# $F$-method for constructing equivariant differential operators*

*Dedicated to Professor Sigurdur Helgason for his 85th birthday*

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**Abstract**

Using an algebraic Fourier transform of operators, we develop a method ($F$-method) to obtain explicit highest weight vectors in the branching laws by differential equations. This article gives a brief explanation of the $F$-method and its applications to a concrete construction of some natural equivariant operators that arise in parabolic geometry and in automorphic forms.

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# 1 Introduction

The aim of this article is to give a brief account of a method that helps us to find a closed formula of highest weight vectors in the branching laws of certain generalized Verma modules, or equivalently, to construct explicitly equivariant differential operators from generalized flag varieties to subvarieties.

This method, which we call the $F$-method, transfers an algebraic problem of finding explicit highest weight vectors to an analytic problem of solving differential equations (of second order) via the algebraic Fourier transform of operators (Definition 3.1). A part of the ideas of the $F$-method has grown in a detailed analysis of the Schrödinger model of the minimal representation of indefinite orthogonal groups [8].

The $F$-method provides a conceptual understanding of some natural differential operators which were previously found by a combinatorial approach based on recurrence formulas. Typical examples that we have in mind are the Rankin–Cohen bidifferential operators

$$R_n^{k_1,k_2}(f_1,f_2) = \sum_{j=0}^{n}(-1)^j \binom{n}{j} \frac{(k_1+n-1)!(k_2+n-1)!}{(k_1+n-j-1)!(k_2+j-1)!} \frac{\partial^{n-j} f_1}{\partial x^{n-j}} \frac{\partial^j f_2}{\partial y^j}\bigg|_{x=y}$$

in automorphic form theory [2, 3, 11], and Juhl's conformally equivariant operators [4] from $C^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^{n-1})$:

$$T_{\lambda,\nu} = \sum_{2j+k=\nu-\lambda} \frac{1}{2^j j!(\nu-\lambda-2j)!} \left( \prod_{i=1}^{\frac{\nu-\lambda}{2}-j} (\lambda+\nu-n-1+2i) \right) \Delta^j_{\mathbb{R}^{n-1}} \left(\frac{\partial}{\partial x_n}\right)^k.$$

These examples can be reconstructed by the $F$-method by using a special case of the *fundamental differential operators*, which are commuting family of second order differential operators on the isotropic cone, see [8, (1.1.3)].

In recent joint works with B. Ørsted, M. Pevzner, P. Somberg and V. Souček [9, 10], we have developed the $F$-method to more general settings, and have found new explicit formulas of equivariant differential operators in parabolic geometry, and also have obtained a generalization of the Rankin–Cohen operators. To find those nice settings where the $F$-method works well, we can apply the general theory [6, 7] that assures discretely decomposable and multiplicity-free restrictions of representations to reductive subalgebras.

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# 2 Preliminaries

## 2.1 Induced modules

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, and $U(\mathfrak{g})$ its universal enveloping algebra. Suppose that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and $V$ is an $\mathfrak{h}$-module. We define the induced $U(\mathfrak{g})$-module by

$$\operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V.$$

If $\mathfrak{h}$ is a Borel subalgebra and if $\dim_{\mathbb{C}} V = 1$, then $\operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ is the standard Verma module.

## 2.2 Extended notion of differential operators

We understand clearly the notion of differential operators between two vector bundles over the same base manifold. We extend this notion in a more general setting where there is a morphism between two base manifolds.

**Definition 2.1.** Let $\mathcal{V}_X \to X$ and $\mathcal{W}_Y \to Y$ be two vector bundles with a smooth map $p : Y \to X$ between the base manifolds. Denote by $C^\infty(X, \mathcal{V}_X)$ and $C^\infty(Y, \mathcal{W}_Y)$ the spaces of smooth sections to the vector bundles. We say that a linear map $T : C^\infty(X, \mathcal{V}_X) \to C^\infty(Y, \mathcal{W}_Y)$ is a *differential operator* if there exists a differential operator $Q$ acting on sections of the vector bundle over $Y$ such that the following diagram commutes:

$$\begin{tikzcd}
C^\infty(Y, p^*\mathcal{V}_X) \arrow[dr, "Q"] & \\
C^\infty(X, \mathcal{V}_X) \arrow[u, "p^*"] \arrow[r, "T"] & C^\infty(Y, \mathcal{W}_Y).
\end{tikzcd}$$

We write $\operatorname{Diff}(\mathcal{V}_X, \mathcal{W}_Y)$ for the vector space of such differential operators.

## 2.3 Equivariant differential operators

Let $G$ be a real Lie group, $\mathfrak{g}(\mathbb{R}) = \mathrm{Lie}(G)$ and $\mathfrak{g} = \mathfrak{g}(\mathbb{R}) \otimes \mathbb{C}$. Analogous notations will be applied to other Lie groups denoted by uppercase Roman letters.

Let $dR$ be the representation of $U(\mathfrak{g})$ on the space $C^\infty(G)$ of smooth complex-valued functions on $G$ generated by the Lie algebra action:

$$(dR(A)f)(x) := \frac{d}{dt}\Big|_{t=0} f(xe^{tA}) \quad \text{for } A \in \mathfrak{g}(\mathbb{R}). \tag{2.1}$$

Let $H$ be a closed subgroup of $G$. Given a finite dimensional representation $V$ of $H$ we form a homogeneous vector bundle $\mathcal{V}_X := G \times_H V$ over the homogeneous space $X := G/H$. The space of smooth sections $C^\infty(X, \mathcal{V}_X)$ can be seen as a subspace of $C^\infty(G) \otimes V$.

Let $V^\vee$ be the (complex linear) dual space of $V$. Then the $(G \times \mathfrak{g})$-invariant bilinear map $C^\infty(G) \times U(\mathfrak{g}) \to C^\infty(G)$, $(f, u) \mapsto dR(u)f$ induces a commutative diagram of $(G \times \mathfrak{g})$-bilinear maps:

$$\begin{array}{ccc} C^\infty(G) \otimes V \times U(\mathfrak{g}) \otimes_{\mathbb{C}} V^\vee & \longrightarrow & C^\infty(G) \\ \uparrow \qquad \downarrow & & \| \\ C^\infty(X, \mathcal{V}_X) \times \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) & \longrightarrow & C^\infty(G). \end{array}$$

In turn, we get the following natural $\mathfrak{g}$-homomorphism:

$$\mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee) \longrightarrow \mathrm{Hom}_G(C^\infty(X, \mathcal{V}_X), C^\infty(G)). \tag{2.2}$$

Next, we take a connected closed subgroup $H'$ of $H$. For a finite dimensional representation $W$ of $H'$ we form the homogeneous vector bundle $\mathcal{W}_Z := G \times_{H'} W$ over $Z := G/H'$. Taking the tensor product of (2.2) with $W$, and collecting all $\mathfrak{h}'$-invariant elements, we get an injective homomorphism:

$$\mathrm{Hom}_{\mathfrak{h}'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \longrightarrow \mathrm{Hom}_G(C^\infty(X, \mathcal{V}_X), C^\infty(Z, \mathcal{W}_Z)), \quad \varphi \mapsto D_\varphi. \tag{2.3}$$

Finally, we take any closed subgroup $G'$ containing $H'$ and form a homogeneous vector bundle $\mathcal{W}_Y := G' \times_{H'} W$ over $Y := G'/H'$. We note that $\mathcal{W}_Y$ is obtained from $\mathcal{W}_Z$ by restricting the base manifold $Z$ to $Y$.

Let $R_{Z\to Y} : C^\infty(Z, \mathcal{W}_Z) \to C^\infty(Y, \mathcal{W}_Y)$ be the restriction map. We set

$$D_{X\to Y}(\varphi) := R_{X\to Y} \circ D_\varphi. \tag{2.4}$$

Since there is a natural ($G'$-equivariant but not necessarily injective) morphism $Y \to X$, the extended notion of differential operators between $\mathcal{V}_X$ and $\mathcal{W}_Y$ makes sense (see Definition 2.1). We then have:

**Theorem 2.2.** *The operator $D_{X\to Y}$ (see (2.4)) induces a bijection:*

$$D_{X\to Y} : \mathrm{Hom}_{\mathfrak{h}'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y). \tag{2.5}$$

*Remark* 2.3. We may consider a holomorphic version of Theorem 2.2 as follows. Suppose $G_{\mathbb{C}}$, $H_{\mathbb{C}}$, $G'_{\mathbb{C}}$ and $H'_{\mathbb{C}}$ are connected complex Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{g}'$ and $\mathfrak{h}'$, and $\mathcal{V}_{X_{\mathbb{C}}}$ and $\mathcal{W}_{Y_{\mathbb{C}}}$ are homogeneous holomorphic vector bundles over $X_{\mathbb{C}} := G_{\mathbb{C}}/H_{\mathbb{C}}$ and $Y_{\mathbb{C}} := G'_{\mathbb{C}}/H'_{\mathbb{C}}$, respectively. Then Theorem 2.2 implies that we have a bijection:

$$D_{X_{\mathbb{C}}\to Y_{\mathbb{C}}} : \mathrm{Hom}_{\mathfrak{h}'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \mathrm{Diff}^{\mathrm{hol}}_{G'_{\mathbb{C}}}(\mathcal{V}_{X_{\mathbb{C}}}, \mathcal{W}_{Y_{\mathbb{C}}}). \tag{2.6}$$

Here $\mathrm{Diff}^{\mathrm{hol}}_{G'_{\mathbb{C}}}$ denotes the space of $G'_{\mathbb{C}}$-equivariant holomorphic differential operators with respect to the holomorphic map $Y_{\mathbb{C}} \to X_{\mathbb{C}}$. By the universality of the induced module, (2.6) may be written as

$$D_{X_{\mathbb{C}}\to Y_{\mathbb{C}}} : \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{h}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \mathrm{Diff}^{\mathrm{hol}}_{G'_{\mathbb{C}}}(\mathcal{V}_{X_{\mathbb{C}}}, \mathcal{W}_{Y_{\mathbb{C}}}). \tag{2.7}$$

The isomorphism (2.7) is well-known when $X_{\mathbb{C}} = Y_{\mathbb{C}}$ is a complex flag variety. The proof of Theorem 2.2 is given in [10] in the generality that $X \neq Y$.

## 2.4 Multiplicity-free branching laws

Theorem 2.2 says that if $\mathrm{Hom}_{\mathfrak{h}'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee))$ is one-dimensional then $G'$-equivariant differential operators from $\mathcal{V}_X$ to $\mathcal{W}_Y$ are unique up to scalar. Thus we may expect that such unique operators should have a natural meaning and would be given by a reasonably simple formula. Then we may be interested in finding systematically the examples where $\mathrm{Hom}_{\mathfrak{h}'}(W^\vee, \mathrm{ind}_{\mathfrak{h}}^{\mathfrak{g}}(V^\vee))$ is one-dimensional. This is a special case of the *branching problems* that asks how representations decompose when restricted to subalgebras. In the setting where $\mathfrak{h}$ is a parabolic subalgebra (to be denoted by $\mathfrak{p}$) of a reductive Lie algebra $\mathfrak{g}$, we have the following theorem:

**Theorem 2.4.** *Assume the nilradical $\mathfrak{n}_+$ of $\mathfrak{p}$ is abelian and $\tau$ is an involutive automorphism of $\mathfrak{g}$ such that $\tau\mathfrak{p} = \mathfrak{p}$. Then for any one-dimensional* *representation* $\mathbb{C}_\lambda$ *of* $\mathfrak{p}$ *and for any finite dimensional representation* $W$ *of* $\mathfrak{p}^\tau := \{X \in \mathfrak{p} : \tau X = X\}$, *we have*

$$\dim \operatorname{Hom}_{\mathfrak{p}^\tau}(W^\vee, \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda^\vee)) \leq 1.$$

There are two known approaches for the proof of Theorem 2.4. One is geometric — to use the general theory of the *visible action* on complex manifolds [5, 6], and the other is algebraic — to work inside the universal enveloping algebra [7].

*Remark* 2.5. Branching laws in the setting of Theorem 2.4 are explicitly obtained in terms of 'relative strongly orthogonal roots' on the level of the Grothendieck group, which becomes a direct sum decomposition when the parameter $\lambda$ of $V$ is 'generic' or sufficiently positive, [6, Theorems 8.3 and 8.4] or [7]. The $F$-method will give a finer structure of branching laws by finding explicitly highest weight vectors with respective reductive subalgebras. The two prominent examples in Introduction, i.e. the Rankin–Cohen bidifferential operators and the Juhl's conformally equivariant differential operators, can be interpreted in the framework of the $F$-method as a special case of Theorem 2.4.

# 3 A recipe of the $F$-method

The idea of the $F$-method is to work on the branching problem of representations by taking the Fourier transform of the nilpotent radical. We shall explain this method in the complex setting where $H_\mathbb{C}$ is a parabolic subgroup $P_\mathbb{C}$ with abelian unipotent radical (see Theorem 2.2 and Remark 2.3) for simplicity. A detaild proof will be given in [10] (see also [9] for a somewhat different formulation and normalization).

## 3.1 Weyl algebra and algebraic Fourier transform

Let $E$ be an $n$-dimensional vector space over $\mathbb{C}$. The Weyl algebra $\mathcal{D}(E)$ is the ring of holomorphic differential operators on $E$ with polynomial coefficients.

**Definition 3.1** (algebraic Fourier transform)**.** We define an isomorphism of two Weyl algebras on $E$ and its dual space $E^\vee$:

$$\mathcal{D}(E) \to \mathcal{D}(E^\vee), \qquad T \mapsto \widehat{T}, \tag{3.1}$$

which is induced by

$$(3.2) \qquad \widehat{\frac{\partial}{\partial z_j}} := -\zeta_j, \quad \widehat{z_j} := \frac{\partial}{\partial \zeta_j} \quad (1 \le j \le n),$$

where $(z_1, \ldots, z_n)$ are coordinates on $E$ and $(\zeta_1, \ldots, \zeta_n)$ are the dual coordinates on $E^\vee$.

*Remark* 3.2. (1) The isomorphism (3.1) is independent of the choice of coordinates.

(2) An alternative way to get the isomorphism (3.1) or its variant is to use the Euclidean Fourier transform $\mathcal{F}$ by choosing a real form $E(\mathbb{R})$ of $E$. We then have

$$\widehat{\frac{\partial}{\partial z}} = \sqrt{-1}\mathcal{F} \circ \frac{\partial}{\partial x} \circ \mathcal{F}^{-1}, \quad \widehat{z} = -\sqrt{-1}\mathcal{F} \circ z \circ \mathcal{F}^{-1}$$

as operators acting on the space $\mathcal{S}'(E^\vee)$ of Schwartz distributions. This was the approach taken in [9]. In particular, $\widehat{T} \neq \mathcal{F} \circ T \circ \mathcal{F}^{-1}$ in our normalization here. The advantage of our normalization (3.2) is that the commutative diagram in Theorem 3.5 does not involve any power of $\sqrt{-1}$ that would otherwise depend on the degrees of differential operators. As a consequence, the final step of the $F$-method (see Step 5 below) as well as actual computations becomes simpler.

## 3.2 Infinitesimal action on principal series

Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ be a Levi decomposition of a parabolic subalgebra of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{l} + \mathfrak{n}_+$ the Gelfand–Naimark decomposition. Since the following map

$$\mathfrak{n}_- \times \mathfrak{l} \times \mathfrak{n}_+ \to G_{\mathbb{C}}, \quad (X, Z, Y) \mapsto (\exp X)(\exp Z)(\exp Y)$$

is a local diffeomorphism near the origin, we can define locally the projections $p_-$ and $p_o$ from a neighbourhood of the identity to the first and second factors $\mathfrak{n}_-$ and $\mathfrak{l}$, respectively. Consider the following two maps:

$$\begin{aligned} \alpha : \mathfrak{g} \times \mathfrak{n}_- \to \mathfrak{l}, \qquad & (Y, X) \mapsto \frac{d}{dt}\Big|_{t=0} p_o\left(e^{tY} e^X\right), \\ \beta : \mathfrak{g} \times \mathfrak{n}_- \to \mathfrak{n}_-, \qquad & (Y, X) \mapsto \frac{d}{dt}\Big|_{t=0} p_-\left(e^{tY} e^X\right). \end{aligned}$$

We may regard $\beta(Y,\cdot)$ as a vector field on $\mathfrak{n}_-$ by the identification $\beta(Y,X) \in \mathfrak{n}_- \simeq T_X\mathfrak{n}_-$.

For $\mathfrak{l}$-module $\lambda$ on $V$, we set $\mu := \lambda^\vee \otimes \Lambda^{\dim}\mathfrak{n}_+$. Since $\Lambda^{\dim \mathfrak{n}_+}\mathfrak{n}_+$ is one-dimensional, we can and do identify the representation space with $V^\vee$. We inflate $\lambda$ and $\mu$ to $\mathfrak{p}$-modules by letting $\mathfrak{n}_+$ act trivially. Consider a Lie algebra homomorphism

$$d\pi_\mu : \mathfrak{g} \to \mathcal{D}(\mathfrak{n}_-) \otimes \operatorname{End}(V^\vee), \tag{3.3}$$

defined for $F \in C^\infty(\mathfrak{n}_-, V^\vee)$ as

$$(d\pi_\mu(Y)F)(X) := \mu(\alpha(Y,X))F(X) - (\beta(Y,\cdot)F)(X). \tag{3.4}$$

If $(\mu, V^\vee)$ lifts to the parabolic subgroup $P_\mathbb{C}$ of a reductive group $G_\mathbb{C}$ with Lie algebras $\mathfrak{p}$ and $\mathfrak{g}$ respectively, then $d\pi_\mu$ is the differential representation of the induced representation $\operatorname{Ind}_{P_\mathbb{C}}^{G_\mathbb{C}}(V)$ (without $\rho$-shift). We note that the Lie algebra homomorphism (3.4) is well-defined without integrality condition of $\mu$. The $F$-method suggests to take the algebraic Fourier transform (3.1) on the Weyl algebra $\mathcal{D}(\mathfrak{n}_-)$. We then get another Lie algebra homomorphism

$$\widehat{d\pi_\mu} : \mathfrak{g} \to \mathcal{D}(\mathfrak{n}_+) \otimes \operatorname{End}(V^\vee). \tag{3.5}$$

Then we have (see [10])

**Proposition 3.3.** *There is a natural isomorphism*

$$F_c : \operatorname{ind}_\mathfrak{p}^\mathfrak{g}(\lambda^\vee) \xrightarrow{\sim} \operatorname{Pol}(\mathfrak{n}_+) \otimes V^\vee$$

*which intertwines the left $\mathfrak{g}$-action on $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^\vee$ with $\widehat{d\pi_\mu}$.*

## 3.3 Recipe of the $F$-method

Our goal is to find an explicit form of a $G'$-intertwining differential operator from $\mathcal{V}_X$ to $\mathcal{W}_Y$ in the upper right corner of Diagram 3.1. Equivalently, what we call the $F$-method yields an explicit homomorphism belonging to $\operatorname{Hom}_{\mathfrak{g}'}(\operatorname{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \operatorname{ind}_\mathfrak{p}^\mathfrak{g}(V^\vee)) \simeq \operatorname{Hom}_{\mathfrak{p}'}(W^\vee, \operatorname{ind}_\mathfrak{p}^\mathfrak{g}(V^\vee))$ in the lower left corner of Diagram 3.1 in the setting that $\mathfrak{n}_+$ is abelian.

The recipe of the $F$-method in this setting is stated as follows:

Step 0. Fix a finite dimensional representation $(\lambda, V)$ of $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$.

Step 1. Consider a representation $\mu := \lambda^\vee \otimes \Lambda^{\dim \mathfrak{n}_+}\mathfrak{n}_+$ of the Lie algebra $\mathfrak{p}$. Consider the restriction of the homomorphisms (3.3) and (3.5) to the subalgebra $\mathfrak{n}_+$:

$$\begin{aligned} d\pi_\mu : \mathfrak{n}_+ &\to \mathcal{D}(\mathfrak{n}_-) \otimes \mathrm{End}(V^\vee), \\ \widehat{d\pi_\mu} : \mathfrak{n}_+ &\to \mathcal{D}(\mathfrak{n}_+) \otimes \mathrm{End}(V^\vee). \end{aligned}$$

Step 2. Take a finite dimensional representation $W$ of the Lie algebra $\mathfrak{p}'$. For the existence of nontrivial solutions in Step 3 below, it is necessary and sufficient for $W$ to satisfy

$$\mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \neq \{0\}. \tag{3.6}$$

Choose $W$ satisfying (3.6) if we know a priori an abstract branching law of the restriction of $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ to $\mathfrak{g}'$. See [6, Theorems 8.3 and 8.4] or [7] for some general formulae. Otherwise, we take $W$ to be any $\mathfrak{l}'$-irreducible component of $S(\mathfrak{n}_+) \otimes V^\vee$ and go to Step 3.

Step 3. Consider the system of partial differential equations for $\psi \in \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee \otimes W$ which is $\mathfrak{l}'$-invariant under the diagonal action:

$$\widehat{d\pi_\mu}(C)\psi = 0 \qquad \text{for } C \in \mathfrak{n}'_+. \tag{3.7}$$

Notice that the equations (3.7) are of second order. The solution space will be one-dimensional if we have chosen $W$ in Step 2 such that

$$\dim \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) = 1. \tag{3.8}$$

Step 4. Use invariant theory and reduce (3.7) to another system of differential equations on a lower dimensional space $S$. Solve it.

Step 5. Let $\psi$ be a polynomial solution to (3.7) obtained in Step 4. Compute $(\mathrm{Symb} \otimes \mathrm{Id})^{-1}(\psi)$. Here the symbol map

$$\mathrm{Symb} : \mathrm{Diff}^{\mathrm{const}}(\mathfrak{n}_-) \xrightarrow{\sim} \mathrm{Pol}(\mathfrak{n}_+)$$

is a ring isomorphism given by the coordinates

$$\mathbb{C}[\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}] \to \mathbb{C}[\xi_1, \cdots, \xi_n], \quad \frac{\partial}{\partial z_j} \mapsto \xi_j.$$

In case the Lie algebra representation $(\lambda, V)$ lifts to a group $P_{\mathbb{C}}$, we form a $G_{\mathbb{C}}$-equivariant holomorphic vector bundle $\mathcal{V}_{X_{\mathbb{C}}}$ over $X_{\mathbb{C}} = G_{\mathbb{C}}/P_{\mathbb{C}}$. Likewise, in case $W$ lifts to a group $P'_{\mathbb{C}}$, we form a $G'_{\mathbb{C}}$-equivariant holomorphic vector bundle $\mathcal{W}_{Y_{\mathbb{C}}}$ over $Y_{\mathbb{C}} = G'_{\mathbb{C}}/P'_{\mathbb{C}}$. Then $(\mathrm{Symb} \otimes \mathrm{Id})^{-1}(\psi)$ in Step 5 gives an explicit formula of a $G'_{\mathbb{C}}$-equivariant differential operator from $\mathcal{V}_{X_{\mathbb{C}}}$ to $\mathcal{W}_{Y_{\mathbb{C}}}$ in the coordinates of $\mathfrak{n}_-$ owing to Theorem 3.5 below. This is what we wanted.

*Remark* 3.4. In Step 2 we can find all such $W$ if we know a priori (abstract) explicit branching laws. This is the case, e.g., in the setting of Theorem 2.4. See Remark 2.5.

Conversely, the differential equations in Step 3 sometimes give a useful information on branching laws even when the restrictions are not completely reducible, see [9].

For concrete constructions of equivariant differential operators by using the $F$-method in various geometric settings, we refer to [9, 10]. A further application of the $F$-method to the construction of *non-local* operators will be discussed in another paper.

The key tool for the F-method is summarized as:

**Theorem 3.5** ([10]). *Let $P'_{\mathbb{C}}$ be a parabolic subgroup of $G'_{\mathbb{C}}$ compatible with a parabolic subgroup $P_{\mathbb{C}}$ of $G_{\mathbb{C}}$. Assume further the nilradical $\mathfrak{n}_+$ of $\mathfrak{p}$ is abelian. Then the following diagram commutes:*

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathbb{C}}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) & \simeq & \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W) & \xleftarrow[\sim]{\mathrm{Symb} \otimes \mathrm{Id}} & \mathrm{Diff}^{\mathrm{const}}(\mathfrak{n}_-) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W) \\
\cup & & \circlearrowleft & & \cup \\
\mathrm{Hom}_{\mathfrak{p}'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) & & \xrightarrow{D_{X_{\mathbb{C}} \to Y_{\mathbb{C}}}} & & \mathrm{Diff}_{G'_{\mathbb{C}}}(\mathcal{V}_{X_{\mathbb{C}}}, \mathcal{W}_{Y_{\mathbb{C}}}).
\end{array}$$

Diagram 3.1

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