Existence of closed G_2-structures on 7-manifolds

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Abstract

In this note we propose a new way of constructing compact 7-manifolds with a closed G_2-structure. As a result we find a first example of a closed G_2-structure on $S^3 \times S^4$. We also prove that any integral closed G_2-structure on a compact 7-manifold M^7 can be obtained by embedding M^7 to a universal space $(W^{3(80+8.C_8^3)}, h)$.

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1 Introduction.

Let $\Lambda^k V^n$ be the space of k-linear anti-symmetric forms on a given linear space V^n. For each $\omega \in \Lambda^k(V^n)$ we denote by I_ω the linear map

$$I_\omega : V^n \to \Lambda^{k-1}(V^n),\ x \mapsto (x \rfloor \omega) := \omega(x, \cdots).$$

A k-form ω is called **multi-symplectic**, if I_ω is a monomorphism.

The classification (under the action of $Gl(V^n)$) of multi-symplectic 3-forms in dimension 7 has been done by Bures and Vanzura [B-V2002]. There are together 8 types of these forms, among them there two generic classes of G_2-form ω_1^3 and $\tilde{G}_2$-form ω_2^3. They are generic in the sense of $Gl(V^7)$-action, more precisely the orbits $Gl(V^7)(\omega_i^3), i = 1, 2$, are open sets in $\Lambda^3(V^7)$. The corresponding isotropy groups are the compact group G_2 and its dual non-compact group $\tilde{G}_2$.

We shall write here a canonical expression of the G_2-form ω_1^3 (see e.g. [B-V2002] or [Joyce1996])

$$\omega_1^3 = \theta_1 \wedge \theta_2 \wedge \theta_3 + \alpha_1 \wedge \theta_1 + \alpha_2 \wedge \theta_2 + \alpha_3 \wedge \theta_3. \tag{1.1}$$

Here α_i are 2-forms on V^7 which can be written as

$$\alpha_1 = y_1 \wedge y_2 + y_3 \wedge y_4,\ \alpha_2 = y_1 \wedge y_3 - y_2 \wedge y_4,\ \alpha_3 = y_1 \wedge y_4 + y_2 \wedge y_3$$

and $(\theta_1, \theta_2, \theta_3, y_1, y_2, y_3, y_4)$ is an oriented basis of $(V^7)^*$.

A 7-dimensional manifold M^7 is said to be provided with **a G_2-structure**, if there is given differential 3-form ϕ^3 on it such that at every point $x \in M^7$ the form $\phi^3(x)$ is of G_2-type.

- A G_2-structure ϕ is called **closed**, if $d\phi = 0$. The closedness of a G_2-structure ϕ is a necessary condition for a G_2-structure to be flat, i.e. the Ricci curvature of the associated Riemannian metric $g(\phi)$ (via the canonical embedding $G_2 \to SO(7)$) vanishes (see e.g. [Bryant2005]). We notice that the first examples of a Riemannian metric with G_2 holonomy has been constructed by Joyce [Joyce1996] by deforming certain closed G_2-structures. Closed 3-forms have been also used by Severa and Weinstein to deform Poisson structures [V-W2001].

- We shall call that a closed structure G_2 **integral**, if the cohomology of the G_2-form ϕ is an integral class in $H^3(M^7, \mathbb{Z}) \subset H^3(M^7, \mathbb{R})$.

Without additional conditions the existence of a G_2-structure is a purely topological question (see [Gray1969]). On the other hand the existence of a flat G_2-structure is really "exceptional" in the sense that this structure is a solution to an overdetermined PDE (see e.g. [Bryant2005]). The intermediate class of closed G_2-structures is nevertheless has not been investigated in deep. We know only few examples of these structures on homogeneous spaces [Fernandez1987], and their local geometry [C-I2003]. The examples of flat G_2-structures on M^7 obtained by Joyce [Joyce1996] and Kovalev [Kovalev2001] have a common geometrical flavor, that they begin with M^7 with simple (or well understood) holonomy and then modify topologically these manifolds.

In this note we propose a new way to construct a closed G_2-structure by embedding a closed manifold M^7 into a semi-simple group G. The motivation for this construction is the fact that there exists a closed multi-symplectic bi-invariant 3-form on G, so "generically" the restriction of this 3-form to any 7-manifold in G must be a G_2-form. We shall show different ways to get a closed G_2-structure on $S^3 \times S^4$ by this method (Theorem 2.2 and Theorem 2.10). In Theorem 3.6 we prove that any closed integral G_2-structure ϕ on a compact M^7 can be "multi-embedded" in a finite product of $S^3 = SU(2)$ with a canonical closed 3-form h such that the pull-back of h is equal to ϕ. This theorem is close to the Tits theorem on the embedding of compact integral symplectic manifold to $\mathbb{C}P^n$. We prove theorem 3.6 by using Gromov H-principle. We also showed in Theorem-Remark 3.15 that the existence of a closed G_2-structure on an open manifold M^7 is purely a topological question. This can be done in the same way as Gromov proved the analogous theorem for open symplectic manifolds. Theorem 3.15 is also called a remark, because it is a direct consequence of the Eliashberg-Mishachev holonomy appoximation theorem.

2 Two ways to get a closed G_2-structure on $S^3 \times S^4$.

Our examples (Theorem 2.2 and Theorem 2.10) are closed submanifolds $S^3 \times S^4$ in semi-simple Lie groups $SU(3)$ and $G \times (SU(2))^N$, $N = 80 + 8 \times C_8^3$. On each semi-simple Lie group G there exists a natural bi-invariant 3-form ϕ_0^3 which is defined at the Lie algebra $g = T_eG$ as follows

$$\phi_0^3(X, Y, Z) = < X, [Y, Z] >,$$

where $<,>$ denotes the Killing form on g.

2.1. Lemma. *The form ϕ_0^3 is multi-symplectic.*

Proof. We need to show that $I_{\phi_0^3}$ is monomorphism. We notice that if $X \in \ker I_{\phi_0^3}$ then

$$< X, [Y, Z] >= 0 \text{ for all } Y, Z \in g.$$

But this condition contradicts the semi-simplicity of g. □

Let us consider the group $G = SU(3)$. For each $1 \leq i \leq j \leq 3$ let $g_{ij}(g)$ be the complex function on $SU(3)$ induced from the standard unitary representation ρ of $SU(3)$ on $\mathbb{C}^3$: $g_{ij}(g) :=< \rho(g) \circ e_i, \bar{e}_j >$.

Here $\{e_1 = (1,0,0),\, e_2 = (0,1,0),\, e_3 = (0,0,1)\}$ is a unitary basis of $\mathbb{C}^3$. Now we denote by X^7 the co-dimension 1 subset in $SU(3)$ which is defined by the equation $Im(g_{11}(g)) = 0$.

2.2. Theorem. *The subset X^7 is diffeomorphic to the manifold $S^3 \times S^4$. Moreover X^7 is provided with a closed G_2-form ω^3 which is the restriction of ϕ_0^3 to X^7.*

Proof. Let $SU(2)$ be the subgroup in $SU(3)$ consisting of all $g \in SU(3)$ such that $\rho(g) \circ e_1 = e_1$. We denote by π the natural projection

$$\pi : SU(3) \to SU(3)/SU(2).$$

We identify $SU(3)/SU(2)$ with the sphere $S^5 \subset \mathbb{C}^3$ via the standard representation ρ of $SU(3)$ on $\mathbb{C}^3$. This identification denoted by $\tilde{\rho}$ is expressed as follows.

$$\tilde{\rho}(g \cdot SU(2)) = g \circ e_1.$$

We denote by Π the composition $\tilde{\rho} \circ \pi : SU(3) \to SU(3)/SU(2) \to S^5$. Let $S^4 \subset S^5$ be the great circle which consists of points $v \in S^5$ such that $Im\, e^1(v) = 0$. Here $\{e^i, i = 1,2,3\}$ are the complex 1-forms on $\mathbb{C}^3$ which are dual to $\{e_i\}$. The pre-image $\Pi^{-1}(S^4)$ consists of all $g \in SU(3)$ such that

$$Im\, e^1(g \circ e_1) = 0.$$

$$\Longleftrightarrow Im\,(g_{11}) = 0.$$

So X^7 is $SU(2)$-fibration over S^4. But this fibration is the restriction of the $SU(2)$-fibration $\Pi^{-1}(D^5)$ over the half-sphere D^5 to the boundary $\partial D^5 = S^4$. So it is a trivial fibration. This proves the first statement of Theorem 2.2.

We fix now a subgroup $SO(2)^1$ in $SU(3)$ where $SO(2)^1$ is the orthogonal group of the real space $\mathbb{R}^2 \subset \mathbb{C}^3$ such that $\mathbb{R}^2$ is the span of e_1 and e_2 over $\mathbb{R}$.

We denote by $m_L(g)$ (resp. $m_R(g)$) the left multiplication (resp. the right multiplication) by an element $g \in SU(3)$.

2.3. Lemma. *X^7 is invariant under the action of $m_L(SU(2)) \cdot m_R(SU(2))$. For each $v \in S^4$ there exist an element $\alpha \in SO(2)^1$ and an element $g \in SU(2)$ such that $\Pi(g \cdot \alpha) = v$. Consequently for any point $x \in X^7$ there are $g_1, g_2 \in SU(2)$ and $\alpha \in SO(2)^1$ such that*

$$x = g_1 \cdot \alpha \cdot g_2, \tag{2.3.1}$$

Proof. The first statement follows from straightforward calculations, (our realization that $X^7 = \Pi^{-1}(S^4)$ implies that the orbit of $m_R(SU(2))$-action on X^7 are the fiber $\Pi^{-1}(v)$). Let $v = (\cos\alpha, z_2, z_3) \in S^4$, where $z_i \in \mathbb{C}$. We choose $\alpha \in SO(2)^1$ so that

$$\rho(\alpha) \circ e_1 = (\cos\alpha, \sin\alpha) \in \mathbb{R}^2. \tag{2.4}$$

Clearly α is defined by v uniquely up to sign $\pm$. We set

$$w := (\sin\alpha, 0) \in \mathbb{C}^2 =< e_2, e_3 >_{\otimes\mathbb{C}} .$$

We notice that

$$|z_2|^2 + |z_3|^2 = \sin^2\alpha.$$

Since $SU(2)$ acts transitively on the sphere S^3 of radius $|\sin\alpha|$ in $\mathbb{C}^2 =< e_2, e_3 >_{\otimes\mathbb{C}}$, there exists an element $g \in SU(2)$ such that $\rho(g) \circ w = (z_2, z_3)$. Clearly

$$\Pi(g \cdot \alpha) = v.$$

The last statement of Lemma 2.3 follows from the second statement and the fact that $X^7 = \Pi^{-1}(S^4)$ □

Using (2.3.1) to complete the proof of Theorem 2.2 it suffices to check that the value of ω at any $\alpha \in SO(2)^1 \subset X^7$ is a G_2-form, since ϕ_0^3 is a bi-invariant form on $SU(3)$. We divide the remaining part of the proof of Theorem 2.2 into two steps. In the first step we shall compute that value ω^3 at $\alpha = e$ and in the second step we shall compute the value ω^3 at any $\alpha \in SO(2)^1$.

Step 1. Let us first compute the value $\omega^3(e) \in X^7$. We shall use the Killing metric to identify the Lie algebra $su(3)$ with its co-algebra g. Thus in what follows we shall not distinguish co-vectors and vectors, poly-vector and exterior forms on $su(3)$. Clearly we have

$$T_e X^7 = \{v \in su(3) : Im\, g_{11}(v) = 0\}.$$

Now we identify $gl(\mathbb{C}^3)$ with $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$ and we denote by e_{ij} the element of $gl(\mathbb{C}^3)$ of the form $e_i \otimes (e_j)^*$.

A straightforward calculation gives us

$$\omega^3(x_0) = \sqrt{2}\delta_1 \wedge \delta_2 \wedge \delta_3 + \frac{1}{\sqrt{2}}\omega_1 \wedge \delta_1 + \frac{1}{\sqrt{2}}\omega_2 \wedge \delta_2 + \frac{1}{\sqrt{2}}\omega_3 \wedge \delta_3, \tag{2.5}$$

where δ_i are 1-forms in T_eX^7 which are defined as follows:

$$\delta_1 = \frac{i}{\sqrt{2}}(e_{22} - e_{33}), \delta_2 = \frac{1}{\sqrt{2}}(e_{23} - e_{32}), \delta_3 = \frac{i}{\sqrt{2}}(e_{23} + e_{32}).$$

Furthermore, ω_i are 2-forms on T_eX^7 which have the following expressions:

$$2\omega_1 = -(e_{12} - e_{21}) \wedge i(e_{12} + e_{21}) + (e_{13} - e_{31}) \wedge i(e_{13} + e_{31}),$$

$$2\omega_2 = -(e_{12} - e_{21}) \wedge (e_{13} - e_{31}) - i(e_{12} + e_{21}) \wedge i(e_{13} + e_{31}),$$

$$2\omega_3 = -(e_{12} - e_{21}) \wedge i(e_{13} + e_{31}) + i(e_{12} + e_{21}) \wedge (e_{13} - e_{31}).$$

Now compare (2.5) with (1.1) we observe that these two 3-forms are $Gl(\mathbb{R}^7)$ equivalent (e.g. by rescaling δ_i with factor (1/2)). This proves that $\omega^3(x_0)$ is a G_2-form. This completes the step 1.

Step 2. Using step 1 it suffices to show that

$$D\, m_L(\alpha^{-1})(T_\alpha X^7) = T_eX^7 \tag{2.6}$$

for any $\alpha \in SO(2)^1 \subset X^7$, $\alpha \neq e$.

Since $X^7 \supset \alpha \cdot SU(2)$, we have

$$su(2) \subset D\, m_L(\alpha^{-1})(T_\alpha X^7). \tag{2.7}$$

Denote by $SO(3)$ the standard orthogonal group of $\mathbb{R}^3 \subset \mathbb{C}^3$. Since $\alpha \in SO(3) \subset X^7$, we have $D\, m_L(\alpha^{-1})(T_\alpha SO(3)) \subset D\, m_L(\alpha^{-1})(T_\alpha X^7)$. In particular we have

$$< (e_{12} - e_{21}), (e_{13} - e_{31}) >_{\otimes\mathbb{R}} \subset D\, m_L(\alpha^{-1})(T_\alpha X^7). \tag{2.8}$$

Since $SU(2) \cdot \alpha \subset X^7$, we have

$$Ad(\alpha^{-1})su(2) \subset D\, m_L(\alpha^{-1})(T_\alpha X^7). \tag{2.9}$$

Using the formula

$$Ad(\alpha^{-1}) = \exp(-ad(t \cdot \frac{e_{12} - e_{21}}{\sqrt{2}})),\ t \not\equiv 0$$

we get immediately from (2.7), (2.8), (2.9) the following inclusion

$$< i(e_{12} + e_{21}), i(e_{13} + e_{31}) >_{\otimes\mathbb{R}} \subset D\, m_L(\alpha^{-1})(T_\alpha X^7))$$

which together with (2.7), (2.8) imply the desired equality (2.6). □

This completes the proof of Theorem 2.2. □

Our constructed subsmanifold $S^3 \times S^4$ in $SU(3)$ is quite symmetric. The symmetry of a submanifold helps us to compute a lot of things on it easily. On the other side, the notion of symmetry is quite far from the notion of genericity. That is why it takes a lot of time for me in searching another nontrivial (that means not via a representation of $SU(3)$ into another compact Lie group) example of a submanifold M^7 with a closed (and induced) G_2-structure in a Lie group G which has a lot of symmetries. The idea is how to integrate the G_2-structure distribution to a compact submanifold. It is not hard to find that distribution in the Lie algebra g, but is is hard (like in calibration geometry) to integrate that distribution to an explicit symmetric and compact submanifold.

So we chose another way to construct a closed G_2-structure on $S^3 \times S^4$ by combining Theorem 2.2 with the technique of our proof of Theorem 3.6.

Theorem 2.10. *For any given simply-connected compact semi-simple Lie group G, and any given integral closed G_2-structure ϕ on $S^3 \times S^4$ (e.g. that from Theorem 2.2) there exists an embedding $f : S^3 \times S^4 \to G' = G \times (SU(2))^{80+4\cdot C_8^3}$ such that the restriction of the standard bi-invariant form ϕ_0^3 from G' to $f(S^3 \times S^4)$ is equal to ϕ. Moreover we can require that the pull-back (via the projection) of a given non-decomposable element $\alpha \in H^3(M, \mathbb{Z})$ to the image $f(S^3 \times S^4)$ is equal to $[\phi] \in H^3(M, \mathbb{Z})$.*

Proof. Using the fact that $H^3(S^3 \times S^4, \mathbb{Z}) = \pi_3(S^3) = \mathbb{Z}$, and taking into account for a Lie group G as in Theorem 2.10 the following identity: $H^3(G, \mathbb{Z}) = \pi_3(G)$ we can find a map $f_1 : M^7 \to G$ such that the second condition in Theorem 2.10 holds. Now I shall modify this map f_1 to the required embedding f by using the same H-principle as in our proof of Theorem 3.6. The only thing we can improve in this proof is the dimension of the target manifold. Instead of number 8 of special coverings on M^7 (using in the step 2 of the proof of Theorem 3.6) we can chose 4 open disks which cover $S^3 \times S^4$. □

2.11. Remark. It remains also an interesting question, if we can deform closed G_2-structures on $S^3 \times S^4$ to a flat one.

3 Universal space for closed G_2-structures.

In this section we shall show that any integral closed G_2-structure ϕ on a compact 7-dimensional smooth manifold M^7 can be induced from an embedding M^7 to a universal space $(\bar{W}, \bar{h})$, see Theorem 3.6.

Our definition of the universal space $(\bar{W}, \bar{h})$ is based on the work of Dold and Thom [D-T1958].

Let $SP^q(X)$ be the q-fold symmetric product of a locally compact, paracompact Hausdorff pointed space $(X, 0)$, i.e. $SP^q(X)$ is the quotient space of the q-fold Cartesian $(X^q, 0)$ over the permutation group σ_q. We shall denote by $SP(X, 0)$ the inductive limit of $SP^q(X)$ with the inclusion

$$X = SP^1(X) \xrightarrow{i_1} SP^2(X) \xrightarrow{i_2} \cdots \to SP^q(X) \xrightarrow{i_q} \cdots ,$$

where

$$SP^q(X) \xrightarrow{i_q} SP^{q+1}(X) : \ [x_1, x_2, \cdots, x_q] \mapsto [0, x_1, x_2, \ldots, x_q].$$

Equivalently we can write

$$SP(X, 0) = \sum_q SP^q(X)/([x_1, x_2, \cdots, x_q] \sim [0, x_1, x_2, \cdots, x_q]).$$

So we shall also denote by i_q the canonical inclusion $SP^q(X) \to SP(X, 0)$.

3.1. Theorem (see [D-T1958, Satz 6.10]). *There exist natural isomorphisms* $j : H_q(X, \mathbb{Z}) \to \pi_q(SP(X, 0))$ *for* $q > 0$.

3.2. Corollary. ([D-T1958]) *The space* $SP(S^n, 0)$ *is the Eilenberg-McLane complex* $K(\mathbb{Z}, n)$.

3.3. Lemma. *Any continuous map* f *from* M^7 *to* $SP(S^3, 0)$ *is homotopic equivalent to a continuous map* $\tilde{f}$ *from* M^7 *to* $i_3(SP^3(S^3)) \subset SP(S^3, 0)$.

Proof. We fix the following simplicial decomposition: $S^3 = \mathbb{R}^3 \cup \{0\}$. Then $SP^q(S^3)$ has the following simplicial decomposition

(3.3.1) $$SP^q(S^3) = \{0\} \cup_{p=1}^{q} (\mathbb{R}^3)^p.$$

It follows that

(3.3.2) $$SP(S^3, 0) = \{0\} \cup_{p=1}^{\infty} (\mathbb{R}^3)^p.$$

Denote by $\Sigma^7(SP(S^3,0))$ the 7-dimensional skeleton of $SP(X^3,0)$. Clearly any continuous map $f : M^7 \to SP(S^3,0)$ is homotopic to a map $\tilde{f} : M^7 \to \Sigma^7(SP(S^3,0))$. Now using the canonical inclusion $i_3 : SP^3(S^3) \to SP(S^3,0)$ we get Lemma 3.3 immediately from (3.3.1) and (3.3.2). $\square$

Using the diffeomorphism $S^3 = SU(2)$ we choose a canonical parallelization of TS^3 by the left multiplication on S^3. Let $dx^i_j, i = 1,2,3$ denote the left invariant 1-forms on S^3_j dual to $\delta_1, \delta_2, \delta_3$ in section 2, see (2.5). Then

(3.4) $dx^1_j = \sqrt{2} dx^2_j \wedge dx^3_j,\ dx^2_j = -\sqrt{2} dx^1_j \wedge dx^3_j,\ dx^3_j = \sqrt{2} dx^1_j \wedge dx^2_j.$

Let $\pi_j : \Pi^k_{j=1} S^3_j \to S^3_j = S^3$ denote the canonical projection. We shall abbreviate $\Pi^*_i(dx^i_j)$ also by dx^i_j.

3.5. Lemma. *The differential form $h = \sum_{j=1}^k dx^1_j \wedge dx^2_j \wedge dx^3_j$ is closed. It descends to a differential form $\bar{h}$ on $SP^k(S^3)$. This form $\bar{h}$ is the generator of $H^3(SP^k(S^3), \mathbb{R}) = H^3(SP^k(S^3), \mathbb{Z}) = \mathbb{Z}$.*

Proof. Clearly h is closed. Furthermore the form h is invariant under the action of the permutation group σ_k on $\Pi^k_{i=j} S^3_j$. This proves the second statement.

To prove the last statement we notice that the integration of $\bar{h}$ over the image of $i_{k-1} \circ \cdots \circ i_1(S^3 = SP^1(S^3)) \subset SP^k(S^3)$ is equal to 1. $\square$

Now we state the main theorem of this section. Put $N = 80 + 8 \cdot C^3_8$. Set $W = \Pi^N_{i=1} S^3_i$ and $\bar{W} = SP^N(S^3)$.

3.6. Theorem. *Suppose that ϕ is a closed integral G_2-form on a compact smooth manifold M^7. Then there is an embedding $f : M^7 \to (\bar{W}, \bar{h})$ such that $f^*(\bar{h}) = \phi$.*

Proof of Theorem 3.6. The proof of Theorem 3.6 is based on the Gromov H-principle.[1] Let us quickly recall several notions introduced by Gromov in [Gromov1986].

Let V and W be smooth manifolds. We denote by $(V,W)^{(r)}$, $r \geq 0$, the space of r-jets of smooth mappings from V to W. We shall think of each map $f : V \to W$ as a section of the fibration $V \times W = (V,W)^{(0)}$

[1] to avoid confusing between the original notion h-principle of Gromov and his notion of h as a differential form, we decide to use the capital H for H-principle.

over V. Thus $(V,W)^{(r)}$ is a fibration over V, and we shall denote by p^r the canonical projection $(V,W)^{(r)}$ to V, and by p^s_r the canonical projection $(V,W)^{(s)} \to (V,W)^{(r)}$.

We also say that a differential relation $\mathcal{R} \subset (V,W)^{(r)}$ satisfies the **H-principle near a map** $f_0 : V \to W$, if every continuous section $\phi_0 : V \to \mathcal{R}$ which lies over f_0, (i.e. $p^r_0 \circ \phi_0 = f_0$) can be brought to a holonomic section ϕ_1 by a homotopy of sections $\phi_t : V \to \mathcal{R}_U$, $t \in [0,1]$, for an arbitrary small neighborhood U of $f_0(V)$ in $V \times W$ [Gromov1986, 1.2.2]. Here for an open set $U \subset V \times W$, we write

$$\mathcal{R}_U := (p^r_0)^{-1}(U) \cap \mathcal{R} \subset (V,W)^r.$$

The H-principle is called C^0**-dense**, if it holds true C^0-near every map $f : V \to W$.

Let h be a smooth differential k-form on W. A subspace $T \subset T_w W$ is called $h(w)$**-regular**, if the composition of $I_{h(w)}$ with the restriction homomorphism $\Lambda^{k-1}T_w W \to \Lambda^{k-1}T$ sends $T_w W$ onto $\Lambda^{k-1}T$.

An immersion $f : V \to W$ is called **h-regular**, if for all $v \in V$ the subspace $Df(T_v V)$ is $h(f(v))$-regular.

Let G be a finite group acting effectively on W. It is well-known (see e.g. [Pflaumen2001]) that the deRham complex on an orbifold $\bar{W} = W/G$ can be identified with the complex of G-invariant differential forms on W. Furthermore the cohomology of the deRham complex on $\bar{W}$ coincides with the singular cohomology of $\bar{W}$ with coefficients in $\mathbb{R}$.

Any map $\bar{f} : V \to \bar{W}$ can be seen (or lifted to) as a G-invariant multi-map $f : V \to W$. Conversely, any G-invariant multi-map $f : V \to W$ descends to a map $\bar{f} : V \to \bar{W}$. We say that a G-invariant multi-map $f : V \to W$ is a **h-regular multi-immersion**, if it is a h-regular immersion at every branch of f. This definition descends to $\bar{W}$ and gives us the notion of $\bar{h}$**-regular immersion**. In the same way we define the notion of H-principle for G-invariant multi-maps $V \to W$ which is equivalent to the notion of H-principle for a map $\bar{f} : V \to \bar{W}$. For treating general stratified spaces $\bar{W}$ we refer the reader to [Pflaumen2001].

We also use the notions of a flexible sheaf and a microflexible sheaf introduced by Gromov in order to study the H-principle.

Suppose we are given a differential relation $\mathcal{R} \subset (V,W)^{(r)}$. Fix an integer $k \geq r$ and denote by $\Phi(U)$ the space of C^k-solution of $\mathcal{R}$ over

U for all open $U \subset V$. This set equipped with the natural restriction $\Phi(U) \to \Phi(U')$ for all $U' \subset U$ makes Φ a sheaf. We shall say that Φ **satisifes the H-principle**, if $\mathcal{R}$ satisfies the H-principle. [2]

A sheaf Φ is called **flexible (microflexible)**, if the restriction map $\Phi(C) \to \Phi(C')$ is a fibration (microfibration) for all pair of compact subsets C and $C' \subset C$ in M. We recall that the map $\alpha : A \to A'$ is called **microfibration**, if the lifting homotopy property for a homotopy $\psi : P \times [0,1] \to A'$ is valid only "micro", e.g. there exists $\varepsilon > 0$ such that ψ can lift to a $\bar{\psi} : P \times [0, \varepsilon] \to A$.

Now we suppose that M^7 is a compact manifold with a closed G_2-form ϕ. Because ϕ is nowhere vanishing, $[\phi]$ represents a non-trivial cohomology class in $H^3(M^7, \mathbb{R})$. Let us consider a manifold $M^8 = M^7 \times (-1, 1)$ provided with a form $g = \phi \oplus 0$. Denote by Φ_{reg} the sheaf of $\bar{h}$-regular immersion $\bar{f}$ of M^8 to $(\bar{W}, \bar{h})$ such that $\bar{f}^*[\bar{h}] = [g]$.

3.7. Proposition. *The sheaf Φ_{reg} is microflexible.*

Proof. Let $\bar{f}_0$ be a $\bar{h}$-regular immersion from M^8 to $\bar{W}$ such that $\bar{f}_0^*[\bar{h}] = [g]$. We denote by $\bar{F}_0$ the corresponding section of $M^8 \times \bar{W} \to M^8$, i.e. $\bar{F}_0(v) = (v, \bar{f}_0(v))$. Denote by $\Gamma_0 \subset M^8 \times \bar{W}$ the graph of $\bar{f}_0$ (i.e. it is the image of $\bar{F}_0$), and let $p(g)$ and $p(\bar{h})$ be the pull-back of the forms g and $\bar{h}$ to $M^8 \times \bar{W}$ under the obvious projection. Take a small neighborhood $\bar{Y} \supset \Gamma_0$ in $M^8 \times \bar{W}$. Using the Whitney local triviality property of the orbifolds (see e.g. [Pflaum2001]) we get immediately

3.8. Lemma. *The graph Γ_0 is a deformation retract of $\bar{Y}$.*

Thus we can write

$$p(\bar{h}) - p(g) = d\hat{h}$$

for some smooth 2-form $\hat{h}$ on $\bar{Y}$. Alternatively by working on the covering space Y, we notice that, if $p(g)$ and $p(h)$ are G-invariant cohomologuous differential forms, then $p(g) - p(h) = dh_1$, where h_1 is a G-invariant differential form on Y. In our case G is the permutation group Σ_k.

Our next observation is

3.9. Lemma. *Suppose a map $\bar{F} : M^8 \to \bar{Y}$ corresponds to a $\bar{h}$-regular immersion $\bar{f} : M^8 \to \bar{W}$. Then $\bar{F}$ is a $d\hat{h}$-regular immersion.*

[2]The reader can look at [Gromov 1986, 2.2.1] for a more general definition.

Proof. We work now on the covering space Y. We need to show that for all $y \in \{F(z)\} \subset Y$, $z \in M^8$, the composition ρ of the maps

$$T_y Y \overset{I_{p(h)-p(g)}}{\to} \Lambda^2 T_y Y \to \Lambda^2(dF^y(T_{(z)}(M^8))$$

is onto, where F^y is a branch of F such that $F^y(z) = y$. This follows from the consideration of the restriction of ρ to the subspace $S \subset T_y Y$ which is tangent to the fiber W in $M^8 \times W \supset Y$. $\square$

Now for a map $\bar{F} : M^8 \to \bar{Y}$ and 1-form ϕ on M^8 we set

$$\mathcal{D}(\bar{F}, \phi) := \bar{F}^*(\hat{h}) + d\phi.$$

Since the lifted form $\hat{h}_1$ is invariant under the action of $G = \Sigma_k$ we see easily that $\bar{F}^*(\hat{h})$ is a smooth differential form on M^8.

With this notation the maps $\bar{f} : M^8 \to \bar{W}$ corresponding to $\bar{F} : M^8 \to Y$ satisfy

$$\bar{f}^*(\bar{h}) = \bar{F}^*(p(\bar{h})) = g + \bar{F}^*(d\hat{h}) = g + d\mathcal{D}(\bar{F}, \phi),$$

for any ϕ. Hence follows that the space of sections $\bar{F} : M^8 \to Y$ for which $\bar{f}^*(\bar{h}) = g + dg_1$ for a given 2-form g_1 has the same homotopy type as the space of solutions to the equation

$$\mathcal{D}(\bar{F}, \phi) = \tilde{g}_1.$$

In particular the equation $\bar{f}^*(\bar{h}) = g$ reduces to the equation $\mathcal{D}(\bar{F}, \phi) = 0$ in so far as the unknown map $\bar{f}$ is C^0-close to $\bar{f}_0$ (so that its graph lies inside Y).

3.10. Lemma. *The differential operator $\mathcal{D}$ is infinitesimal invertible at those pairs $(\bar{F}, \phi)$ for which the underlying map $\bar{f}$ is a $\bar{h}$-regular immersion.*

Proof. First we shall show that the fibration $(\bar{F}^*(T\bar{W}, \bar{h}))$ is a smooth vector fibration over manifold M^8 provided with a smooth form h. In our notation $\bar{F}^*(T\bar{W})$ denotes the Σ_k-invariant multi-(vector) bundle $\{F^*(TW)\}$ provided with Σ_k-invariant form $F^*(h)$. We define an action ν of $g \in G = \Sigma_k$ on this multi-bundle as follows. For each $z \in M^8$ the fiber of this multi-bundle consists of the set $\{(F^{w_k})^*(T_{w_k}W),\, w_k \in F(z)\}$. We set

$$\text{if } g(w_k) = w_k, \text{ then } g(V) = V \text{ for } V \in T_{w_k}W,$$

if $w_k \neq g(w_k) = w_s$, then $g(V) = g_*(V) \in T_{w_s}W$ for $V \in T_{w_k}W$.

This action of g is smooth. The quotient of this multi- (vector) bundle over Σ_k is clearly a smooth vector bundle over M^8 provided with h. Any section $\bar{s}$ of this smooth vector bundle can be lifted to the $\nu(\Sigma_k)$-invariant section s of the above Σ_k-invariant multi-bundle. It is easy to see that the space of $\nu(\Sigma_k)$-invariant sections corresponding to the space of infinitesimal variations (or tangent bundle) of the Σ_k-invariant multi-section F.

Now we shall work on the covering space. The linearized operator $L_{\bar{f}} := L_{\bar{F}}\mathcal{D}$ applies to the pairs $(\partial, \tilde{\phi})$ where ∂ is a $\nu(\Sigma_k)$-invariant section of the bundle $f^*(TW)$ over M^8 and $\tilde{\phi}$ is a 1-form on M^8. Downstairs we shall identify ∂ with its image denoted by $\bar{\partial}$ which is a (local) vector field in Y along $\bar{F}(M^8)$. To prove Lemma 3.10 it suffices to show that the equation

$$L_{\bar{f}}(\bar{\partial}, \tilde{\phi}) = \tilde{g} \tag{3.10.1}$$

has a solution $(\bar{\partial}, \tilde{\phi})$ for any given smooth differential 2-form $\tilde{g}$ on M^8. Clearly

$$L_{\bar{f}}(\bar{\partial}, \tilde{\phi}) = \bar{F}^*((\bar{\partial}\rfloor d\hat{h}) + d(\bar{\partial}\rfloor\hat{h})) + d\tilde{\phi}.$$

By Lemma 3.9 the map $\bar{F}_0$ is a $d\hat{h}$-regular immersion. Hence the system

$$\bar{F}^*(\bar{\partial}\rfloor d\hat{h}) = \tilde{g}, \tag{3.10.2}$$

$$\bar{F}^*(\bar{\partial}\rfloor\hat{h}) + \tilde{\phi} = 0 \tag{3.10.3}$$

is solvable for all 2-form $\tilde{g}$ on M^8. Clearly every solution $(\bar{\partial}, \tilde{\phi})$ of (3.10.2) and (3.10.1) satisfies (3.10.1). $\square$

Now using A.4 and Lemma 3.10 we complete the proof of Lemma 3.9. $\square$

Completion of the proof of Theorem 3.6.

Using Corollary 3.2, Lemma 3.3 and Lemma 3.4 we can find a map $\bar{f} : M^7 \to \bar{W}$ such that $\bar{f}^*([\bar{h}]) = [\phi] \in H^3(M^7, \mathbb{Z})$. Since M^7 is a deformation retract of M^8 the map f extends to a map $\bar{F} : M^8 \to \bar{W}$ such that $\bar{F}^*[\bar{h}] = [g]$.

For each $z \in M^8$ we denote by $Mono((T_zM^8, g), (T_{F(z)}W, h))$ the set of all monomorphisms $\rho : T_zM^8 \to T_{F(z)}W$ such that the restriction of $h(F(z))$ to $dF(T_vM^8)$ is equal to $(dF^{-1})^*g$. To save the notation, whenever we consider the restriction of the form g to an open subset $U \subset M^8$ we shall denote also by g this restriction. The following Proposition is crucial in our proof in order to use the H-principle.

3.11. Proposition. *There exists a section s of the fibration $Mono((TM^8, g), (\bar{F}^*(T\bar{W}, h))$ such that $s(z)(T_z M^8)$ is h-regular subspace for all $z \in M^8$.*

In our case $W = (S^3)^N$. The tangential bundle TS^3 is paralellizable, hence $\bar{F}^*(T\bar{W}) = M \times \mathbb{R}^{3N}$.

Proof of Proposition 3.11. The proof of Proposition 3.11 consists of 3 steps.

Step 1. In the first step we show the existence of a section $s_1 \in Mono(TM^8, M \times \mathbb{R}^{3N_0})$ such that the image of s_1 is h-regular subbundle of dimension 8 in $M \times \mathbb{R}^{3N_0}$. To save notation we also denote by h the following 3-form on $\mathbb{R}^{3N_0}$

$$h = \sum_{j=1}^{N_0} dx_j^1 \wedge dx_j^2 \wedge dx_j^3.$$

It is easy to see that h is multi-symplectic. Furthermore we shall assume that $(w_j^i), 1 \le i \le 3$, is some fixed vector basis in $\mathbb{R}_i^3$.

3.12. Lemma. *For each given $k \ge 3$ there there exists a k-dimensional subspace V^k in $\mathbb{R}^{3N_0}$ such that V^k is h-regular subspace, provided that $N_0 \ge 5 + (k/2 - 2)(3 + k/2)$, if k is even, and $N_0 \ge 6 + ([k/2] - 2)(3 + [k/2]) + [k/2]$, if k is odd.*

Proof. We shall construct a linear embedding $f : V^k \to \mathbb{R}^{3N_0}$ whose image satisfies the condition of Lemma 3.12. Each linear map f can be written as

$$f = (f_1, f_2, \cdots, f_N),\ f_i : V^k \to \mathbb{R}_i^3, i = \overline{1, N_0}.$$

Now we can assume that $V^3 \subset V^4 \subset \cdots \subset V^k$ is a chain of subspaces in V^k which is generated by some vector basis $(e_1, \cdots e_k)$ in V^k. We denote by $(e_1^*, \cdots, e_k^*)$ the dual basis of $(V^k)^*$. By construction, the restriction of $(e_1^*, \cdots, e_i^*)$ to V^i is the dual basis of $(e_1, \cdot s, e_i) \in V^i$. For the simplicity we shall denote the restriction of any v_j^* to these subspaces also by v_j^* (if the restriction is not zero). We shall construct f_i inductively on the dimension k of V^k such that the following condition holds for all $3 \le i \le k$

$$(3.13) \quad < f_1^*(\Lambda^2(\mathbb{R}^3_1)), f_2^*(\Lambda^2(\mathbb{R}^3_2)), \cdots, f^*_{\delta(i)}(\Lambda^2(\mathbb{R}^3_{\delta(i)}) >_{\otimes\mathbb{R}} = \Lambda^2(V^i).$$

The condition (3.13) implies that $f(V^i)$ is h-regular, since the image

$$I_h(\mathbb{R}^3_1 \times \cdots \times \mathbb{R}^3_{\delta(i)}) = \oplus_{j=1}^{\delta(i)} \Lambda^2(\mathbb{R}^3_j).$$

For $i = 3$ we can take $f_1 = Id$, and $\delta(1) = 1$. Suppose that $f_{\delta(i)}$ is already constructed. To find f_j, $\delta(i) + 1 \le j \le \delta(i+1)$, we need to find a linear embeddings $f_{\delta(i)+1}, \cdots, f_{\delta(i+1)}$ such that

$$(3.14) \quad < f^*_{\delta(i)+1}\Lambda^2(\mathbb{R}^3_{\delta(i)+1}), \cdots, f^*_{\delta(i+1)}\Lambda^2(\mathbb{R}^3_{\delta(i+1)}) >_{\otimes\mathbb{R}} \supset e^*_{i+1} \wedge \Lambda^1(V^i).$$

We can proceed as follows. We let

$$f_j(e_{i+1}) = w_j^1 \in \mathbb{R}^3_j, \text{ if } j \ge \delta(i) + 1, \; f_j(e_{i+1}) = 0, \text{ if } j \le \delta(i).$$

To complete the construction of f_j we need to specify $f_j(e_l)$, for $1 \le l \le i$ and $j \ge \delta(i) + 1$. For such l and j we shall define $f_j(e_l) = 0$ or $f_j(e_l) = w_j^2$ or $f_j(e_l) = w_j^3$ so that (3.14) holds. A simple combinatoric calculation shows that the most economic " distribution" of $f_j(e_l)$ satisfies the estimate for $\delta(i)$ as in Lemma 3.12. $\square$

Now once we have chosen a h-regular subspace V^{17} in $\mathbb{R}^{80}$ by Lemma 3.12, we shall find a section s_1 for the step 1 by require that s_1 is a section of $Mono(TM^8, M \times V^{17})$. This section exists, since the fiber $Mono(T_xM^8, \mathbb{R}^{17})$ is homotopic equivalent to $SO(17)/SO(9)$ which has all homotopy groups π_j vanishing, if $j \le 8$. This completes the step 1.

Step 2. Once a section s_1 in Step 1 is specified we put the following form g_1 on TM^8:

$$g_1 = g - s_1^*(h).$$

In this step we show the existence of a section s_2 of the fibration $End((TM^8, g_1), (M \times \mathbb{R}^{3N_1}, h))$ (we do not require that s_2 is a monomorphism).

Using the Nash trick [Nash1956] we can find a finite number of open coverings U_i^j, $j = \overline{1,8}$ of M^8 which satisfy the following properties:

$$(3.14) \quad N_i^j \cap N_k^j = \emptyset, \; \forall j = \overline{1,8} \text{ and } i \ne k,$$

and moreover U_i^j is diffeomorphic to an open disk for all i, j. Since U_i^j satisfy the condition (3.14), for a fixed j we can embed the union $\cup_i U_i^j$ into $\mathbb{R}^8$. Thus for each j on the union $\cup_i U_i^j$ we have local coordinates x_j^r, $r = \overline{1,8}$, $j = \overline{1,8}$. Using partition of unity functions $f_j(z)$ corresponding to $\cup_i U_i^j$ we can write

$$g_1(z) = \sum_{j=1}^{8} f_j(z) \cdot \mu_j^{r_1 r_2 r_3}(z) \cdot dx_j^{r_1} \wedge dx_j^{r_2} \wedge dx_j^{r_3},\ r_i \in (1, \cdots, 8).$$

We numerate (i.e. find a map θ to $\mathbb{N}^+$) the set $(j, r_1 r_2 r_3)$. Let $N_1 = 8 \cdot C_8^3$. Next we find the section s_2 of form

$$s_2(z) = (s_1(z), \cdots, s_{N_1}(z)),\ s_q(z) \in End(T_z M^8, \mathbb{R}_q^3)$$

such that

$$s_{\theta(j, r_1 r_2 r_3)}(z) = f_j(z) \cdot \mu_j^{r_1 r_2 r_3}(z) \cdot A_{r_1, r_2, r_3},\ \text{where } A(\partial x_{r_l}) = \delta_l^i e_i.$$

Here (e_1, e_2, e_3) is a vector basis in $\mathbb{R}_q^3$ for $q = \theta(j, r_1 r_2 r_3)$. Clearly the map s_2 satisfies the condition $s_2(h) = g_1$. This completes the second step.

Step 3. We put

$$s = (s_1, s_2),$$

where s_1 is constructed in Step 1 and s_2 is constructed in Step 2. Clearly s satisfies the condition of Lemma 3.11. □

Theorem 3.6 now follows from Proposition 3.7, Proposition 3.11, Appendix A.2 and the following observation [Gromov1986, 3.4.1.B'] that M^7 is a sharply movable submanifold by strictly exact diffeotopies in M^8. □

3.15. Theorem-Remark. It follows directly from the Eliashberg-Mishachev Theorem on the approximation of given differential form by a closed form [E-M2002,10.2.1] and form the openess and invariance of the space of G_2 structures, that any G_2 structure on an open manifold M^7 is homotopic to a closed G_2-structure on M^7.

4 Appendix: Flexibility, microflexibility and Nash-Gromov implicit function theorem.

In this appendix we recall Gromov theorems on the relation between flexibility as well as microflexibility and H-principle.

A1. H-principle and flexibility[Gromov1986, 2.2.1.B].*If V is a locally compact countable polyhedron (e.g.) manifold, then every flexible sheaf over V satisfies the H-principle.* (Actually the parametric H-principle which implies the H-principle.)

To formulate the relation between the flexibility and microflexibility (of solution sheafs) under certain conditions in [A2] we need to describe these conditions with the notion of acting in (solution) sheaf diffeotopies, which move sharply a set.

Suppose that $U \subset U' \subset V$ are open subsets in V. We say that diffeotopies $\delta_t : U \to U', t \in [0,1], \delta_0 = Id$, **act in a sheaf Φ on subset** $\Phi' \subset \Phi(U')$, if δ_t assigns each section $\phi \in \Phi'$ a homotopy of sections in $\Phi(U)$ which we shall call $\delta_t^*\phi$ such that the following conditions hold

- $\delta_0^*\phi = \phi_{|U}$
- If two sections $\phi_1, \phi_2 \in \Phi'$ coincide at some point $u_0' \in U'$ and if $\delta_{t_0}(u_0) = u_0'$ for some $u_0 \in U$ and $t_0 \in [0,1]$, then $(\delta_{t_0}^*\phi_1)(u) = (\delta_{t_0}\phi_2)(u_0)$. This allows us to write $\phi(\delta_t(u))$ instead of $(\delta_t^*\phi)(u), u \in U$.
- Let $U_0 \subset U$ be the maximal open subset where $(\delta_t)_{|U} = Id$. Then The homotopy $\delta_t^*(\phi)$ is constant in t over U_0.
- If the diffeotopy δ_t is constant in t for $t \geq t_0$ over all U, then the homotopy $\delta_t^*\phi$ is also constant in t for $t \geq t_0$.
- If $\phi_p \in \Phi'$, $p \in P$ is a continuous family of sections, then the family $\delta_t\phi_p$ is jointly continuous in t and p.

Let V_0 be a closed subset of the above $U' \subset V$. Suppose that V is provided with some metric. Let $\mathcal{A}$ be a set of diffeotopies $\delta_t : U' \supset$

$\mathcal{O}pV_0 \to U'$. We call $\mathcal{A}$ **strictly moving** a given subset $S \subset V_0$, if $dist(\delta_t(S), V_0) \geq \mu > 0$ for $t \geq 1/2$ and for all $\delta_t \subset \mathcal{A}$.

Further we call $\mathcal{A}$ **sharp** at S, if for every $\mu > 0$ there exists $\delta_t \in \mathcal{A}$ such that

- $(\delta_t)_{|\mathcal{O}p(v)} = Id, t \in [0,1]$ for all points $v \in V_0$ such that $dist(v, S) \geq \mu$, where $\mathcal{O}p(v)$ is an (arbitrary) small neighborhood of v
- $\delta_t = \delta_{1/2}$ for $t \geq 1/2$.

For a given sheaf Φ on V and for a given action of the set $\tilde{\mathcal{A}}$ of diffeotopies δ_t on subset $\Phi'_{\delta_t} \subset \Phi(U')$, we say that acting diffeotopies **sharply move** V_0 **at** $S \subset V_0$, if for each compact family of sections $\Phi_p \in \Phi(U')$ there exists a subset $\mathcal{A} \subset \tilde{\mathcal{A}}$ which is strictly moving S and sharp at S such that $\phi_p \in \Phi'_{\delta_t}$ for all $\delta_t \in \mathcal{A}$.

We say that acting in Φ diffeotopies **sharply moves a submanifold** $V_0 \subset V$, if each point $v \in V_0$ admits a neighborhood $U' \subset V$ such that acting diffeotopies $\delta_t : V'_0 = V_0 \cap U' \to U'$ sharply move V'_0 at any given closed hypersurface $S \subset V'_0$.

A.2. A criterion on flexibility.[Gromov1986, 2.2.3.C"] *Let Φ be a microflexible sheaf over V and let a submanifold $V_0 \subset V$ be sharply movable by acting in Φ diffeotopies. Then the sheaf $\Phi_0 = \Phi_{|V_0}$ is flexible and hence it satisfies the h-principle.*

Before stating the Nash-Gromov implicit Function Theorem in A2 we need to introduce several new notions. Let X be a C^∞-fibration over an n-dimensional manifold V and let $G \to V$ be a smooth vector bundle. We denote by $\mathcal{X}^\alpha$ and $\mathcal{G}^\alpha$ respectively the spaces of C^α-sections of the fibrations X and G for all $\alpha = 0, 1, \cdots, \infty$. Let $\mathcal{D} : \mathcal{X}^r \to \mathcal{G}^0$ be a differential operator of order r. In other words the operator $\mathcal{D}$ is given by a map $\triangle : X^{(r)} \to G$, namely $\mathcal{D}(x) = \triangle \circ J^r_x$, where $J^r_x(v)$ denotes the r-jet of x at $v \in V$. We assume below that $\mathcal{D}$ is a C^∞-operator and so we have continuous maps $\mathcal{D} : \mathcal{X}^{\alpha+r} \to \mathcal{G}^\alpha$ for all $\alpha = 0, 1, \cdots, \infty$.

We say that the operator $\mathcal{D}$ is **infinitesimal invertible over a subset** $\mathcal{A}$ in the space of sections $x : V \to X$ if there exists a family of linear differential operators of certain order s, namely $M_x : \mathcal{G}^s \to \mathcal{Y}^0_x$, for $x \in \mathcal{A}$, such that the following three properties are satisfied.

1. There is an integer $d \geq r$, called **the defect of the infinitesimal inversion** M, such that $\mathcal{A}$ is contained in $\mathcal{X}^d$, and furthermore, $\mathcal{A} = \mathcal{A}^d$ consists (exactly and only) of C^d-solutions

of an open differential relation $A \subset X^{(d)}$. In particular, the sets $\mathcal{A}^{\alpha+d} = \mathcal{A} \cap \mathcal{X}^{\alpha+d}$ are open in $\mathcal{X}^{\alpha+d}$ in the respective fine $C^{\alpha}+d$-topology for all $\alpha = 0, 1, \cdots, \infty$.

2. The operator $M_x(g) = M)x, g)$ is a (non-linear) differential operator in x of order d. Moreover the global operator

$$M : \mathcal{A}^d \times \mathcal{G}^s \to \mathcal{J}^0 = T(\mathcal{X}^0)$$

is a differential operator, that is given by a C^{∞}-map $A \oplus G^{(s)} \to T_{vert}(X)$.

3. $L_x \circ M_x = Id$ that is

$$L(x, M(x, g)) = g \text{ for all } x \in \mathcal{A}^{d+r} \text{ and } g \in \mathcal{G}^{r+s}.$$

Now let $\mathcal{D}$ admit over an open set $\mathcal{A} = \mathcal{A}^d \subset \mathcal{X}^d$an infinitesimal inversion M of order s and of defect d. For a subset $\mathcal{B} \subset \mathcal{X}^0 \times \mathcal{G}^0$ we put $\mathcal{B}^{\alpha,\beta} := \mathcal{B} \cap (\mathcal{X}^{\alpha} \times \mathcal{G}^{\beta})$. Let us fix an integer σ_0 which satisfies the following inequality

$$(*) \qquad \sigma_0 > \bar{s} = \max(d, 2r+s).$$

Finally we fix an arbitrary Riemannian metric in the underlying manifold V.

A.3. Nash-Gromov implicit function theorem.[Gromov1986, 2.3.2]. *There exists a family of sets $\mathcal{B}_x \subset \mathcal{G}^{\sigma_0+s}$ for all $x \in \mathcal{A}^{\sigma_0+r+s}$, and a family of operators $\mathcal{D}_x^{-1} : \mathcal{B}_x \to \mathcal{A}$ with the following five properties.*

1. *Neighborhood property: Each set $\mathcal{B}_x$ contains a neighborhood of zero in the space $\mathcal{G}^{\sigma_0+s}$. Furthermore, the union $\mathcal{B} = \{x\} \times \mathcal{B}_x$ where x runs over $\mathcal{A}^{\sigma_0+r+s}$, is an open subset in the space $\mathcal{A}^{\sigma_0+r+s} \times \mathcal{G}^{\sigma_0+s}$.*

2. *Normalization Property: $\mathcal{D}_x^{-1}(0) = x$ for all $x \in \mathcal{A}^{\sigma_0+r+s}$.*

3. *Inversion Property: $\mathcal{D} \circ \mathcal{D}_x^{-1} - \mathcal{D}(x) = Id$, for all $x \in \mathcal{A}^{\sigma_0+r+s}$, that is*

$$\mathcal{D}(\mathcal{D}_x^{-1}(g)) = \mathcal{D}(x) + g,$$

for all pairs $(x, g) \in \mathcal{B}$.

4. *Regularity and Continuity: If the section $x \in \mathcal{A}$ is C^{η_1+r+s}-smooth and if $g \in \mathcal{B}_x$ is C^{σ_1+s}-smooth for $\sigma_0 \le \sigma_1 \le \eta_1$, then the section $\mathcal{D}_x^{-1}(g)$ is C^{σ}-smooth for all $\sigma < \sigma_1$. Moreover the operator $\mathcal{D}^{-1} : \mathcal{B}^{\eta_1+r+s,\sigma_1+s} \to \mathcal{A}^{\sigma}$, $\mathcal{D}^{-1}(x,g) = \mathcal{D}_x^{-1}(g)$, is jointly continuous in the variables x and g. Furthermore, for $\eta_1 > \sigma_1$, the section $\mathcal{D}^{-1} : \mathcal{B}^{\eta_1+r+s,\sigma_1+s} \to \mathcal{A}^{\sigma_1}$ is continuous.*

5. *Locality: The value of the section $\mathcal{D}_x^{-1}(g) : V \to X$ at any given point $v \in V$ does not depend on the behavior of x and g outside the unit ball $B_v(1)$ in V with center v, and so the equality $(x,g)_{|B_v(1)} = (x',g')_{|B_v(1)}$ implies $\mathcal{D}_x^{-1}(g))(v) = (\mathcal{D}_{x'}^{-1}(g'))(v)$.*

A.3'. Corollary. Implicit Funtion Theorem. *For every $x_0 \in \mathcal{A}^{\infty}$ there exists fine $C^{\bar{s}+s+1}$-neighborhood $\mathcal{B}_0$ of zero in the space of $\mathcal{G}\bar{s}+s+1$, where $\bar{s} = \max(d, 2r+s)$, such that for each $C^{\sigma+s}$-section $g \in \mathcal{B}_0$, $\sigma \ge \bar{s}+1$, the equation $\mathcal{D}(x) = \mathcal{D}(x_0 = +g$ has a C^{σ}-solution.*

Finally we shall define the solution sheaf Φ whose flexibility is a consequence of the Nash-Gromov implicit function theorem.

Let us fix a C^{∞}-section $g : V \to G$ and call a C^{∞}-germ $x : \mathcal{O}p(v) \to X$, $v \in V$, **an infinitesimal solution of order α of the equation** $\mathcal{D}(x) = g$, if at the point v the germ $g' = g - \mathcal{D}(x)$ has zero α-jet , i.e. $J^{\alpha}_{g'}(v) = 0$. We denote by $\mathcal{R}^{\alpha}(\mathcal{D}, g) \subset X^{(r+\alpha)}$ the set of all jets represented by these infinitesimal solutions of order α over all points $v \in V$. Now we recall the open set $A \subset X^{(d)}$ defining the set $\mathcal{A} \subset X^{(d)}$ and for $\alpha \ge d - r$ we put

$$\mathcal{R}_{\alpha} = \mathcal{R}_{\alpha}(A, \mathcal{D}, g) = A^{r+\alpha-d} \cap \mathcal{R}^{\alpha}(\mathcal{D}, g) \subset X^{(r+\alpha)},$$

where $A^{r+\alpha-d} = (p_d^{r+\alpha})^{-1}(A)$ for $p_d^{r+\alpha} : X^{r+\alpha} \to X^d$.

A $C^{r+\alpha}$-section $x : V \to X$ satisfies $\mathcal{R}_{\alpha}$, iff $\mathcal{D}(x) = g$ and $x \in \mathcal{A}$.

Now we set $\mathcal{R} = \mathcal{R}_{d-r}$ and denote by $\Phi = \Phi(\mathcal{R}) = \Phi(A, \mathcal{D}, g)$ the sheaf of C^{∞}-solutions of $\mathcal{R}$.

A.4. Microflexibility of the sheaf of solutions and Nash-Gromov implicit functions.[Gromov1986 2.3.2.D"] *The sheaf Φ is microflexible.*

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