

Minimal representations and reductive dual pairs in conformal field theory*

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Abstract

A *minimal representation* of a simple non-compact Lie group is obtained by “quantizing” the minimal nilpotent coadjoint orbit of its Lie algebra. It provides context for Roger Howe’s notion of a reductive dual pair encountered recently in the description of global gauge symmetry of a (4-dimensional) conformal observable algebra. We give a pedagogical introduction to these notions and point out that physicists have been using both minimal representations and dual pairs without naming them and hence stand a chance to understand their theory and to profit from it.

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1 Introduction

“... for me the motivation is mostly the desire to understand the hidden machinery in a striking concrete example, around which one can build formalisms”.¹

The concept of a minimal representation, introduced in the mid 1970’s by A. Joseph [J74] [J76] (triggered by the physicists’ excitement with spectrum generating algebras) keeps attracting attention of mathematicians and (mathematical) physicists (for a review up to the late 1990’s – see [Li00]; for subsequent developments – see e.g. [G01, G06] [K02] [GS05] [KM] [F09] [Li09] [FG09] and references therein). We shall review the identification of the minimal orbit – the classical counterpart of the minimal representation – in Section 2. We will also display, in Section 3, an old physical example – the *oscillator* (called in the 1960’s *ladder representations* of $U(2, 2)$ which describe zero-mass particles [MT69] (constructed before the concept of a minimal representation was coined).

The second item of our survey, the notion of a (*reductive*) *dual pair*, was introduced by Roger Howe in an influential preprint of the 1970’s that was eventually published as [H89]. It was previewed in a lecture [H85] highlighting its applications to physics. The principle underlying such applications was, in fact, anticipated by the founders of the axiomatic approach to quantum field theory who introduced the concept of superselection sectors $[W^3]$ and developed it systematically in the work of Doplicher, Haag and Roberts (for a review and further references – see [H]). In this framework the algebra of observables and the global (compact) gauge group are mutual commutants in the (extended) field algebra, thus providing a generic example of a dual pair. A more elementary example is provided by a pair of subgroups, G_1, G_2 , of a group G , each being the centralizer of the other in G . One can then ask how does an irreducible representation (IR) of G split into IRs of the pair (G_1, G_2) . This question is, in general, not easy to answer. It becomes manageable if the given IR of G is a minimal one. We review in Section 4 the concept of a reductive dual pair and display Howe’s classification of such *irreducible pairs* acting by automorphisms on a *complex symplectic space*.

In Section 5 we prepare the stage for an extension and application of the above notions to an infinite dimensional framework: *globally conformal invariant quantum field theory* in four spacetime dimensions. The assumption of global conformal invariance (GCI) [NT] allows to extend the concept of *vertex algebra* – the mathematicians’ way of viewing the chiral observable algebra of 2-dimensional conformal field theory (2D CFT) – to any higher even dimensional theory [N]. We also introduce there the notion of a *bilocal field* which opens the way to use infinite dimensional Lie algebras in the context of 4D CFT. In Section 6 we review earlier work [B07, B08] which demonstrates that the physical (unitary, positive energy) representations of the observable algebra $\mathcal{L}$ is a

¹M. Kontsevich, Beyond numbers, in: *The Unravelers, Mathematical Snapshots*, ed. by J.-F. Dars, A. Lesne, A. Papillaut, A.K. Peters, Wellesley, MA 2008, p. 182.

Fock space representation that splits into IRs whose multiplicities are dictated by a compact gauge group U , the pair $(\mathcal{L}, U)$ appearing as a dual pair in the field algebra extension of $\mathcal{L}$ envisaged by Haag et al. [H]. Before concluding we also review in Section 6 the basic properties and the oscillator representations of the non-compact orthogonal group $SO^*(4n)$ (combining material of [BB] and of [B08]).

2 Minimal coadjoint orbits

A coadjoint orbit of a Lie group G is an orbit of the coadjoint representation Ad'_G of G acting on the vector space $\mathcal{G}'$ dual to the Lie algebra $\mathcal{G}$ of G . Each coadjoint orbit $\mathcal{O}$ admits a G -invariant symplectic form and hence is an (even dimensional) symplectic manifold which can be interpreted as the phase space of a classical Hamiltonian system. The *orbit method* reflects the expectation that the unitary IRs of G can be obtained by “quantizing” such classical symplectic orbits. It has been introduced by A.A. Kirillov in 1962 and further developed (along with the *geometric quantization*, chiefly) by Kirillov, Kostant, Duflou, and Souriau – see for a readable introduction and references [K99].

We recall some basic notions and facts of the theory of *semisimple Lie algebras* $\mathcal{G}$. An *element* $s \in \mathcal{G}$ is called *semisimple* if it is conjugate to an element of a Cartan subalgebra of $\mathcal{G}$ (or, equivalently, if ad_s is diagonalizable). An $n \in \mathcal{G}$ is said to be *nilpotent* if ad_n is nilpotent (i.e., if a finite power of ad_n vanishes).

Fact. Every element $x \in \mathcal{G}$ has a unique representation of the form $x = s + n$ where s is semisimple, n is nilpotent, and $[s, n] = 0$. The non-degenerate Killing form $B(X, Y)$ on $\mathcal{G}$ allows to define an isomorphism of $\mathcal{G}$ -spaces $\mathcal{G} \leftrightarrow \mathcal{G}'$. The invariance of B implies that there exists a bijection between adjoint and coadjoint orbits that preserves semi-simplicity and nilpotency. (We say that a coadjoint orbit $\mathcal{O}$ is semisimple/nilpotent if every (equivalently some) element of $\mathcal{O}$ has this property.)

A complex semisimple Lie algebra $\mathcal{G}_{\mathbb{C}}$ has a unique (up to conjugacy) Cartan subalgebra $\mathfrak{h}$. The set of semisimple coadjoint orbits of $\mathcal{G}_{\mathbb{C}}$ is isomorphic to $\mathfrak{h}/W$ where W is the Weyl group of $\mathcal{G}_{\mathbb{C}}$; hence, there is a continuum of such orbits. By contrast, the (partially ordered) set $\mathcal{N}$ of *nilpotent orbits* is finite (see [CMcG] for details and proofs – also of Proposition 2.1 below – and Section 5 of [F09] for a summary).

Proposition 2.1. – (a) *There exists a unique principal nilpotent orbit $\mathcal{O}_{\text{princ}}$ of maximal dimension*

$$|\mathcal{O}_{\text{princ}}| = |\mathcal{G}| - \text{rank } \mathcal{G}, \quad (2.1)$$

where by $|\mathcal{M}|$ we denote the dimension of the manifold $\mathcal{M}$.

(b) *There is a unique subregular nilpotent orbit, $\mathcal{O}_{\text{subreg}}$, of dimension*

$$|\mathcal{O}_{\text{subreg}}| = |\mathcal{G}| - \text{rank } \mathcal{G} - 2. \quad (2.2)$$

(c) *There exists a unique non-zero nilpotent orbit $\mathcal{O}_{\min}$ of minimal dimension.*

Remark 2.1. The principal orbit is dense in $\mathcal{N}$ with respect to the *Zariski topology*. Its closure consists of all lower dimensional nilpotent orbits. The subregular orbit is Zariski dense in $\mathcal{N} \setminus \mathcal{O}_{\text{princ}}$. The *Zariski closure* of $\mathcal{O}_{\min}$ is $\mathcal{O}_{\min} \cup \{0\}$.

Example. The semisimple orbits of $\mathfrak{sl}_2$ are

$$\mathcal{O}_k = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix}, x, y, z \in \mathbb{C}; x^2 + yz = k \neq 0 \right\}. \quad (2.3)$$

For $k = 0$, the corresponding $\mathcal{O}_0$ is the unique non-trivial nilpotent orbit: $\mathcal{O}_0 = \mathcal{O}_{\text{princ}} = \mathcal{O}_{\min}$ ($\mathcal{O}_{\text{subreg}}$ is trivial in this case – consisting of the 0-element of $\mathfrak{sl}_2$).

To construct the minimal orbit of a simple Lie algebra $\mathcal{G}$ we observe that all simple complex Lie algebras have an essentially unique 5-grading (as exploited in [G01, K02, G06]):

$$\mathcal{G} = \mathcal{G}_{-2} \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2, \quad (2.4)$$

given by the eigenvalues of the Cartan generator H_θ , corresponding to the highest root θ . Indeed the relations

$$[H_\theta, E_\alpha] = 2 \frac{(\alpha | \theta)}{(\theta | \theta)} E_\alpha, \quad 2 \frac{(\alpha | \theta)}{(\theta | \theta)} \in \mathbb{Z}, \quad |(\alpha | \theta)| \leq (\theta | \theta) \quad (2.5)$$

imply that the coefficient to E_α in the right hand side belongs to the set $\{-2, -1, 0, 1, 2\}$. The spaces $\mathcal{G}_{\pm 2}$ have each dimension 1 and are generated by $E_{\pm\theta}$ satisfying $[E_\theta, E_{-\theta}] = H_\theta$. The triple $\{E_\theta, H_\theta, E_{-\theta}\}$ thus spans a distinguished subalgebra $\mathfrak{sl}_2$ of $\mathcal{G}$. Let $\mathcal{H}$ be the maximal Lie subalgebra of $\mathcal{G}$ commuting with this principal $\mathfrak{sl}_2$. The minimal orbit in the complexification $\mathcal{G}_{\mathbb{C}}$ of $\mathcal{G}$, isomorphic to the adjoint orbit of E_θ , has dimension

$$|\mathcal{O}_{\min}| = |\mathcal{G}_1| + 2 =: 2 \dim \pi_\theta \quad (2.6)$$

where π_θ is the corresponding *minimal representation* and $\dim \pi_\theta$ is its *Gelfand-Kirillov dimension* – roughly speaking, the number of variables on which depends the wave function of the associated Schrödinger picture [KM]. Knowing $\mathcal{G}$ and $\mathcal{H}$ we can compute $|\mathcal{G}_1|$ and hence $\dim \pi_\theta$

$$|\mathcal{G}| = |\mathcal{H}| + 2 |\mathcal{G}_1| + 3. \quad (2.7)$$

In Table 1 we list the type of $\mathcal{H}$ and the values of $|\mathcal{G}_1| = \frac{1}{2}(|\mathcal{G}| - |\mathcal{H}| - 3)$ and of $\dim \pi_\theta$ for all complex simple Lie algebras (cf. [G06]).

$\mathcal{G}_\mathbb{C}$	$ \mathcal{G} $	$\mathcal{H}$	$ \mathcal{G}_1 = \frac{1}{2}(\mathcal{G} - \mathcal{H} - 3)$	$\dim_\mathbb{C} \pi_\theta$
$sl_n = A_{n-1}$	$n^2 - 1$	gl_{n-2}	$2(n-2)$	$n-1$
$so(2n+1) = B_n$	$n(2n+1)$	$B_{n-2} \oplus sl_2$	$2(2n-3)$	$2n-2$
$sp(2n) = C_n$	$n(2n+1)$	C_{n-1}	$2(n-1)$	n
$so(2n) = D_n$	$n(2n-1)$	$D_{n-2} \oplus sl_2$	$4(n-2)$	$2n-3$
E_6	78	sl_6	20	11
E_7	133	D_6	32	17
E_8	248	E_7	56	29
F_4	52	C_3	14	8
G_2	14	sl_2	4	3

Table 1. – List of $\mathcal{G}_\mathbb{C} \supset sl_2 \oplus \mathcal{H}$; dimensions.

The next step, the “quantization of $\mathcal{O}_{\min}$ ”, is not canonical. It has been best studied for $sp(2n, \mathbb{R})$: it is realized in the familiar Fock space (or oscillator) representation of the Heisenberg algebra (of n creation and n annihilation operators) that will be reviewed in the next section. It is also referred to as the Segal-Shale-Weil representation, made popular by the work of Weil [W] with its applications to θ -functions and automorphic forms – an important chapter in number theory (for a systematic pedagogical review of this case – see [F]. The case of $O(p, q)$ is developed more recently and surveyed in the monograph [KM].)

3 Examples: $u(n, n) \subset sp(4n, \mathbb{R})$. Massless particles

We begin, following Howe [H85], with the Heisenberg algebra of n annihilation and n creation operators $c = (c_i, i = 1, \dots, n)$ and $c^* = (c_j^*, j = 1, \dots, n)$ satisfying the *canonical commutation relations* (CCR)

$$[c_i, c_j] = 0 = [c_i^*, c_j^*], \quad [c_i, c_j^*] = \delta_{ij}, \quad i, j = 1, \dots, n. \quad (3.1)$$

The CCR are left invariant by the $n(2n+1)$ dimensional real symplectic group $Sp(2n, \mathbb{R})$ whose elements g can be written in terms of a pair of complex $n \times n$ matrices U and V :

$$g = \begin{pmatrix} U & \bar{V} \\ V & \bar{U} \end{pmatrix}, \quad gc = Uc + \bar{V}c^*, \quad gc^* = Vc + \bar{U}c^*, \quad (3.2)$$

where $\bar{U}$ and $\bar{V}$ are complex conjugate to U and V and we have

$$UV^* = \bar{V}^t U, \quad UU^* - \bar{V}^t V = \mathbb{I} \quad (3.3)$$

(U^* is the hermitean conjugate to U , $U^* = {}^t \bar{U}$, ${}^t A$ standing for the transposed of the matrix A). We note that the pair (c, c^*) provides a complex variable

parametrization of a real $2n$ dimensional vector space; a similar remark applies to the above parametrization of the real symplectic group $Sp(2n, \mathbb{R})$.

A more symmetric way of writing the CCR consists in introducing a $2n$ -component vector $\xi = (\xi_a, a = 1, \dots, 2n)$ and a *symplectic form* $J = (J_{ab})$ such that

$$\begin{aligned} [\xi_a, \xi_b] &= J_{ab} \quad (= -J_{ba}) \\ JJ^* &= -J^2 = \mathbb{I} \quad (= (\delta_{ab})). \end{aligned} \quad (3.4)$$

The invariance of the CCR (3.4) under symplectic transformations then reads

$$g J^t g = J \quad \text{for } g \in Sp(2n, \mathbb{R}). \quad (3.5)$$

The relations (3.1) correspond to the choice $\xi = (c, c^*)$, $J = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix}$.

The *oscillator representation* of the Lie algebra $sp(2n, \mathbb{R})$ is spanned by the anticommutators $\frac{1}{2}(\xi_a \xi_b + \xi_b \xi_a)$. The *Fock space representation* of the Heisenberg algebra (with *ket* and *bra* vacuum vectors $|0\rangle$ and $\langle 0|$ such that $c_i |0\rangle = 0 = \langle 0| c_i^*$, $i = 1, \dots, n$; $\langle 0|0\rangle = 1$) splits into two most degenerate IRs of $sp(2n, \mathbb{R})$. They can be only exponentiated to unitary IRs of the double cover of $Sp(2n, \mathbb{R})$, the *metaplectic group* $Mp(2n)$ (that is not a matrix group). The resulting Fock space representation of $Mp(2n)$ corresponds to a minimal coadjoint orbit, thus providing an example of a minimal representation. We shall exploit this in the physically interesting case of $n = 4$, displaying on the way the reduction of the minimal representation of $Mp(8)$ into the zero mass representations of the (helicity extended) spinorial conformal group $U(2, 2)$ (reviewing the construction of [MT69]).

To make explicit the inclusion $u(2, 2) \subset sp(8, \mathbb{R})$ we shall label the annihilation (and the creation) operators by a_1, a_2, b_1, b_2 (and $a_1^*, \dots, b_2^*$). With this notation the $u(2, 2)$ sub (Lie) algebra of $sp(8, \mathbb{R})$ is defined as the centralizer of (twice) the helicity operator,

$$h := a_1^* a_1 + a_2^* a_2 - b_1^* b_1 - b_2^* b_2. \quad (3.6)$$

If we introduce the (Dirac conjugate, operator valued) spinors

$$\begin{aligned} \varphi &= \begin{pmatrix} a_1 \\ a_2 \\ b_1^* \\ b_2^* \end{pmatrix}, \quad \tilde{\varphi} = \varphi^* \beta = (a_1^*, a_2^*, -b_1, -b_2) \quad (\beta = \text{diag}(1, 1, -1, -1)), \\ [\varphi^\alpha, \tilde{\varphi}_{\alpha'}] &= \delta_{\alpha'}^\alpha \quad (\alpha, \alpha' = 1, 2, 3, 4) \end{aligned} \quad (3.7)$$

then the oscillator realization of $u(2, 2)$ is given by operators of the form $\hat{X} = \tilde{\varphi} X \varphi$ where X are 4×4 matrices satisfying the antihermiticity condition $X^* \beta + \beta X = 0$. A Chevalley-Cartan basis of $su(2, 2)$ is given by

$$E_1 = a_1^* a_2, \quad E_2 = a_2^* b_1^*, \quad E_3 = -b_1 b_2^*, \quad F_1 = a_2^* a_1, \quad F_2 = -b_1 a_2, \quad F_3 = -b_2 b_1^*,$$

$$H_i = [E_i, F_i], \quad H_1 = a_1^* a_1 - a_2^* a_2, \quad H_2 = a_2^* a_2 + b_1 b_1^*, \quad H_3 = b_2^* b_2 - b_1^* b_1 \quad (3.8)$$

reproducing the standard Cartan matrix (c_{ij}) for $sl_4 : [H_i, E_j] = c_{ij} E_j$. The positive roots correspond to the raising operators E_i , $[E_1, E_2] = a_1^* b_1^*$, $[E_2, E_3] = a_2^* b_2^*$ and the highest root

$$E_\theta = [[E_1, E_2], E_3] = a_1^* b_2^*. \quad (3.9)$$

The two other generators of the distinguished $sl_2 (= su(1, 1))$ are

$$(E_{-\theta} \equiv) F_\theta = -b_2 a_1, \quad H_\theta = [E_\theta, F_\theta] = a_1^* a_1 + b_2 b_2^* \quad ([H_\theta, E_\theta] = 2E_\theta). \quad (3.10)$$

Its commutant $\mathcal{H}$ in $su(2, 2)$ is the 4-dimensional $u(1, 1)$ subalgebra spanned by products of (a_2^*, b_1) with (a_2, b_1^*) .

Remark 3.1. Splitting, more generally, the annihilation (creation) operators $c_1^{(*)}, \dots, c_{2n}^{(*)}$ into “positive and negative charge” ones, $a_1^{(*)}, \dots, a_n^{(*)}, b_1^{(*)}, \dots, b_n^{(*)}$ we can realize, similarly, the embedding $u(n, n) \subset sp(4n)$ where $u(n, n)$ is the centralizer of the *charge operator*

$$Q := a_1^* a_1 + \dots + a_n^* a_n - b_1^* b_1 - \dots - b_n^* b_n.$$

The non-compact Chevalley generators are then $E_n = a_n^* b_1^*$, $F_n = -b_1 a_n$ while the highest root operator becomes $E_\theta = a_1^* b_n^*$. The commutant $\mathcal{H}$ of the distinguished sl_2 , spanned by $E_\theta, F_\theta = -b_n a_1, H_\theta = a_1^* a_1 + b_n b_n^*$, is $u(n-1, n-1)$ and the minimal coadjoint orbit is $4n-2$ dimensional.

Returning to the case $n = 2$, the 6-dimensional minimal coadjoint orbit is spanned by E_θ, H_θ and by the elements $E_1, E_3, [E_1, E_2]$ and $[E_2, E_3]$ of $\mathcal{G}_1$ (of (2.3)). We shall introduce light-cone “momentum space” variables that give rise to a Schrödinger like realization of the minimal representation (in the state space of massless particles). To this end we introduce two complex conjugate pairs of variables z_α and $\bar{z}_\alpha$, $\alpha = 1, 2$, setting:

$$\begin{aligned} a_\alpha^* &= \frac{1}{\sqrt{2}} (z_\alpha - \bar{\partial}_\alpha), \quad b_\alpha^* = \frac{1}{\sqrt{2}} (\bar{z}_\alpha - \partial_\alpha), \quad \partial_\alpha := \frac{\partial}{\partial z_\alpha}, \quad \bar{\partial}_\alpha := \frac{\partial}{\partial \bar{z}_\alpha}, \\ a_\alpha &= \frac{1}{\sqrt{2}} (\bar{z}_\alpha + \partial_\alpha), \quad b_\alpha = \frac{1}{\sqrt{2}} (z_\alpha + \bar{\partial}_\alpha) \end{aligned} \quad (3.11)$$

or

$$z_\alpha = \frac{1}{\sqrt{2}} (a_\alpha^* + b_\alpha), \quad \bar{z}_\alpha = \frac{1}{\sqrt{2}} (a_\alpha + b_\alpha^*), \quad \partial_\alpha = \frac{1}{\sqrt{2}} (a_\alpha - b_\alpha^*), \quad \bar{\partial}_\alpha = \frac{1}{\sqrt{2}} (b_\alpha - a_\alpha^*).$$

Then the 4-vector

$$p_\mu = z \sigma_\mu \bar{z} \quad \mu = 0, 1, 2, 3 \quad \left(\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \quad (3.12)$$

($\sigma_1, \sigma_2, \sigma_3$ being the Pauli matrices) is light-like ($p_0 = |\mathbf{p}|$) and has been proved to correspond to the translation generators of the Poincaré subalgebra of $su(2, 2)$

– i.e. to the 4-momentum (see [MT69]). Thus, our representation indeed corresponds to 0-mass particles and can be realized in a space of functions of three real variables $\mathbf{p} = (p_1, p_2, p_3)$, in accord with the value $\dim \pi_\theta = 3$ ($= \frac{1}{2} |\mathcal{O}_{\min}|$) of its Gelfand-Kirillov dimension. The irreducible components of this representation are labeled by the (integer) eigenvalues of the operator $h = \frac{1}{2} (\tilde{\varphi} \varphi + \varphi \tilde{\varphi})$ (3.4) (which, according to the conventions of [MT69], are equal to minus twice the helicity). The vacuum vector belongs to the zero helicity IR and has the form

$$|0\rangle = \frac{2}{\pi} e^{-z\bar{z}} \left(= \frac{2}{\pi} e^{-p_0} \right), \quad z\bar{z} = z_1\bar{z}_1 + z_2\bar{z}_2, \\ \langle 0 | 0 \rangle = \frac{4}{\pi^2} \int d^2 z_1 \int d^2 z_2 e^{-2z\bar{z}} = 4 \int_0^\infty p_0 e^{-2p_0} dp_0 = 1. \quad (3.13)$$

It can be defined as the unique lowest weight vector (satisfying, by definition, $F_i |0\rangle = 0$, $i = 1, 2, 3$) that transforms under a 1-dimensional representation of the maximal compact subgroup $S(U(2) \times U(2))$ of $SU(2, 2)$:

$$E_1 |0\rangle = 0 = H_1 |0\rangle = E_3 |0\rangle = H_3 |0\rangle, \\ (H_1 + 2H_2 + H_3) |0\rangle = (z\bar{z} - \bar{\partial}\partial) \frac{2}{\pi} e^{-z\bar{z}} = 2 |0\rangle \quad (3.14)$$

($H_1 + 2H_2 + H_3$ being the generator of the centre, $u(1)$, of $s(u(2) \oplus u(2))$). The vacuum thus plays the role of a *spherical vector*.

The Fock space ($\mathcal{F}$ -) representation of the Heisenberg algebra of $(a_\alpha^{(*)}, b_\beta^{(*)})$ splits into two IRs of $sp(8, \mathbb{R})$ which, in turn, split into IRs of $u(2, 2)$ involving either, only even or only odd eigenvalues of h (cf. [I67]). The minimal coadjoint orbit of $Mp(8, \mathbb{R})$, the orbit of the highest root $\frac{1}{2} a_1^{*2}$, is 8 dimensional (see Table 1), so that the associated Gelfand-Kirillov dimension is four (the 4th quantum number being the helicity).

The subgroup $U(2, 2)$ and its centre $U(1)$ provide an example of a reductive dual pair in $Mp(8)$, $U(1)$ playing the role of a (global) gauge group.

Remark 3.2. The subspace of $\mathcal{F}$ spanned by functions $e^{-z\bar{z}} P(z_1, z_2)$ where P is a polynomial provides a model for the IRs of $SU(2)$, its Hilbert space closure being isomorphic to the Bargmann space of analytic functions (see [B]).

4 Two types of reductive dual pairs

Let us look more closely at the example of $U(1)$, $U(2, 2)$ as a dual pair in $Mp(8)$. The symplectic group $Sp(8, \mathbb{R})$ acts by symplectomorphisms on the 8-dimensional vector space $\mathcal{W}$ spanned by $(a_1, a_2, b_1, b_2; a_1^*, a_2^*, b_1^*, b_2^*)$: it preserves both their commutation relations and their conjugation properties. (So does its double cover $Mp(8)$ which, however, does not act faithfully on $\mathcal{W}$.) The “4-spinors” φ and $\tilde{\varphi}$ (3.7) define a (complex) *Lagrangian polarization* in $\mathcal{W}$:

$$\mathcal{W} = \mathcal{V} \oplus \mathcal{V}', \quad \varphi^\alpha \in \mathcal{V}, \quad \tilde{\varphi}_{\alpha'} \in \mathcal{V}' \quad (4.1)$$

as $[\varphi^\alpha, \varphi^\beta] = 0 = [\tilde{\varphi}_{\alpha'}, \tilde{\varphi}_{\beta'}]$, while the CCR $[\varphi^\alpha, \tilde{\varphi}_{\alpha'}] = \delta_{\alpha'}^\alpha$, allow to regard the elements of $\mathcal{V}'$ as linear functionals on $\mathcal{V}$; thus, the subspaces $\mathcal{V}$ and $\mathcal{V}'$ of $\mathcal{W}$ are dual to each other. The subgroup $U(2, 2)$ of $Mp(8)$, viewed as an automorphism group of $\mathcal{W}$, preserves separately the subspaces $\mathcal{V}$ and $\mathcal{V}'$ – i.e., the Lagrangian polarization. Sure, $U(1) \subset U(2, 2)$ (the subgroup of automorphisms $u_t(\varphi) = e^{-it} \varphi$, $u_t(\tilde{\varphi}) = e^{it} \tilde{\varphi}$, generated by the helicity invariant h (3.6)) also preserves both $\mathcal{V}$ and $\mathcal{V}'$.

A simple example of a (reductive) dual pair which does not preserve any Lagrangian subspace of $\mathcal{W}$ is given by the entire group of symplectomorphism ($Sp(8)$, in our example) and its centre which changes the sign of all creation and annihilation operators. The 2-element group $\mathbb{Z}/(2)$ of the centre of $Sp(2n)$ can be viewed as the 1-dimensional orthogonal group $O(1)$. It turns out that our two examples are simple prototypes of a general classification which we proceed to describe.

Let G and G' be a dual pair of subgroups of $Mp(2n)$. Then they act by symplectomorphisms on the $2n$ dimensional space $\mathcal{W}$. They are said to be *reductive* iff for each $G^{(\prime)}$ -invariant subspace $\mathcal{V} \subset \mathcal{W}$ its complement $\tilde{\mathcal{V}}$ (such that $\mathcal{W} = \mathcal{V} \oplus \tilde{\mathcal{V}}$) is also $G^{(\prime)}$ invariant. We shall only deal with reductive dual pairs and will skip the qualification “reductive”. If $\mathcal{W}_1$ and $\mathcal{W}_2$ are two symplectic vector spaces and if $(G_1, G'_1), (G_2, G'_2)$ are two dual pairs in the corresponding metaplectic groups, then the direct products $(G = G_1 \times G_2, G' = G'_1 \times G'_2)$ will form a dual pair for the symplectic vector space $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$. Such a dual pair is called *reducible*. A dual pair (G, G') acting on $\mathcal{W}$ is called *irreducible* if it cannot be represented that way. An irreducible dual pair (G, G') acting on $\mathcal{W}$ is said to be of *type II* if both G and G' leave a Lagrangian subspace of $\mathcal{W}$ invariant and of *type I* otherwise. These two types exhaust all possibilities of irreducible dual pairs acting on a *complex* symplectic space $\mathcal{W}$ (or, more generally, if the ground field is algebraically closed). For the case of a real $\mathcal{W}$ a classification appears in the second reference [H89]. We shall content ourselves by giving some generic examples that generalize those discussed in the beginning of this section.

We first consider the tensor product $\mathcal{W}_{2mn} := \mathbb{R}^m \otimes \mathcal{W}_{2n}$ where $\mathcal{W}_{2n}$ is a (real) $2n$ dimensional space with symplectic form $J = (J_{ab})$ (3.4). Viewing $\mathbb{R}^m$ as an orthogonal vector space with a $O(m)$ -invariant scalar product we can equip $\mathcal{W}_{2mn}$ with a symplectic form setting in an appropriate basis ξ_{ia} , $i = 1, \dots, m$, $a = 1, \dots, 2n$

$$[\xi_{ia}, \xi_{jb}] = \delta_{ij} J_{ab}. \quad (4.2)$$

Then $(O(m), Mp(2n))$ form a dual pair of type I in $Mp(2mn)$ (our $O(1)$ example above appearing as a special case).

To generalize our type II example, $(U(1), U(2, 2))$, we consider the tensor product space

$$\mathcal{W}_{8n} := \mathbb{C}^n \otimes (\mathcal{V} \oplus \mathcal{V}') \quad (4.3)$$

where $\mathcal{V}$ and $\mathcal{V}'$ are the dual spaces of Dirac conjugate spinors φ and $\tilde{\varphi}$ (see 4.1)). Regarding $\mathbb{C}^n$ as a unitary space equipped with a $U(n)$ invariant (positive)

hermitean form we can introduce a symplectic structure on $\mathcal{W}_{8n}$ by defining (in a suitable basis $\varphi_i^\alpha, \tilde{\varphi}_{j\beta}$) the non-trivial CCR as

$$[\varphi_i^\alpha, \tilde{\varphi}_{j\beta}] = \delta_{ij} \delta_\beta^\alpha, \quad i, j = 1, \dots, n; \quad \alpha, \beta = 1, \dots, 4. \quad (4.4)$$

The subgroups $U(n)$ and $U(2, 2)$ of $Mp(8n)$ form in this case a dual pair, the Weyl generators of $u(2, 2)$,

$$\sum_{i=1}^n \tilde{\varphi}_{i\alpha} \varphi_i^\beta \quad (4.5)$$

being $U(n)$ invariant. The Lagrangian subspaces $\mathbb{C}^n \otimes \mathcal{V}$ and $\mathbb{C}^n \otimes \mathcal{V}'$ (spanned by φ_i^α and $\tilde{\varphi}_{j\beta}$, respectively) are both $U(n)$ and $U(2, 2)$ invariant, so that our dual pair is indeed of type II.

One of the elements of each dual pair ($O(m)$ for type I and $U(n)$ for type II) in the above examples is a compact group. This is, sure, not the general case but our choice is motivated physically: it allows to interpret this element as a global *gauge group*. We shall encounter infinite dimensional generalization of these examples in the quantum field theory context in Section 6.

5 Global conformal invariance and infinite dimensional Lie algebras

It has been often argued that 2D CFT methods are much too special to allow an extension to higher dimensions:

- (i) the appearance of a chiral current algebra with meromorphic correlation functions is usually related to the splitting of the $(1+1)$ -dimensional light-cone into a product of two light rays;
- (ii) the representation theory of Kac-Moody and of Virasoro algebras is playing a crucial role in constructing soluble 2D CFT models; there are, on the other hand, no local Lie fields in higher dimensions (see [B08] and [T08] for a more detailed discussion and for references to the no-go theorem of Robinson and Baumann).

It was revealed, however, that the existence of a current algebra with rational correlation functions is due to a symmetry property: *global* (rather than infinitesimal) *conformal invariant* (GCI) that implies a strong (Huygens) locality condition and thereby rationality of correlation functions [NT] in any even number of space-time dimensions. Indeed, the fact that global conformal transformations can map a space-like onto a time-like pair of events, together with local commutativity for space-like separations imply that local commutators are supported on the light cone

$$(x_{12}^2)^n [\phi_1(x_1), \phi_2(x_2)] = 0 \quad \text{for} \quad x_{12}^2 := (\mathbf{x}_1 - \mathbf{x}_2)^2 - (x_1^0 - x_2^0)^2, \quad n \gg 0 \quad (5.1)$$

($n \gg 0$ meaning n sufficiently large). Combined with the remaining Wightman axioms (in particular, with the energy positivity condition) this implies rationality of correlation functions of local GCI fields [NT]; more precisely, n -point

Wightman functions are proven to have the form

$$\mathcal{W}_n(x_1, \dots, x_n) = \frac{P(x_{ij})}{\prod_{i < j} (x_{ij}^2)^{\mu_{ij}}}, \quad \mu_{ij} \in \mathbb{N}, \quad P \text{ a polynomial.} \quad (5.2)$$

Furthermore, although there are no local scalar Lie fields in four (or higher) dimensions, infinite dimensional Lie algebras are generated by *bifields* $V(x_1, x_2)$ that naturally arise in the operator product expansions of a finite set of (say, hermitean, scalar) fields ϕ_i of dimension d (> 1):

$$(x_{12}^2)^d \phi_i(x_1) \phi_j(x_2) = N_{ij} + x_{12}^2 V_{ij}(x_1, x_2) + O((x_{12}^2)^2), \quad (5.3)$$

where V_{ij} are defined as infinite sums of contributions of (twist two) local conserved tensor currents (and the real symmetric matrix N_{ij} is positive definite). These bifields are proven (as a consequence of the local conservation laws) to be harmonic in each variable ([NST03]):

$$\square_1 V_{ij}(x_1, x_2) = 0 = \square_2 V_{ij}(x_1, x_2), \quad \square_k = \frac{\partial}{\partial x_k^\mu} \frac{\partial}{\partial x_{k\mu}}. \quad (5.4)$$

It can be deduced from the analysis of 4-point functions that the commutator algebra of the harmonic bifields V_{ij} appearing in (5.1) only closes on the V 's and the unit operator if the local GCI scalar fields ϕ_j have dimension $d = 2$ ($d = 1$ corresponding to a free field theory). In that case the bifields V are proven to be Huygens bilocal [NRT].

The infinite Lie algebras generated by such bilocal fields are characterized by the following result of [NRT].

Theorem 5.1. – *The harmonic bilocal fields V arising in the expansion (5.3) of the GCI hermitean scalar fields ϕ_j of dimension $d = 2$ can be labeled by elements M of a unital real matrix algebra $\mathcal{M} \subset \text{Mat}(L, \mathbb{R})$ closed under transposition $M \rightarrow {}^t M$ in such a way that the commutator of V_M and $V_{M'}$ be given by*

$$\begin{aligned} [V_M(x_1, x_2), V_{M'}(x_3, x_4)] &= \Delta_{13} V_{tMM'}(x_2, x_4) + \\ &+ \Delta_{24} V_{M {}^t M'}(x_1, x_3) + \Delta_{23} V_{MM'}(x_1, x_4) + \Delta_{14} V_{M' M}(x_3, x_2) + \\ &+ \text{tr}(MM') \Delta_{12,34} + \text{tr}({}^t MM') \Delta_{12,43}. \end{aligned} \quad (5.5)$$

Here Δ_{ij} is the free field commutator, $\Delta_{ij} := \Delta_{ij}^+ - \Delta_{ji}^+$, and $\Delta_{12,ij} = \Delta_{1i}^+ \Delta_{2j}^+ - \Delta_{1j}^+ \Delta_{2i}^+$ where

$$\Delta_{12}^+ = \int_{p_0=|\mathbf{p}|} e^{-ipx_{12}} \frac{d^3 p}{(2\pi)^3 2p_0} = \frac{1}{4\pi^2 (x_{12}^2 + ix_{12}^0)}, \quad x_{12} = x_1 - x_2 \quad (5.6)$$

is the 2-point Wightman function of a free massless scalar field.

The matrix algebra $\mathcal{M}$ is an example of a (finite dimensional) *real star algebra*; we shall call it a *t-algebra* as the star operation is just matrix transposition.

It can be equipped with a *Frobenius inner product*

$$\langle M_1, M_2 \rangle = \text{tr}({}^t M_1 M_2) = \sum_{ij} M_{1ij} M_{2ij} \quad (5.7)$$

which is symmetric, positive definite and has the property

$$\langle M_1 M_2, M_3 \rangle = \langle M_1, M_3 {}^t M_2 \rangle. \quad (5.8)$$

This implies that for every right ideal $\mathcal{I} \subset \mathcal{M}$ its orthogonal complement is again a right ideal while its transposed ${}^t \mathcal{I}$ is a left ideal. It follows that the algebra $\mathcal{M}$ is semi-simple so that every $\mathcal{M}$ -module is a direct sum of irreducible modules.

Let now $\mathcal{M}$ be irreducible. It then follows from Schur's lemma (whose real version, [L], is more involved than the better known complex one) that its commutant $\mathcal{M}'$ in $\text{Mat}(L, \mathbb{R})$ coincides with one of the *three real division rings*: the fields of real and complex numbers, $\mathbb{R}$ and $\mathbb{C}$, or the (noncommutative) ring $\mathbb{H}$ of quaternions. In each case the Lie algebra of bilocal fields is a central extension of an infinite dimensional Lie algebra that admits a discrete series of highest weight representations. Moreover, it was proven (first in the theory of a single scalar field of dimension 2 [NST] and eventually for an arbitrary set of such fields [NRT]) that in each case we are dealing with a minimal representation appearing as a subrepresentation of the Heisenberg algebra. More precisely, the bilocal field V_M can always be written as a linear combination of normal products of free massless scalar fields φ . For each of the above irreducible types V_M has a canonical form:

$$\begin{aligned} \mathbb{R} : \quad V(x_1, x_2) &= \sum_{i=1}^N : \varphi_i(x_1) \varphi_i(x_2) : \\ \mathbb{C} : \quad W(x_1, x_2) &= \sum_{j=1}^N : \varphi_j^*(x_1) \varphi_j(x_2) : \\ \mathbb{H} : \quad Y(x_1, x_2) &= \sum_{m=1}^N : \varphi_m^+(x_1) \varphi_m(x_2) :, \end{aligned} \quad (5.9)$$

where φ_i are real, φ_j are complex and φ_m are quaternionic valued fields (corresponding to $L = N, 2N$, and $4N$, respectively). We shall denote the corresponding infinite dimensional Lie algebra by $\mathcal{L}(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

In order to identify the types of Lie algebras $\mathcal{L}(\mathbb{F})$ it is convenient to introduce a discrete basis. There is a natural expansion of the free massless field φ into discrete modes since φ can be viewed (as any GCI field) as defined on the conformal compactification $\bar{M}$ of Minkowski space M . To this end it is handy to use the complex variable parametrization

$$\begin{aligned} \bar{M} &= \left\{ z = (z_\alpha, \alpha = 1, \dots, 4); z_\alpha = \ell^{it} u_\alpha, t, u_\alpha \in \mathbb{R}, u^2 := \sum_{\alpha=1}^4 u_\alpha^2 = 1 \right\} \\ &= \mathbb{S}^3 \times \mathbb{S}^1 / \mathbb{Z}/(2), \end{aligned} \quad (5.10)$$

introduced in [T86] (and later explored in [N] and [NT05]). M is embedded in $\bar{M}$ by setting

$$z = \frac{\mathbf{x}}{\omega(x)} (\mathbf{x} = (x^1, x^2, x^3)), \quad z_4 = \frac{1 - x^2}{2\omega(x)}, \quad 2\omega(x) = 1 + x^2 - 2ix^0 \quad (5.11)$$

$$\left(x^2 := \mathbf{x}^2 - (x^0)^2, \quad \mathbf{x}^2 = \sum_{i=1}^3 (x^i)^2 \right), \quad \varphi_M(x) = \frac{1}{2\pi\omega(x)} \varphi_{\bar{M}}(z(x)).$$

Remark 5.1. Factoring $\mathbb{Z}/(2) = \{id, (t, u) \rightarrow (t + \pi, -u)\}$ in (5.10) makes sense if we view $\bar{M}$ as a (real) pseudo-Riemannian manifold with metric

$$ds^2 = \frac{dz^2}{z^2} = du^2 - dt^2 \quad \left(\text{for } u du = \sum_{\alpha=1}^4 u_\alpha du_\alpha = 0 \right) \quad (5.12)$$

or if we equip it with a group structure, setting $\bar{M} = U(2)$ (which is only possible in four space-time dimensions – cf. [U]). Viewed as just a smooth manifold $\mathbb{S}^3 \times \mathbb{S}^1 / \mathbb{Z}/(2)$ is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$.

A free massless field $\varphi(z)$ on $\bar{M}$ admits an expansion of the form

$$\varphi(z) = \sum_{n=1}^{\infty} \sum_{\ell=0}^{n-1} \sum_{m=-\ell}^{\ell} \left(\frac{a_{n\ell m}}{(z^2)^n} + a_{n\ell m}^* \right) h_{n\ell m}(z), \quad (5.13)$$

where $h_{n\ell m}$ are (suitably normalized) homogeneous (of degree $n - 1$) harmonic polynomials, the *spherical harmonics*, satisfying

$$(H - n) h_{n\ell m}(z) = 0 \quad \text{for} \quad H = z \cdot \frac{\partial}{\partial z} + 1 \quad (5.14)$$

(H is the *conformal Hamiltonian* that generates translations in t),

$$\mathbf{L}^2 h_{n\ell m}(z) = \ell(\ell + 1) h_{n\ell m}(z), \quad (L_3 - m) h_{n\ell m}(z) = 0 \quad (5.15)$$

$\mathbf{L} = (L_1, L_2, L_3)$ being the angular momentum vector operator,

$$L_j = i \varepsilon_{jkl} z_\ell \frac{\partial}{\partial z_k} \quad \left(L_3 = i \left(z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2} \right) \right), \quad (5.16)$$

while $a_\nu^{(*)}$ (where ν is the triple index, $\nu = (n, \ell, m)$) obey the CCR

$$[a_\nu, a_{\nu'}^*] = \delta_{\nu\nu'} (= \delta_{nn'} \delta_{\ell\ell'} \delta_{mm'}). \quad (5.17)$$

The bilocal fields (5.9) are then expanded into products of $a_\nu^{(*)}, a_{\nu'}^{(*)}$. Cutting this expansion up to some finite energy level n we can then identify the CR (5.5) with the CRs of a classical (real) Lie algebra.

6 The dual pairs $(\mathcal{L}(\mathbb{F}), U(\mathbb{F}, N))$ and their Fock space representations. The Lie algebra $so^*(4n)$

The analysis of [B07] and [B08] yields the following

Proposition 6.1. *The algebras $\mathcal{L}(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ are 1-parameter central extensions of appropriate completions of the inductive limits of classical real matrix algebras:*

$$\begin{aligned} \mathbb{R} &: sp(2\infty, \mathbb{R}) = \lim_{n \rightarrow \infty} sp(2n, \mathbb{R}) \\ \mathbb{C} &: u(\infty, \infty) = \lim_{n \rightarrow \infty} u(n, n) \\ \mathbb{H} &: so^*(4\infty) = \lim_{n \rightarrow \infty} so^*(4n). \end{aligned} \quad (6.1)$$

In the free field realization (5.9) the suitably normalized central charge coincides with the positive integer N . Moreover, in that case the corresponding bilocal field (V, W, Y) is left invariant under a transformation of the constituent fields φ by the compact gauge group $U(N, \mathbb{F})$ where

$$U(N, \mathbb{R}) = O(N), \quad U(N, \mathbb{C}) = U(N), \quad U(N, \mathbb{H}) = Sp(2N) \quad (\equiv USp(2N)). \quad (6.2)$$

It is remarkable (albeit disappointing from a physical point of view) that the free field realization holds in general.

Theorem 6.2. (i) *In any unitary irreducible positive energy representation (UIPER) of $\mathcal{L}(\mathbb{F})$ the central charge N is a positive integer.*

(ii) *All UIPERs of $\mathcal{L}(\mathbb{F})$ are realized (with multiplicities) in the Fock space $\mathcal{F}$ of $N \times \dim_{\mathbb{R}} \mathbb{F}$ free hermitean massless scalar fields.*

(iii) *The ground states of equivalent UIPERs in $\mathcal{F}$ form irreducible representations of the gauge group $U(N, \mathbb{F})$ (6.2). This establishes a one-to-one correspondence between UIPERs of $\mathcal{L}(\mathbb{F})$ in the Fock space and the (finite dimensional) irreducible representations of $U(N, \mathbb{F})$.*

The *proof* of this theorem for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ is given in [B07] (the proof of (i) for the case of a single scalar field ϕ is already contained in [NST]); the proof for $\mathbb{F} = \mathbb{H}$ is given in [B08].

The Lie algebras $sp(2n, \mathbb{R})$ and $u(n, n)$ were introduced in Section 3. The Lie algebra $so^*(4n)$, associated with the quaternions, can be defined as a subalgebra of the algebra $u(2n, 2n)$ of $4n \times 4n$ complex matrices,

$$u(2n, 2n) = \left\{ X \in \text{Mat}(4n, \mathbb{C}); X^* \beta = -\beta X, \text{ for } \beta = \begin{pmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{pmatrix} \right\} \quad (6.3)$$

such that²

$$so^*(4n) = \left\{ X \in u(2n, 2n); {}^t X \sigma = -\sigma X, \sigma = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix} \right\} \quad (6.4)$$

($\mathbb{I}$ standing for the $2n \times 2n$ unit matrix). It consists of $4n \times 4n$ matrices that can be written in the following $2n \times 2n$ block form:

$$X = \begin{pmatrix} u & v \\ v^* & -{}^t u \end{pmatrix}, \quad u \in u(2n) \text{ (i.e. } u + u^* = 0), \quad v \in so(2n, \mathbb{C}), \text{ i.e. } v + {}^t v = 0. \quad (6.5)$$

(In particular, $\text{tr } X = 0$, so that $so^*(4n) \subset su(2n, 2n)$.) We again view $u(2n, 2n)$ (and hence $so^*(4n)$) as a sub Lie algebra of $sp(8n, \mathbb{R})$ and consider its oscillator representation in which the $4n \times 4n$ matrices X (6.3) are mapped into the Fock space operators

$$\tilde{\varphi} X \varphi, \quad \varphi = \begin{pmatrix} a_1 \\ \dots \\ a_{2n} \\ b_1^* \\ \dots \\ b_{2n}^* \end{pmatrix}, \quad \tilde{\varphi} = \varphi^* \beta = (a_1^*, \dots, a_{2n}^*, -b_1, \dots, -b_{2n}) \quad (6.6)$$

(cf. Remark 3.1. This construction corresponds to $N = 1$ in Proposition 6.1 and Theorem 6.2.) The Chevalley-Cartan basis of the subalgebra $so^*(4n)$ in this realization reads:

$$\begin{aligned} E_i &= a_i^* a_{i+1} + b_{i+1} b_i^*, \quad F_i = E_i^*, \quad i = 1, \dots, 2n-1, \\ E_{2n} &= a_{2n-1}^* b_{2n}^* - a_{2n}^* b_{2n-1}^*, \quad F_{2n} = a_{2n} b_{2n-1} - a_{2n-1} b_{2n}, \\ H_i &= a_i^* a_i - a_{i+1}^* a_{i+1} + b_i^* b_i - b_{i+1}^* b_{i+1}, \quad i = 1, \dots, 2n-1, \\ H_{2n} &= a_{2n-1}^* a_{2n-1} + a_{2n}^* a_{2n} + b_{2n-1}^* b_{2n-1} + b_{2n}^* b_{2n}. \end{aligned} \quad (6.7)$$

The centre $u(1)$ of $u(2n, 2n)$ is generated by the *charge operator*

$$Q = \sum_{i=1}^{2n} (a_i^* a_i - b_i^* b_i) =: a^* \cdot a - b^* \cdot b. \quad (6.8)$$

The (non-compact) raising operator E_θ corresponding to the highest root $\theta = \alpha_{2n-2} + \sum_{i=1}^{2n} \alpha_i$ of $so^*(4n)$ is

$$E_\theta = E_{12}, \text{ for } E_{ij} = a_i^* b_j^* - a_j^* b_i^*, \quad 1 \leq i < j \leq 2n \quad (6.9)$$

²As explained in [BB] the resulting Lie algebra is isomorphic to the algebra of complex skew symmetric matrices $Y \in so(4n, \mathbb{C})$ such that $Y^* \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ -\mathbb{I} & \mathbb{O} \end{pmatrix} = - \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} Y$. We share the feelings of Barut and Bracken who call this algebra *remarkable*.

($E_{2n-12n} \equiv E_{2n}$). The commutant $\mathcal{H}$ of the principal sl_2 subalgebra of $so^*(4n)$ is $so^*(4n-4) \oplus su(2)$.

Proposition 6.3. *The Lie algebras $sp(2) \simeq su(2)$ (of the compact group $Sp(2) (\equiv USp(2)) \simeq SU(2)$) with generators*

$$E = a^* \cdot b \left(:= \sum_{i=1}^{2n} a_i^* b_i \right), \quad F = b^* \cdot a, \quad Q = [E, F] = a^* a - b^* b \quad (6.10)$$

and $so^*(4n)$ form a dual pair in $sp(8n)$.

This proposition enters as part in the argument (for a special case) of Theorem 6.2. Its proof uses the fact that the second order Casimir operator C^{so} of $so^*(4n)$ is majorized by the Casimir C^u of its maximal compact subalgebra $u(2n)$; more precisely, we have the relation

$$C^{so} = C^u - \sum_{1 \leq i < j \leq 2n} E_{ij} E_{ij}^* \quad \text{for} \quad C^u = C^{su} + \frac{H^2}{2n} \quad (6.11)$$

where C^{su} is the standard quadratic Casimir operator of $su(2n)$ while

$$H = a^* \cdot a + b \cdot b^* \quad (6.12)$$

is the generator of the centre $u(1)$ of $u(2n)$ (see [B08]). Given the fundamental weights Λ_i of $so^*(4n)$ (such that $(\Lambda_i | \alpha_j) = \delta_{ij}$ for $i, j = 1, \dots, 2n$, where α_j are the simple roots, $E_j = E_{\alpha_j}$), the Fock space vacuum can be identified with $|2 \Lambda_{2n}\rangle$:

$$(H_i - 2 \delta_{i2n}) |2 \Lambda_{2n}\rangle = 0 = F_i |2 \Lambda_{2n}\rangle = a_i |2 \Lambda_{2n}\rangle = b_i |2 \Lambda_{2n}\rangle. \quad (6.13)$$

In accord with Proposition 6.3 (and Theorem 6.2) the vacuum is non-degenerate (it is an $SU(2)$ singlet), the UIPERs of lowest weight $2 \Lambda_{2n} - \Lambda_{2n-1}$ form an isospin doublet:

$$\begin{aligned} F_i b_{2n}^* |2 \Lambda_{2n}\rangle &= 0 = F_i a_{2n}^* |2 \Lambda_{2n}\rangle, \quad i = 1, \dots, 2n, \\ E b_j^* |2 \Lambda_{2n}\rangle &= a_j^* |2 \Lambda_{2n}\rangle, \quad j = 1, \dots, 2n; \end{aligned} \quad (6.14)$$

$(a_{2n}^{*2}, a_{2n}^* b_{2n}^*, b_{2n}^{\times 2}) |2 \Lambda_{2n}\rangle$ span an isotriplet of ground state vectors of the type $|2 \Lambda_{2n} - 2 \Lambda_{2n-1}\rangle$ etc.

Remark 6.1. For $n = 2$ the group $so^*(8)$ is isomorphic to the conformal group in six dimensions [BB],

$$so^*(8) \simeq so(6, 2). \quad (6.15)$$

The degeneracy of the oscillator representation is reflected in the fact that the non-compact raising operators E_{ij} satisfy second order equations of the type

$$E_{12} E_{34} + E_{14} E_{23} = E_{13} E_{24}. \quad (6.16)$$

(For $so^*(8)$ this “nilpotent cone relation” is unique.)

Theorem 6.2 provides a link between two parallel developments, one in the study of highest weight modules of reductive Lie groups, the other in the work of Doplicher-Haag-Roberts [H] on the theory of global gauge symmetry and superselection sectors (see [B07] and [T08] for a discussion and references). The infinite Lie algebra $\mathcal{L}(\mathbb{F})$ and the compact gauge group $U(N, \mathbb{F})$ appear as a rather special (limiting) case of a reductive dual pair (Section 4). It would be interesting to explore whether other (inequivalent) pairs appear in the study of the commutator algebra of (spin) tensor bifield that would naturally arise in the study of (say, scalar) fields of higher (than two) scale dimension.

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