CATEGORIFICATION OF THE JONES-WENZL PROJECTORS

BENJAMIN COOPER AND VYACHESLAV KRUSHKAL*

Abstract. The Jones-Wenzl projectors p_n play a central role in quantum topology, underlying the construction of SU(2) topological quantum field theories and quantum spin networks. We construct chain complexes P_n, whose graded Euler characteristic is the "classical" projector p_n in the Temperley-Lieb algebra.We show that the P_n are homotopy idempotents and uniquely defined up to homotopy. Our results fit within the general framework of Khovanov's categorification of the Jones polynomial. Consequences of our construction include families of knot invariants corresponding to higher representations of $\mathrm{U}_q\,\mathfrak{su}(2)$ and a categorification of quantum spin networks. We introduce $6j$-symbols in this context.

1. Introduction

In [Kho00] Mikhail Khovanov introduced a categorification of the Jones polynomial, giving rise to a new conceptual framework for quantum invariants of links in the 3-sphere. The results in [Kho00] fit in the context of categorification of the Temperley-Lieb algebra [Kho02], [BN05]. Roughly speaking, categorification associates to an algebra A a category $\mathcal{C}$ whose Grothendieck group $\mathrm{K}_0(\mathcal{C})$ is isomorphic to A. Moreover, multiplication by generators of A gives rise to functors acting on $\mathcal{C}$ and satisfying natural properties [KMS09]. An extension from planar Temperley-Lieb diagrams to tangles is achieved by passing from additive to triangulated categories. The resulting link homology theory satisfies functoriality under surface cobordisms in 4-space, an important feature that was not apparent at the level of its graded Euler characteristic, the Jones polynomial.

An important open problem in the subject is to extend categorification from links in the 3-sphere to quantum invariants of 3-manifolds. The constructions of the SU(2) quantum invariants by Reshetikhin-Turaev [Tur94] and Turaev-Viro [TV92] rely on the Jones-Wenzl projectors p_n [Jon97, Wen87], certain special elements of the Temperley-Lieb algebra. In the Reshetikhin-Turaev theory, one uses the Jones-Wenzl projectors to label the components of the link in a surgery presentation of the 3-manifold. In the Turaev-Viro approach, a triangulation of the 3-manifold is assigned a state sum involving the $6j$-symbols, an important ingredient in the theory of quantum spin networks. (An additional key feature of the 3-manifold invariants, closely related to the properties of the Jones-Wenzl projectors, is that the "quantum"

* Partially supported by the NSF and by the I.H.E.S.

parameter q has to be specialized to a root of unity in order to get a semisimple topological quantum field theory).

The main goal of this paper is to introduce a categorification of the Jones-Wenzl projectors. The Temperley-Lieb algebra TL_n is an algebra over $\mathbb{Z}[q, q^{-1}]$, additively generated by planar diagrams connecting n points at the top and at the bottom of a rectangle and the multiplication is defined on generators by vertical stacking of such diagrams (see section 2 below for more details). The Jones-Wenzl projector p_n is an idempotent element of TL_n, uniquely characterized by the following two properties: (1) the coefficient of the unit element, corresponding to n vertical strands, in the expression for p_n is 1 and (2) p_n is "killed by turnbacks", that is $p_n D = D p_n = 0$ where D is any planar diagram generator of TL_n other than the unit element.

Note that unlike the Jones polynomial and various other link invariants that have been previously categorified, the coefficients in the expansion of p_n in terms of the generators of TL_n are *rational*, rather than polynomial, functions of q, q^{-1}. This suggests that categorification of the projectors cannot be achieved by chain complexes of finite length.

We use Bar-Natan's formulation of Khovanov's theory: the objects in this category are the Temperley-Lieb diagrams and morphisms are surface cobordisms in 3-space between such diagrams, see [BN05] and section 2 below. In this framework, for each n we construct a chain complex P_n whose graded Euler characteristic is the formal power series corresponding to p_n. For example, the power series for $n = 2$ is

$$p_2 \;=\;)(\;-\; \frac{1}{q+q^{-1}} \;\asymp\; = \;)(\;+\; \sum_{i=1}^{\infty} (-1)^i q^{2i-1} \;\asymp$$

We show that the chain complexes P_n are uniquely characterized up to homotopy by properties analogous to those of the Jones-Wenzl projectors $p_n \in \mathrm{TL}_n$: (1) the identity diagram appears in the chain complex P_n only once, in degree zero and (2) P_n is contractible "under turnbacks", see the definition of a *universal projector* and theorem 3.2 in section 3. It follows from these properties that P_n is a "homotopy idempotent": $P_n \otimes P_n \simeq P_n$. We write down the chain complexes explicitly for $n = 2, 3$, see section 4. The main technical part of the paper is the inductive construction of the chain complex P_n for larger n in section 7, modeled on the Frenkel-Khovanov recursion [FK97] for the Jones-Wenzl projectors. The universality properties satisfied by P_n and the invariance under Reidemeister moves, discussed further below, suggest the naturality of the construction proposed in this paper.

An immediate consequence of our construction is a categorification of quantum spin networks. That is, to a spin network G we associate a chain complex whose graded Euler characteristic is a Laurent series in q corresponding to the quantum evaluation of G. Some interesting phenomena are observed here. In the simplest example the rational homology of the trace of the second projector, $\mathrm{Tr}(P_2)$ has the expected graded Euler characteristic $[3] = q^{-2} + 1 + q^2$, but the homology itself has infinite rank (with extra generators canceling in pairs in the Euler characteristic). Further, there is 2-torsion when the homology is taken with integer coefficients, see 4.3.1. In section 6.3 we formulate a categorified analogue of the $6j$-symbols. It takes the form of an iterated cone construction, giving rise to a "homotopy change of basis" in the category of chain complexes.

Our construction also gives rise to an invariant of tangles, leading to a categorification of the colored Jones polynomial, see section 5. Note that the included computations imply that our work is different from the previously defined categorification of the colored Jones polynomial [Kho05] (see also [BW08]). See 4.1 for further discussion.

We would like to mention that while preparing this manuscript for publication, during the MSRI workshop "Homology Theories of Knots and Links" in March 2010 we learned that an alternative, representation-theoretic, approach to categorifying the Jones-Wenzl projectors has been pursued by Igor Frenkel, Catharina Stroppel and Joshua Sussan [FSS]. In light of the universality properties of our construction (see section 3), it seems reasonable to believe that the two approaches are equivalent, although the methods are quite different. One advantage of working in Khovanov's and Bar-Natan's framework for categorification of the Temperley-Lieb algebra is that our construction of the categorified projectors is explicit and it is readily available for topological applications.

We would like to add that more recently Lev Rozansky [Roz10] has proposed an elegant idea on categorification of the Jones-Wenzl projectors, based on the properties of the infinite torus braid. Our construction is based on the Frenkel-Khovanov recursive formula, however it seems reasonable to believe that the two approaches may be related (and more generally the universality properties satisfied by the projectors imply that the different constructions are homotopy equivalent).

2. The Temperley-Lieb Algebra and the Jones-Wenzl Projectors

This section summarizes the relevant background on definition and categorification of the Temperley-Lieb algebra. Section 2.5 states a version of the Gaussian elimination lemma which will be used throughout the paper.

2.1. The **Temperley-Lieb Algebra** is the unital $\mathbb{Z}[q, q^{-1}]$-algebra of $\mathrm{U}_q\,\mathfrak{su}(2)$-equivariant maps between n-fold tensor powers of the fundamental representation V.

$$\mathrm{TL}_n = \mathrm{Hom}_{\mathrm{U}_q\,\mathfrak{su}(2)}(V^{\otimes n}, V^{\otimes n})$$

There is an explicit presentation given by the standard generators 1 and e_i, $0 < i < n$, satisfying the relations:

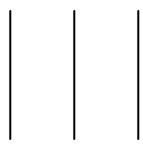
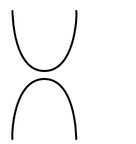
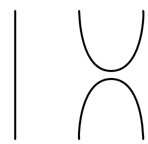
(1) $e_i e_j = e_j e_i$ if $|i - j| \geq 2$.

(2) $e_i e_{i\pm 1} e_i = e_i$
(3) $e_i^2 = -[2]e_i$

where the quantum integer $[n]$ is defined to be

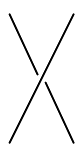
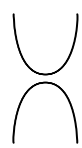
$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-(n-1)} + q^{-(n-3)} + \cdots + q^{n-3} + q^{n-1}$$

Each generator e_i can be pictured as a diagram consisting of n chords between two collections of n points on two horizontal lines in the plane. All strands are vertical except for two, connecting the ith and the $(i+1)$-st points in each collection. For instance, when $n = 3$ we have the following diagrams,

$1 =$  , $e_1 =$  and $e_2 =$ 

The multiplication is given by vertical composition of diagrams and planar isotopy induces relations 1 and 2 between the generators above. The third relation says that any circles which are created may be removed at the cost of multiplication by $-[2] = -q - q^{-1}$.

This algebra is well-known in low-dimensional topology in particular due to its natural extension from planar diagrams to tangles, captured by the Kauffman bracket relation:

 $\mapsto q$ $-$ q^2 

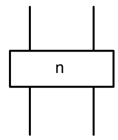
which yields the Jones Polynomial [KL94, Jon97].

2.2. The **Jones-Wenzl Projectors** $p_n \in \mathrm{TL}_n$ are idempotent elements of the Temperley-Lieb algebra which have proven to be fundamental to its study and applications. The projectors appear in the study of spin networks or the graphical calculus of higher $\mathrm{U}_q\,\mathfrak{su}(2)$ representations, the colored Jones polynomial and many constructions of Chern-Simons theory [KL94, TV92, Tur94, BHMV95, BK01, Wal].

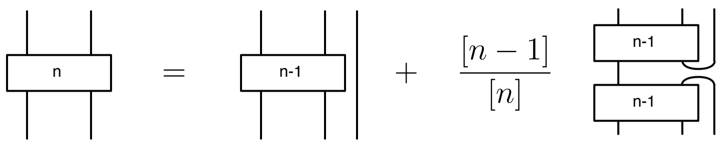
The projectors were originally [Wen87] defined by the recurrence relation,

$$p_1 = 1$$
$$p_n = p_{n-1} + \frac{[n-1]}{[n]} p_{n-1} e_n p_{n-1}$$

If we depict p_n graphically by a box with n incoming and outgoing chords:

$p_n =$ 

then the formula may be illustrated as follows:



It can be shown that the Jones-Wenzl projectors are uniquely characterized by the following properties:

(1) $p_n \in \mathrm{TL}_n$ considered as a $\mathbb{Z}[\![q]\!]$-algebra.
(2) $p_n - 1$ belongs to the subalgebra generated by $\{e_1, e_2, \ldots, e_{n-1}\}$
(3) $e_i p_n = p_n e_i = 0$ for all $i = 1, \ldots, n-1$.

See also [KL94, Lic97].

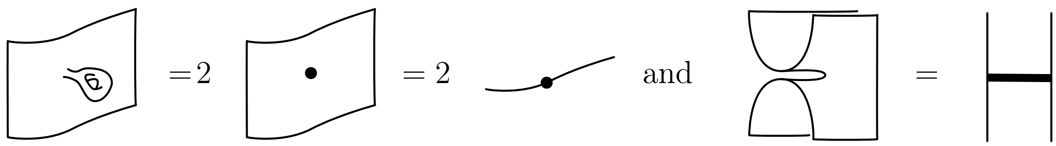
2.3. Categorification of the Temperley-Lieb algebra. Work by a number of authors on the existence of integral bases in Lie group theory led to a categorification of the Temperley-Lieb algebra by Mikhail Khovanov in which integer coefficients were replaced by the dimensions of graded vector spaces and polynomials were replaced by graded Euler characteristics [KMS09, FK97, Kho00]. This construction extends to tangles and there is a corresponding functoriality with respect to cobordisms between these tangles [Jac04, BN05].

In this section we recall Dror Bar-Natan's graphical formulation [BN05] of the Khovanov categorification. It will be used throughout the remainder of this paper. Using the Bar-Natan formulation has the advantage of allowing our constructions to apply to a number of variant categorifications which exist in the literature.

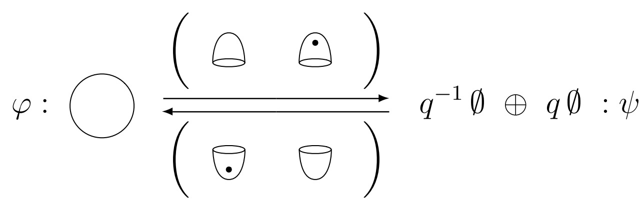
There is an additive category Pre-Cob(n) whose objects are isotopy classes of formally q-graded Temperley-Lieb diagrams between $2n$ boundary points. The morphisms are given by the free $\mathbb{Z}$-module spanned by isotopy classes of orientable cobordisms bounded in $\mathbb{R}^3$ between any two planes containing such diagrams. If $\chi(S)$ is the Euler characteristic of a surface S, then a cobordism $C : q^i A \to q^j B$ has *degree* given by

$$|C| = \chi(C) - n + j - i$$

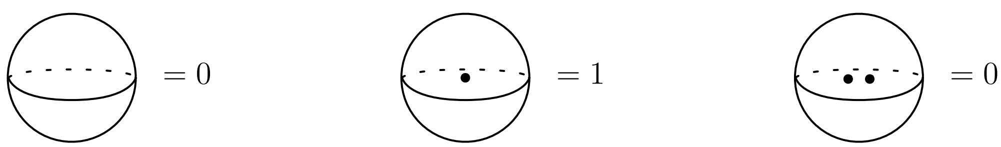
It has become a common notational shorthand to represent a handle by a dot and a saddle by a flattened diagram containing a dark line.

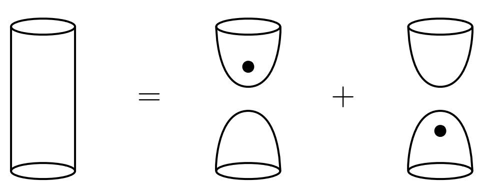


We would like a category $\mathcal{C}$ such that $\mathrm{K}_0(\mathcal{C}) \cong \mathrm{TL}_n$ and the only relation of TL_n which is not given by isotopy, and so not automatically accounted for, is the third. We require that the object represented by a closed circle be isomorphic to sum of two empty objects in degrees ± 1 respectively. If such maps are to be degree preserving then the most natural choice for these maps is given below.



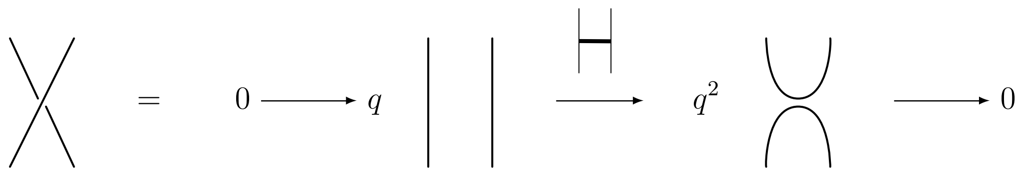
We force $\varphi \circ \psi = 1$ and $\psi \circ \varphi = 1$ by forming a new category $\mathrm{Cob}(n) = \mathrm{Cob}^3_{\cdot/l}(n)$ obtained as a quotient of the category Pre-Cob(n) by the relations given below:





The cylinder or neck cutting relation implies that closed surfaces Σ_g of genus $g > 3$ must evaluate to 0. In what follows we will let Σ_3 be a free variable and absorb it into our base ring. One can think of Σ_3 as a deformation parameter. Some authors use α to denote Σ_3.

In this categorification the skein relation becomes



Definition 2.4. Let $\mathrm{Kom}(n) = \mathrm{Kom}(\mathrm{Mat}(\mathrm{Cob}^3_{\cdot/l}(n)))$ be the category of chain complexes of matrices of objects in $\mathrm{Cob}^3_{\cdot/l}(n)$.

The skein relation allows us to associate to any tangle diagram D with $2n$ boundary points an object in $\mathrm{Kom}(n)$ which is unique up to chain homotopy equivalence.

There is an inclusion $- \sqcup 1^{m-n} : \mathrm{Kom}(n) \to \mathrm{Kom}(m)$ whenever $n \leq m$ obtained by unioning each diagram with $m-n$ disjoint vertical intervals to each object and $m-n$ disjoint disks to each morphism. If $m = n$ then the empty set is used instead of either intervals or disks.

Given two objects $C, D \in \mathrm{Kom}(n)$ we will use $C \otimes D$ to denote the categorified Temperley-Lieb multiplication $\otimes : \mathrm{Kom}(n) \otimes \mathrm{Kom}(n) \to \mathrm{Kom}(n)$ obtained by gluing all diagrams and morphisms along the n boundary points and n boundary intervals respectively.

2.5. Chain Homotopy Lemmas. We will make frequent use of the following standard lemma in this paper, see [BN05, MN08].

Lemma 2.6. *(Gaussian Elimination) Let K_* be a chain complex in an additive category $\mathcal{A}$ containing a summand of the form given below:*

$$A \xrightarrow{\begin{pmatrix} \cdot \\ \alpha \end{pmatrix}} \begin{matrix} B \\ \oplus \\ C \end{matrix} \xrightarrow{\begin{pmatrix} \varphi & \lambda \\ \mu & \eta \end{pmatrix}} \begin{matrix} D \\ \oplus \\ E \end{matrix} \xrightarrow{\begin{pmatrix} \cdot & \epsilon \end{pmatrix}} F$$

Then if $\varphi : B \to D$ is an isomorphism there is a homotopy equivalence from K_* to a smaller complex containing the summand below obtained by removing B and D terms via φ:

$$A \xrightarrow{\alpha} C \xrightarrow{\eta - \mu\varphi^{-1}\lambda} E \xrightarrow{\epsilon} F$$

The following result is a direct generalization which will be very useful in our context.

Lemma 2.7. *(Simultaneous Gaussian Elimination) Let K_* be a chain complex in an additive category $\mathcal{A}$ of the form*

$$K_* = \begin{matrix} A_0 \\ \oplus \\ C_0 \end{matrix} \xrightarrow{M_0} \begin{matrix} A_1 \\ \oplus \\ B_1 \\ \oplus \\ C_1 \end{matrix} \xrightarrow{M_1} \begin{matrix} A_2 \\ \oplus \\ B_2 \\ \oplus \\ C_2 \end{matrix} \xrightarrow{M_2} \begin{matrix} A_3 \\ \oplus \\ B_3 \\ \oplus \\ C_3 \end{matrix} \xrightarrow{M_3} \cdots$$

where

$$M_0 = \begin{pmatrix} a_0 & c_0 \\ d_0 & f_0 \\ g_0 & j_0 \end{pmatrix} \quad \text{and} \quad M_i = \begin{pmatrix} a_i & b_i & c_i \\ d_i & e_i & f_i \\ g_i & h_i & j_i \end{pmatrix} \text{ for all } i > 0$$

If $a_{2i} : A_{2i} \to A_{2i+1}$ and $e_{2i+1} : B_{2i+1} \to B_{2i+2}$ are isomorphisms for $i \geq 0$ then the chain complex K_* is homotopy equivalent to the smaller chain complex D_* obtained by removing all A_i and B_i terms via the isomorphisms a_{2i} and e_{2i+1}:

$$D_* = C_0 \xrightarrow{q_0} C_1 \xrightarrow{q_1} C_2 \xrightarrow{q_2} C_3 \xrightarrow{q_3} \cdots$$

where $q_{2i} = j_{2i} - g_{2i}a_{2i}^{-1}c_{2i}$ and $q_{2i+1} = j_{2i+1} - h_{2i+1}e_{2i+1}^{-1}f_{2i+1}$.

Proof. First apply Gaussian elimination to each isomorphism a_{2i} in order to obtain the chain complex

$$C_0 \xrightarrow{X} \begin{matrix} B_1 \\ \oplus \\ C_1 \end{matrix} \xrightarrow{Y_1} \begin{matrix} B_2 \\ \oplus \\ C_2 \end{matrix} \xrightarrow{Y_2} \begin{matrix} B_3 \\ \oplus \\ C_3 \end{matrix} \xrightarrow{Y_3} \begin{matrix} B_4 \\ \oplus \\ C_4 \end{matrix} \longrightarrow \cdots$$

where $X = \begin{pmatrix} f_0 - c_0a_0^{-1}d_0 \\ j_0 - c_0a_0^{-1}g_0 \end{pmatrix}$ and

$$Y_{2i} = \begin{pmatrix} e_{2i} - d_{2i}a_{2i}^{-1}b_{2i} & f_{2i} - d_{2i}a_{2i}^{-1}c_{2i} \\ h_{2i} - g_{2i}a_{2i}^{-1}b_{2i} & j_{2i} - g_{2i}a_{2i}^{-1}c_{2i} \end{pmatrix} \qquad Y_{2i+1} = \begin{pmatrix} e_{2i+1} & f_{2i+1} \\ h_{2i+1} & j_{2i+1} \end{pmatrix}$$

Now apply Gaussian elimination to each isomorphism e_{2i+1} in order to obtain the chain complex D_* above. $\square$

We will also need the following

Lemma 2.8. *(Big Collapse) A chain complex K_* of contractible chain complexes K_i is contractible.*

3. Universal Projectors and Statement of the Main Theorem

The projectors defined in this paper satisfy a universal property making them unique up to homotopy.

Definition 3.1. A chain complex $(P_*, d_*) \in \mathrm{Kom}(n)$ is a *universal projector* if

(1) It is positively graded with degree zero differential.
   (a) $P_k = 0$ for all $k < 0$ and $\deg_q(P_k) \geq 0$ for all $k > 0$.
   (b) d_k is a matrix of degree zero maps for all $k \in \mathbb{Z}$.
(2) The identity diagram appears only in degree zero and only once.
   (a) $P_0 \cong 1$
   (b) $P_k \ncong 1 \oplus D$ for any $D \in \mathrm{Mat}(\mathrm{Cob}(n))$ for all $k > 0$.
(3) The chain complex P_* is contractible "under turnbacks," that is for any generator $e_i \in \mathrm{TL}_n$, $0 < i < n$,
   (a) $P_* \otimes e_i \simeq 0$
   (b) $e_i \otimes P_* \simeq 0$

Compare these axioms to the axioms in section 2.2 characterizing the Jones-Wenzl projectors $p_n \in \mathrm{TL}_n$. The first two axioms are non-triviality conditions. The first excludes uninteresting variants of the definition obtained by degree shifting and symmetry. For instance, we can require a negative q-grading and reverse all of the arrows in this paper to obtain a different chain complex. The second excludes contractible complexes from consideration. The third axiom implies that composing the projector with any Temperley-Lieb diagram which is not identity yields an object in $\mathrm{Kom}(n)$ which is homotopic to the zero complex.

A more subtle consequence of the first axiom above is that together with the constraints on degrees of maps imposed by the categorification in section 2.3 it follows that all chain complexes defined below must be monotonic in the q-degree and that the naive graded Euler characteristic of objects in our construction will always yield formal power series. It is in this sense that the Jones-Wenzl projectors and spin networks are images of our constructions under the Grothendieck group.

Our construction is essentially universal across reasonable categorifications of the Temperly-Lieb algebra. If a categorification contains a Frobenius algebra compatible with the one given in section 2.3 then our construction of the universal projectors can be performed. For a more detailed discussion see [Kho06].

It is important to note that our definition *disagrees* with some previous categorifications based on different axiomizations of the Jones-Wenzl projectors such as the dimension axiom [Kho05] (for related work see [BW08, GW]):

$$\mathrm{Tr}(p_n) = [n+1]$$

This is implied by the homotopy uniqueness corollary below and the computation of $\mathrm{H}_*(\mathrm{Tr}(P_2))$ contained in the next section.

We can now state the main theorem of the paper.

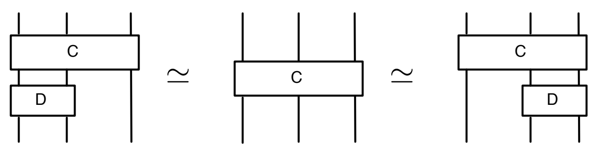
Theorem 3.2. *For each $n > 0$, there exists a chain complex $C \in \mathrm{Kom}(n)$ that is a universal projector.*

We summarize some immediate consequences of the axioms in definition 3.1. See also sections 5 and 6.

Proposition 3.3. *If $C \in \mathrm{Kom}(n)$ is a universal projector and $D \in \mathrm{Kom}(m)$ is a universal projector such that $0 \leq m \leq n$ then*

$$C \otimes (D \sqcup 1^{n-m}) \simeq C \simeq (D \sqcup 1^{n-m}) \otimes C$$

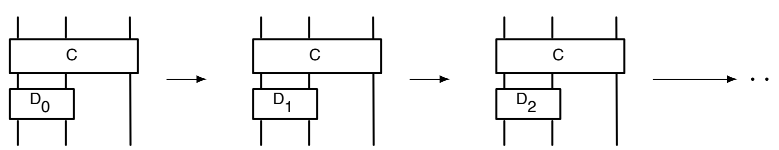
Pictorially,



Proof. The tensor product of chain complexes $C_* \otimes (D_* \sqcup 1^{n-m})$ is the total complex of a bicomplex which can be written as a chain complex of chain complexes:

$$C_* \otimes (D_0 \sqcup 1^{n-m}) \to C_* \otimes (D_1 \sqcup 1^{n-m}) \to C_* \otimes (D_2 \sqcup 1^{n-m}) \to \cdots$$

Or graphically,



By the second axiom $D_0 = 1$ in degree 0 and so the identity diagram cannot be found as a summand of $D_k \sqcup 1^{n-m}$ for any $k > 1$. In addition, C satisfies axiom 3 so it follows that:

(1) The degree 0 portion of this chain complex is isomorphic to C_*
(2) All chain complexes in degree above zero are contractible.

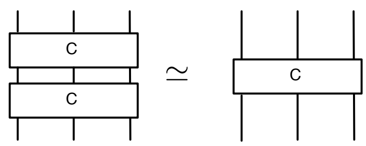
Lemma 2.8 (big collapse) implies that there is a homotopy equivalence $C_* \simeq C_* \otimes (D_* \sqcup 1^{n-m})$. The other equivalence $(D_* \sqcup 1^{n-m}) \otimes C_* \simeq C_*$ is proven in the same manner. $\square$

A special case of the above proposition with $m = n$ implies that universal projectors we have defined behave like idempotent elements[1].

Corollary 3.4. *(Idempotency) If $C \in \mathrm{Kom}(n)$ is a universal projector then*

$$C \otimes C \simeq C$$

This is represented diagrammatically as



The proposition above also implies that the universal projectors are unique up to homotopy.

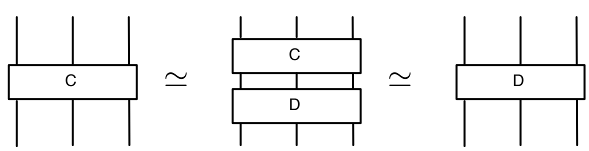
Corollary 3.5. *(Homotopy Uniqueness) If $C, D \in \mathrm{Kom}(n)$ are universal projectors then*

$$C \simeq D$$

Proof. Let $C, D \in \mathrm{Kom}(n)$ be universal projectors. The previous proposition holds when $n = m$ so that $1^0 = \emptyset$,

$$C \simeq C \otimes (D \sqcup \emptyset) \cong (C \sqcup \emptyset) \otimes D \simeq D$$

In pictures,



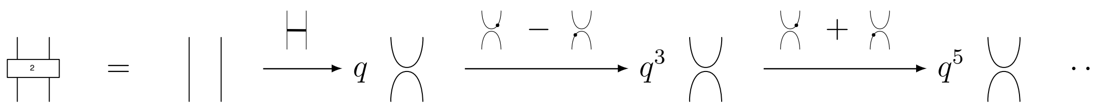
□

4. Explicit Formulae and Computations

We now give some explicit examples of lower order projectors. The second projector below will play a role in the proof of the main theorem.

[1]Technically, if C is a universal projector then the endofunctors $C \otimes -$ and $- \otimes C$ are idempotent on the homotopy category of $\mathrm{Kom}(n)$

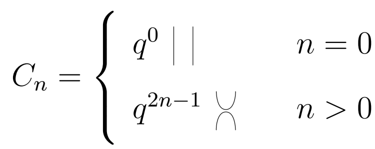
4.1. The Second Projector. The second projector is defined to be the chain complex



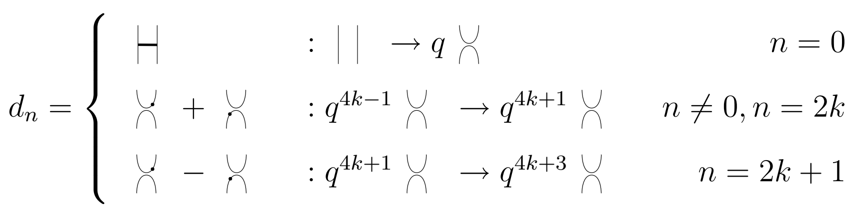
in which the last two maps alternate ad infinitum. More explicitly,

$$P_2 = (C_*, d_*)$$

The chain groups are given by

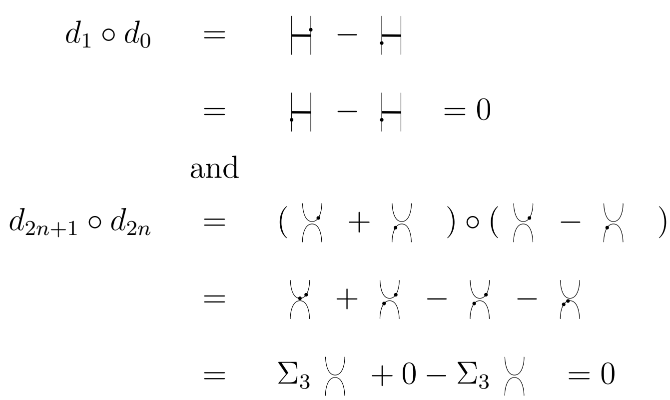


The differential is given by



Proposition 4.2. *P_2 defined above is a chain complex, that is successive compositions of the differential are equal to zero.*

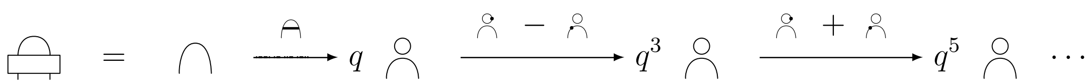
Proof. Since $d_{2n+1} \circ d_{2n} = d_{2n} \circ d_{2n-1}$ there are only two cases,



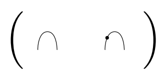
□

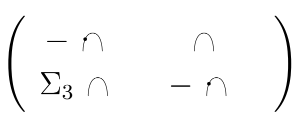
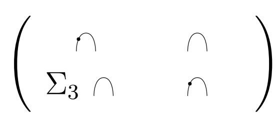
Theorem 4.3. *The chain complex $P_2 \in \mathrm{Kom}(2)$ defined above is a universal projector.*

Proof. Since the identity object only appears in degree 0 and the chain complex is positively graded with degree zero differentials, axioms 1 and 2 are satisfied by definition. For axiom 3, note that there is only one standard generator $e_1 \in \mathrm{TL}_2$ and the vertical symmetry in the definition of P_2 implies $P_2 \otimes e_1 \cong e_1 \otimes P_2$. Consider $e_1 \otimes P_2$:



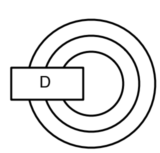
We "deloop" and conjugate our differentials by the isomorphism φ in section 2.3 to obtain the isomorphic complex

where $A =$ ,

$B =$  $C =$ 

Applying lemma 2.7 (simultaneous Gaussian elimination) by using the identity map in the first component of the first map and the identity in the upper righthand component of each successive matrix shows that the complex is homotopic to the zero complex. ∎

4.3.1. *Homology of the Trace.* In the Temperley-Lieb algebra the trace of any diagram $D \in \mathrm{TL}_n$ is defined to be the element $\mathrm{Tr}(D) \in \mathbb{Z}[q, q^{-1}]$ associated with the diagram obtained by connecting each of the bottom boundary points to the corresponding top points by parallel arcs in the plane:

$\mathrm{Tr}(D) =$ 

The Jones-Wenzl projectors, $p_n \in \mathrm{TL}_n$ are commonly known to satisfy

$$\mathrm{Tr}(p_n) = [n+1]$$

In fact, they can be characterized by this property together with the turnback axiom 3 of definition 3.1. One would expect then that the graded Euler characteristic of the complex given by the trace of the universal projectors defined in this document are given by the polynomials $[n+1]$.

This is true when the coefficient ring is rational and the surface $\Sigma_3 = 0$. It is however *not true* that the homology of $\mathrm{Tr}(P_2)$ is spanned *only* by classes that correspond to coefficients of the graded Euler characteristic; the homology contains infinitely many terms which cancel in the graded Euler characteristic.

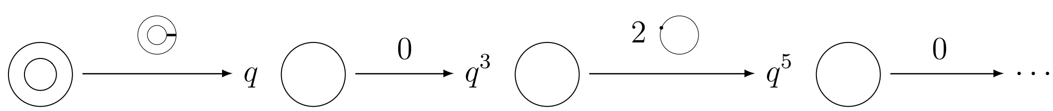
When $\Sigma_3 = 0$ and coefficients are rational,

$$\mathrm{H}_n(\mathrm{Tr}(P_2);\mathbb{Q}) = \begin{cases} q^{-2}\mathbb{Q} \oplus q^0\mathbb{Q} & n = 0 \\ 0 & n = 1 \\ q^{4k-2}\mathbb{Q} & n = 2k \text{ and } k > 0 \\ q^{4k+2}\mathbb{Q} & n = 2k+1 \text{ and } k > 0 \end{cases}$$

Note that the graded Euler characteristic equals $[3] = q^{-2} + 1 + q^2$. All other terms cancel in pairs.

If $\Sigma_3 = 0$ and the coefficient ring is integral then there is an additional infinite family of 2-torsion. If $\Sigma_3 \neq 0$ and the coefficient ring is $\mathbb{Q}$ then the homology of P_2 is 2-dimensional and isomorphic for any choice of Σ_3. If $\Sigma_3 \in \mathbb{Z}_+$ and the coefficient ring is $\mathbb{Z}$ then there is an infinite family of 2-torsion *and* an infinite family of $2\Sigma_3$-torsion. In particular, the homotopy type of the projectors is not constant with respect to the deformation parameter Σ_3.

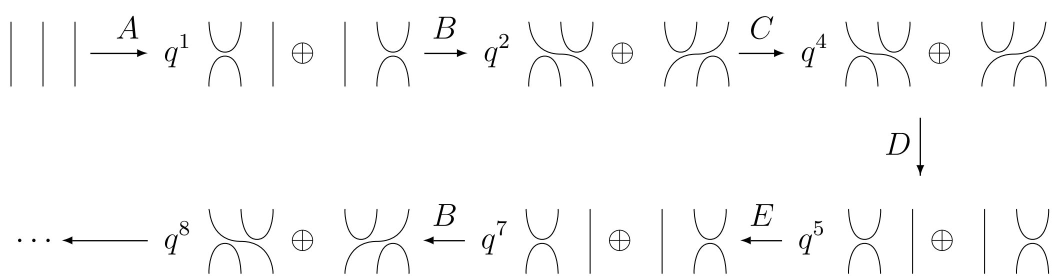
Taking the trace of our projector yields a complex with alternating differential:

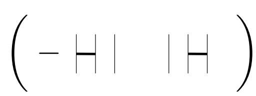


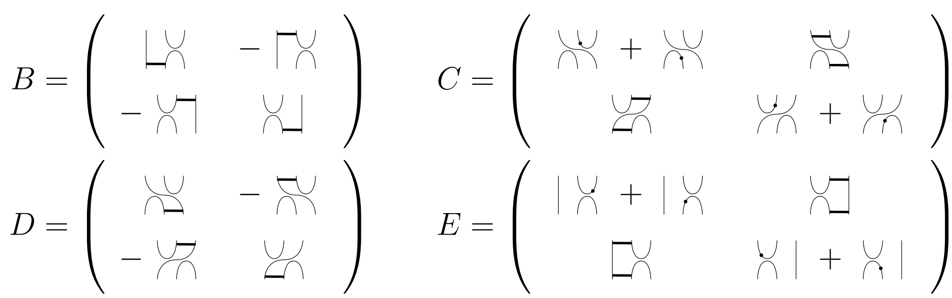
The homology of this complex is in general given by

$$\mathrm{H}_n(\mathrm{Tr}(P_2)) = \begin{cases} q^{-2}\mathbb{Z} \oplus q^0\mathbb{Z} & n = 0, \Sigma_3 = 0 \text{ or } \Sigma_3 \neq 0 \\ 0 & n = 1, \Sigma_3 = 0 \text{ or } \Sigma_3 \neq 0 \\ q^{4k-2}\mathbb{Z} & n = 2k, \Sigma_3 = 0 \\ q^{4k+2}\mathbb{Z} \oplus q^{4k}\mathbb{Z}/2 & n = 2k+1, \Sigma_3 = 0 \\ 0 & n = 2k, \Sigma_3 \neq 0 \\ q^{4k+2}\mathbb{Z}/(2\Sigma_3) \oplus q^{4k}\mathbb{Z}/2 & n = 2k+1, \Sigma_3 \neq 0 \end{cases}$$

4.4. The Third Projector. We give an inductive definition of the chain complex for the nth projector in section 7 below, with the second projector defined above as the base of the induction. The third projector P_3 can therefore be deduced from that inductive definition. In this section we present a minimal (in the sense that it cannot be reduced by a chain homotopy) chain complex for P_3. The third projector is the last which can be written down in a short and explicit diagrammatic form. After the initial identity term the complex below becomes 4 periodic.



Where $A =$  and



Theorem 4.5. *The definition of P_3 given above is a chain complex that satisfies the axioms of the universal projector. In particular,*

$$e_i \otimes P_3 \simeq 0 \simeq P_3 \otimes e_i \qquad\qquad i = 1, 2.$$

The proof is analogous to the proof of theorem 4.3. The main theorem also produces a universal projector P_n for $n = 3$. We give applications of our construction of the projectors to tangles and spin networks in sections 5, 6, postponing the proof of the main theorem to section 7.

5. Reidemeister Moves and Graphical Calculus

In this section we define homotopy invariants of tangles obtained by applying the mth projector to the strands of a cabling and showing that the result is Reidemeister invariant. These are categorifications of the invariants of higher representations of $\mathrm{U}_q\,\mathfrak{su}(2)$ corresponding to the colored Jones polynomial.

Definition 5.1. Given $m \in \mathbb{N}$ consider the homotopy category $\mathrm{Kom}_m(n)$ with objects

$$\mathrm{Ob}(\mathrm{Kom}_m(n)) = \mathrm{Ob}(\mathrm{Kom}(n))$$

To any object $D \in \mathrm{Ob}(\mathrm{Kom}(n))$ associate a chain complex $F(D)$ in the category $\mathrm{Kom}(mn)$ by replacing each strand in each diagram with m parallel strands composed with the mth projector.

If A and $B \in \mathrm{Kom}_m(n)$ are two objects then we define

$$\mathrm{Hom}_{\mathrm{Kom}_m(n)}(A, B) = \mathrm{Hom}_{\mathrm{Kom}(mn)}(F(A), F(B))$$

This can be illustrated by

We use the word homotopy category above because in order to define the composition of two cobordisms between edges the relation $P_n \otimes P_n \simeq P_n$ may need to be invoked.

In the remainder of this section we wish to prove that the Reidemeister moves and some standard graphical relations are satisfied up to homotopy.

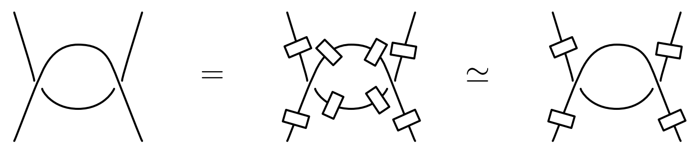
Lemma 5.2. *(Projector Isotopy) A free strand can be moved over or under a projector up to homotopy. In pictures,*

Proof. The proof is similar to the proof of proposition 3.3 and corollary 3.5 in section 3. Specifically, observe that both the chain complex for the diagram with the projector below the strand and the chain complex for the diagram with the projector above the strand are chain homotopy equivalent to the chain complex C for the diagram with *two* projectors: one above the strand and one below the strand. This is true because expanding either of the two projectors in C gives the identity diagram in degree zero and every other term involves a turnback, which is contractible when combined with the second copy of the projector. $\square$

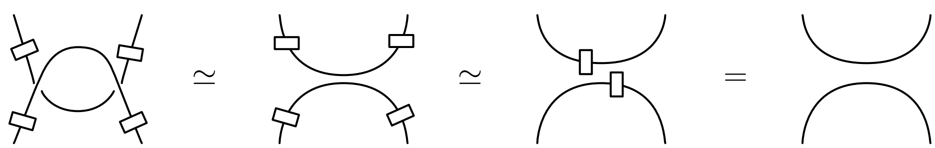
Applications of this lemma allow us to show that the Reidemeister moves are satisfied by the category $\mathrm{Kom}_m(n)$.

Theorem 5.3. *The category* $\mathrm{Kom}_m(n)$ *contains invariants of tangles.*

Proof. For the second Reidemeister move,

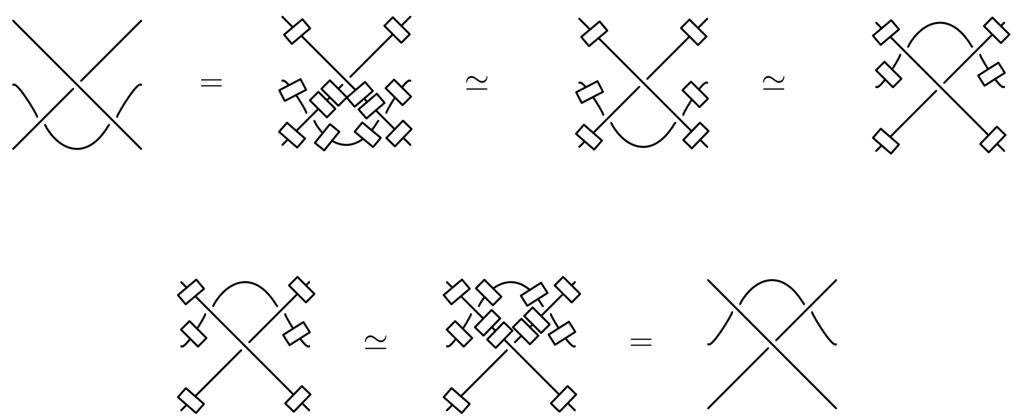


The first equality is by definition. The homotopy equivalence follows from the projector isotopy lemma and $P_n \otimes P_n \simeq P_n$.



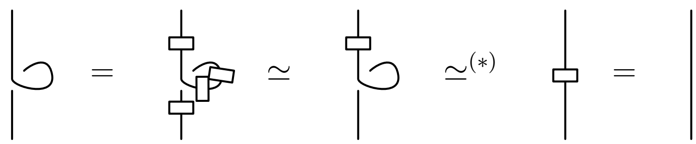
The first homotopy equivalence follows from the second Reidemeister move in $\mathrm{Kom}(mn)$, the second follows from another application of $P_n \otimes P_n \simeq P_n$.

The argument for the third Reidemeister move features the same ideas.



Applying the definition to the standard Reidemeister 3 diagram we obtain a diagram that looks like spaghetti which simplifies considerably up to homotopy to a diagram in which the standard Reidemeister 3 homotopy in $\mathrm{Kom}(mn)$ holds.

The first Reidemeister move is saved for last,



Where the applications of the first Reidemeister move in (*) above hold on the nose after appropriate degree shifting. If L is a tangle with no open strands then the chain complex $L \in \mathrm{Kom}_n(0)$ is a categorification of the nth colored Jones polynomial.

$\square$

6. Spin Networks

In this section we describe how to associate to any spin network a chain complex in a category defined using the universal projector of section 3. Constructions involving four projectors are then explored more thoroughly leading to a categorification of the 6j symbols.

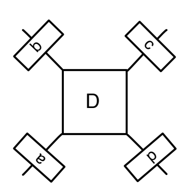
6.1. Categories and Invariants. Let $I = \mathbb{N}$ be the set indexing the finite dimensional irreducible representations of $\mathrm{U}_q\,\mathfrak{su}(2)$. For any n-tuple $\mathbf{t} = (i_1, \ldots, i_n) \in I^n$ define the invariants of the n-fold tensor product by

$$\mathrm{Inv}(\mathbf{t}) = \mathrm{Inv}(V_{i_1} \otimes \cdots \otimes V_{i_n}) = \mathrm{Hom}_{\mathrm{U}_q\,\mathfrak{su}(2)}(V_{i_1} \otimes \cdots \otimes V_{i_n}, 1)$$

This space is described by Temperly-Lieb diagrams with boundary labeled by Jones-Wenzl projectors: $p_{i_1} \sqcup p_{i_2} \sqcup \cdots \sqcup p_{i_n}$ [KL94, Kup96].

For any such $\mathbf{t} \in I^n$ the main theorem allows us to construct a category $\mathrm{Kom}(\mathbf{t})$ with objects given by chain complexes obtained from Temperly-Lieb diagrams with boundary labeled by universal projectors and morphisms given by chain maps. When $\mathbf{t} = (a, b, c, d)$ there is an associated picture,

Ob(Kom(**t**)) = {

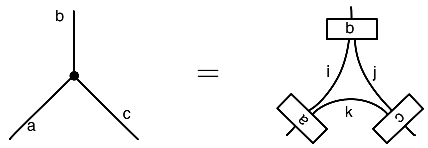


: D is a Temperly-Lieb diagram }

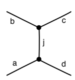
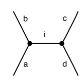
The axiomatic correspondence between the Jones-Wenzl projector and the universal projectors in this paper implies the following theorem.

Theorem 6.2. *The category* $\mathrm{Kom}(\mathbf{t})$ *categorifies the invariants* $\mathrm{Inv}(\mathbf{t})$*.*

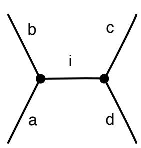
6.3. 6j Symbols. There is a standard way to resolve a trivalent vertex with edges labeled by $a, b, c \in I$,

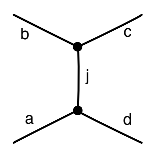


Where $i = (a+b-c)/2,\ j = (a+c-b)/2,\ k = (b+c-a)/2$. We say that a diagram is *admissible* if all trivalent vertices can be resolved using the assigned labels or equivalently $a+b+c$ is even and the triangle inequalites hold for a, b and c. Using this notation we can describe two bases for $\mathrm{Inv}(a,b,c,d)$,

$V = \Bigg\{$  $:$ $j \in I$ admissible $\Bigg\}$ and $H = \Bigg\{$  $:$ $i \in I$ admissible $\Bigg\}$

The base change coefficients are called *6j symbols*,



$$= \sum_j \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\}$$


which determine the change of basis map $S : H \to V$. S is the matrix of $6j$ symbols,

$$S_{ij} = \left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\}$$

Outline. Our goal is to categorify S as a functor $\mathrm{Kom}(a,b,c,d) \to \mathrm{Kom}(a,b,c,d)$. Our construction is modeled on the linear-algebraic proof ([KL94] chapter 7.2) that the "vertical" and "horizontal" collections V, H above are indeed bases for the space of Temperley-Lieb diagrams $\mathrm{Inv}(a,b,c,d)$, pictured on the previous page. The key point is that the identity diagram appears only once in the chain complex for the projector P_n, in degree zero (axiom (1) in definition 3.1 of the universal projector). Therefore, the identity diagram may be represented up to homotopy as the cone of the inclusion of the positive degree part into the chain complex P_n. However the positive degree part may in turn be inductively represented as an iterated cone on lower order projectors. This is made precise in the proof of theorem 6.5 below. We begin by introducing a categorical analogue of a linear basis.

Before proceeding we recall a number of definitions. The concept we wish to capture is that of a category that is homotopy equivalent to some subcategory that sits inside of it. In our case this amounts to a category of complexes in which every chain complex is homotopy equivalent to a chain complex of chain complexes contained within the subcategory of interest.

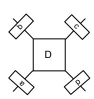
A subcategory $\mathcal{C} \subset \mathcal{D}$ is *full* if for all pairs of objects $A, B \in \mathrm{Ob}(\mathcal{C})$,

$$\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{D}}(A, B)$$

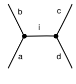
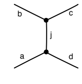
A category $\mathcal{C}$ is *differential graded* if for all objects $A, B \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a chain complex. Two functors $F, G : \mathcal{C} \to \mathcal{D}$ between differential graded categories are homotopic, $F \simeq G$ if there are natural transformations $\varphi : F \to G$ such that for φ_A is a homotopy equivalence for all $A \in \mathrm{Ob}(\mathcal{C})$. Two differential graded categories $\mathcal{C}$ and $\mathcal{D}$ are *homotopy equivalent* if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG \simeq 1_{\mathcal{D}}$ and $GF \simeq 1_{\mathcal{C}}$.

Definition 6.4. If $\mathcal{A}$ is an additive category and $\mathcal{C} = \mathrm{Kom}(\mathcal{A})$ is the category of chain complexes of formal direct sums of objects in $\mathcal{A}$ then a full subcategory $\mathcal{B} \subset \mathcal{C}$ *spans* $\mathcal{C}$ if the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is a homotopy equivalence of categories.

Since projectors with turnbacks are contractible and any disjoint circles can be removed by isomorphisms, $\mathrm{Kom}(a, b, c, d)$ is naturally spanned by the full subcategory $\mathcal{N}$:

$\mathrm{Ob}(\mathcal{N}) = \Bigg\{$  : D is a TL diagram which contains no disjoint circles and no projector is capped by a turnback $\Bigg\}$

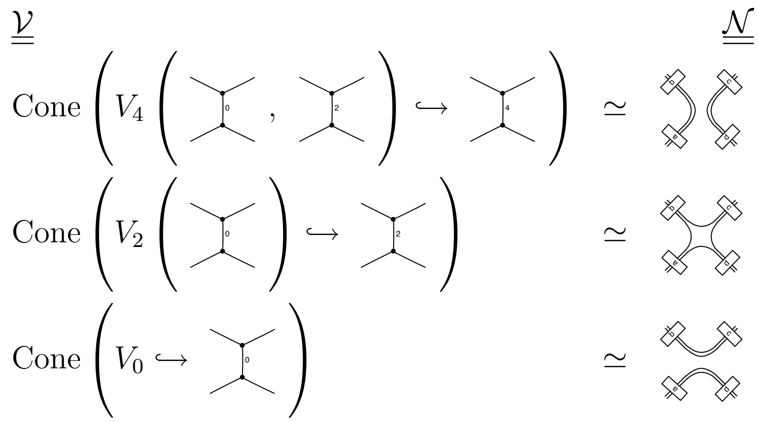
There are two other categories we would like to consider: $\mathcal{H}$ and $\mathcal{V}$. These are the full subcategories of $\mathrm{Kom}(a, b, c, d)$ with objects given by horizontal and vertical diagrams respectively.

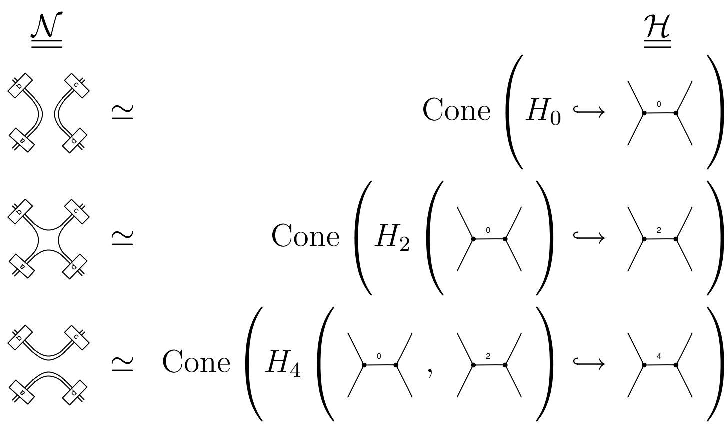
$\mathrm{Ob}(\mathcal{H}) = \Bigg\{$  : $i \in I$ admissible $\Bigg\}$ and $\mathrm{Ob}(\mathcal{V}) = \Bigg\{$  : $j \in I$ admissible $\Bigg\}$

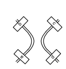
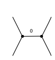
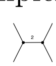
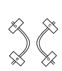
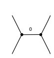
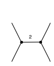
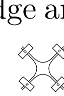
We can now state our theorem,

Theorem 6.5. *For any $a, b, c, d \in I$ the full subcategories $\mathcal{H}$ and $\mathcal{V}$ defined above span the category* $\mathrm{Kom}(a, b, c, d)$.

The proof consists of constructing a family of chain complexes V_n and H_n each of which comes from the positive degree part of the chain complex defining the nth universal projector P_n constructed above. The gist of the proof is captured by the two tables below in which $(a, b, c, d) = (2, 2, 2, 2)$.



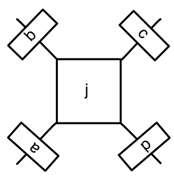


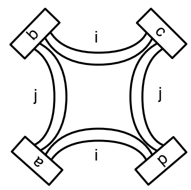
In the second table the first chain complex  can be constructed as cone on the horizontal spin network  . When $(a, b, c, d) = (2, 2, 2, 2)$, $H_0 = 0$ and the spin network itself is equal to the first chain complex in $\mathcal{N}$. In the next line, H_2 is obtained from expanding the central projector of  and then carefully substituting instances of  with Cone($H_0 \hookrightarrow$ ). The chain complex H_2 consists only of chain complexes using the $(2, 2, 2, 2)$ network with the 0 labeled edge and contractible terms. The cone on the inclusion of H_2 into  is homotopic to . The last line states that the last object in $\mathcal{N}$ is homotopic to the cone on a third chain complex

H_4 consisting only of horizontal networks labeled 0 and 2. We now give a proof of theorem 6.5.

Proof. We will explain the construction only for $\mathcal{V}$ since the argument is the same for $\mathcal{H}$.

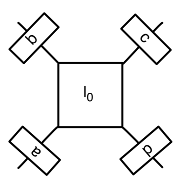
For fixed $a, b, c, d \in I$ each diagram D defining an object in the spanning subcategory $\mathcal{N}$ defined above must be of the form

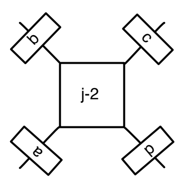
$D_j =$ 

$:=$ 

where j is the number of vertical strands and i is the number of horizontal strands. Notice that i depends on j. In order for the diagram to be admissible the number j assumes either odd or even integer values between two non-negative integers l_0 and l_N.

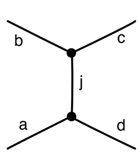
For each $a, b, c, d \in I$ if j is admissible then we define the chain complex V_j,

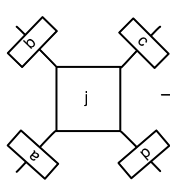
$V_j = V_j\Big($ 

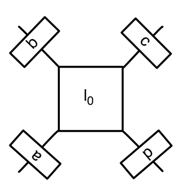
$, \cdots ,$ 

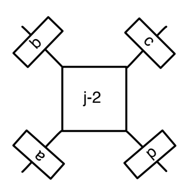
$\Big)$

to be the tail of the chain complex obtained by expanding the central projector P_j in the a, b, c, d, j labeled diagram:



$=$ 

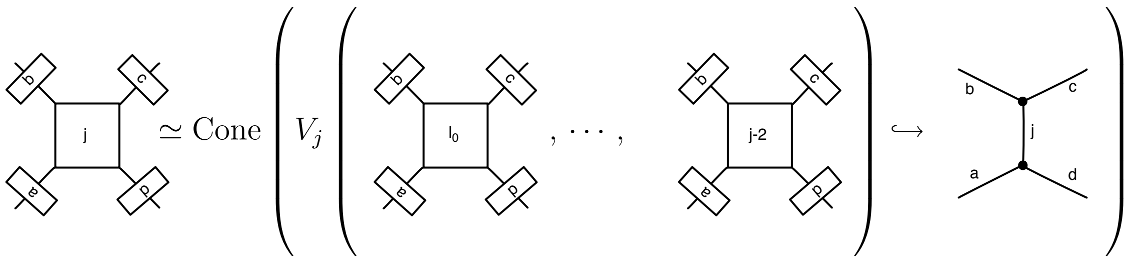
$\rightarrow V_j\Big($ 

$, \cdots ,$ 

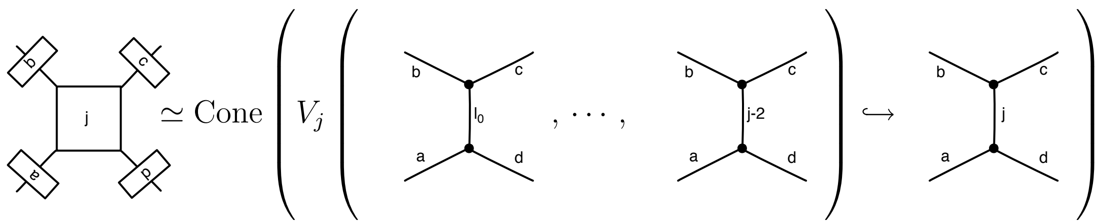
$\Big) \quad (*)$

As our notation suggests V_j is a chain complex containing only contractible terms and objects of $\mathcal{N}$ defined by diagrams $D_{l_0}, \ldots, D_{j-2}$.

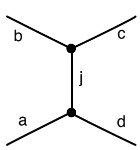
The preceding diagram implies that the homotopy equivalence below is tautological:



We now seek to construct the equivalence below this line from the equivalence above this line by substitution of (∗) two lines above.



The proof that substitution works is an application of the change of basis isomorphism used in the Gaussian elimination lemma in section 2.5 to the first vertical identity map below.



$$= \mathrm{Cone}\left(\begin{tikzcd} D_j \arrow[r, "d_1"] & C_1 \arrow[r, "d_2"] & C_2 \arrow[r] & \cdots \\ 0 \arrow[u] \arrow[r] & C_1 \arrow[u, "1"'] \arrow[r, "d_2"] & C_2 \arrow[u, "1"'] \arrow[r] & \cdots \end{tikzcd}\right)$$

$$\cong \mathrm{Cone}\left(\begin{tikzcd} D_j \arrow[r, "0"] & C_1 \arrow[r, "d_2"] & C_2 \arrow[r] & \cdots \\ 0 \arrow[u] \arrow[r] & C_1 \arrow[u, "1"'] \arrow[r, "0"] & C_2 \arrow[u, "1"'] \arrow[r] & \cdots \end{tikzcd}\right)$$

Note that in the base case, expanding the projector p_{l_0} yields only contractible terms in degree greater than zero. The a, b, c, d, l_0 labeled network is homotopy equivalent to the cone on a nullhomotopic map of the form above.

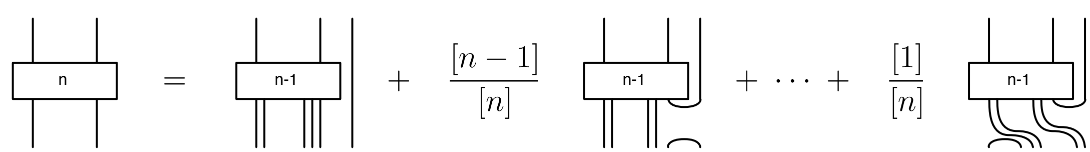
□

Corollary 6.6. *There is a naturally defined homotopy equivalence of categories* $\mathcal{S} : \mathcal{H} \to \mathcal{V}$ *which categorifies the matrix of 6j symbols* $S : H \to V$ *defined in section 6.3.*

The quantum reader is invited to prove the homotopy Biedenharn-Elliot identity.

7. Proof of the Main Theorem

The two term recurrence relation satisfied by the Jones-Wenzl projectors in section 2.2 is quadratic in the sense that in order to define p_n the $n-1$st projector p_{n-1} appears twice in the second term. One obtains the linear recurrence of Frenkel and Khovanov [FK97] by expanding the bottom p_{n-1} term completely and removing terms containing a turnback $p_{n-1}e_i$ for any $0 < i < n-1$. Keeping track of the coefficients in this process gives the recurrence



This can be shown to satisfy the axioms (1)-(3) in section 2.2 and so is equal to the Jones-Wenzl projector. In this section we prove the main theorem of the paper by constructing a chain complex in the category $\mathrm{Kom}(n)$, motivated by the Frenkel-Khovanov recursive formula above, satisfying the axioms of the universal projector given in section 3.

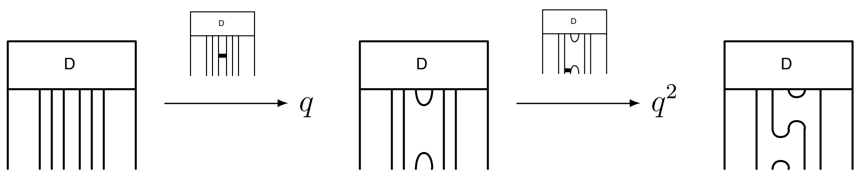
7.1. Triples and Quadruples. We will begin by examining some situations in which local cancelations can be made in a chain complex containing a turnback: $C_* \otimes e_i$. There are two important cases: either a sequence of three terms can be canceled after delooping the middle term or a sequence of four terms can be canceled after delooping the two middle terms. We will call the first case a *triple* and the second a *quadruple*. Both cases are necessary to prove $P_3 \otimes e_i \simeq 0$ and, as we will see, they suffice to prove the general case.

7.1.1. *The Triple.*

Definition 7.2. If $D \in \mathrm{Kom}(n)$ is any chain complex and e_i is a standard generator of TL_n then an *i-triple* or *triple* is a sequence of maps in $\mathrm{Kom}(n)$ of the form

$$D \longrightarrow qD \otimes e_i \longrightarrow q^2 D \otimes e_i \otimes e_{i\pm 1}$$

Where the maps are given by saddles. Pictorially,



Applying the functor $- \otimes e_i$ to the above yields the top sequence in the following commutative diagram:

$$\begin{tikzcd}
D \otimes e_i \arrow[r] \arrow[d, "\cong"'] & qD \otimes e_i \otimes e_i \arrow[r] \arrow[d, "\cong"'] & q^2 D \otimes e_i \otimes e_{i\pm 1} \otimes e_i \arrow[d, "\cong"'] \\
D \otimes e_i \arrow[r, "1 \oplus -"] & (D \otimes e_i) \oplus q^2 (D \otimes e_i) \arrow[r, "- \oplus 1"] & q^2 D \otimes e_i
\end{tikzcd}$$

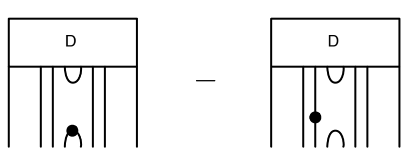
After applying $- \otimes e_i$ to an i-triple, the middle term can be "delooped" and the last term satisfies the categorified planar isotopy relation 2 of section 2, yielding the bottom sequence in the diagram above. Note that if this triple is part of a chain complex, then it can be canceled using two applications of the Gaussian elimination (lemma 2.6 in section 2.5) or by a single application of the simultaneous Gaussian Elimination (lemma 2).

7.2.1. *The Quadruple.*

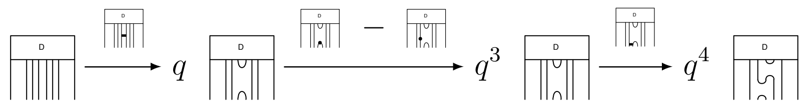
Definition 7.3. If $D \in \mathrm{Kom}(n)$ is any chain complex and e_i an elementary generator of TL_n then an *i-quadruple* or *quadruple* is a sequence of maps in $\mathrm{Kom}(n)$ of the form

$$D \longrightarrow qD \otimes e_i \xrightarrow{A} q^3 D \otimes e_i \longrightarrow q^4 D \otimes e_i \otimes e_{i\pm 1}$$

All maps are given by saddles except for the q-degree 2 map A,

$A =$ 

A is given by subtracting the addition of a handle on one strand from the addition of a handle on an adjacent strand. Although this will not affect the arguments below, we will fix the convention that the dot is placed either to the left or the right of the saddle in the second term, depending upon whether e_{i+1} or e_{i-1} saddle is used at the end of the quadruple.The entire sequence of maps can be pictured as



Applying the functor $-\otimes e_i$ yields the top row of the following diagram:

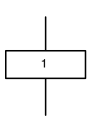
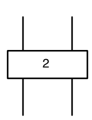
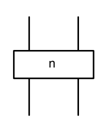
$$\begin{tikzcd}
D\otimes e_i \arrow[r] \arrow[d, "\cong"'] & qD\otimes e_i\otimes e_i \arrow[r, "A"] \arrow[d, "\cong"'] & q^3 D\otimes e_i\otimes e_i \arrow[r] \arrow[d, "\cong"'] & q^4 D\otimes e_i\otimes e_{i\pm 1}\otimes e_i \arrow[d, "\cong"'] \\
D\otimes e_i \arrow[r, "1\oplus -"] & (D\otimes e_i)\oplus q^2(D\otimes e_i) \arrow[r, "B"] & q^2(D\otimes e_i)\oplus q^4(D\otimes e_i) \arrow[r, "-\oplus 1"] & q^4 D\otimes e_i
\end{tikzcd}$$

in which B is of the form[2],

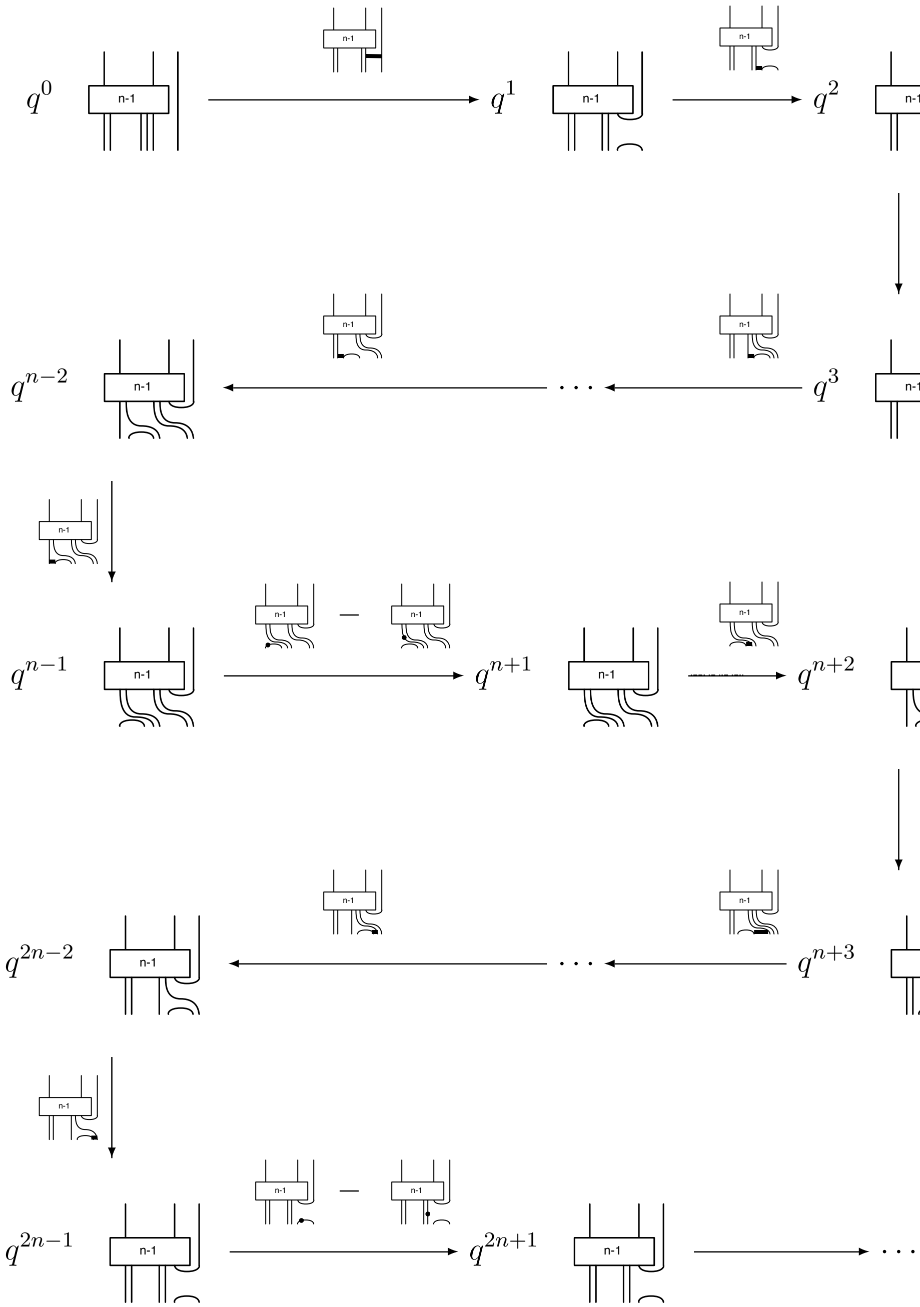
$$B = \begin{pmatrix} - & 1 \\ - & - \end{pmatrix}$$

Again note that if this quadruple is part of a chain complex, then it can be canceled using three applications of the Gaussian elimination (or a single application of the simultaneous Gaussian Elimination).

7.4. The Frenkel-Khovanov Sequence. The Frenkel-Khovanov formula from the beginning of this section suggests the following recursive definition

 $=$,  is defined in section 4.1 and for $n > 2$  $=$

[2]Compare to delooping in Section 4.1



Although the proposition below implies that this definition behaves "correctly" with respect to the turnback axiom (3) of the universal projector (see definition 3.1 in section 3), the composition of two saddles is not equal to zero[3]. The technical heart of this paper consists of a detour taken purely for the purpose of arriving at an

[3]Although it is homotopic to zero.

actual chain complex. The final formulation obtained in section 7.10 will amount to a version of the above which has been carefully thickened by contractible summands. The reader is encouraged to check that the graded Euler characteristic of the sequence illustrated above is a formal power series corresponding to the Frenkel-Khovanov recursion formula for $p_n \in \mathrm{TL}_n$.

In order to formalize the definition above consider the category $\mathbb{N}$ determined by the graph

$$0 \xrightarrow{d^0} 1 \xrightarrow{d^1} 2 \xrightarrow{d^2} \cdots$$

The objects of $\mathbb{N}$ are non-negative integers and the morphisms are generated by compositions of identity maps $1_i : i \to i$ and differentials $d^i : i \to i+1$.

Definition 7.5. A *sequence* F in $\mathrm{Kom}(n)$ is a commutative diagram determined by a functor:

$$F : \mathbb{N} \to \mathrm{Kom}(n)$$

For each $n \geq 1$, we will define a sequence $\mathrm{FK}_n : \mathbb{N} \to \mathrm{Kom}(n)$. $\mathrm{FK}_n(k)$ will correspond to the bottom of each diagram in the illustration on the previous page[4]. After the initial identity diagram, FK_n is $2(n-1)$ periodic. The following is an algebraic definition of the diagrams pictured above.

Definition 7.6. If $m \in \mathbb{Z}_+$ write $m = 2(n-1)q + r$ with $0 < r \leq 2(n-1)$ then the mth diagram of the nth *Frenkel-Khovanov sequence* is defined by

$$\mathrm{FK}_n(m) = \begin{cases} 1 & \text{if } m = 0 \\ q^m e_{n-1} \otimes \ldots \otimes e_{n-m} & \text{if } 1 \leq m < n \\ q^{2(m-n+1)}\, \mathrm{FK}_n(2n-m-1) & \text{if } n \leq m \leq 2(n-1) \\ q^{2n}\, \mathrm{FK}_n(r) & \text{otherwise} \end{cases}$$

For a given length $l \geq 0$ the nth *truncated Frenkel-Khovanov sequence* is given by

$$\mathrm{FK}_{n,l}(m) = \begin{cases} \mathrm{FK}_n(m) & \text{if } m \leq l \\ 0 & \text{otherwise} \end{cases}$$

Above we use the multiplicativity of the formal q-grading: $q^i(q^j D) = q^{i+j} D$ for any $D \in \mathrm{Kom}(n)$. The differential between any two objects whose q-degree differs by one is given by a saddle map. In each period the two q-degree 2 differentials are defined to be those illustrated in the diagram above. (The degree 2 maps in the sequence are separated by $n-2$ saddle maps). In what follows f_m will be used to denote the differential, $f_m\colon\ \mathrm{FK}_n(m) \longrightarrow \mathrm{FK}_n(m+1)$.

The following proposition and its corollary are key ingredients in the proof of contractibility under turnbacks contained in section 7.10.

[4] The illustration itself equals $(P_{n-1} \sqcup 1) \otimes \mathrm{FK}_n$

Proposition 7.7. *Let* $\mathrm{FK}_n : \mathbb{N} \to \mathrm{Kom}(n)$ *be the* n*th Frenkel-Khovanov sequence defined above. Then for any standard generator* e_i, $i = 1, \ldots, n-1$,

$$\mathrm{FK}_n(-) \otimes e_i : \mathbb{N} \to \mathrm{Kom}(n)$$

is a sequence such that for every $k \in \mathbb{N}$ *the diagram* $\mathrm{FK}_n(k) \otimes e_i \in \mathrm{Kom}(n)$ *either*

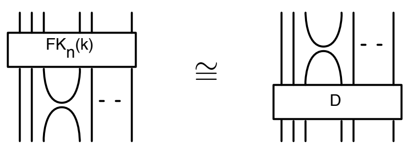
(1) *Satisfies a commutativity condition: there exists a Temperley-Lieb element* $D \in \mathrm{Cob}(n)$ *such that*

$$\mathrm{FK}_n(k) \otimes e_i \cong e_j \otimes D$$

where $j = i, i-1,$ or $i-2$ *or*

(2) *Is contained in an* i*-triple or* i*-quadruple sequence.*

The case (1) above when $j = i$ can be pictured by



Before giving the proof, we note that once FK_n is part of the chain complex for the universal projector (defined further below), both conclusions (1) and (2) above imply that all terms in $(P_{n-1} \sqcup 1) \otimes \mathrm{FK}_n \otimes e_i$ may be contracted. In the case (1) this will follow by the inductive contractibility of P_{n-1} under turnbacks. The contractibility in case (2) follows from the analysis of triples and quadruples in section 7.1.

Proof. The periodicity of the diagram FK_n ensures that inspecting the first $2n+1$ terms of $\mathrm{FK}_n \otimes e_i$ is sufficient. Geometrically inclined readers are invited to prove the proposition by examining the illustration at the beginning of this section. Expanding FK_n allows us to write the first period of $\mathrm{FK}_n \otimes e_i$ as follows:

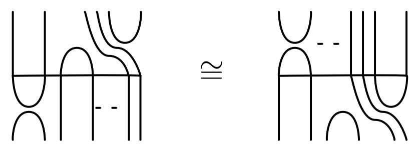
$$\begin{tikzcd}
1 \otimes e_i \arrow[r] & e_{n-1} \otimes e_i \arrow[r] & (e_{n-1} \otimes e_{n-2}) \otimes e_i \arrow[r] & \cdots \arrow[r] & (e_{n-1} \otimes \cdots \otimes e_1) \otimes e_i \arrow[d, "2"'] \\
e_{n-1} \otimes e_i & e_{n-1} \otimes e_i \arrow[l, "2"'] & \cdots \arrow[l] & (e_{n-1} \otimes \cdots \otimes e_2) \otimes e_i \arrow[l] & (e_{n-1} \otimes \cdots \otimes e_1) \otimes e_i \arrow[l]
\end{tikzcd}$$

We have dropped the q-grading because it is implied by the requirement that the first axiom (in definition 3.1) holds. We write "2" above arrows in order to indicate which maps are of q-degree 2 and all other maps are given by saddles.

There are several cases to consider. The first two are boundary cases $i = 1$ and $i = n-1$ and the last is the generic case for $1 < i < n-1$.

(1) If $i = 1$ then consider $-\otimes e_1$. If $n-k > 2$ then because of the far commutativity relation,

$$(e_{n-1} \otimes e_{n-2} \otimes \cdots \otimes e_{n-k}) \otimes e_1 \cong e_1 \otimes (e_{n-1} \otimes e_{n-2} \otimes \cdots \otimes e_{n-k})$$



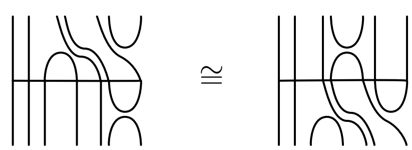
The terms corresponding to $n-k = 1$ and $n-k = 2$ fit in the following four term sequence:

$$\alpha \otimes e_1 \longrightarrow \alpha \otimes e_1 \otimes e_1 \xrightarrow{2} \alpha \otimes e_1 \otimes e_1 \longrightarrow \alpha \otimes e_1,$$

where $\alpha = e_{n-1} \otimes e_{n-2} \otimes \cdots \otimes e_2$. This sequence forms a 1-quadruple (see 7.2.1).

(2) If $i = n-1$ then when $k > 2$ we have

$$(e_{n-1} \otimes e_{n-2} \otimes \cdots \otimes e_{n-k}) \otimes e_{n-1} \cong e_{n-3} \otimes (e_{n-1} \otimes e_{n-2} \otimes \cdots \otimes e_{n-k})$$



When $k \leq 2$, the first three terms form an $(n-1)$-triple (see 7.1.1)

$$1 \otimes e_{n-1} \longrightarrow e_{n-1} \otimes e_{n-1} \longrightarrow e_{n-1} \otimes e_{n-2} \otimes e_{n-1}.$$

After the first period there is an $(n-1)$-quadruple surrounding every other degree 2 map:

$$e_{n-1} \otimes e_{n-2} \otimes e_{n-1} \longrightarrow e_{n-1} \otimes e_{n-1} \xrightarrow{2} e_{n-1} \otimes e_{n-1} \longrightarrow e_{n-1} \otimes e_{n-2} \otimes e_{n-1}$$

(3) If $i \neq 1$ and $i \neq n-1$, each term has the form

$$(e_{n-1} \otimes e_{n-2} \otimes \cdots \otimes e_{n-k}) \otimes e_i$$

for some k such that $2 \leq k < n-1$. Depending on k there are several cases to consider,

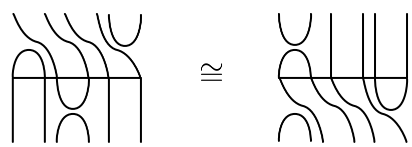
(a) If $n-k > i+1$ then e_i commutes with e_j for all j, $n-k \leq j \leq n-1$ because of the far commutativity relation. It follows that

$$(e_{n-1} \otimes e_{n-2} \otimes \cdots \otimes e_{n-k}) \otimes e_i \cong e_i \otimes (e_{n-1} \otimes e_{n-2} \otimes \cdots \otimes e_{n-k})$$

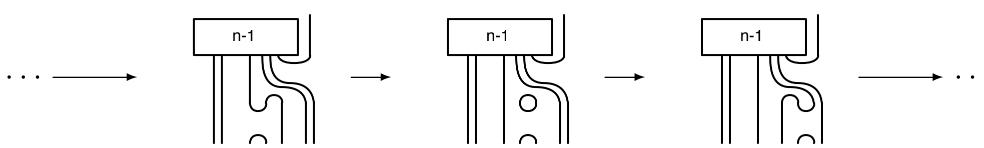
This can be pictured in the same way as the other application of the far commutativity relation in (1).

(b) If $n-k < i-1$ then similarly,

$$(e_{n-1} \otimes e_{n-2} \otimes \cdots \otimes e_{n-k}) \otimes e_i \cong e_{i-2} \otimes (e_{n-1} \otimes e_{n-2} \otimes \cdots \otimes e_{n-k})$$



(c) The terms in which $n-k = i-1$, $n-k = i$ and $n-k = i+1$ form an i-triple. For instance,



□

Let f_m be the mth map in the Frenkel-Khovanov sequence,

$$f_m\colon\ \mathrm{FK}_n(m) \longrightarrow \mathrm{FK}_n(m+1)$$

(see definition 7.6). Recall that each map f_m has q-degree equal to either 1 or 2. All degree 1 maps are given by saddle cobordisms and the degree 2 maps are shown in the diagram at the beginning of section 7.4.

Corollary 7.8. *Suppose $P_{n-1} \in \mathrm{Kom}(n-1)$ is an $n-1$st universal projector and let $l \in \mathbb{N}$ be such that f_l is a degree 1 map. Then each term in the truncated sequence*

$$f_l\big((P_{n-1} \sqcup 1) \otimes \mathrm{FK}_{n,l-1}\big)$$

either is the projector P_{n-1} capped with a turnback or is contained in a triple or a quadruple.

The proof of corollary 7.8 follows from proposition 7.7 since the map f_l is assumed to be a degree 1 map, that is a saddle cobordism. Therefore the sequence $f_l((P_{n-1} \sqcup 1) \otimes \mathrm{FK}_{n,l})$ equals $((P_{n-1} \sqcup 1) \otimes \mathrm{FK}_{n,l}) \otimes e_i$ for some i. $\square$

The point of this corollary is that if $f_l\big((P_{n-1} \sqcup 1) \otimes \mathrm{FK}_{n,l}\big)$ is part of a chain complex, then it can be contracted. (In the first case, the projector P_{n-1} capped with a turnback is contractible by axiom (1) of the universal projector P_{n-1}. In the second case, each triple or quadruple is contractible according to the analysis in sections 7.1.1, 7.2.1). This will play an important role in the proof of the main theorem below.

7.8.1. *The homotopy projector.* If an $n-1$st universal projector P_{n-1} exists then corollary 7.8 shows that the Frenkel-Khovanov sequence can be used to define a sequence which satisfies the axioms for an nth universal projector (definition 3.1) up to homotopy.

Definition 7.9. The nth *homotopy projector* $HP_n : \mathbb{N} \to \mathrm{Kom}(n)$ is the sequence defined by

$$\begin{aligned} HP_1 &= 1 \\ HP_n &= \mathrm{Tot}\left((P_{n-1} \sqcup 1) \otimes \mathrm{FK}_n\right) \end{aligned}$$

For a given length $l \geq 0$ the *truncated homotopy projector* is defined using the truncated Frenkel-Khovanov sequence:

$$HP_{n,l} = (P_{n-1} \sqcup 1) \otimes \mathrm{FK}_{n,l}$$

A picture of HP_n is given at the beginning of this section. For each e_i, $0 < i < n$ by the above proposition $HP_n(k) \otimes e_i$ is either a term containing $(HP_{n-1} \otimes e_j) \sqcup 1$ where $0 < j < n-1$ or fits into an i-triple or i-quadruple. If this were a chain complex then it would be contractible by the lemmas of section 2.5.

7.10. Construction of the chain complex: fattening the Frenkel-Khovanov sequence. The remark at the end of section 7.8.1 implies that the sequence HP_n (definition 7.9) behaves like a universal projector. However it is not a chain complex: the composition of any two successive saddle maps is not zero[5].

In order to obtain a chain complex and so complete the proof of the main theorem, we thicken the FK sequence by contractible pieces. Specifically, we consider the truncated Frenkel-Khovanov sequence $\mathrm{FK}_{n,l}$ of length l and our construction is inductive in l.

[5]Although it is not difficult to see that all compositions are homotopic to zero.

Let $P_{n-1} \in \operatorname{Kom}(n-1)$ be a chain complex representing the $n-1$st universal projector. We will now define a chain complex $\mathrm{CFK}_{n,l}$ inductively in length l using the maps of the Frenkel-Khovanov sequence $\{f_k\}_0^\infty$ above.

$\mathrm{CFK}_{n,l}$ is constructed inductively either as a two term chain complex defined in terms of $\mathrm{CFK}_{n,l-1}$ and f_l (in case the following map f_{l+1} has degree 1) or as a three term chain complex defined in terms of $\mathrm{CFK}_{n,l-1}$, f_l and f_{l+1} (in case f_{l+1} has degree 2).

Definition 7.11. Let $\{f_k\}_0^\infty$ be the maps in the Frenkel-Khovanov sequence above and $\deg_q(f_k)$ denote the q-degree of the map f_k. Set $\mathrm{CFK}_{n,0} = P_{n-1} \sqcup 1$ and if $\deg_q(f_l) = 1$ then set,

$$\mathrm{CFK}_{n,l} = \begin{cases} \mathrm{CFK}_{n,l-1} \xrightarrow{f_l} q\, f_l\, \mathrm{CFK}_{n,l-1} & \text{if } \deg_q(f_{l+1}) = 1 \\ \mathrm{CFK}_{n,l-1} \xrightarrow{f_l} q\, f_l\, \mathrm{CFK}_{n,l-1} \xrightarrow{f_{l+1}} q^3\, f_{l+1} f_l\, \mathrm{CFK}_{n,l-1} & \text{if } \deg_q(f_{l+1}) = 2 \end{cases}$$

Otherwise, $\deg_q(f_l) = 2$ and set,

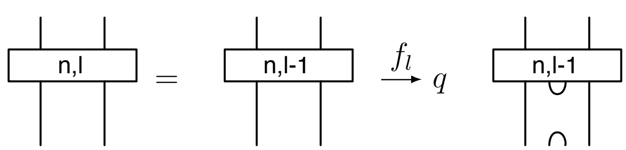
$$\mathrm{CFK}_{n,l} = \mathrm{CFK}_{n,l-1}$$

In this second step we do not change the complex $\mathrm{CFK}_{n,l}$ after having just used a degree 2 map in order to avoid a degree shift.

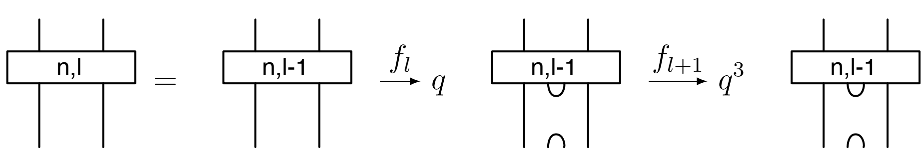
Here we follow the convention that $f_l(D \oplus D) = f_l(D) \oplus f_l(D)$, $q^i(q^j D) = q^{i+j} D$ and $f_{l+1}(D) = D$ if $\deg_q(f_{l+1}) = 2$. Note that the three term sequence in the last case is indeed a chain complex, that is $f_{l+1} f_l = 0$.

The recursive step can be visualized as follows.

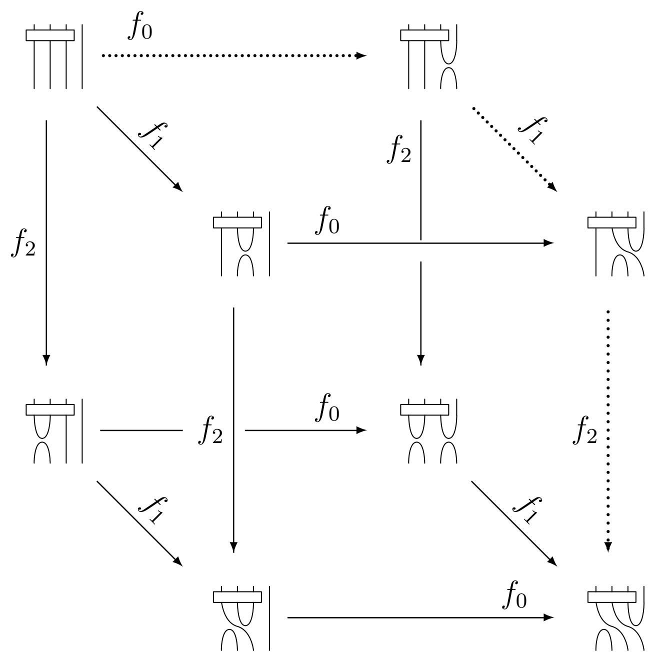
(1) If $\deg_q(f_l) = 1$ and $\deg_q(f_{l+1}) = 1$,



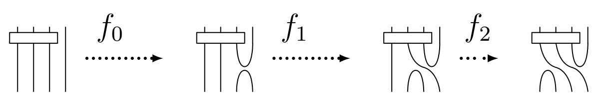
(2) If $\deg_q(f_{l+1}) = 2$,



Consider the chain complex $\mathrm{CFK}_{4,3}$



Recall that the truncated Frenkel-Khovanov sequence $\mathrm{FK}_{4,3}$ is given by



Here we use dotted arrows to help the reader find the relevant information. The sequence $\mathrm{FK}_{4,3}$ starts in the upper left hand corner of the cube, travels to the right then to the front face of the cube and finally lands in the lower right hand corner. This is precisely the first four terms of the diagram pictured at the beginning of section 7.4 together with four contractible terms.

The proofs contained in the remainder of this section are rooted in the observation that $\mathrm{CFK}_{n,l}$ will always decompose as $\mathrm{FK}_{n,l}$ plus a contractible subcomplex K_l consisting of truncated Frenkel-Khovanov sequences containing turnbacks.

Lemma 7.12. *(Structure of* $\mathrm{CFK}_{n,l}$*) For each* $l \geq 0$ *the chain complex* $\mathrm{CFK}_{n,l}$ *admits a decomposition,*

$$\mathrm{CFK}_{n,l} \cong (P_{n-1} \sqcup 1) \otimes (\mathrm{FK}_{n,l} \oplus K_l), \tag{1}$$

where the second summand $(P_{n-1} \sqcup 1) \otimes K_l$ *is contractible.*

More specifically, the contractibility of $(P_{n-1} \sqcup 1) \otimes K_l$ *is a consequence of simultaneous Gaussian elimination of some of the terms in* K_l*, so that each remaining term (in* *the notation of lemma 2.7)* $(P_{n-1} \sqcup 1) \otimes C_i$ *is contractible. Moreover, the off-diagonal component*

$$(P_{n-1} \sqcup 1) \otimes K_l \xrightarrow{\beta} (P_{n-1} \sqcup 1) \otimes \mathrm{FK}_{n,l}$$

of the $\mathrm{CFK}_{n,l}$ *differential with respect to the decomposition (1) vanishes on the domain of the isomorphisms underlying the simultaneous Gaussian elimination in* K_l*.*

Proof. The proof is by induction on l using the recurrence defining $\mathrm{CFK}_{n,l}$ (definition 7.11). If $l = 0$ or $\deg_q(f_l) = 2$ then there is nothing to prove. If $\deg_q(f_{l+1}) \neq 2$ then $\mathrm{CFK}_{n,l}$ is defined as a two term sequence,

$$\mathrm{CFK}_{n,l} \;=\; \mathrm{CFK}_{n,l-1} \xrightarrow{f_l} q f_l \, \mathrm{CFK}_{n,l-1}$$

By induction we may assume that

$$\mathrm{CFK}_{n,l-1} \cong (P_{n-1} \sqcup 1) \otimes (\mathrm{FK}_{n,l-1} \oplus K_{l-1})$$

where $K_{l-1} \simeq 0$ satisfies the conclusion of the lemma. We claim that there is a decomposition

$$\mathrm{CFK}_{n,l} \cong (P_{n-1} \sqcup 1) \otimes \Big(\mathrm{FK}_{n,l} \oplus q f_l \, \mathrm{FK}_{n,l-2} \oplus K_{l-1} \oplus q f_l K_{l-1} \Big) \tag{2}$$

This can be observed by writing the most important part of the recursion defining $\mathrm{CFK}_{n,l}$ in a helpful way[6],

$$\begin{tikzcd}
\Big[\mathrm{FK}_{n,l-1}(0) \arrow[r, "f_0"] \arrow[d, "f_l"'] & \cdots \arrow[r, "f_{l-2}"] & \mathrm{FK}_{n,l-1}(l-2) \arrow[r, "f_{l-1}"] \arrow[d, "f_l"'] & \mathrm{FK}_{n,l-1}(l-1) \Big] \arrow[d, "f_l"'] & \subset \mathrm{CFK}_{n,l-1} \arrow[d, "f_l"'] \\
\Big[f_l \, \mathrm{FK}_{n,l-1}(0) \arrow[r, "f_0"] & \cdots \arrow[r, "f_{l-2}"] & f_l \, \mathrm{FK}_{n,l-1}(l-2) \Big] \arrow[r, "\beta_l"] & f_l \, \mathrm{FK}_{n,l-1}(l-1) \arrow[d, equal] & \subset f_l \, \mathrm{CFK}_{n,l-1} \\
& & & \mathrm{FK}_{n,l}(l) &
\end{tikzcd}$$

The terms in the top row are $\mathrm{FK}_{n,l-1}$, the truncated Frenkel-Khovanov sequence of length $l-1$. By the inductive assumption, this sequence is a summand in $\mathrm{CFK}_{n,l-1}$ and the remaining part - the contractible summand K_{l-1} - is not included in the diagram. The terms on the left in the bottom row are of the form $f_l \, \mathrm{FK}_{n,l-2}$. Observe

[6]In this diagram the lower order projector $(P_{n-1} \sqcup 1)$ is omitted to simplify the notation.

that by definition the last term $f_l\,\mathrm{FK}_{n,l-1}(l-1)$ is equal to the next term $\mathrm{FK}_{n,l}(l)$ in the Frenkel-Khovanov sequence. The $\mathrm{FK}_{n,l}$ summand in (2) is seen in the diagram above as $\mathrm{FK}_{n,l-1}$ in the top row followed by the vertical map f_l to $f_l\,\mathrm{FK}_{n,l-1}(l-1)$. The equation (2) follows immediately.

To prove that $(P_{n-1} \sqcup 1) \otimes K_l$ is contractible, where

$$K_l := qf_l\,\mathrm{FK}_{n,l-2} \oplus K_{l-1} \,\oplus\, qf_lK_{l-1},$$

further analysis of the differential $\mathrm{CFK}_{n,l}$ is necessary. Note that $(P_{n-1} \sqcup 1) \otimes f_l\,\mathrm{FK}_{n,l-2}$ is contractible by corollary 7.8, more precisely all terms in this sequence are either contained in triples or quadruples or are projectors capped by turnbacks. By the inductive hypothesis on the off-diagonal entry of the differential in the statement of the lemma, when the terms participating in the Gaussian eliminations in K_{l-1}, qf_lK_{l-1} and $qf_l\,\mathrm{FK}_{n,l-2}$ are grouped together, (in the notation of lemma 2.7) the isomorphisms underlying the Gaussian eliminations in the summands, $a_{2i} : A_{2i} \to A_{2i+1}$ and $e_{2i+1} : B_{2i+1} \to B_{2i+2}$ remain isomorphisms because the matrices are lower triangular. After removing all triples and quadruples, the remaining chain complex consisting of contractible terms may be contracted by lemma 2.8 (big collapse). This concludes the proof that $(P_{n-1} \sqcup 1) \otimes K_l$ is contractible.

To propagate the inductive hypothesis on the differential, note that the only new off-diagonal component, of the form in the statement of the lemma, introduced during the inductive step, is the map β_l in the diagram above. The Gaussian eliminations take place in the sequence $qf_l\,\mathrm{FK}_{n,l-2}$ in the bottom row and since β_l is defined on the last term of that sequence, clearly the component of the differential on the domain of the isomorphisms in the Gaussian eliminations is trivial.

The proof in the second case (when $\deg_q(f_{l+1}) = 2$) is almost exactly the same. Instead of one row of contractible terms there are two new rows of contractible terms. By definition,

$$\mathrm{CFK}_{n,l} \quad = \quad \mathrm{CFK}_{n,l-1} \xrightarrow{f_l} qf_l\,\mathrm{CFK}_{n,l-1} \xrightarrow{f_{l+1}} q^3 f_{l+1}f_l\,\mathrm{CFK}_{n,l-1}$$

Again induction we may assume that

$$\mathrm{CFK}_{n,l-1} \cong (P_{n-1} \sqcup 1) \otimes (\mathrm{FK}_{n,l-1} \oplus K_{l-1})$$

and $C_{l-1} \simeq 0$. In this case the claim is that there is a decomposition

$$\mathrm{CFK}_{n,l} \cong (P_{n-1} \sqcup 1) \otimes (\mathrm{FK}_{n,l} \oplus K_l)$$

where

$$K_l := qf_l\,\mathrm{FK}_{n,l-2} \oplus K_{l-1} \oplus qf_lK_{l-1} \oplus q^3f_{l+1}f_l\,\mathrm{FK}_{n,l-1} \oplus q^3f_{l+1}f_lK_{l-1}$$

Again this can be observed by writing the most important part of the recursion as follows:

$$
\begin{tikzcd}
\Big[\ \mathrm{FK}_{n,l-1}(0) \arrow[r, "f_0"] \arrow[d, "f_l"'] & \cdots \arrow[r, "f_{l-2}"] & \mathrm{FK}_{n,l-1}(l-2) \arrow[r, "f_{l-1}"] \arrow[d, "f_l"'] & \mathrm{FK}_{n,l-1}(l-1)\Big] \arrow[d, "f_l"'] & \subset & \mathrm{CFK}_{n,l-1} \arrow[d, "f_l"'] \\
\Big[\ f_l\,\mathrm{FK}_{n,l-1}(0) \arrow[r, "f_0"] \arrow[d, "f_{l+1}"'] & \cdots \arrow[r, "f_{l-2}"] & f_l\,\mathrm{FK}_{n,l-1}(l-2)\Big] \arrow[r, "\beta_l"] \arrow[d, "f_{l+1}"'] & f_l\,\mathrm{FK}_{n,l-1}(l-1) \arrow[d, "f_{l+1}"'] & \subset & f_l\,\mathrm{CFK}_{n,l-1} \arrow[d, "f_{l+1}"'] \\
\Big[f_{l+1}f_l\,\mathrm{FK}_{n,l-1}(0) \arrow[r, "f_0"] & \cdots \arrow[r, "f_{l-2}"] & f_{l+1}f_l\,\mathrm{FK}_{n,l-1}(l-2)\Big] \arrow[r, "\beta_{l+1}"] & f_{l+1}f_l\,\mathrm{FK}_{n,l-1}(l-1) & \subset & f_{l+1}f_l\,\mathrm{CFK}_{n,l-1}
\end{tikzcd}
$$

As in the previous case, the summands in K_l are contractible. $\square$

Remark. The proof of the lemma above used the recursive definition of the chain complex $\mathrm{CFK}_{n,l}$. Completely expanding the recursion gives the following decomposition:

$$\mathrm{CFK}_{n,l} = \bigoplus_I q^{l(I)+\tau(I)} f_I \,\mathrm{FK}_{n,l-l(I)}$$

where I are k-tuples indexing maps in the sequence FK_n, $l(I)$ is the cardinality of I, $f_I = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}$ when $I = (i_1, i_2, \ldots, i_k)$ and $\tau(I)$ is the number of degree 2 maps in f_I. Moreover, if f_m is the differential of FK_n then the differential of in each summand, $f_I \,\mathrm{FK}_{n,l-l(I)}$, is $f_I(f_m)$. Each summand (except for $I = \emptyset$) is contractible by corollary 7.8. In the notation of lemma 7.12, $K_l = \bigoplus_{I\neq\emptyset} q^{l(I)+\tau(I)} f_I \,\mathrm{FK}_{n,l-l(I)}$ and the summand $FK_{n,l}$ corresponds to $I = \emptyset$.

The following statement is important for establishing the properties of a universal projector:

Lemma 7.13. *(Contractibility under turnbacks)*
Let $n > 2$, $l \geq 0$ and $j \in \{1, \ldots, n-1\}$. Then all terms in the chain complex

$$\mathrm{CFK}_{n,l} \otimes e_j$$

may be contracted, except possibly for the lth term,

$$(P_{n-1} \sqcup 1) \otimes FK_{n,l}(l)$$

Proof. By lemma 7.12,

$$\mathrm{CFK}_{n,l} \otimes e_j \cong \Big((P_{n-1} \sqcup 1) \otimes (\mathrm{FK}_{n,l} \otimes e_j)\Big) \oplus \Big((P_{n-1} \sqcup 1) \otimes (K_l \otimes e_j)\Big)$$

Application of $- \otimes e_j$ does not change the contractibility of the the second summand. By proposition 7.7 all of the terms (besides possibly the last, depending on j) in the first summand are either projectors P_{n-1} capped by turnbacks or contained in triples and quadruples. The rest of the proof is identical to the proof of the inductive step in lemma 7.12. $\square$

As a consequence of lemma 7.12 we have the following definition.

Definition 7.14. (The truncated projector $P_{n,l}$) Contracting $K_l \subset \mathrm{CFK}_{n,l}$ yields a homotopy equivalence

$$\mathrm{CFK}_{n,l} \longrightarrow P_{n,l}$$

onto a *chain complex* $P_{n,l}$ that consists of the first l terms of the Frenkel-Khovanov sequence pictured at the beginning of section 7.4.

Note that the chain complex $P_{n,l}$ may be thought of as a completed version of the truncated homotopy chain complex $HP_{n,l}$ (definition 7.9). Given a homotopy chain complex there is a standard obstruction theoretic approach to constructing a chain complex in which new components corresponding to nullhomotopies and Massey products of nullhomotopies are added to the differential [GM03]. The extra maps in $P_{n,l}$ are precisely those corresponding to these homotopies and Massey products. Our axioms (definition 3.1) guarantee that any such choice of Massey products yields a unique chain complex up to homotopy.

The universal projector P_n will be defined as the limit of $P_{n,l}$ as $l \to \infty$. Its contractibility under turnbacks (to show that P_n satisfies the axioms of a universal projector in definition 3.1) follows from lemma 7.13. The remaining property, ensuring that the limit exists, is the "stability" of the sequence $\{P_{n,l}\}$, proved in the following proposition 7.15.

We now show that the chain complex $P_{n,l+1}$ is obtained from the chain complex $P_{n,l}$ by adding the next term in the picture at the beginning of section 7.4 and only adding maps to the differential from the old terms to the new term. The maps between those terms in $P_{n,l+1}$ which come from $P_{n,l}$ are exactly the same as the maps between terms in $P_{n,l}$. We may conclude from this together with the previous proposition that there is a chain complex $P_n = P_{n,\infty}$ which is a universal projector.

Proposition 7.15. *(Stability of construction) The inclusion*

$$P_{n,l} \hookrightarrow P_{n,l+1}$$

is an isomorphism onto its image. Moreover,

$$P_{n,l+1} \cong P_{n,l} \oplus \big((P_{n-1} \sqcup 1) \otimes \mathrm{FK}_{n,l}(l)\big)$$

$d_{P_{n,l+1}}$ *is lower triangular with respect to this decomposition and* $d_{P_{n,l+1}}|_{P_{n,l}} = d_{P_{n,l}}$ *.*

Proof. This follows from an analogue[7] of the first commutative diagram in the proof of proposition 7.12. $P_{n,l+1}$ is obtained from $P_{n,l}$ by,

$$\begin{tikzcd}
\Big[\, P_{n,l}(0) \arrow[r, "d_0"] \arrow[d, "f_l"'] & \cdots \arrow[r, "d_{l-2}"] \arrow[d, phantom, "\cdots"] & P_{n,l}(l-2) \arrow[r, "d_{l-1}"] \arrow[d, "f_l"'] & P_{n,l}(l-1)\,\Big] \arrow[d, "f_l"'] \\
\Big[\, f_lP_{n,l}(0) \arrow[r, "d_0"] & \cdots \arrow[r, "d_{l-2}"] & f_lP_{n,l}(l-2)\,\Big] \arrow[r, "\beta_l"] & f_lP_{n,l}(l-1)
\end{tikzcd}$$

The terms in the lower lefthand corner are again contractible. Contracting them does not change the maps d_i along the top row. $\square$

7.16. A Doubling Construction. In the proof of the main theorem we only concerned ourselves with what could be called right contractibility or the statement that for $C_* \in \mathrm{Kom}(n)$ and $0 < i < n$,

$$C_* \otimes e_i \simeq 0$$

If C_* is right contractible then define $\bar{C}_* \in \mathrm{Kom}(n)$ to be the chain complex in which each diagram and morphism is flipped upside down. Now define a new chain complex D_* by

$$D_n = \bar{C}_n \otimes C_n$$

The contractibility of D_* by turnbacks on both sides now follows from that of C_* on one side. The first two axioms of the universal projector are satisfied by D_* provided that they are satisfied by C_*.

[7]For the sake of clarity we have omitted from the diagram the parts of the differential $d_{P_{n,l}}$ between non-consecutive terms.

References

[BHMV95] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quantum field theories derived from the Kauffman bracket. *Topology*, 34(4):883–927, 1995.

[BK01] Bojko Bakalov and Alexander Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.

[BN05] Dror Bar-Natan. Khovanov's homology for tangles and cobordisms. *Geom. Topol.*, 9:1443–1499 (electronic), 2005.

[BW08] Anna Beliakova and Stephan Wehrli. Categorification of the colored Jones polynomial and Rasmussen invariant of links. *Canad. J. Math.*, 60(6):1240–1266, 2008.

[FK97] Igor B. Frenkel and Mikhail G. Khovanov. Canonical bases in tensor products and graphical calculus for $U_q\,\mathfrak{su}(2)$. *Duke Math. J.*, 87(3):409–480, 1997.

[FSS] I. Frenkel, C. Stroppel, and J. Sussan. In preparation.

[GM03] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.

[GW] J. Grigsby and S. Wehrli. On the colored jones polynomial, sutured floer homology and knot floer homology. *Preprint*.

[Jac04] Magnus Jacobsson. An invariant of link cobordisms from Khovanov homology. *Algebr. Geom. Topol.*, 4:1211–1251 (electronic), 2004.

[Jon97] Vaughan F. R. Jones. A polynomial invariant for knots via von Neumann algebras [MR0766964 (86e:57006)]. In *Fields Medallists' lectures*, volume 5 of *World Sci. Ser. 20th Century Math.*, pages 448–458. World Sci. Publ., River Edge, NJ, 1997.

[Kho00] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[Kho02] Mikhail Khovanov. A functor-valued invariant of tangles. *Algebr. Geom. Topol.*, 2:665–741 (electronic), 2002.

[Kho05] Mikhail Khovanov. Categorifications of the colored Jones polynomial. *J. Knot Theory Ramifications*, 14(1):111–130, 2005.

[Kho06] Mikhail Khovanov. Link homology and Frobenius extensions. *Fund. Math.*, 190:179–190, 2006.

[KL94] Louis H. Kauffman and Sóstenes L. Lins. *Temperley-Lieb recoupling theory and invariants of 3-manifolds*, volume 134 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1994.

[KMS09] Mikhail Khovanov, Volodymyr Mazorchuk, and Catharina Stroppel. A brief review of abelian categorifications. *Theory Appl. Categ.*, 22:No. 19, 479–508, 2009.

[Kup96] Greg Kuperberg. Spiders for rank 2 Lie algebras. *Comm. Math. Phys.*, 180(1):109–151, 1996.

[Lic97] W. B. Raymond Lickorish. *An introduction to knot theory*, volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.

[MN08] Scott Morrison and Ari Nieh. On Khovanov's cobordism theory for $U_q\,\mathfrak{su}(3)$ knot homology. *J. Knot Theory Ramifications*, 17(9):1121–1173, 2008.

[Roz10] Lev Rozansky. An infinite torus braid yields a categorified jones-wenzl projector. *arXiv:1005.3266*, 2010.

[Tur94] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[TV92] V. G. Turaev and O. Ya. Viro. State sum invariants of 3-manifolds and quantum $6j$-symbols. *Topology*, 31(4):865–902, 1992.

[Wal] Kevin Walker. On witten' 3-manifold invariants. *http://canyon23.net/math/*.

[Wen87] Hans Wenzl. On sequences of projections. *C. R. Math. Rep. Acad. Sci. Canada*, 9(1):5–9, 1987.

Department of Mathematics, University of Virginia, Charlottesville, VA 22904

E-mail address: bjc4n@virginia.edu, krushkal@virginia.edu