distance between the spirals of the domain D_2 will naturally increase, what will prohibit from the appearance of a semi-stable limit cycle in this domain for $\mu_2 < \mu_2^{12}$.

Thus, there are no such values of the parameters, $\mu_1^* > 0$, $\mu_2^* < 0$, $\mu_3^* > 0$, for which system (4.6) would have an additional semi-stable limit cycle. Therefore, the fourth case cannot be realized. The third case is considered absolutely similarly. It follows from the first two cases that system (4.6) can have at most two limit cycles.

Suppose that system (4.6) has two limit cycles, Γ_1 and Γ_2, fix the parameters $\mu_1 > 0$, $\mu_2 < 0$, $\mu_3 > 0$ and input the fourth parameter, $\mu_4 < 0$, into this system supposing that μ_4 rotates its vector field clockwise:

$$\dot{x} = P_n(x, y, \mu_1, \ldots, \mu_4, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \ldots, \mu_4, 0, \ldots, 0). \tag{4.8}$$

The most interesting logical possibility here is that when the third (stable) limit cycle, Γ_3, appears from the origin and then, under preservation of the cycles Γ_1 and Γ_2, in the domain D_3 bounded on the inside by the cycle Γ_3 and on the outside by the cycle Γ_2, a semi-stable limit cycle, Γ_{45}, appears and then splits into a stable cycle, Γ_4, and an unstable cycle, Γ_5, i. e., when system (4.8) for some set of values of the parameters, μ_1^*, μ_2^*, μ_3^*, μ_4^*, has at least five limit cycles. Logically, such a semi-stable limit cycle could also appear in the domain D_1 bounded on the inside by the cycle Γ_1, since, under decreasing μ_4, the spirals of the trajectories of (4.8) will twist and the distance between their coils will decrease. On the other hand, in the domain D_2 bounded on the inside by the cycle Γ_2 and on the outside by the cycle Γ_1 and also in the domain D_4 bounded by the origin and Γ_3, a semi-stable limit cycle cannot appear, since, under decreasing μ_4, the spirals will untwist and the distance between their coils will increase. To prove impossibility of the appearance of a semi-stable limit cycle in the domains D_3 and D_1, suppose the contrary, i. e., for some set of values of the parameters, $\mu_1^* > 0$, $\mu_2^* < 0$, $\mu_3^* > 0$, and $\mu_4^* < 0$, such a semi-stable cycle exists. Return to system (4.2) again, input first the parameters $\mu_4 < 0$, $\mu_2 < 0$ and then the parameter $\mu_1 > 0$:

$$\dot{x} = P_n(x, y, \mu_1, \mu_2, \mu_4, 0, \ldots, 0), \quad \dot{y} = Q_n(x, y, \mu_1, \mu_2, \mu_4, 0, \ldots, 0). \tag{4.9}$$

Fix the parameters μ_4, μ_2 under the values μ_4^*, μ_2^*, respectively. Under increasing μ_1, a separatrix cycle is formed around the origin generating a stable limit cycle, Γ_1. Fix μ_1 under the value μ_1^* and input the parameter $\mu_3 > 0$ into (4.9) getting system (4.8).

Since, on our assumption, system (4.8) has three limit cycles for $\mu_3 < \mu_3^*$, there exists some value of the parameter μ_3^{23} $(0 < \mu_3^{23} < \mu_3^*)$ for which a semi-stable limit cycle, Γ_{23}, appears in this system and then splits into an unstable cycle, Γ_2, and a stable cycle ,Γ_3, under further increasing μ_3. The formed domain D_3 bounded by the limit cycles Γ_2, Γ_3 and also the domain D_1 bounded on the inside by the limit cycle Γ_1 will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there.

All other combinations of the parameters μ_1, μ_2, μ_3, and μ_4 are considered in a similar way. It follows that system (4.8) has at most three limit cycles. If we continue the procedure of successive inputting the field rotation parameters, $\mu_5, \mu_6, \ldots, \mu_k$, into system (4.2), it is possible to conclude that system (4.1) can have at most $k-1$ limit cycles surrounding the origin. The theorem is proved.

5 The Wintner–Perko termination principle

For the global analysis of limit cycle bifurcations in [15], we used the Wintner–Perko termination principle which was stated for relatively prime, planar, analytic systems and which connected the main bifurcations of limit cycles [27], [31]. Let us formulate this principle for the polynomial system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \tag{5.1$\boldsymbol{\mu}$}$$

where $\boldsymbol{x} \in \mathbf{R}^2$; $\boldsymbol{\mu} \in \mathbf{R}^n$; $\boldsymbol{f} \in \mathbf{R}^2$ ($\boldsymbol{f}$ is a polynomial vector function).

Theorem 5.1 (Wintner–Perko termination principle). *Any one-parameter family of multiplicity-m limit cycles of relatively prime polynomial system* (5.1$\boldsymbol{\mu}$) *can be extended in a unique way to a maximal one-parameter family of multiplicity-m limit cycles of* (5.1$\boldsymbol{\mu}$) *which is either open or cyclic.*

If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of (5.1$\boldsymbol{\mu}$), *which is typically a fine focus of multiplicity* m, *or on a* (*compound*) *separatrix cycle of* (5.1$\boldsymbol{\mu}$), *which is also typically of multiplicity* m.

The proof of the Wintner–Perko termination principle for general polynomial system (5.1$\boldsymbol{\mu}$) with a vector parameter $\boldsymbol{\mu} \in \mathbf{R}^n$ parallels the proof of the planar termination principle for the system

$$\dot{x} = P(x, y, \lambda), \quad \dot{y} = Q(x, y, \lambda) \tag{5.1$_\lambda$}$$

with a single parameter $\lambda \in \mathbf{R}$ (see [15], [27]), since there is no loss of generality in assuming that system (5.1$\boldsymbol{\mu}$) is parameterized by a single parameter λ; i. e., we can assume that there exists an analytic mapping $\boldsymbol{\mu}(\lambda)$ of $\mathbf{R}$ into $\mathbf{R}^n$ such that (5.1$\boldsymbol{\mu}$) can be written as (5.1 $_{\boldsymbol{\mu}(\lambda)}$) or even (5.1$_\lambda$) and then we can repeat everything, what had been done for system (5.1$_\lambda$) in [27]. In particular, if λ is a field rotation parameter of (5.1$_\lambda$), it is valid the following Perko's theorem on monotonic families of limit cycles.

Theorem 5.2. *If* L_0 *is a nonsingular multiple limit cycle of* (5.1$_0$), *then* L_0 *belongs to a one-parameter family of limit cycles of* (5.1$_\lambda$); *furthermore*:

1) *if the multiplicity of* L_0 *is odd, then the family either expands or contracts monotonically as* λ *increases through* λ_0;

2) *if the multiplicity of* L_0 *is even, then* L_0 *befurcates into a stable and an unstable limit cycle as* λ *varies from* λ_0 *in one sense and* L_0 *disappears as* λ *varies from* λ_0 *in the opposite sense; i. e., there is a fold bifurcation at* λ_0.

Using Theorems 5.1 and 5.2 in [12]–[16], we proved a theorem on three limit cycles around a singular point for canonical systems (2.2) and (2.3). Let us prove the same theorem using systems (2.4) and (2.5).

Theorem 5.3. *There exists no quadratic system having a swallow-tail bifurcation surface of multiplicity-four limit cycles in its parameter space. In other words, a quadratic system cannot have neither a multiplicity-four limit cycle nor four limit cycles around a singular point* (*focus*), *and the maximum multiplicity or the maximum number of limit cycles surrounding a focus is equal to three.*

Proof. The proof of this theorem is carried out by contradiction. Consider canonical systems (2.4) and (2.5), where system (2.5) represents two limit cases of (2.4).

Suppose that system (2.4) with four field rotation parameters, λ, α, β, and γ, has four limit cycles around the origin (system (2.5) is considered in a similar way). Then we get into some domain of the field rotation parameters being restricted by definite conditions on two other parameters, a and c, corresponding to one of the six cases of finite singularities which we considered in [15]. Without loss of generality, we can fix both of these parameters. Thus, there is a domain bounded by three fold bifurcation surfaces forming a swallow-tail bifurcation surface of multiplicity-four limit cycles in the space of the field rotation parameters λ, α, β, and γ.

The corresponding maximal one-parameter family of multiplicity-four limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity five (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-five limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-five limit cycles by a field-rotation parameter, according to Theorem 5.2, we will obtain a monotonic curve which, by the Wintner–Perko termination principle (Theorem 5.1), terminates either at the origin or on some separatrix cycle surrounding the origin. Since we know absolutely precisely at least the cyclicity of the singular point (Bautin's result [1]) which is equal to three, we have got a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate.

If the maximal one-parameter family of multiplicity-four limit cycles is not cyclic, on the same principle (Theorem 5.2), this again contradicts to Bautin's result not admitting the multiplicity of limit cycles higher than three. This contradiction completes the proof.

As was shown in [15], to complete the solution of *Hilbert's Sixteenth Problem* for quadratic systems, it is sufficient to prove impossibility of the $(2:2)$-distribution of limit cycles only in the case of two finite foci and a saddle at infinity. In [4] (see also [15]), using canonical system (2.1) with two field rotation parameters, α and γ, in the case of two foci and a saddle at infinity, we constructed a quadratic system with at least four limit cycles in the $(3:1)$-distribution. If to let this system have only three limit cycles in the $(2:1)$-distribution, i. e., two cycles around the focus $(0,0)$ and the only one around the focus $(1,0)$, it is easy to show impossibility of obtaining the second limit cycle around $(1,0)$ by means of the parameters α and γ. Logically, we can suppose only that a semi-stable cycle appears around the focus $(1,0)$ under the variation of a field rotation parameter, for example, α. Then, applying the Wintner–Perko termination principle, we can show that the maximal one-parameter family of multiplicity-three limit cycles parameterized by another field rotation parameter, γ, cannot terminate in the focus $(1,0)$, since it will be a rough focus for any $\alpha \neq 0$ (see [4], [15]). The same proof could be given for canonical system (2.4). Thus, we have given one more proof of Theorem 2.3 on at most four limit cycles in the only $(3:1)$-distribution for quadratic systems.

In [18], we considered a generalized Liénard's cubic system of the form:

$$\dot{x} = y, \quad \dot{y} = -x + (\lambda - \mu)\, y + (3/2)\, x^2 + \mu\, xy - (1/2)\, x^3 + \alpha\, x^2 y\,. \tag{5.2}$$

This system has three finite singularities: a saddle $(1,0)$ and two antisaddles — $(0,0)$ and $(2,0)$. At infinity system (5.2) can have either the only nilpotent singular point of fourth order with two closed elliptic and four hyperbolic domains or two singular points: one of them is a hyperbolic saddle and the other is a triple nilpotent singular point with two elliptic and two hyperbolic domains. We studied global bifurcations of limit and separatrix cycles of (5.2), found possible distributions of its limit cycles and carried out a classification of its separatrix cycles. We proved also the following theorems.

Theorem 5.4. *The foci of system* (5.2) *can be at most of second order.*

Theorem 5.5. *System* (5.2) *has at least three limit cycles.*

Using the results obtained in [18] and applying the approach developed in this paper, we can easily prove a much stronger theorem.

Theorem 5.6. *System* (5.2) *has at most three limit cycles with the following their distributions*: $((1,1),1)$, $((1,2),0)$, $((2,1),0)$, $((1,0),2)$, $((0,1),2)$, *where the first two numbers denote the numbers of limit cycles surrounding each of two anti-saddles and the third one denotes the number of limit cycles surrounding simultaneously all three finite singularities.*

Theorem 5.6 agrees, for example, with the earlier results by Iliev and Perko [20], but it does not agree with a quite recent result by Dumortier and Li [7] published in the same journal. The authors of both papers use very similar methods: small perturbations of a Hamiltonian system. In [20], the zeros of the Melnikov functions are studied and, in particular, it is proved that at most two limit cycles can bifurcate from either the interior or exterior period annulus of the Hamiltonian under small parameter perturbations giving a generalized Liénard system. In [7], zeros of the Abelian integrals are studied and it is "proved" that at most four limit cycles can bifurcate from the exterior period annulus. Thus, Dumortier and Li "obtain" a configuration of four big limit cycles surrounding three finite singularities together with the fifth small limit cycle which surrounds one of the anti-saddles.

The result by Dumortier and Li [7] also does not agree with the Wintner–Perko termination principle for multiple limit cycles [15], [27]. Applying the method as developed in [3], [12]–[17], we can show that system (5.2) cannot have neither a multiplicity-three limit cycle nor more than three limit cycles in any configuration. That will be another proof of Theorem 5.6 (the same approach can be applied to proving Theorems 3.2 and 4.1 as well).

Proof of Theorem 5.6. The proof is carried out by contradiction. Suppose that system (5.2) with three field rotation parameters, λ, μ, and α, has three limit cycles around, for example, the origin (the case when limit cycles surround another focus is considered in a similar way). Then we get into some domain in the space of these parameters which is bounded by two fold bifurcation surfaces forming a cusp bifurcation surface of multiplicity-three limit cycles.

The corresponding maximal one-parameter family of multiplicity-three limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity four (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-four limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-four limit cycles by a field-rotation parameter, according to Theorem 5.2, we will obtain a monotonic curve which, by the Wintner–Perko termination principle (Theorem 5.1), terminates either at the origin or on some separatrix cycle surrounding the origin. Since we know absolutely precisely at least the cyclicity of the singular point (Theorem 5.4) which is equal to two, we have got a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate.

If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, on the same principle (Theorem 5.1), this again contradicts to Theorem 5.4 not admitting the multiplicity of limit cycles higher than two. Moreover, it also follows from the termination principle that neither the ordinary separatrix loop nor the eight-loop cannot have the multiplicity (cyclicity) higher than two (in that way, it can be proved that the cyclicity of three other separatrix cycles [18] is at most two). Therefore, according to the same principle, there are no more than two limit cycles in the exterior domain surrounding all three finite singularities of (5.2). Thus, system (5.2) cannot have neither a multiplicity-three limit cycle nor more than three limit cycles in any configuration. The theorem is proved.

6 Conclusion

In [15], applying the methods of catastrophe theory and the Wintner–Perko termination principle for multiple limit cycles, we have developed the global bifurcation theory of planar polynomial dynamical systems and, basing on this theory, we have suggested a program on the complete solution of *Hilbert's Sixteenth Problem* in the case of quadratic systems. In principal, the program has been realized in [15] (see Section 5). In this paper, we have presented a new (geometric) approach to its realization (Section 2). Besides, we have applied this approach to solving the *Problem* in the case of Liénard's polynomial system (it is *Smale's Thirteenth Problem*) and to completing the global qualitative analysis of a generalized Liénard's cubic system with three finite singularities (Section 3 and Section 5). Generalizing the obtained results, we have also presented a solution of the *Problem* on the maximum number of limit cycles surrounding a singular point for an arbitrary polynomial system (Section 4).

Our program on the solution of *Hilbert's Sixteenth Problem* is an alternative to the program which is put forward in [8], [9] and which is often called as "Roussarie's program" by the name of its ideological inspirer [28]. Roussarie's program is reduced to the classification of separatrix cycles, determining their cyclicity and finding an upper bound of the number of limit cycles for quadratic systems. Unfortunately, there are some serious problems in the realization of this program: for example, it is not clear how to determine the cyclicity of non-monodromic separatrix cycles when there is no return map in the neighborhood of these cycles and there is no general approach to the study of the cyclicity of separatrix cycles in the case of center when the return map is identical zero. Besides, even in the case of realization of the program, as its authors note themselves [8], the obtained upper bound of the number of limit cycles obviously can not be optimal, since the used pure analytic methods cannot ensure neither the global control of limit cycle bifurcatins around a singular point nor, especially, the simultaneous control of the bifurcations around different singular points.

Thereupon, it makes sense to say some words on Roussarie's review MR2023976 (2005d:37102) on [15]. The only concrete remark in this "awkward" review is the following: "I just mention the hazardous claim made in Theorem 4.12, page 137, that there exists no quadratic system having a swallow-tail bifurcation surface of multiplicity-four limit cycles. Looking at the proof, it seems that the author unfortunately confuses two different notions: paths of limit cycles, as defined in Definition 4.7, page 112, and lines of multiple limit cycles, as defined by Perko (and recalled in Definition 4.13, page 127). In fact, there is nothing forbidding that a path begin at a parameter value with a multiplicity-four limit cycle and end at a focus point". So, Roussarie's remark is related to a swallow-tail bifurcation surface of multiplicity-four limit cycles. However, Definition 4.13, page 127, is a definition of a cusp bifurcation surface of multiplicity-three limit cycles. This is an evident lack of correspondence! Maybe, the reviewer means Definition 4.14, pages 128-129? Then it seems that he did not pay attention for our remark on page 132, following just after Theorem 4.10, which could perhaps settle his doubts. Moreover, there is a reference to the corresponding work by Perko in this remark (see also [27]). Or the reviewer has complaints against Perko's work, too? Besides, his "claim" that "there is nothing forbidding that a path begin at a parameter value with a multiplicity-four limit cycle and end at a focus point" says that he unfortunately does not see (or does not want to see) Bautin's result [1] (Theorem 2.1, page 45) on the cyclicity of a singular point of the focus or center type, which is an obstacle on such a path. Or, maybe, Bautin's result is also "questionable"?

So, we have found two approaches to solving *Smale's Thirteenth and Hilbert's Sixteenth Problems.* Both these approaches are based on the application of field rotation parameters which determine limit cycle bifurcations of polynomial systems.

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