HIGHLY TRANSITIVE ACTIONS OF $\mathrm{Out}(\mathbf{F_n})$

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Abstract. An action of a group on a set is called *k-transitive* if it is transitive on ordered k-tuples and *highly transitive* if it is k-transitive for every k. We show that for $n \geq 4$ the group $\mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$ admits a faithful highly transitive action on a countable set.

1. Introduction

1.1. **Notation.** Throughout the paper G will denote a finitely generated simple group, with $d = d(G)$ its minimal number of generators. Given an n-tuple $\underline{g} = (g_1, g_2, \ldots, g_n) \in G^n$ we denote by $\langle \underline{g} \rangle = \langle g_1, g_2, \ldots, g_n \rangle < G$ the subgroup generated by it. Let $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ be the nonabelian free group on n generators. There is a natural identification of G^n with $\mathrm{Hom}(F_n, G)$; associating the n-tuple $\underline{g}$ with the homomorphism

$$\begin{aligned} \alpha_{\underline{g}} : F_n &\to G \\ \alpha_{\underline{g}}(x_i) &\mapsto g_i. \end{aligned}$$

We will use $\mathrm{Hom}(F_n, G)$ and G^n interchangeably. For example we will identify the set of epimorphisms $\mathrm{Epi}(F_n, G)$ with the set

$$V_n(G) = \{\underline{g} \in G^n : \langle \underline{g} \rangle = G\},$$

of all generating n-tuples.

The group $\mathrm{Aut}(G)$ acts on $\mathrm{Hom}(F_n, G)$ (from the left) by post-composition and $\mathrm{Aut}(F_n)$ acts on $\mathrm{Hom}(F_n, G)$ (from the right) by pre-composition. One easily verifies that in the corresponding action of $\mathrm{Aut}(F_n)$ on G^n the standard Nielsen transformations act as follows:

$$\begin{aligned} R^{\pm}_{i,j} &: (g_1, \ldots, g_i, \ldots, g_n) \to (g_1, \ldots, g_i \cdot g_j^{\pm 1}, \ldots, g_n), \\ L^{\pm}_{i,j} &: (g_1, \ldots, g_i, \ldots, g_n) \to (g_1, \ldots, g_j^{\pm 1} \cdot g_i, \ldots, g_n), \\ P_{i,j} &: (g_1, \ldots, g_i, \ldots, g_j, \ldots, g_n) \to (g_1, \ldots, g_j, \ldots, g_i, \ldots, g_n), \\ I_i &: (g_1, \ldots, g_i, \ldots, g_n) \to (g_1, \ldots, g_i^{-1}, \ldots, g_n). \end{aligned}$$

We denote by $\bar{V}_n(G)$ the set of all $\mathrm{Aut}(G)$-orbits on $V_n(G)$. Given $\underline{g} \in V_n(G)$ we denote by $[\underline{g}]$ the corresponding equivalence class in $\bar{V}_n(G)$. Since the $\mathrm{Aut}(F_n)$ action preserves $V_n(G)$ and commutes with the $\mathrm{Aut}(G)$ action it descends to an action of $\Gamma := \mathrm{Out}(F_n) = \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n)$ on $\bar{V}_n(G)$.

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1.2. **The main theorem.** A Tarski monster group is a noncyclic group G, all of whose proper subgroups are cyclic. It was shown by A. Yu. Olshanskii that for every large enough prime p there exist uncountably many Tarski monsters all of whose subgroups are isomorphic to $\mathbf{Z}/p\mathbf{Z}$, as well as Tarski monsters all of whose subgroups are infinite cyclic (see [Ol′80]).

Establishing the existence of infinite Tarski monsters is difficult, but once such a group is given many structural results follow directly from the definition. A Tarski monster G is necessarily simple. For every $n \geq 2$, the collection of generating n-tuples is given by

$$V_n(G) = G^n \setminus \left\{ \underline{g} \in G^n \mid \langle \underline{g} \rangle \text{ is cyclic} \right\}.$$

Definition. An action of a group on a set $\Gamma \curvearrowright \Omega$ is called *k-transitive* if it is transitive on ordered k-tuples. It is called *highly transitive* if it is k-transitive for every $k \in \mathbf{N}$.

Definition. A group G is said to satisfy a *group law*, if there exists some m and some non-trivial word in the free group $w \in F_m$ such that

$$w(\underline{g}) = 1, \qquad \forall \underline{g} \in G^m.$$

For example every abelian group satisfies the law given by the commutator $[x, y] \in F_2$ and every group of exponent p satisfies the word $x^p \in \mathbf{Z}$. The goal of this paper is to prove the following

Theorem. *Let G be a Tarski monster and $n \geq 4$ then the action of $\Gamma = \mathrm{Out}(F_n)$ on the character variety $\bar{V}_n(G)$ is highly transitive. Moreover, this action is faithful if and only if G does not satisfy a group law.*

Clearly every Tarski monster of finite exponent satisfies a group law. As for torsion free Tarski monsters, some of them do satisfy a group law while others do not, as was recently proved by Zusmanovich [Zus10]. Combining Zusmanovich's theorem with our theorem, and a few well known results concerning permutation groups we obtain the following:

Corollary. *For $n \geq 4$ let $\Gamma_n = \mathrm{Out}(F_n)$ and let $\Gamma_n \curvearrowright \Omega$ the faithful highly transitive action constructed above. Let $\Delta < \Gamma_n$ be any subgroup containing a non-trivial subnormal subgroup. Then the following holds*

1. *The action of Δ on Ω is highly transitive,*
2. *Δ contains a maximal subgroup of infinite index,*
3. *Δ does not satisfy any group law, in particular it cannot be finite, abelian, or even virtually solvable.*
4. *If $\Delta' < \Gamma$ is another group containing a non-trivial subnormal subgroup then $\Delta \cap \Delta' \neq \langle e \rangle$.*

Proof. Saying that $\Gamma_n \curvearrowright \Omega$ is highly transitive is equivalent to saying that the corresponding embedding $\Gamma_n < \mathrm{Sym}(\Omega)$ has a dense image; with respect to the pointwise convergence topology on $\mathrm{Sym}(\Omega)$. But then $\overline{N} \lhd \lhd \overline{\Gamma_n} = \mathrm{Sym}(\Omega)$ for any subnormal subgroup $N \lhd \lhd \Gamma_n$ and since $\mathrm{Sym}(\Omega)$ is topologically simple N is also dense, proving (1). Now (2) follows, since a highly transitive action is automatically primitive so $\Delta_\omega < \Delta$ is a maximal subgroup for every $\omega \in \Omega$. Since $\mathrm{Sym}(\Omega)$ contains a free subgroup it cannot satisfy any group law itself and this is automatically inherited by any dense

subgroup, which establishes (3). Finally for (4) assume by way of contradiction that $N, N' \lhd\lhd \Gamma_n$ are two non-trivial subnormal subgroups that intersect trivially. We claim that there are two (possibly equal) non-trivial subnormal subgroups M, M' that commute. Indeed let $N = N_l \lhd N_{l-1} \lhd \ldots \lhd N_0 = \Gamma_n$ and assume first that $N' \lhd \Gamma_n$ is normal. If j is the first index such that $N' \cap N_j = \langle e \rangle$ we set $M = N_j$ and $M' = N' \cap N_{j-1}$. The argument is concluded by induction on the minimal length of a subnormal series for N'. Now since M, M' are both dense in the topology induced from $\mathrm{Sym}(\Omega)$ it turns out that $\mathrm{Sym}(\Omega)$ is abelian which is absurd. □

Item (2) above can be viewed as an analogue for $\mathrm{Out}(F_n)$ of the theorems due to Margulis and Soĭfer [MS79, MS81] and to Ivanov [Iva92, Theorem 5] concerning the existence of maximal subgroups of infinite index. Margulis and Soĭfer prove that a finitely generated linear group admits an infinite index maximal subgroup if and only if it is not virtually solvable. Ivanov's theorem states that finitely generated subgroup of the mapping class group of a surface admits a maximal subgroup of infinite index if and only if it is not virtually abelian. Both theorems were then generalized in [GG08] to general countable subgroups. On a certain level our current result is much stronger because highly transitive actions are rarer than primitive actions. On the other hand our current method is restricted to very special classes of subgroups of $\mathrm{Out}(F_n)$.

1.3. **Gilman's work on the Wiegold conjecture.** Transitivity of the action of $\Gamma \curvearrowright \bar{V}_n(G)$, for various groups G was extensively studied in various different settings in the last few decades. We refer the readers to a comprehensive survey article on this subject by Alex Lubotzky [Lub]. In particular it is conjectured by Wiegold that the action of $\mathrm{Out}(F_3) \curvearrowright \bar{V}_3(G)$ is transitive for every finite simple group G.

Of particular interest from our point of view is the work of Gilman [Gil77] who proved the Wiegold conjecture in the case $G = \mathrm{PSL}_2(\mathbf{F}_p)$ for every prime $p \geq 5$ and $n \geq 3$. Gilman showed, in fact, that the image of $\mathrm{Out}(F_n)$ in $\mathrm{Sym}(\bar{V}_n(\mathrm{PSL}_2(\mathbf{F}_p))$ is either the full symmetric group, or the alternating group. Thus proving a much stronger statement.

Gilman has further proved that if G is a finite simple nonabelian group and $n \geq 4$ then $\mathrm{Out}(F_n)$ acts as a symmetric or alternating group on at least one of its orbits in $\bar{V}_n(G)$. This result was extended to $n = 3$ by Evans [Eva93].

The current paper grew out of an attempt to find an infinite setting in which Gilman's proof can be implemented. Technically there is one qualitative difference between the finite and the infinite case. In the former in order to prove that a finite permutation group contains the alternating group it is sufficient to prove primitivity and then establish the existence of one long cycle. In the latter k-transitivity has to be verified directly, by induction, for every $k \in \mathbf{N}$.

1.4. **A hierarchy of actions.** One can define a hierarchy of transitivity properties, for example as follows: transitive actions, quasiprimitive actions, primitive actions, 2-transitive actions, ... highly transitive actions

etc. Where we recall that an action is called *quasiprimitive* if every normal subgroup acts either trivially or transitively and *primitive* if there is no Γ-invariant equivalence relation on the set. In this context the following question seems very natural.

Question. Given a group Γ, how transitively can Γ act on a set?

The few results that are known seem to hint that the existence of an action with good transitivity properties entails non-trivial structural information about the group. For example Margulis and Soĭfer prove that a finitely generated linear group admits a primitive action on an infinite set if and only if it is not virtually solvable. A complete classification of countable linear groups that admit a faithful primitive action on a set is obtained in [GG08]. As far as higher transitivity properties not much is known, and in particular only few examples are known of groups that admit highly transitive actions. Examples include non abelian free groups (see [McD77, Dix90] as well as fundamental groups of surfaces of genus at least 2 (see [Kit09]).

There is a big gap between our understanding of primitive permutation representations, such as these achieved in [MS81, GG08] and highly transitive representations constructed in [McD77, Dix90, Kit09]. The former use structural properties of linear groups, while the latter rely on the great flexibility available in constructing representations of free groups, or more generally of limit groups. It seems that a key example that should be considered is that of arithmetic groups such as $\mathrm{PSL}_3(\mathbf{Z})$. While this group does admit a faithful primitive action, it usually exhibits a very rigid behavior which might hint that it does not admit actions with higher transitivity properties. In this context it came to us as a surprise that the group $\mathrm{Out}(F_n)$, which is sometimes considered as a non-abelian version of $\mathrm{PSL}_n(\mathbf{Z})$, admits such a nice highly transitive permutation representation.

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2. Preliminary results

2.1. **Generation of powers.** We denote by $M_{n,k}(G)$ the collection of $n \times k$ matrices with entries taken from the group G. Let

$$A = \begin{pmatrix} g_1^1 & \dots & g_1^k \\ g_2^1 & \ddots & g_2^k \\ \vdots & & \vdots \\ g_n^1 & \dots & g_n^k \end{pmatrix} = \begin{pmatrix} - & \underline{g}_1 & - \\ - & \underline{g}_2 & - \\ & \vdots & \\ - & \underline{g}_n & - \end{pmatrix} = \begin{pmatrix} | & & | \\ \underline{g}^1 & \dots & \underline{g}^k \\ | & & | \end{pmatrix}, \tag{1}$$

be such a matrix, with $\underline{g}_1, \dots, \underline{g}_n \in G^k$ and $\underline{g}^1, \dots \underline{g}^k \in G^n$ denoting the corresponding row and column vectors respectively. The following result was originally used by P. Hall [Hal36] in the realm of finite simple groups. Compare for example [KL90, Prop. 6]. For the convenience of the reader we add a proof.

Proposition. *Let G be a nonabelian simple finitely generated group, $k \geq d(G)$ and $A \in M_{n,k}(G)$ a matrix as above, all of whose columns generate: $\underline{g}^i \in V_n(G)$, $\forall 1 \leq i \leq k$. Denote by $\alpha^1, \ldots, \alpha^k \in \mathrm{Hom}(F_n, G)$ the corresponding epimorphisms, by $K^1, \ldots, K^k$ their respective kernels and by $A = (\alpha^1, \alpha^2, \ldots, \alpha^k)$ the combined homomorphism defined by*

$$\begin{array}{rcl} A : F_n & \to & G^k \\ w & \mapsto & (w(\underline{g}^1), w(\underline{g}^2), \ldots, w(\underline{g}^k)). \end{array}$$

Then the following are equivalent

1. *$\{[\underline{g}^i] \mid 1 \leq i \leq k\}$ are all different as elements of $\bar{V}_n(G)$.*
2. *$\alpha^j\left(\cap_{i \neq j} K^i\right) = G$, for every $1 \leq j \leq k$.*
3. *$A : F_n \to G^k$ is surjective.*

Proof. (2) $\Longrightarrow$ (3): is obvious.
(3) $\Longrightarrow$ (1): Assume (1) fails. By definition this means that there is some $\sigma \in \mathrm{Aut}(G)$ and indices i, j such that $\sigma \circ \alpha^j = \alpha^i$. But then $A(F_n) < \{(g_1, \ldots, g_k) \mid g_i = \sigma(g_j)\}$ contradicting (3).
(1) $\Longrightarrow$ (2). By symmetry we argue for $j = k$. Since $\alpha^k : F_n \to G$ is surjective and $\cap_{i=1}^{k-1}(K^i) \lhd F_n$ is normal, it follows that $\alpha^k\left(\cap_{i=1}^{k-1}(K^i)\right) \lhd G$. By simplicity of G we need only rule out the possibility that the latter group is trivial. If $k = 2$ this implies that $K^1 < K^2$ and hence there is a commutative diagram:

$$\begin{tikzcd} F_n \arrow[r, "\alpha^1"] \arrow[dr, "\alpha^2"'] & G \arrow[d, "\eta"] \\ & G \end{tikzcd}$$

Since G is simple η must be an isomorphism, contradicting (1).

We proceed by induction on k. Recall that by assumption, G is nonabelian and let $x, y \in G$ be two elements with $[x, y] \neq e$. By our induction assumption we can find $\tilde{x}, \tilde{y} \in F_n$ such that

$$\begin{array}{rcl} A(\tilde{x}) & = & (*, e, e, \ldots, e, x), \\ A(\tilde{y}) & = & (e, *, e, \ldots, e, y). \end{array}$$

It follows that $A([\tilde{x}, \tilde{y}]) = (e, e, \ldots, e, [x, y])$, contradicting the triviality of $\alpha^k\left(\cap_{i=1}^{k-1}(K^i)\right)$. $\square$

Note. We assumed implicitly in the above proposition that k is finite. If $k = \infty$ is infinite it is no longer possible for A to be surjective, but the above proposition remains correct upon replacing (3) by the assumption that $\overline{A(F_n)} = G^{\times\infty}$ where the closure is taken with respect to the product topology.

2.2. **Powers of Tarski monsters.** Applying the proposition above in the specific case of Tarski monsters we obtain the following:

Corollary. *Let G be a Tarski monster then G^k is 3-generated for every $k \in \mathbf{N}$.*

Remark. In other words for every $k \in \mathbf{N}$ there exists a matrix $A \in M_{3\times k}(G)$ satisfying the equivalent conditions of Proposition 2.1.

Proof. Indeed let $x, y \in G$ be a generating tuple, and consider the matrix:

$$A = \begin{pmatrix} x & x & \dots & x \\ y & y & \dots & y \\ z_1 & z_2 & \dots & z_k \end{pmatrix}$$

Clearly every column generates. But if two columns, say $\underline{g}^1$ and $\underline{g}^2$ are in the same $\mathrm{Aut}(G)$ orbit then we have an automorphism $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(x) = x; \sigma(y) = y; \sigma(z_1) = z_2$. Since x, y generate this implies that $z_1 = z_2$. Thus condition (1) of Proposition 2.1 will be satisfied if all z_i are different. □

2.3. **Spread.**

Definition. We say that a 2-generated group G has *spread greater or equal to* k if for every $\underline{g} = (g_1, g_2, \dots, g_k) \in G^k$ there exists some $h \in G$ such that

$$G = \langle h, g_i \rangle \quad \forall 1 \le i \le k.$$

Lemma. *A Tarski monster group G has spread greater or equal to k for every $k \in \mathbf{N}$.*

Proof. Let $k \in \mathbf{N}$ and $\underline{g} = (g_1, g_2, \dots, g_k) \in G^k$ be as above. Since G is finitely generated, it follows from Zorn's lemma that every g_i is contained in a maximal proper subgroup $g_i \in H_i \lneqq G$. By the definition of a Tarski monster every H_i is cyclic and hence of infinite index. But an infinite group is never a union of a finite number of subgroups of infinite index. Thus, any $h \in G \setminus \cup_{i=1}^k H_i$ will satisfy the condition required in the definition of the spread. □

2.4. **Stronger generation properties.** In order to prove Corollary 2.2 we constructed a matrix $3 \times k$ all of whose columns represent different elements of $\bar{V}_n(G)$. In the sequel we will need a matrix satisfying a stronger condition, which is somewhat technical but useful.

Lemma. *For any $k, n \in \mathbf{N}$ with $n \ge 4$ there exists a matrix $A \in M_{n\times k}(G)$, as in Equation (1) in Section 2.1, with the following properties:*

(1) *Every pair of entries generates G:*

$$(i, j) \ne (l, m) \Rightarrow \langle g_j^i, g_m^l \rangle = G,$$

(2) *Every three rows generate G^k:*

$$1 \le i < j < l \le n \Rightarrow \langle \underline{g}_i, \underline{g}_j, \underline{g}_l \rangle = G^k.$$

(3) *The following configuration will never appear as a 4×4 minor of the matrix, for any $\sigma, \tau, \eta, \theta \in \mathrm{Aut}(G)$ and $p, q, r, s \in G$.*

$$\begin{pmatrix} \cdot & p^\tau & p^\eta & p^\theta \\ q^\sigma & \cdot & q^\eta & q^\theta \\ r^\sigma & r^\tau & \cdot & r^\theta \\ s^\sigma & s^\tau & s^\eta & \cdot \end{pmatrix}$$

Proof. We construct the matrix entries one by one according to the following order

$$g_1^1, g_2^1, g_3^1, \dots, g_1^2, g_2^2, \dots, g_{n-1}^k, g_n^k,$$

namely column by column. Making sure in the process that all three desired properties hold:

(1) Every pair of elements generates: Upon adding the element g_l^m one can make sure that it generates G with every previous entry by Lemma 2.3. In fact the proof of that lemma makes it clear that there are infinitely many possible choices of an element that will satisfy this condition. Thus we can guarantee the validity of (1) even if we require later in the proof to exclude finitely many possibilities at every stage.

General discussion: Assuming from now on that (1) is indeed satisfied we notice that for a given 2×2 minor

$$\begin{pmatrix} g_s^i & g_s^j \\ g_t^i & g_t^j \end{pmatrix}$$

there is at most one $\sigma \in \mathrm{Aut}(G)$ such that $g_s^j = \left(g_s^i\right)^\sigma$ and $g_t^j = \left(g_t^i\right)^\sigma$. If this holds for a given $1 \le i < j \le k$ and for some choice of $1 \le s < t \le n$ we say that the columns i, j are *σ-near*. Next we extend the notion of near columns to be an equivalence relation - declaring two columns (i, j) to be *σ-related* if there is a sequence of distinct columns $i = i_0, i_1, i_2, \dots, i_r = j$ such that i_α and $i_{\alpha+1}$ are σ_α-near and $\sigma = \sigma_0\sigma_1\sigma_2 \dots \sigma_{r-1}$. We will denote the set of all automorphisms relating two columns $1 \le i < j \le k$ by:

$$\Xi_{i,j} := \{\sigma \in \mathrm{Aut}(G) \mid j \text{ is } \sigma\text{-related to } i\},$$

it is clear from the definition, and from the fact that any two matrix elements generate G that this set is finite. Finally let us note that these relations can be defined even for matrices that are only partially defined, namely for matrices with some missing entries.

Assume that we are now adding the element g_l^m namely that we have already constructed the matrix

$$\begin{pmatrix} g_1^1 & g_1^2 & \dots & g_1^{m-1} & g_1^m & \square \\ \vdots & \vdots & \ddots & \vdots & \vdots & \square \\ g_{l-1}^1 & \vdots & \dots & g_{l-1}^{m-1} & g_{l-1}^m & \square \\ \vdots & \vdots & \ddots & \vdots & \square & \square \\ g_n^1 & g_n^2 & \dots & g_n^{m-1} & \square & \square \end{pmatrix}$$

We choose an element g_l^m which will satisfy condition (1) while excluding the finite set of possibilities:

$$\bigcup_{i=1}^{m-1} \bigcup_{\sigma \in \Xi_{i,m}} \left(g_l^i\right)^\sigma$$

(2) Every three rows generate $\mathbf{G^k}$: By Proposition 2.1 all we have to do in order to prove that $G^k = \langle \underline{g}_i, \underline{g}_j, \underline{g}_l \rangle$, is to exclude the possibility that

for some $1 \leq s < m \leq k$ and some $\sigma \in \mathrm{Aut}(G)$ we have

$$\begin{pmatrix} g_i^s \\ g_j^s \\ g_l^s \end{pmatrix} = \begin{pmatrix} g_i^m \\ g_j^m \\ g_l^m \end{pmatrix}^\sigma,$$

where the automorphism applied to the vector just means that it is applied to each entry separately. But such a configuration is not possible by construction because at the time of the choice of the element g_l^m, the columns s, m are already σ-related (in fact they are even σ-near) and thus the choice of $g_l^m = (g_l^s)^\sigma$ is ruled out.

(3) Excluding cyclic configurations: A configuration such as the one appearing in (3) is excluded because at the time construction of the element labeled r^θ this choice is in fact invalid. Indeed at that time the first and the last column are already $\sigma^{-1}\theta = (\sigma^{-1}\eta)(\eta^{-1}\theta)$ related so that the choice $r^\theta = (r^\sigma)^{\sigma^{-1}\theta}$ is invalid. Equivalently, the same choice can be ruled out, by the fact that the second and fourth columns are $\tau^{-1}\theta$-related. □

3. Highly transitive actions.

Let G be a Tarski monster and $n \geq 4$. We argue by induction on k that the action of $\Gamma = \mathrm{Out}(F_n)$ on $\bar{V}_n(G)$ is k-transitive.

3.1. **The induction basis.**

Proposition. *For every $n \geq 3$, $\mathrm{Aut}(F_n)$ acts transitively on $V_n(G)$, where G is a Tarski monster group.*

Proof. Let us fix a basepoint $\underline{g} = (g_1, g_2, \ldots, g_n) \in V_n(G)$. We are at liberty to choose a convenient base point - and using Lemma 2.3 repeatedly we impose the condition $\langle g_i, g_j \rangle = G \; \forall i \neq j$. Now given any $\underline{h} = (h_1, h_2, \ldots, h_n) \in V_n(G)$ we have to exhibit a sequence of Nielsen transformations taking $\underline{g}$ to $\underline{h}$. Since any generating set contains a generating pair we may assume, after renumbering the indices, that $\langle h_1, h_2 \rangle = G$. Since G has spread greater or equal to 2 there exists some $z \in G$ such that

$$\langle g_2, z \rangle = \langle h_1, z \rangle = G. \tag{2}$$

We proceed with the following sequence of Nielsen transformations:

$$\underline{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{pmatrix} \looparrowright \begin{pmatrix} g_1 \\ g_2 \\ z \\ \vdots \\ g_n \end{pmatrix} \looparrowright \begin{pmatrix} h_1 \\ g_2 \\ z \\ \vdots \\ g_n \end{pmatrix} \looparrowright \begin{pmatrix} h_1 \\ h_2 \\ z \\ \vdots \\ g_n \end{pmatrix} \looparrowright \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_n \end{pmatrix} = \underline{h}. \tag{3}$$

The first $\looparrowright$ stands for a sequence of Nielsen transformations of the form $w(R_{3,1}, R_{3,2})$ where w is any free word on two generators satisfying $w(g_1, g_2) = g_3^{-1}z$. The existence of such a word is guaranteed by the fact that $\langle g_1, g_2 \rangle = G$. The next three $\looparrowright$ use the exact same argument, but instead of using the fact that $\langle g_1, g_2 \rangle = G$ they appeal in turn to the two parts of Equation (2) and then to our assumption that $\langle h_1, h_2 \rangle = G$. □

Since $\mathrm{Aut}(F_n) \curvearrowright V_n(G)$ is transitive, so is the quotient action $\Gamma \curvearrowright \bar{V}_n(G)$, which is exactly what we require for the basis of our induction.

3.2. **The induction step.** We choose a base k-tuple

$$\Big([\underline{g}^1],[\underline{g}^2],[\underline{g}^3],\dots,[\underline{g}]=[\underline{g}^k]\Big)$$

of distinct elements in $\bar{V}_n(G)$. In order to establish the induction step we have to show that for any $[\underline{h}]\notin\{[\underline{g}^1],[\underline{g}^2],[\underline{g}^3],\dots,[\underline{g}^{k-1}]\}$ there is a group element $\gamma\in\Gamma$ such that $\gamma[\underline{g}^i]=[\underline{g}^i],\ \forall 1\le i\le k-1$ and $\gamma[\underline{g}]=[\underline{h}]$.

Again we have a lot of freedom in the choice of our basis k-tuple. We make our choice by picking a matrix

$$A=\begin{pmatrix} g_1^1 & \dots & g_1^k\\ g_2^1 & \ddots & g_2^k\\ \vdots & & \vdots\\ g_n^1 & \dots & g_n^k\end{pmatrix}=\begin{pmatrix} - & \underline{g}_1 & -\\ - & \underline{g}_2 & -\\ & \vdots & \\ - & \underline{g}_n & -\end{pmatrix}=\begin{pmatrix} | & & |\\ \underline{g}^1 & \dots & \underline{g}^k\\ | & & |\end{pmatrix},$$

satisfying the conditions guaranteed by Lemma 2.4, and taking its columns as representatives. The element $\underline{h}=(h_1,h_2,\dots,h_n)\in V_n(G)$ on the other hand is dictated to us. But, since $\langle\underline{h}\rangle=G$ we may assume, after possibly reordering the indices that $\langle h_1,h_2\rangle=G$.

We wish to proceed in much the same way as we did in Equation (3), taking $\underline{g}$ to $\underline{h}$, but this time we have to be careful not to touch the elements $\{\underline{g}^i\mid 1\le i\le k-1\}$. Let us find an element $z\in G$ such that all the $3\times k$ matrices A_2,A_3,A_4 below satisfy the equivalent conditions of Proposition 2.1

$$\begin{aligned} A_2 &= \begin{pmatrix} - & - & \underline{g}_2 & - & -\\ - & - & \underline{g}_3 & - & -\\ g_4^1 & g_4^2 & \dots & g_4^{k-1} & z\end{pmatrix}\\ A_3 &= \begin{pmatrix} g_1^1 & g_1^2 & \dots & g_1^{k-1} & h_1\\ - & - & \underline{g}_3 & - & -\\ g_4^1 & g_4^2 & \dots & g_4^{k-1} & z\end{pmatrix}\\ A_4 &= \begin{pmatrix} g_1^1 & g_1^2 & \dots & g_1^{k-1} & h_1\\ g_2^1 & g_2^2 & \dots & g_2^{k-1} & h_2\\ g_4^1 & g_4^2 & \dots & g_4^{k-1} & z\end{pmatrix}\end{aligned}$$

For the matrices A_2,A_4 this can be achieved by avoiding finitely many bad values of z, by an argument identical to that used in the proof of Lemma 2.4. For the matrix A_3 we can use the same argument assuming $\langle h_1,g_3\rangle=G$. On the other hand if $\langle h_1,g_3\rangle$ is a cyclic subgroup then the desired condition $[h_1,g_3,z]\neq[g_1^i,g_3^i,g_4^i]\ \ \forall 1\le i<k$ is satisfied automatically, for any choice of z which makes (h_1,g_3,z) into a generating set, just because all the other columns satisfy the condition that every pair of elements generates G and the last column does not.

We now proceed by applying the following sequence of Nielsen transformations:

$$\underline{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ \vdots \\ g_n \end{pmatrix} \looparrowright \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ z \\ \vdots \\ g_n \end{pmatrix} \looparrowright \begin{pmatrix} h_1 \\ g_2 \\ g_3 \\ z \\ \vdots \\ g_n \end{pmatrix} \looparrowright \begin{pmatrix} h_1 \\ h_2 \\ g_3 \\ z \\ \vdots \\ g_n \end{pmatrix} \looparrowright \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ z \\ \vdots \\ g_n \end{pmatrix} \looparrowright \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ \vdots \\ h_n \end{pmatrix} = \underline{h}.$$

This time the first $\looparrowright$ corresponds to $w(R_{4,1}, R_{4,2}, R_{4,3})$, where w is chosen so as to satisfy the two conditions:

- $w(g_1, g_2, g_3) = g_4^{-1} z$,
- $w(g_1^i, g_2^i, g_3^i) = \mathbf{Id}, \quad \forall 1 \leq i < k$.

The existence of such a word follows directly from condition (2) of Proposition 2.1, combined with our assumptions on the matrix A. The existence of Nielsen transformations realizing the next three $\looparrowright$, while fixing $\underline{g}^i \ \ \forall 1 \leq i < k$ follows, in the exact same way using our assumptions on the matrices A_2, A_3, A_4 respectively.

The last $\looparrowright$ can be treated in a similar fashion as long as the matrix:

$$A_5 = \begin{pmatrix} g_1^1 & g_1^2 & \dots & g_1^{k-1} & h_1 \\ g_2^1 & g_2^2 & \dots & g_2^{k-1} & h_2 \\ g_3^1 & g_3^2 & \dots & g_3^{k-1} & h_3 \end{pmatrix}$$

satisfies the conditions of Proposition 2.1.

Note that if there is a pair of indices such that $\langle h_i, h_j \rangle \neq G$ these conditions are automatically satisfied, perhaps after rearrangement of the indices. Indeed after rearranging the indices so that

$$\langle h_2, h_3 \rangle \neq G$$

the last column is the only one in A_5 with this property and thus it cannot be in the $\mathrm{Aut}(G)$ orbit of any of the other columns. Thus we can assume that every pair of h entries generates the entire group.

Finally if the conditions of Proposition 2.1 are not satisfied for A_5. Not even after we change the order of the indices, by making arbitrary permutations of the first four rows of A. This means that there are four automorphisms $\sigma, \tau, \eta, \theta \in \mathrm{Aut}(G)$ and four columns $1 \leq i < j < l < m \leq k$ such that the corresponding 4×4 minor admits the forbidden configuration described in condition (3) of Lemma 2.4

$$\begin{pmatrix} g_1^i & g_1^j & g_1^l & g_1^m \\ g_2^i & g_2^j & g_2^l & g_2^m \\ g_3^i & g_3^j & g_3^l & g_3^m \\ g_4^i & g_4^j & g_4^l & g_4^m \end{pmatrix} = \begin{pmatrix} \cdot & (h_1)^\tau & (h_1)^\eta & (h_1)^\theta \\ (h_2)^\sigma & \cdot & (h_2)^\eta & (h_2)^\theta \\ (h_3)^\sigma & (h_3)^\tau & \cdot & (h_3)^\theta \\ (h_4)^\sigma & (h_4)^\tau & (h_4)^\eta & \cdot \end{pmatrix},$$

contradicting our construction of the matrix A.

4. Concerning faithfulness of the action

4.1. **A theorem of Magnus.** We will require the following, well known, theorem of Wilhelm Magnus from 1930:

Theorem. [MKS76, Theorem N5 page 172]. *Let $a, b \in F_m$ be elements of the free group and assume that they generate the same normal subgroup*

$$\langle a \rangle^{F_m} = \langle b \rangle^{F_m},$$

then $a = gb^{\epsilon}g^{-1}$ for some $g \in F_m$ and some $\epsilon \in \{\pm 1\}$.

Namely the normal closure of a cyclic group, determines the generator up to conjugation and inversion.

4.2. The action $\mathrm{Out}(F_n)$ on the redundant locus of F_{n-1}.

Definition. If $n > d(G)$ the redundant locus of $V_n(G)$ is defined as:

$$R_n(G) = \{\phi \in V_n(G) \mid \langle \phi(x_1), \dots, \phi(x_{n-1}) \rangle = G, \text{ for some basis } x_1, \dots, x_n < F_n\}$$

and $\bar{R}_n(G) < \bar{V}_n(G)$ is the image of this (invariant) set, modulo $\mathrm{Aut}(G)$.

Lemma. *For every $n \geq 3$, $\mathrm{Out}(F_n)$ acts faithfully on $\bar{R}_n(F_{n-1})$.*

Proof. It is easy to verify that $\phi : F_n \to F_{n-1}$ corresponds to an element of $R_n(F_{n-1})$ if and only if it is surjective and the kernel is generated, as a normal subgroup, by a primitive element of F_n.

Let $\alpha \in \mathrm{Aut}(F_n)$ represent an element of $\mathrm{Out}(F_n)$ that is in the kernel of the action on $\bar{R}_n(F_{n-1})$. Thus by definition for every $\phi \in R_n(F_{n-1})$ there exists some $\sigma \in \mathrm{Aut}(F_{n-1})$ such that $\phi \circ \alpha = \sigma \circ \phi$, and in particular

$$\alpha^{-1}(\ker(\phi)) = \ker(\phi \circ \alpha) = \ker(\sigma \circ \phi) = \ker(\phi).$$

So α acts trivially on the collection of normal subgroups generated by a primitive element. By Magnus' theorem stated above, for every primitive element $x \in F_n$ there exist $g_x \in F_n, \epsilon_x \in \{\pm 1\}$ such that

$$\alpha x = g_x x^{\epsilon} g_x^{-1}. \tag{4}$$

In particular setting $g_i = g_{x_i}, \epsilon_i = \epsilon_{x_i}$ for some basis $\underline{x} = \{x_1, x_2, \dots, x_n\}$ we have:

$$\alpha(\underline{x}) = \begin{pmatrix} \alpha_1(\underline{x}) \\ \alpha_2(\underline{x}) \\ \vdots \\ \alpha_n(\underline{x}) \end{pmatrix} = \begin{pmatrix} x_1^{\epsilon_1} \\ g_2 x_2^{\epsilon_2} g_2^{-1} \\ \vdots \\ g_n x_n^{\epsilon_n} g_n^{-1} \end{pmatrix}.$$

Here, since to begin with, α is defined only up to inner automorphisms, we have assumed without loss of generality that $g_1 = 1$.

Let Y be the $2n$-regular Cayley tree of F_n with respect to the given set of generators. We label the oriented edges of Y by elements of the corresponding symmetric generating set $\{x_1, x_1^{-1}, x_2, \dots, x_n^{-1}\}$ in such a way that F_n is identified with the group of color preserving automorphisms of the tree. Every element $x \in F_n$ acts on Y as a hyperbolic automorphism with translation length $\ell(x) := \min\{d(v, xv) \mid v \in Y\} > 0$ and axis $X_x = \{v \in Y \mid d(v, xv) = \ell(x)\}$. Equation (4) implies that $\alpha(x_i)$ is a hyperbolic element of translation length 1 whose axis is $g_i X_{x_i}$ (with either of its two possible orientations). In particular the axis of $\alpha(x_i)$ is labeled either $\dots x_i, x_i, x_i, \dots$ or with the inverse of this sequence. Similarly the translation length and axis labeling is preserved for any primitive element of $\mathrm{Aut}(F_n)$.

We claim that $X_{\alpha(x_i)} \cap X_{\alpha(x_j)} = g_i X_i \cap g_j X_j \neq \emptyset, \quad \forall i \neq j$. Indeed, assuming the contrary, since α preserves the translation length of primitive elements we have

$$2 = \ell(x_i x_j) = \ell(\alpha(x_i)\alpha(x_j)) = 2 + 2d(X_{\alpha(x_i)}, X_{\alpha(x_j)});$$

so $d(X_{\alpha(x_i)}, X_{\alpha(x_j)}) = 0$ which is a contradiction. Now, by the version of Helly's theorem for trees (see for example [Ser80, I.6.5 Lemma 10]) this implies that there is a point $o \in \cap_{i=1\dots n} X_{\alpha(x_i)}$. After conjugation by an appropriate power of $x_1 = \alpha(x_1)$ we may assume that

$$o = \cap_{i=1\dots n} X_{\alpha(x_i)} = \cap_{i=1\dots n} X_i.$$

But now $X_i = X_{\alpha(x_i)}$ since these two axes share a point and, up to orientation, they have the same coloring; hence upon replacing g_i by $g_i x_i^{m_i}$ for an appropriate choice of power m_i, we can assume that $g_i o = o$. This immediately implies that $g_i = 1 \quad \forall i$.

We still have to show that $\epsilon_i = 1 \quad \forall i$. But if, say, $\epsilon_1 = -1$ then the primitive element $x_1 x_2 x_3$ will map to $x_1^{-1} x_2^{\epsilon_2} x_3^{\epsilon_3}$. A short verification will show that, regardless of the values of ϵ_2, ϵ_3, this element is neither conjugate to $x_1 x_2 x_3$ nor to $(x_1 x_2 x_3)^{-1}$. This completes the proof of the lemma. □

4.3. **Group laws on two letters are universal.**

Lemma. *Assume that G is a finitely generated group that satisfies a group law. Then G already satisfies a non-trivial group law on two letters.*

Proof. Assume that G satisfies a group law $w \in F_m$ for some $m > 2$. By [BG09, Corollary 3.3] there is a homomorphism $\phi = (\phi_1, \phi_2, \dots, \phi_n) : F_m \to F_2$ such that $\phi(w) \neq 1$. Now G satisfies the non-trivial group law $\phi(w)$ as

$$\phi(w)(g,h) = w(\phi_1(g,h), \phi_2(g,h), \dots, \phi_n(g,h)) = 1, \quad \forall g,h \in G.$$

□

4.4. **When G satisfies a group law the action is not faithful.**

Proposition. *If G is any finitely generated group which satisfies a group law, then for any $n > \max\{3, d(G)\}$, the action of $\mathrm{Out}(F_n)$ on $\bar{V}_n(G)$ is not faithful.*

Proof. As we saw in Section 4.3 we may assume that G satisfies a group law on two letters, i.e. there exists a word $w \in F_2$ such that $w(g,h) = 1, \ \forall g,h \in G$.

Consider the automorphism $\alpha = w(R_{n,1}, R_{n,2})$ given explicitly by

$$\alpha(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n \cdot w(x_1, x_2)).$$

α is not the trivial automorphism because, by assumption w is a non-trivial word in F_2 and $R_{n,1}, R_{n,2}$ generate a free subgroup of $\mathrm{Out}(F_n)$. But by construction α acts trivially on G^n. □

4.5. **When the action is not faithful G satisfies a group law.** The following completes the proof of our main theorem.

Proposition. *Let G be a Tarski monster and $n \geq 3$. Then the action of $\mathrm{Out}(F_n)$ on $\bar{V}_n(G)$ is faithful if and only if G satisfies no group law.*

Proof. The if part is treated in Proposition 4.4. Assume that G is a Tarski monster group and that the action of $\mathrm{Out}(F_n)$ on $\bar{V}_n(G)$ is not faithful. We will show that G satisfies a group law.

It is enough to show that there exists some non-trivial word $w = w(x, y) \in F_2$ such that $w(a, b) = 1$ for any $(a, b) \in V_2(G)$. Indeed, assume that w can be written in reduced form as $w = z_1 \dots z_n$, where $z_1, \dots, z_n \in \{x^{\pm 1}, y^{\pm 1}\}$, and let

$$v = wzw^{-1}z^{-1} = z_1 \dots z_n z z_n^{-1} \dots z_1^{-1} z^{-1},$$

where $z \in \{x^{\pm 1}, y^{\pm 1}\}$ satisfies that $z \neq z_1^{-1}, z_n, z_n^{-1}$. Then $v = v(x, y)$ is a non-trivial word in F_2. Moreover, if $(a, b) \in V_2(G)$ then $v(a, b) = 1$ since $w(a, b) = 1$. If a and b do not generate G, then they belong to the same cyclic group, and so there exist some $c \in G$ and $i, j \in \mathbf{Z}$ s.t. $a = c^i$ and $b = c^j$. In this case, $w(a, b) = w(c^i, c^j) = c^k$ for some $k \in \mathbf{Z}$, and so $v(a, b) = v(c^i, c^j) = c^k c^l c^{-k} c^{-l} = 1$ (where $l \in \{\pm i, \pm j\}$). Hence, G satisfies a group law with the word v.

By assumption, there exists some automorphism $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathrm{Aut}(F_n)$, which is not an inner automorphism, such that for any $\underline{g} \in V_n(G)$ there exists some $\sigma \in \mathrm{Aut}(G)$ such that.

$$\begin{aligned} \alpha_1(\underline{g}) &= \sigma(g_1), \\ \alpha_2(\underline{g}) &= \sigma(g_2), \\ &\vdots \\ \alpha_n(\underline{g}) &= \sigma(g_n). \end{aligned}$$

In particular for any $(a, b) \in V_2(G)$ and for any word $u \in F_2$, we can apply the above to the the n-tuple $(a, b, u(a, b), 1, 1, \dots) \in V_n(G)$ obtaining the equation

$$\alpha_3(a, b, u(a, b)) = \sigma(u(a, b)) = u(\sigma(a), \sigma(b)) = u\big(\alpha_1(a, b, u(a, b)), \alpha_2(a, b, u(a, b))\big).$$

Here we used $\alpha_1(a, b, u(a, b))$ as a short for $\alpha_1(a, b, u(a, b), 1, 1, \dots)$.

Now consider the three words on two letters

$$\begin{aligned} w_3^u(x, y) &= u\left(\alpha_1(x, y, u(x, y)), \alpha_2(x, y, u(x, y))\right) \cdot \alpha_3(x, y, u(x, y))^{-1}, \\ w_2^u(x, y) &= u\left(\alpha_1(x, u(x, y), y), \alpha_3(x, u(x, y), y)\right) \cdot \alpha_2(x, u(x, y), y)^{-1}, \\ w_1^u(x, y) &= u\left(\alpha_2(u(x, y), x, y), \alpha_3(u(x, y), x, y)\right) \cdot \alpha_1(u(x, y), x, y)^{-1}. \end{aligned}$$

By permuting the role of $a, b, u(a, b)$ among the first three coordinates in the above argument we know that $w_i^u(a, b) = 1 \quad \forall (a, b) \in V_2(G), \forall u \in F_2(x, y)$. If one of the w_i^u is a non-trivial word in F_2 then we have our group law. But if w_i^u represents the trivial word in F_2 for every $i = 1, 2, 3$ and every $u \in F_2$ then $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ gives rise to an element of $\mathrm{Out}(F_3)$ in the kernel of the action on $\bar{R}_3(F_2)$ and hence a contradiction to Lemma 4.2.

□

5. Final remarks and questions

We conclude with three natural questions:

5.1. **Lower rank groups.**

Question. What about $\mathrm{Out}(F_2)$ and $\mathrm{Out}(F_3)$? Do they admit a highly transitive action on a set?

There is a chance that the action that we study in this paper, of $\mathrm{Out}(F_n)$ on $\bar{V}_n(G)$ still has very good transitivity properties for $n = 3$. Even if this is true it seems that the proof would be much harder as it would require a much better understanding of the Tarski monster G and its automorphisms. A topic that we carefully avoided in this paper.

On the other hand it is plausible that one can construct completely different actions in this lower rank setting. For example since $\mathrm{Out}(F_2) = \mathrm{PSL}_2(\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$, there is a good chance that one can construct a highly transitive action for this group using the methods of [Dix90]. It is important to note though that this is not automatic. It is not in general true that if a finite index subgroup admits a highly transitive action then so does the group itself. Even though the other direction is true of course.

As for $n = 3$. At least for the group $\mathrm{Aut}(F_3)$ it follows from [GL09, Corollary 1.2] that there is a finite index subgroup $\Delta < \mathrm{Aut}(F_3)$ that maps onto a finitely generated free group, and hence admits a (non-faithful) highly transitive action. Again it is plausible that one might be able to construct a highly transitive non-faithful action of $\mathrm{Aut}(F_3)$ from this.

5.2. **Mapping class groups.**

Question. Does $\mathrm{Mod}(S)$, the mapping class group of a closed orientable surface S of a high enough genus admit a highly transitive action on a set? If not, how about k-transitive actions for various values of k?

Recall that from [GG08] it follows that these groups do admit faithful primitive actions.

5.3. **Faithfulness.**

Our proof of the faithfulness statement in section 4 gives rise to the following general theorem:

Theorem. *For any finitely generated group G, the following are equivalent:*

- *The action of* $\mathrm{Out}(F_n)$ *on the* $\mathrm{Aut}(G)$*-classes of* $\mathrm{Hom}(F_n, G)$ *is faithful for all large enough n.*
- *The group G does not satisfy a group law.*

It is a very natural question if the above still holds if one replaces $\mathrm{Hom}(F_n, G)$ by $\bar{V}_n(G)$.

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