# Polylogarithms and multizeta values in massless Feynman amplitudes*

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**Abstract**

The last two decades have seen a remarkable development of analytic methods in the study of Feynman amplitudes in perturbative quantum field theory. The present lecture offers a physicists' oriented survey of Francis Brown's work on singlevalued multiple polylogarithms, the associated multizeta periods and their application to Schnetz's graphical functions and to $x$-space renormalization. To keep the discussion concrete we restrict attention to explicit examples of primitively divergent graphs in a massless scalar QFT.

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# 1 Introduction

It is refreshing for mathematically minded theorists that computer calculations in perturbative Quantum Field Theory (QFT) far from making analytic methods obsolete go in effect hand in hand with their developments. It took nine years before an error in the first numerical calculation of the $\alpha^2$ contribution to the anomalous magnetic moment $(g-2)$ of the electron was corrected by Petermann (and independently by Sommerfield) while computing the relevant seven Feynman diagrams analytically. The answer involves a $\zeta(3)$. (For a historical review – see [St]; for the expressions of the $\alpha^2$ and $\alpha^3$ contributions to $g-2$ in terms of zeta values of weight three and five, respectively, and for references to the original work of the late 1950's on the $\alpha^2$ term and the mid 1990's on the $\alpha^3$ graphs – see [Sch].) It was in the course of a calculation of the electron form-factors that multiple polylogarithms were used by Remiddi et al. and subsequently surveyed under the name of *harmonic polylogarithms* in [RV]. (Later computer aided higher order calculations of $g-2$ took over – see the entertaining review of the field up to 2010 by Kinoshita [K].)

Mathematicians were attracted to the beauty of the dilogarithm and the enigma of *multiple zeta values* (MZVs) since the work of Euler – see [Z, C01, W, D] for reviews. The *singlevalued multiple polylogarithms* (SVMP), introduced and studied by Brown [B04] were soon recognized to play a central role in euclidean calculations of scattering amplitudes – see the systematic elaboration and application to the study of graphical functions in massless QFT in [S13] as well as a choice of influential recent papers [GSVV, GMSV, DGR, DDEHPS, DDDP] and references to earlier work cited there.

The notion of a *Feynman period* [Sch], identified as *residue* of a primitively divergent graph, was used systematically in [NST13, NST] in the study of $x$-space renormalization of massless Feynman amplitudes. Such residues/periods appear in the perturbative expansion of the renormalization group beta function. They were studied by Broadhurst and Kreimer [BK] back in 1996 up to nine loops in the $\varphi^4$ theory and found to be given in most cases by MZVs – i.e. by rational linear combinations of multiple (convergent) series

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \le k_1 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \qquad (n_i \in \mathbb{N}\,, \quad n_r > 1)\,. \tag{1.1}$$

The multiple polylogarithms were first encountered as multiple power series of a similar type, convergent in the unit circle. They admit an analytic continuation to the punctured projective plane $(z \in)\, \mathbb{CP}^1 \backslash \{0, 1, \infty\}$ given by multiple *iterated integrals* [C], [B] labeled by words of two letters $\{0, 1\}$. The MZVs appear as values of the multipolylogarithms at the boundary point $z = 1$. It is remarkable that this family of functions admits a double algebra structure: a *shuffle* and a *stuffle algebra* (both commutative) which incorporate a wide family of identities among them. Moreover, the SVMPs naturally form a shuffle subalgebra. Both algebraic structures pass to the MZV and allow to speak about the algebra of *singlevalued MZV* [B13].

In general, the residue of a primitively (ultra-violet) divergent Feynman amplitude is defined by an integral over a compact projective space (see [NST13], Theorem 2.3). In many cases (for instance for amplitudes involving a conformally invariant integration) the same residue can be computed using integration over a (non-compact) unbounded domain. An example of this type, the wheel with $n$ strokes was considered in [B93] (and later surveyed in Appendix D to [NST13] and in [S13]). All $\varphi^4$ periods considered in [Sch, BS, S13] are of this type. This allows to compute such periods using recursive relations that involve integration over $\mathbb{R}^4$. Furthermore, it offers the possibility to treat graphs with internal vertices and thus to face the large $x$ (infrared) behaviour.

The paper is organized as follows. We start in Sect. 2 with a basic example: integration over an internal vertex in the $\varphi^4$ theory yielding the Bloch-Wigner dilogarithm. The details of the calculation (using Gegenbauer polynomial technics [CKT]) are relegated to Appendix A. Sect. 3 introduces the multipolylogarithms as iterated integrals $L_w$ labeled by words $w$ in two letters $\{0,1\}$ obeying shuffle algebra relations. The (possibly regularized) value of $L_w(z)$ at $z=1$ is identified with the (generalized) MZV $\zeta_w$. The series MZV correspond to a passage from the two letter alphabet to one with an infinite number of letters:

$$\zeta(n_1,\ldots,n_r) = (-1)^r\,\zeta_{10^{n_1-1}\ldots 10^{n_r-1}}\,, \quad n_i = 1,2,\ldots, \quad i=1,\ldots,r \tag{1.2}$$

($n_r = 1$ corresponding to the generalized/regularized MZVs). It is for the MZVs that we also define (in Sect. 3.2) the stuffle relations (which reduce to easily derivable identities for the series (1.1)). The number of arguments $r$ in the MZV (1.1) corresponding to the number of 1's in $w$ is called *length* or *depth* while the number of all letters $\{0,1\}$ of a word $w$ is called its *weight*. We treat systematically the identities among MZV of weight up to five in Appendix B. The study of the monodromy of multipolylogarithms (Sect. 3.3 and Appendix C) is streamlined by the introduction of the generating series $L = L_{e_0\,e_1}(z)$ and $Z = Z_{e_0\,e_1}$ (3.15). It is a prerequisit for the study of the monodromy of $L_w$ and hence for introducing SVMP by Brown's Theorem 3.1. Schnetz's notion of a *graphical function* is reviewed in Sect. 4. As an introduction to the generating series (4.8) for SVMP we work out in Sect. 4.1 the graphical function and the period for the wheel with $n$ spokes which only involves the simpler SVMP of depth one. We return to our main example, the four loop amplitude $G_4$ (Fig. 1), in Sect. 4.2 (and Appendix C). Its residue $I(G_4)$ is expressed as a sum of four pairs of SVMPs evaluated at $z=1$: one of depth one, which reproduces the period of the wheel with four spokes

$$I(W_4) = \binom{6}{3}\zeta(5)\,, \tag{1.3}$$

two of depth two with a negative contribution $(-20\,\zeta(5))$ to $I(G_4)$, and one of depth three whose contribution $(20\,\zeta(5))$ cancels that of the depth two terms. Thus we confirm the expected result $I(G_4) = I(W_4)$ (4.15) demonstrating that integration over internal vertices in a primitively divergent $\varphi^4$ graph commutes with taking the residue.

## 2 An inspiring example: the Bloch-Wigner singlevalued dilogarithm

The main example, on which we shall test the basic concepts and tools, reviewed in this lecture, is the massless 4-point $\varphi^4$-amplitude in (euclidean) position space

$$G_4(x_1,\dots,x_4) = \frac{I(x_1,\dots,x_4)}{x_{12}^2\,x_{23}^2\,x_{34}^2\,x_{14}^2}\,,\quad x_{ij}=x_i-x_j\,,\quad i,j=1,\dots,4\,,$$

$$x_i = (x_i^\alpha,\ \alpha=1,\dots,4)\,,\quad x_{ij}^2=\sum_{\alpha=1}^{4}(x_{ij}^\alpha)^2\,, \tag{2.1}$$

where $I(x_1,\dots,x_4)$ is the (conformally covariant) Feynman integral

$$I(x_1,\dots,x_4) = \int\prod_{i=1}^{4}\frac{1}{(x_i-x)^2}\,\frac{d^4x}{\pi^2} = \frac{\mathcal{F}(u,v)}{x_{13}^2\,x_{24}^2}\,, \tag{2.2}$$

$u$ and $v$ being the two independent cross ratios

$$u=\frac{x_{12}^2\,x_{34}^2}{x_{13}^2\,x_{24}^2}\,,\quad v=\frac{x_{14}^2\,x_{23}^2}{x_{13}^2\,x_{24}^2}\,. \tag{2.3}$$

The amplitude $G_4$ corresponds to the four-loop Feynman graph displayed on Fig. 1

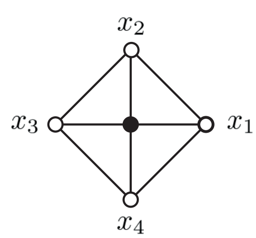



Figure 1.
*Four-point graph; the open circles correspond to external vertices.*

The integral (2.2) can be interpreted both as a $\varphi^4$ integral in position space and as one corresponding to the box diagram of a $\varphi^3$ theory in momentum space (Fig. 2)

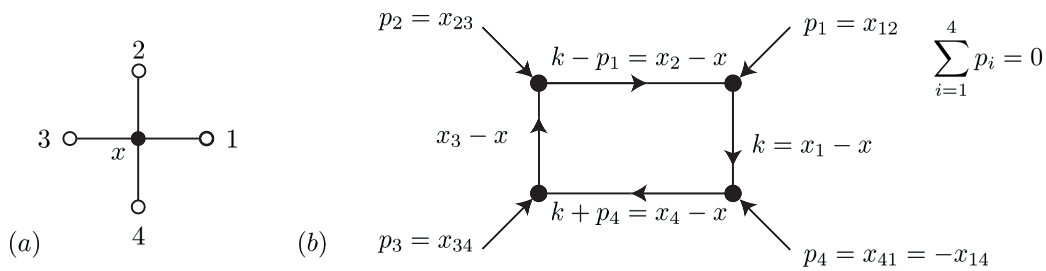



Figure 2.

*Dual interpretation of the integral* (2.2).

The second interpretation provides an elementary example of what came to be called a *dual conformal symmetry* [DHKS]. It was for the momentum space box diagram (as the simplest example of a ladder graph) that the integral (2.2) was first computed [UD] (back in 1993) using Melin transform. A modern computation using Gegenbauer polynomial technics [CKT] is sketched in Appendix A. The result is expressed in terms of a dilogarithm function of a complex variable $z$ and its conjugate $\bar{z}$ related to the conformal cross ratios (2.3) by

$$u = z\,\bar{z}\,, \qquad v = (1-z)(1-\bar{z})\,. \tag{2.4}$$

The derivation of Appendix A uses the fact that the 4-dimensional hyperspherical Gegenbauer polynomial $C_n^1$ is expressed in terms of the Tchebyshev polynomial of the second kind:

$$|z|^n\, C_n^1\left(\frac{z+\bar{z}}{2\,|z|}\right) = \frac{z^{n+1}-\bar{z}^{n+1}}{z-\bar{z}} \qquad (|z|^2 = z\,\bar{z})\,. \tag{2.5}$$

The result is a *singlevalued* real analytic function on $\mathbb{CP}^1 \backslash \{0, 1, \infty\}$ given by

$$\mathcal{F}(u,v) = x_{13}^2\, x_{24}^2\, I(x_1,\ldots,x_4) = \frac{4i\, D(z)}{z-\bar{z}} \tag{2.6}$$

where $D(z)$ is the singlevalued dilogarithm of David Wigner and Spencer Bloch – see [Bl, Z].

$$\begin{aligned} D(z) &= \mathrm{Im}\left(Li_2(z) + \ln|z| \ln(1-z)\right) \\ &= \frac{1}{4i}\left(2Li_2(z) - 2Li_2(\bar{z}) + \ln z\,\bar{z} \ln \frac{1-z}{1-\bar{z}}\right). \end{aligned} \tag{2.7}$$

Here $Li_n(z)$ denotes the polylogarithm given for $|z| < 1$ by the power series

$$Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad (Li_1(z) = -\ln(1-z))\,. \tag{2.8}$$

While $Li_2(z)$ has a *multivalued* analytic continuation to arbitrary complex $z$ given by the integral

$$Li_2(z) = -\int_0^z \ln(1-t)\ \frac{dt}{t}\,, \tag{2.9}$$

that depends on the homotopy class of the path which joins 0 and $z$, the function (2.7) is singlevalued (and continuous) on the entire projective plane.

The symmetries of $D(z)$ can be best described by introducing a (real valued) function of four complex variables that behaves as a (scale invariant) local fermionic 4-point amplitude in a two-dimensional conformal field theory:

$$\widetilde{\mathcal{D}}(z_1, z_2, z_3, z_4) = D\left(\frac{z_{12}\, z_{34}}{z_{13}\, z_{24}}\right) \quad \text{where} \quad z_{ij} = z_i - z_j\,. \tag{2.10}$$

It is invariant under even permutations and changes sign under odd permutations of the variables $(z_1, \dots, z_4)$. This implies

$$\begin{aligned} \mathcal{D}(z) &= \mathcal{D}\left(\frac{z-1}{z}\right) = \mathcal{D}\left(\frac{1}{1-z}\right) = -\mathcal{D}\left(\frac{1}{z}\right) \\ &= -\mathcal{D}(1-z) = -\mathcal{D}\left(\frac{z}{z-1}\right) (= -\mathcal{D}(\bar{z}))\,. \end{aligned} \tag{2.11}$$

The function $\widetilde{\mathcal{D}}$ (2.10) gives the volume of the ideal (oriented) tetraedron with vertices $z_1, \dots, z_4$ on the absolute (also called horosphere) of Lobachevsky space and has already been studied by Lobachevsky himself (cf. [M82]; for background on the Beltrami model of Lobachevsky space – see [L]).

The significance of this example stems from the fact that it displays properties common to low loop calculations in more structured quantum field theory models (such as the $N = 4$ super-Yang-Mills theory [DDEHPS]) as well as in physically relevant calculations in quantum chromodynamics [DDDP]. In particular, the singlevaluedness of euclidean Feynman amplitudes is dictated by general considerations of the *symbol of iterated integrals* [GSVV, GMSV, DGR].

# 3 The shuffle algebra of multipolylogarithms and of multizeta values

It is both fortunate and demanding for a newcomer in the field that the multipolylogarithms (as well as their values at $z = 1$ – the MZV) appear with a rich algebraic structure.

## 3.1 The algebra of words in two letters. Recursive definition of polylogarithms

We start by introducing a family of iterated integrals[1]. Denote by $\{0,1\}^\times$ the set of words $w$ in the two letters 0 and 1, including the empty word $\emptyset$. The multipolylogarithms of a single variable $z$ are defined inductively by the differential equations

$$\frac{d}{dz} L_{wa}(z) = \frac{L_w(z)}{z-a}\,, \quad a \in \{0,1\}\,, \quad L_\emptyset = 1\,, \tag{3.1}$$

and the initial condition

$$L_w(0) = 0 \quad \text{for} \quad w \neq 0^n (= 0 \ldots 0 - n \text{ times}), \quad L_{0^n}(z) = \frac{(\ln z)^n}{n!}\,. \tag{3.2}$$

In particular, for $n, n_i \geq 1$ we have

$$L_{1^n}(z) = \frac{[\ln(1-z)]^n}{n!}\,;\ (-1)^r L_{10^{n_1-1}\ldots 10^{n_r-1}}(z) = Li_{n_1\ldots n_r}(z)$$
$$\left( = \sum_{1 \leq k_1 < \ldots < k_r} \frac{z^{k_r}}{k_1^{n_1} \ldots k_r^{n_r}} \quad \text{for} \quad |z| < 1 \right). \tag{3.3}$$

For any ring $R$ of numbers (which includes the ring $\mathbb{Z}$ of rational integers), we define the $R$-module $R(\{e_0, e_1\}^\times)$ of formal linear combination of words in the alphabet $\{e_0, e_1\}$ and introduce the *shuffle algebra* $\mathrm{Sh}_R(e_0, e_1)$ equipping it with the (commutative) *shuffle product* $w \sqcup\!\sqcup\, w'$ defined recursively by

$$\emptyset \sqcup\!\sqcup\, w = w (= w \sqcup\!\sqcup\, \emptyset)\,, \quad au \sqcup\!\sqcup\, bv = a(u \sqcup\!\sqcup\, bv) + b(au \sqcup\!\sqcup\, v) \tag{3.4}$$

where $u, v, w$ are (arbitrary) words while $a, b$ are letters (note that the empty word is *not* a letter). Extending by ($R$-)linearity the correspondence $w \to L_w(z)$ one proves that the resulting map $\mathrm{Sh}_R(e_0, e_1) \to R(L_w)$ is a homomorphism of shuffle algebras:

$$L_{u \sqcup\!\sqcup\, v}(z) = L_u(z)\, L_v(z)\,. \tag{3.5}$$

In particular, it is easy to verify that the dilogarithm (2.9) disappears from the shuffle product:

$$L_{0 \sqcup\!\sqcup\, 1}(z) := L_{01}(z) + L_{10}(z) = L_0(z)\, L_1(z) = \ln z \ln(1-z)\,.$$

From the uniqueness of the solution of (3.1) under the condition (3.2) it is straightforward to prove that for a general word

$$w_{\vec{n}} = 0^{n_0}\, 10^{n_1-1} \ldots 10^{n_r-1}\,, \quad n_0 = 0, 1, \ldots, n_i = 1, 2, \ldots\,, \tag{3.6}$$

[1]Iterated integrals were introduced in the mid 1950's and developed essentially single-handedly for over 20 years by K.-T. Chen (1923-87) [C] before gaining recognition in both mathematics and QFT – see [B].

we have

$$L_{w_{\vec{n}}}(z) = \sum_{\substack{k_0 \geq 0\; k_i \geq n_i, 1\leq i\leq r \\ k_0+k_1+\ldots+k_r = n_0+\ldots+n_r}} (-1)^{k_0+n_0+r} \prod_{i=1}^{r} \binom{k_i-1}{n_i-1} L_{0^{k_0}}(z)\, L_{k_1 - k_r}(z)\,. \tag{3.7}$$

## 3.2 Multiple zeta values (MZV)

For $n_r > 1$ in (3.3) (and in (3.7)) $L_{n_1 \ldots n_r}(z)$ is convergent at $z = 1$ and we define the MZVs as the values at 1 of the corresponding multipolylogarithms:

$$\begin{aligned} \zeta(n_1, \ldots, n_r) &= Li_{n_1 \ldots n_r}(1)\,, \\ \zeta_{w_{\vec{n}}} &= (-1)^{n_0+r} \sum_{\substack{k_i \geq n_i \\ \sum_1^r k_i = n_0 + \sum_1^r n_i}} \prod_{i=1}^{r} \binom{k_i-1}{n_i-1} \zeta(k_1, \ldots, k_r)\,. \end{aligned} \tag{3.8}$$

We extend this definition to all words by introducing the *regularized* MZV setting

$$\zeta_1 = -\zeta(1) = 0\,(= \zeta_0) \tag{3.9}$$

and postulating that $\zeta_w$ satisfy the shuffle relation

$$\zeta_{u \sqcup\!\sqcup v} = \zeta_u\, \zeta_v\,. \tag{3.10}$$

There is a second *stuffle product*, $\times$, defined on words in the infinite alphabet of positive integers which is suggested by identities for the series expansions of polylogs or MZV. Rather than reproducing the general definition (see [W]) we just give two simple examples: the *Nielsen reflection formula*

$$\zeta(m)\,\zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(m+n) =: \zeta(m \times n) \tag{3.11}$$

and the relation

$$\begin{aligned} \zeta(\ell) \cdot \zeta(m,n) &= \zeta(\ell, m, n) + \zeta(m, \ell, n) + \zeta(m, n, \ell) \\ &+ \zeta(m+\ell, n) + \zeta(m, n+\ell) =: \zeta(\ell \times (m,n))\,, \end{aligned} \tag{3.12}$$

which suggests the general pattern. The stuffle identities that generalize (3.11), (3.12) prove that the product of MZV can be expanded as a linear combination of MZV with integer coefficients. They also allow to extend the notion of MZV to the case when the last entry is 1. The "regularized MZV" cancel in the difference of the two products yielding, in general, non-trivial identities as illustrated in the following example: subtracting the stuffle from the shuffle equation below,

$$\zeta(1)\,\zeta(2) = \zeta_{1 \sqcup\!\sqcup 10} = 2\zeta(1,2) + \zeta(2,1)$$

$$\zeta(1)\,\zeta(2) = \zeta(1 * 2) = \zeta(1,2) + \zeta(2,1) + \zeta(3)\,,$$

we obtain Euler's identity

$$\zeta(1,2) = \zeta(3) \tag{3.13}$$

between two convergent series (see for a more systematic treatment of the resulting relations Appendix B).

The number $r$ of arguments in $\zeta(k_1, \ldots, k_r)$ corresponding to the number of 1's in the word $w_{\vec{n}}$ (3.6) is called *length* (as in [W]) or *depth* (in [B13D] [S13]) of $w_{\vec{n}}$. The number $|w|$ of all letters of the word $w$ in the alphabet $\{0, 1\}$ is called the *weight* of $w$.

For even $n\,(= 2, 4, \ldots)$ the $\zeta(n)$ is a rational multiple of $\pi^n$ (as established by the 27-year-old Euler in 1734 – see detailed historical references in [D]; for a derivation à la Euler of the explicit formula (3.14) below in terms of the Bernoulli numbers $B_{2k}$ – see [C01]):

$$\zeta(2k) = 2^{2k-1} \frac{|B_{2k}|}{(2k)!} \pi^{2k}\,, \quad B_2 = \frac{1}{6}\,,\ B_4 = -\frac{1}{30}\,,\ B_6 = \frac{1}{42}\,, \ldots\,. \tag{3.14}$$

Calculating (by hand!) $\zeta(3)$ up to ten significant digits Euler verified that it is not given by $\pi^3$ times a rational number with a small denominator [D]. There is a far going (widely believed but completely unproven) conjecture that the numbers $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent. All known relations among zeta values of odd weight involve MZVs of the same weight (like in (3.13)). One may call the relations coming from the shuffle and stuffle identities (see Appendix B) *motivic*. (More precisely, starting from an abstract definition involving the fundamental group of $\mathbb{CP}^1 \backslash \{0, 1, \infty\}$ – see [B12] and the review [D] – one proves that these relations are indeed motivic – cf. also [W] and the explicit treatment of the special case of double zeta values in [C12].) It is conjectured that all motivic zeta values are of this type. A further going conjecture (that would imply the above mentioned belief about the algebraic independence of odd zeta values and $\pi$) says that all relations among MZV are motivic.

## 3.3 Single valued multiple polylogarithms (SVMP)

The monodromies $\mathcal{M}_0$ and $\mathcal{M}_1$ around the potential singularities 0 and 1 of the polylogarithms (2.8) and of $L_{0^n}$ (3.2) are given by the unipotent operators

$$\mathcal{M}_0\, Li_n(z) = Li_n(z)\,, \quad \mathcal{M}_0\, L_{0^n}(z) = L_{0^n}(z) + 2\pi i\, L_{0^{n-1}}(z)\,,$$

$$\mathcal{M}_1\, Li_n(z) = Li_n(z) - 2\pi i\, L_{0^{n-1}}(z)\,.$$

More generally, introducing the generating function $L(z)(= L_{e_0\, e_1}(z) = 1 + \ln z\, e_0 + \ln(1-z)\, e_1 + \ldots)$ and its regularized limit $Z(= Z_{e_0\, e_1})$ at $z = 1$ (called the *Drinfeld associator*),

$$\begin{aligned} L(z) &= \sum_w L_w(z)\, w\,,\ Z = \sum_w \zeta_w\, w \\ &= 1 + \zeta(2)[e_0, e_1] + \zeta(3)[[e_0, e_1], e_0 + e_1] + \ldots \end{aligned} \tag{3.15}$$

(cf. Appendix B) we can write (see Appendix C)

$$\mathcal{M}_0 \, L(z) = e^{2\pi i e_0} \, L(z) \,, \quad \mathcal{M}_1 \, L(z) = Z \, e^{2\pi i e_1} \, Z^{-1} \, L(z) \,. \tag{3.16}$$

The first relation follows from the fact that $L(z)$ is the unique solution of the *Knizhnik-Zamolodchikov equation*

$$\frac{d}{dz} \, L(z) = L(z) \left( \frac{e_0}{z} + \frac{e_1}{z-1} \right) \tag{3.17}$$

obeying the asymptotic condition

$$\begin{aligned} L(z) &= e^{e_0 \ln z} h_0(z) \,, \\ h_0(z) &= 1 + e_1 \ln(1-z) + [e_0, e_1] \, Li_2(z) + e_1^2 \, \frac{[\ln(1-z)]^2}{2} + \ldots \end{aligned} \tag{3.18}$$

(i.e. $h_0(0) = 1$, $h_0(z)$ being a formal power series in the words in $\{e_0, e_1\}^\times$ that is holomorphic in $z$ in the neighbourhood of $z = 0$. The second relation (3.16) is implied by the fact that there exists a counterpart $h_1(z)$ of $h_0$, holomorphic around $z = 1$ and satisfying $h_1(1) = 1$ such that

$$L(z) = Z \, e^{e_1 \ln(1-z)} \, h_1(z) \tag{3.19}$$

(see Appendix C). The knowledge of the monodromy allows to construct single-valued linear combinations of products of the type $L_{w'}(\bar{z}) \, L_w(z)$, the SVMP. A practitioner of 2-dimensional (2D) conformal field theory will notice the analogy with constructing monodromy invariant 2D correlation functions out of (multivalued) chiral conformal blocks. It turns out that SVMP have simple characterization in terms of equations of type (3.1) (3.2) and form an interesting subalgebra of the shuffle algebra of multiple polylogarithms. The following result is due to Brown.

**Theorem 3.1.** [B04] (See also Theorem 2.5 of [S13].) *There exists a unique family of single valued functions* $\{P_w(z) \,, \; w \in \{0,1\}^\times \,, \; z \in \mathbb{C} \backslash \{0,1\}\}$ *each of which is a linear combination of* $L_u(\bar{z}) \, L_v(z)$ *of the same total weight,* $|u| + |v| = |w|$*, which satisfy the differential equations*

$$\partial \, P_{wa}(z) = \frac{P_w(z)}{z-a} \,, \quad \partial \equiv \frac{\partial}{\partial z} \,, \tag{3.20}$$

*such that*

$$P_\emptyset = 1 \,, \quad P_{0^n}(z) = \frac{(\ln z \, \bar{z})^n}{n!} \,, \quad P_w(0) = 0 \;\; \textit{for} \;\; w \neq 0^n \; (w \neq \emptyset) \,. \tag{3.21}$$

*The functions* $P_w$ *satisfy the shuffle relations* (3.5) *and are linearly independent over the ring of polynomials* $\mathbb{C}\left[z, \frac{1}{z}, \frac{1}{1-z}; \bar{z}, \frac{1}{\bar{z}}, \frac{1}{1-\bar{z}}\right]$. *Every singlevalued linear combination of functions of the type* $L_{w'}(\bar{z}) \, L_{w''}(z)$ *can be written as a (unique) linear combination of* $P_w(z)$.

The functions $P_w$ can be constructed explicitly in terms of the corresponding generating function (see [S13], the text after Theorem 2.5; a special case of interest is reproduced in Sect. 4 below). The functions

$$P_w^0(z) = \sum_{uv=w} L_{\tilde{u}}(\bar{z})\, L_v(z) \tag{3.22}$$

(where $\tilde{u} = a_n \dots a_1$ for $u = a_1 \dots a_n$, $a_i \in \{0,1\}$) can serve as a first step in the construction of $P_w$ and actually coincide with $P_w$ for words (of any weight but) of length/depth one as well as for all words of weight at most three. We find, in particular,

$$P_{01}(z) = L_{10}(\bar{z}) + L_{01}(z) + L_0(\bar{z})\, L_1(z) = Li_2(z) - Li_2(\bar{z}) + \ln z\,\bar{z} \ln(1-z)$$

$$P_{10}(z) = L_{01}(\bar{z}) + L_{10}(z) + L_1(\bar{z})\, L_0(z) = Li_2(\bar{z}) - Li_2(z) + \ln z\,\bar{z} \ln(1-\bar{z})\,,$$

so that

$$P_{01} + P_{10} = \ln |z|^2\, \ln |1-z|^2 = P_0\, P_1\,,$$

in accord with the shuffle relation, while

$$P_{01}(z) - P_{10}(z) = 2(Li_2(z) - Li_2(\bar{z})) + \ln z\,\bar{z} \ln \frac{1-z}{1-\bar{z}} = 4i\, D(z) \tag{3.23}$$

reproduces the Bloch-Wigner function (2.7) – the only new SVMP of weight two.

The words $w$ for which the SVMP $P_w$ coincide with $P_w^0$ (3.19) include the wheel with $n$-spokes reviewed in Sect. 4.1 below.

As it is precisely the SVMP that appear in the calculation of Feynman amplitudes, it is natural to expect the Feynman periods (or residues) will also belong to the corresponding restricted shuffle subalgebra of "singlevalued MZV", generated by the values of SVMP at $z = 1$ (see [B13]). This set turns out to be generated by the odd zeta values $\zeta(2n+1)$, $n = 1, 2, \dots$. In particular, the Bloch-Wigner function (2.7) (3.20) vanishes for real $z$, hence so does the singlevalued counterpart of $\zeta(2)$:

$$\zeta^{\mathrm{SV}}(2) = D(1) = 0\,. \tag{3.24}$$

# 4 Graphical functions and periods

## 4.1 SVMP of depth one and the wheel. Generating series for the general SVMP

The computation of the integral (2.2) (or of its simplified version (A.3)) can be viewed as a first step in a recurrence in which $f_n(z) = F(z, W_n)$ are defined by

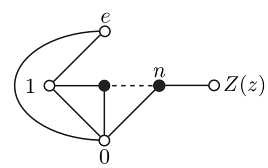
$$\partial\, \bar{\partial}\, f_2(z) = \frac{1}{z(1-\bar{z})} - \frac{1}{\bar{z}(1-z)} \Rightarrow f_2(z) = P_{01}(z) - P_{10}(z)$$

$$\partial\,\bar\partial\, f_{n+1}(z) = \frac{-1}{z\,\bar z}\, f_n(z) \quad \text{for} \quad n = 2, 3, \ldots,\ \partial = \frac{\partial}{\partial z}\,,\ \bar\partial = \frac{\partial}{\partial \bar z}\,, \tag{4.1}$$

whose (unique) SVMP solution is

$$f_{n+1}(z) = (-1)^n \left(P_{0^{n-1}\,10^n}(z) - P_{0^n\,10^{n-1}}(z)\right)\,. \tag{4.2}$$

Here $F(z, W_n)$ gives the Feynman amplitude corresponding to the sequential graph presented on Fig. 3




$$F(z, W_{n+1}) = \int \frac{d^4\, x_1}{\pi^2\, x_1^2} \cdots \int \frac{d^4\, x_n}{\pi^2\, x_n^2}\, \frac{1}{(x_1 - e)^2\, x_{12}^2 \ldots x_{n-m}^2 (x_n - Z)^2}$$

Figure 3.

*Sequential graph for the wheel $W_{n+1}$.*

To prove this identification one uses the 4-dimensional Laplace equation

$$-\frac{1}{4}\,\Delta_Z\, F(z, W_{n+1}) = \frac{1}{z\,\bar z}\, F\,(z, W_n) \tag{4.3}$$

and the expression for $\Delta_Z$ restricted on a function of $z$ and $\bar z$:

$$\frac{1}{4}\,\Delta_Z\, F(z) = \frac{1}{z - \bar z}\,\partial\,\bar\partial\,[(z - \bar z)\, F\,(z)] \quad (Z^2 = z\,\bar z\,,\ (Z - e)^2 = |z - 1|^2)\,. \tag{4.4}$$

The period $I(W_{n+1})$ of the wheel with $n + 1$ spokes is now obtained as the limit of $F(z, W_{n+1})$ for $z \to 1$ ($Z \to e$). (To see this, one should redraw Fig. 3 with the vertices $(e, 1, \ldots, n)$ on a circle and the vertex 0 in its centre.)

For a general word of weight $n_0 + n_1$ and depth one we can use (3.19) to write

$$\begin{aligned} P_{0^{n_0}\,10^{n_1-1}}(z) \quad &= \quad \sum_{k=0}^{n_0} (-1)^{k+1} \binom{n_1 - 1 + k}{n_1 - 1} P_{0^{n_0 - k}}(z)\, Li_{n_1+k}(z) \\ &+ \quad \sum_{k=0}^{n_1-1} (-1)^{k+1} \binom{n_0 + k}{n_0} P_{0^{n_1-1-k}}(z)\, Li_{n_0+k+1}(\bar z) \end{aligned} \tag{4.5}$$

(where $P_{0^n}$ is given in (3.18)). Inserting this expression in (4.2) we obtain

$$F(z, W_{n+1}) = \frac{f_{n+1}(z)}{z - \bar z} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n} P_{0^{n-k}}(z)\, \frac{Li_{n+k}(z) - Li_{n+k}(\bar z)}{z - \bar z}\,. \tag{4.6}$$

In the limit $z \to 1$ only the term with $k = n$ contributes and we find

$$I(W_{n+1}) = F(1, W_{n+1}) = \binom{2n}{n} \zeta(2n-1)\,. \tag{4.7}$$

This result was first derived using a similar recursion by Broadhurst [B93]. The above derivation follows Schnetz [S13].

In general, the generating function of SVMP is given by (see [B04] [S13] and Appendix C below):

$$P_{e_0\, e_1}(z) = \tilde{L}_{e_0\, e_1'}(\bar{z})\, L_{e_0\, e_1}(z) \tag{4.8}$$

where $\tilde{L} = \Sigma\, L_w\, \tilde{w}$ (cf. (3.22)) and $e_1'$ is the unique solution of the equation

$$Z_{-e_0,-e_1'}\, e_1'\, Z^{-1}_{-e_0-e_1'} = Z_{e_0\, e_1}\, e_1\, Z^{-1}_{e_0\, e_1} \tag{4.9}$$

(see Appendix C).

## 4.2 Single valued MZV and the period of $G_4$

The amplitude $G_4$ (2.1) and its period $I(G_4)$ corresponding to the graph on Fig. 1 is of interest as the first strongly connected (or "internally six connected" in the terminology of [S13]) $\varphi^4$-graph that involves integration over an internal vertex. Albeit such an integral is known to be infrared convergent it may interfere with the causal factorization condition for the ultraviolet renormalization (the amplitude $G_4$ being primitively logarithmically divergent). A related question: the period of the amplitude belongs to the wheel series. If we can treat the vertex 0 (with four adjacent lines) as an external one then we should expect to have $I(G_4) = I(W_4) = \binom{6}{3} \zeta(5)$. If we treat it as an internal vertex – see Fig. 4

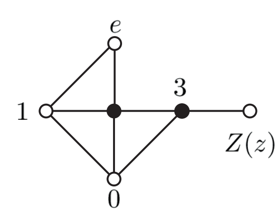



Figure 4.
*Graph for the graphical function $g_4(z)$.*

then we end up with a different graphical function. Indeed, the sequence of differential equations corresponding to the graph on Fig. 4 is

$$\begin{aligned}
g_2(z) &= f_2(z) = P_{01}(z) - P_{10}(z) \\
\partial\,\bar{\partial}\, g_3(z) &= \frac{-g_2(z)}{z\,\bar{z}(1-z)(1-\bar{z})} = \left(\frac{1}{z-1} - \frac{1}{z}\right)\left(\frac{1}{\bar{z}} - \frac{1}{\bar{z}-1}\right) g_2(z) \\
\partial\,\bar{\partial}\, g_4(z) &= \frac{-g_3(z)}{z\,\bar{z}}\,.
\end{aligned} \tag{4.10}$$

The functions

$$\begin{aligned}
g_3^0(z) &= P_{0100}(z) - P_{0010}(z) + P^0_{1010}(z) - P^0_{0101}(z) + P^0_{0011}(z) \\
&- P^0_{1100}(z) + P^0_{1101}(z) - P^0_{1011}(z)\,, \\
g_4^0(z) &= P_{0^31 0^2}(z) - P_{0^21 0^3}(z) + P^0_{0^21010}(z) - P^0_{01010^2}(z) + P^0_{01^20^3}(z) \\
&- P^0_{0^31^20}(z) + P^0_{01 01^2 0}(z) - P^0_{01^2010}(z)
\end{aligned} \tag{4.11}$$

where $P_w^0(z)$ are given by (3.22) provide a multivalued solution of the partial differential equations (4.10). The SVMP $g_3(z)$ is obtained from $g_3^0$ (4.11) by replacing $P_w^0(z)$ by

$$P_w(z) = P_w^0(z) + 2\,\zeta(3)\,\langle w, [[[e_0, e_1]\,e_1]\,e_0 + e_1]\rangle\, L_1(\bar z) \tag{4.12}$$

(see Appendix C). The inner product in $\mathbb{Z}(\{e_0, e_1\}^\times)$ is defined by setting $\langle u \mid v\rangle = 0$ if $u \neq v$ $\langle u \mid u\rangle = 1$ for any two words $u$ and $v$. The period $I(G_4)$ is equal to the derivative $g_4'(z=1)$ given by the limit

$$\begin{aligned}
I(G_4) &= \lim_{z\to 1} \frac{g_4(z)}{z - z'} = P_{0^310}(1) - P_{0^210^2}(1) + P_{0^2101}(1) - P_{01010}(1) \\
&+ P_{01^20^2}(1) - P_{0^31^2}(1) + P_{01 01^2}(1) - P_{01^201}(1)\,.
\end{aligned} \tag{4.13}$$

According to (4.2) and (4.7) the period of the wheel with four strokes is given by just the first two terms of (4.13):

$$I(W_4) = P_{0^310}(1) - P_{0^210^2}(1) = \binom{6}{3}\zeta(5) = 20\,\zeta(5)\,. \tag{4.14}$$

As verified in Appendix C the remaining six terms cancel against each other so that

$$I(G_4) = I(W_4) = 20\,\zeta(5) \tag{4.15}$$

which is a special case of the result of [BS] concerning all zig-zag graphs. This calculation confirms the general argument of Sect. 2 of [Sch] demonstrating that periods in $\varphi^4$ theory are in fact associated to (completed by a "vertex at infinity") 4-point graphs and do not depend on the choice of marked vertices $(\infty, 0, 1, z)$. Thus different (logarithmically divergent) Feynman amplitudes, in our example a 4-point and a 5-point one, may be renormalized by subtracting a pole term with the same residue (multiplied by a 12- and a 16-dimensional $\delta$-function, respectively).

## 4.3 Concluding remarks

In the early days of the development of the "dual resonance model" theoretists were joking about "physics of the red book" – meaning the volumes of the Bateman-Erdileyi classic "Higher Transcendental Functions". There is a marked difference between that old fad and the present day development of analytic methods in perturbative QFT calculations a basic ingredient of which is reviewed in this lecture. Quantum field theory is the language of the standard model of particle physics (which also gives room but is not reduced to speculative dreams that may serve a future theory). The family of multiple polylogarithms, omnipresent in perturbative calculations, far from being just another set of special functions, admits an interesting algebraic structure that passes to the physically relevant subfamily of SVMPs. Residues or periods typically expressed in terms of MZVs are central to our current understanding of ultraviolet renormalization. These developments have transformed QFT from a "reason for divorce between mathematics and physics" [D] half a century ago into a common playing ground for mathematicians and physicists, giving a new vigour to our field.

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## Appendix A. Computation of the integral (2.2)

Using conformal invariance we can send the variable $x_1$ to infinity, $x_4$ to zero, $x_2$ – to a unit 4-vector $e$ and set

$$x_3 = Z\,, \quad \text{where} \quad Z^2 = z\,\bar{z}\,,\ 2Ze = z + \bar{z} \tag{A.1}$$

so that the cross ratios (2.3) assume the form

$$u = Z^2 = z\,\bar{z}\,, \quad v = (Z-e)^2 = (z-1)(\bar{z}-1) \tag{A.2}$$

in accord with (2.4). Then we can write, introducing spherical coordinates $x = r\omega$, $Z = |z|\,\omega_z$,

$$\begin{aligned} \mathcal{F}(u,v) &= F(z) = \frac{1}{\pi^2}\int \frac{d^4\,x}{x^2(x-e)^2(x-Z)^2} \\ &= \frac{1}{\pi^2}\int^{\infty} r\,dr \int_{\mathbb{S}^3} \frac{d^3\,\omega}{(r^2 - 2r\,e\cdot\omega + 1)(r^2 + |z|^2 - 2r\,|z|\,\omega\,\omega_z)}\,. \end{aligned} \tag{A.3}$$

Assuming $|z| < 1$ we can split the radial integral $F$ into three terms $F = F_1 + F_2 + F_3$ corresponding to the domains $r < |z|$, $|z| < r < 1$ and $r > 1$, respectively. In the first one we can write

$$\begin{aligned} (r^2 - 2r\,e\cdot\omega + 1)^{-1} &= \sum_{n=0}^{\infty} r^n\, C_n^1(\omega e)\,, \\ (r^2 + |z|^2 - 2r|z|\,\omega\,\omega_z)^{-1} &= \frac{1}{|z|^2}\sum_{m=0}^{\infty}\left(\frac{r}{|z|}\right)^m C_m^1(\omega\,\omega_z)\ (\text{for } r < |z| < 1) \end{aligned} \tag{A.4}$$

where the hyperspherical (Gegenbauer) polynomials $C_n^1$ can be written as

$$C_n^1(\cos\theta) = \frac{\sin(n+1)\,\theta}{\sin\theta}\,. \tag{A.5}$$

Using further the orthogonality relation

$$\int_{\mathbb{S}^3} C_n^1(\omega\cdot e)\,C_m^1(\omega\,w_z)\,\frac{d^3\omega}{\pi^2} = \frac{2\,\delta_{mn}}{n+1}\,C_n^1(\omega_z\,e) \tag{A.6}$$

where, according to (A.2)

$$\omega_z \cdot e = \frac{z+\bar{z}}{2\,|z|}\,. \tag{A.7}$$

Inserting in $F_1$ and using (A.5) (or (2.5)) and (2.8) we find

$$F_1(z) = \int_0^{|z|} \frac{r\,dr}{|z|^2}\sum_{n=0}^{\infty}\frac{2}{n+1}\,\frac{r^{2n}}{|z|^n}\,C_n^1\left(\frac{z+\bar{z}}{2|z|}\right) = \frac{Li_2(z) - Li_2(\bar{z})}{z-\bar{z}}\,. \tag{A.8}$$

The same result is obtained for $F_3(z)$:

$$F_3(z) = \int_1^\infty \frac{dr}{r^3} \sum_{n=0}^\infty \frac{2}{n+1} \frac{|z|^n}{r^{2n}} C_n^1 \left( \frac{z+\bar z}{2|z|} \right) = \frac{Li_2(z) - Li_2(\bar z)}{z - \bar z} = F_1(z) \,. \tag{A.9}$$

Finally,

$$F_2(z) = 2 \int_{|z|}^1 \frac{dr}{r} \frac{Li_1(z) - Li_1(\bar z)}{z - \bar z} = \ln z\, \bar z \, \frac{\ln(1-z) - \ln(1-\bar z)}{z - \bar z} \,; \tag{A.10}$$

this, together with (A.8), (A.9) completes the proof of (2.6) (2.7) for $|z| < 1$. The same expression can be obtained in a similar fashion for $|z| > 1$; alternatively, it can be deduced from the result for $|z| < 1$ using the symmetry of $F(z)$ implied by (2.11). The result can also be established by verifying that it is single valued and satisfies the first equation (4.1) (in view of the uniqueness of SVMP, Theorem 3.1; cf. [S13]).

## Appendix B. Identities among MZV

Eq. (3.8) which relates the MZV $\zeta_w$ (labeled by words in the two letters $\{0,1\}$) with $\zeta(n_1, \dots, n_r)$, $n_i = 1, 2, \dots$ becomes particularly simple for words of depth one,

$$\zeta_{0^{n_0} 1 0^{n_1 - 1}} = (-1)^{n_0+1} \binom{n_0 + n_1 - 1}{n_1 - 1} \zeta(n_0 + n_1) \,. \tag{B.1}$$

This allows to write the depth one contribution to the generating function $Z$ (3.15) in terms of multiple commutators:

$$\sum_{n=1}^\infty \zeta(n+1) \underbrace{[\dots[}_{n} e_0, e_1], e_0], \dots, e_0] = \sum_{n=1}^\infty \zeta(n+1) \sum_{k=0}^n (-1)^{k+1} \binom{n}{k} e_0^k \, e_1 \, e_0^{n-k} \,, \tag{B.2}$$

which is another way to write down (B.1). It is more interesting – and more difficult – to deduce the relations among $\zeta_w$ for words of higher depth. We shall write down all such relations for depth two and weight $|w| \leq 5$. Note that the number $d_n$ of linearly independent MZV of a given weight $n$ can be read off the generating function conjectured by Don Zagier

$$\frac{1}{1 - t^2 - t^3} = \sum_{n=0}^\infty d_n \, t^n \quad d_2 = d_3 = d_4 = 1 \quad d_5 = d_6 = 2, \dots \tag{B.3}$$

(and proven for the motivic analog of MZV by Brown [B12]; in general, $d_n$ provide an upper bound of the independent MZV).

The Euler's relation (3.13) is a special case of either of the following more general relations which only involve proper (convergent) zeta series:

$$\zeta(\underbrace{1, \dots, 1}_{n-2}, 2) = \zeta(n) \,, \tag{B.4a}$$

$$\sum_{\substack{s_i \geq 1; s_k \geq 2 \\ \Sigma s_i = n}} \zeta(s_1, \ldots, s_k) = \zeta(n) \,. \tag{B.4b}$$

The "unproper" (regularized) zeta value $\zeta(n,1)$ is determined from the stuffle relation:

$$0 = \zeta(1)\,\zeta(n) = \zeta(1,n) + \zeta(n,1) + \zeta(n+1) \,. \tag{B.5}$$

In particular, for $n = 2$, we find

$$\zeta(2,1) = -\zeta(3) - \zeta(1,2) = -2\,\zeta(3) \,. \tag{B.6}$$

From Euler's formula

$$\zeta(2) = \frac{\pi^2}{6} \tag{B.7}$$

(a special case of (3.14)) and from the shuffle and stuffle relations one deduces that all zeta values of weight four are rational multiples of $\pi^4$ (in accord with the Zagier conjecture (B.3)). In particular, the relations for $10 \sqcup\!\sqcup 10$, $10 \times 10$ and (B.4) for $n = 4$,

$$\zeta(2)^2 = \zeta_{10 \sqcup\!\sqcup 10} = 2\,\zeta_{1010} + 4\,\zeta_{1100} = 2\,\zeta(2,2) + 4\,\zeta(1,3) \,,$$

$$\zeta(2)^2 = \zeta_{10\times 10} = 2\,\zeta(2,2) + \zeta(4) \,; \quad \zeta(1,3) + \zeta(2,2) = \zeta(4) \,,$$

allow to express all weight four words of length not exceeding two as integer multiples of $\zeta(1,3)$:

$$\zeta(4) = 4\,\zeta(1,3)(= \zeta(1,1,2)) \,,\ \zeta(2,2) = 3\,\zeta(1,3) \,,\ \zeta(2)^2 = 10\,\zeta(1,3)$$

$$\Rightarrow \zeta(1,3) = \frac{\pi^4}{360} \,. \tag{B.8}$$

Proceeding in a similar fashion with the two products of the words 10 and 100 we find

$$\zeta(2)\,\zeta(3) = 3\,\zeta_{10100} + 6\,\zeta_{11000} + \zeta_{10010} = 3\,\zeta(2,3) + 6\,\zeta(1,4) + \zeta(3,2) \,,$$

$$\zeta(2)\,\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5) \,; \quad \zeta(1,4) + \zeta(2,3) + \zeta(3,2) = \zeta(5) \,.$$

These three equations determine a 2-dimensional space of zeta values of weight five (in accord with (B.3)). Selecting as a basis $\zeta(1,4)$ and $\zeta(2,3)$ we express the remaining convergent $\zeta$-values of weight 5 in terms of this basis with positive integer coefficients

$$\zeta(1,1,3) = \zeta(1,4) \,,\ \zeta(1,2,2) = \zeta(2,3) \,,$$

$$\zeta(5) = 2\,\zeta(2,3) + 6\,\zeta(1,4) \,,\ \zeta(3,2) = \zeta(2,1,2) = \zeta(2,3) + 5\,\zeta(1,4) \,,$$

$$\zeta(2)\,\zeta(3) = 4\,\zeta(2,3) + 11\,\zeta(1,4) \tag{B.9}$$

(while $\zeta(4,1) = -\zeta(1,4) - \zeta(5) = -7\,\zeta(1,4) - 2\,\zeta(2,3)$).

For the study of single valued MZV it is more natural to use the basis $(\zeta(5), \zeta(2)\,\zeta(3))$ instead. Then we find

$$(\zeta(1,1,3) =)\, \zeta(1,4) = 2\,\zeta(5) - \zeta(2)\,\zeta(3)\,,$$

$$(\zeta(1,2,2) =)\, \zeta(2,3) = 3\,\zeta(2)\,\zeta(3) - \frac{11}{2}\,\zeta(5)$$

$$(\zeta(2,1,2) =)\, \zeta(3,2) = \frac{9}{2}\,\zeta(5) - 2\,\zeta(2)\,\zeta(3)\,;\ \zeta(4,1) = \zeta(2)\,\zeta(3) - 3\,\zeta(5)\,. \quad \text{(B.10)}$$

Brown [B12] has demonstrated that a basis for "motivic" MZV for all weights is given by $\zeta(s_1, \ldots, s_k)$, with $s_i \in \{2,3\}$.

From the iterated integral representation of MZV it follows that the generating function (3.15) satisfies:

$$Z^{-1}_{e_0 e_1} = Z_{e_1 e_0} = \tilde{Z}_{-e_0,-e_1}\,. \tag{B.11}$$

(The first equation incorporates, in particular, (B.4a).)

# Appendix C. Monodromy at $z = 1$. Single valued MZV

The representation (3.19) can be obtained from (3.18) by noticing that the substitution $z \to 1-z$ corresponds to the exchange $e_0 \leftrightarrow e_1$ and that the path from 0 to $z$ can be viewed as a composition of two paths: from 0 to 1 and from 1 to $z$. For $0 < z < 1$ one should just set $h_1(z) = h_0(1-z)$. Eq. (3.16) follows from (3.18) (3.19) and the relations

$$\mathcal{M}_0 \ln z = \ln z + 2\pi i\,, \quad \mathcal{M}_1 \ln(1-z) = \ln(1-z) + 2\pi i\,. \tag{C.1}$$

Applying (3.16) one should take into account the relation (B.11)

$$Z^{-1}_{e_0,e_1} = Z_{e_1,e_0} = \tilde{Z}_{-e_0,-e_1} \tag{C.2}$$

where the tilde indicates that each word is replaced by its opposite. We leave it to the reader to verify that the first few terms in the expansion of (3.16) reproduce (C.1) and give

$$\mathcal{M}_1\, L_{01}(z) = L_{01}(z)(= \ln z \ln(1-z) + Li_2(z))$$

$$\mathcal{M}_1\, L_{10}(z)(= \mathcal{M}_1(-Li_2(z))) = L_{10}(z) + 2\pi i \ln z\,. \tag{C.3}$$

We now proceed to the evaluation of the element $e_1'$ defined by Eq. (4.9). To this end we introduce the Lie algebra valued function

$$F(e_0,e_1) = Z_{e_0 e_1} e_1 Z^{-1}_{e_0 e_1} - e_1 = \zeta(2)[[e_0,e_1],e_1] + \zeta(3)[[[e_0,e_1],e_1],e_0+e_1] + \ldots \tag{C.4}$$

Eq. (4.9) can then be solved recursively, writing $e'_1 = \lim_{k\to\infty} e_1^{(k)}$ with

$$e_1^{(0)} = e_1\,,\ e_1^{(k+1)} = e_1 + F(e_0, e_1) + F_0(-e_0, -e_1^{(k)})\,. \tag{C.5}$$

The weight three term with $\zeta(2)$ cancels out and one finds

$$e'_1 = e_1 + 2\,\zeta(3)\,[[[e_0, e_1], e_1], e_0 + e_1] + \zeta(5)(\ldots) + \ldots \tag{C.6}$$

where, according to Schnetz [S13], the $\zeta(5)$ contribution consists of eight bracket words of weight six. (The $\zeta(3)$ contribution will be sufficient to the application that follows.)

The SVMPs in the right hand side of (4.13) are obtained from those in $g_3(z)$ by adding a letter 0 in front and at the end of each labeling word. Evaluating the regularized limit at $z = 1$ (and noting that for $L_{11}(z)$ it is zero) while $\bar{L}_{01}(1) = -\bar{L}_{10}(1) = \zeta(2)$ we find that for each (5-letter) word-label $w$ in (4.13) we obtain the following counterpart of (4.12)

$$\begin{aligned} P_w(1) &= P^0_w(1) + 2\,\zeta(2)\,\zeta(3)\,\langle w, w_{23}\rangle \\ w_{23} &:= [e_0, [[[e_0, e_1], e_1], e_0 + e_1]]\,. \end{aligned} \tag{C.7}$$

We shall see that the role of the second term in the right hand side of (C.7) is to cancel the product $\zeta(2)\,\zeta(3)$ in $P^0_w(1)$, in accord with the observation that $\zeta^{\mathrm{SV}}(2) = 0$.

Indeed the depth one contributions are proportional to $\zeta(5)$:

$$\begin{aligned} P_{0^310}(1)(= P^0_{0^310}(1)) &= L_{0^310}(1) + L_{010^3}(1) = 8\,\zeta(5)\,, \\ P_{0^210^2}(1) &= 2\,L_{0^210^2}(1) = -12\,\zeta(5) \end{aligned}$$

and their difference reproduces (4.14). For depth two we find (after cancelling the products $\zeta(2)\,\zeta(3)$) a negative multiple of $\zeta(5)$:

$$\begin{aligned} P^0_{0^2101}(1) &= \zeta_{0^2101} + \zeta_{1010^2} + \zeta_{100}\,\zeta_{01} \\ &= 3\,\zeta(4,1) + 2\,\zeta(3,2) + 2\,\zeta(2,3) - \zeta(2)\,\zeta(3) = 4\,\zeta(2)\,\zeta(3) - 11\,\zeta(5)\,, \end{aligned}$$

where in the last step we used (B.10), $\langle 0^2101, w_{23}\rangle = -2$ so that $P_{0^2101}(1) = P^0_{0^2101} + 2\,\zeta(2)\,\zeta(3)\langle 0^2101, w_{23}\rangle = -11\,\zeta(5)$; similarly $P_{01010}(1) = 4\,\zeta(5) = P_{0^31^2}(1)$, $P_{01^20^2}(1) = -\zeta(5)$, so that

$$P_{0^2101}(1) - P_{01010}(1) + P_{01^20^2}(1) - P_{0^31^2}(1) = -20\,\zeta(5)\,. \tag{C.8}$$

Finally, the depth three contribution is equal to that of depth one. Indeed we find, using [B13D],

$$\begin{aligned} P_{0101^2}(1) = \zeta_{0101^2} + \zeta_{1^2010} + \zeta_{10}\,\zeta_{01^2} + 6\,\zeta(2)\,\zeta(3) = 11\,\zeta(5)\,, P_{01^201}(1) = -9\,\zeta(5) \\ \Rightarrow P_{0101^2}(1) - P_{01^201}(1) = 20\,\zeta(5)\,. \end{aligned} \tag{C.9}$$

This completes the proof of (4.15). (The expressions (C.8) and (C.9) can be also extracted from the polylog- and polyzeta-procedures of [Schnetz].)

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