Cardy condition for Open-closed field algebras

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Abstract

Let V be a vertex operator algebra satisfying certain reductivity and finiteness conditions such that $\mathcal{C}_V$, the category of V-modules, is a modular tensor category. We study open-closed field algebras over V equipped with nondegenerate invariant bilinear forms for both open and closed sectors. We show that they give algebras over certain $\mathbb{C}$-extension of the Swiss-cheese partial dioperad, and we obtain Ishibashi states easily in such algebras. We formulate Cardy condition algebraically in terms of the action of the modular transformation $S : \tau \mapsto -\frac{1}{\tau}$ on the space of intertwining operators. We then derive a graphical representation of S in the modular tensor category $\mathcal{C}_V$. This result enables us to give a categorical formulation of Cardy condition and modular invariant conformal full field algebra over $V \otimes V$. Then we incorporate the modular invariance condition for genus-one closed theory, Cardy condition and the axioms for open-closed field algebra over V equipped with nondegenerate invariant bilinear forms into a tensor-categorical notion called Cardy $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra. We also give a categorical construction of Cardy $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra in Cardy case.

Contents

0 Introduction

This work is the second half of a study on open-closed field algebra, the first half of which appeared in [Ko2]. It is a part of an extension [H9][HKo1][Ko2] of the program, initiated by I. Frenkel and largely developed by Huang [H1]-[H12], of constructing conformal field theories using the theory of vertex operator algebra. Zhu's work [Z] is also very influencial in this development. In this work, we give only those references on which this work is based. For other references which are related to this extended program, we refer readers to [H9][HKo1][Ko2], in which one can also find more thorough introductions of the program and related things.

In [Ko2], we introduced the notion of open-closed field algebra over a vertex operator algebra V with central charge c. It describes a special class of tree-level open-closed-string interactions, in which arbitrary number of in-coming open strings fuse with arbitrary number of in-coming closed strings into a single out-going open string, in a genus-zero open-closed conformal field theory with boundary conditions preserving the chiral symmetry given by V. We studied some basic properties of such algebra, and gave an operadic formulation and a tensor-categorical formulation of such aglebra when V satisfies certain reductivity and finiteness conditions.

In this work, we study a few more basic open-closed-string interactions in an open-closed conformal field theory (not necessary genus-zero). Our goal is to give an algebraic formulation of open-closed conformal field theory with boundary conditions preserving chiral symmetry V as open-closed field algebra over V equipped with bilinear forms and satisfying finite number of compatibility conditions.

In [Ko1], we studied the tree-level closed-string interactions in which arbitrary number in-coming closed strings fuse and split into arbitrary number out-going closed strings. A few examples of such interactions, which are not studied in [Ko2], are shown in Figure 1 with the convention that in-coming (out-going) strings sit on the left (right). Such interactions can be studied in terms of conformal full field algebra over $V^L \otimes V^R$ [HKo2] equipped with a nondegenerate invariant bilinear form, where V^L and V^R are two vertex operator algebras of central charge c^L and c^R respectively and satisfy certain finiteness and reductivity conditions. In this work, we reformulate this type of interactions operadically by a notion called sphere partial dioperad (see Section 1.1) denoted as $\mathbb{K}$. Then one result in [Ko1] can be restated as follow: a conformal full field algebra over

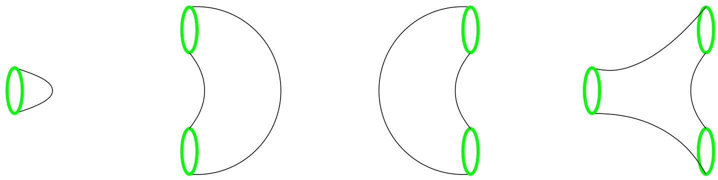


Figure 1: closed-string interactions studied in [Ko1]

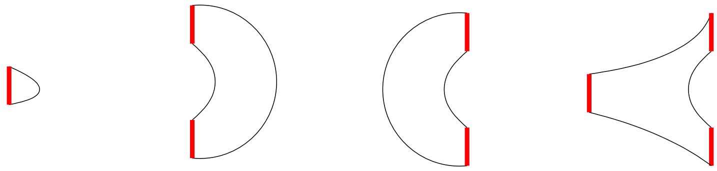


Figure 2: open strings interactions

$V^L \otimes V^R$ equipped with a nondegenerate invariant bilinear form canonically gives an algebra over $\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$, which is a partial dioperad extension of $\mathbb{K}$.

In this work, we also study the tree-level open-string interactions in which arbitrary in-coming open strings fuse and split into arbitrary number of open strings. Some examples of such interactions are shown in Figure 2 with the same convention as that of Figure 1. Such interactions of open strings can be described operadically by the so-called disk partial dioperad (see Section 1.1) denoted as $\mathbb{D}$. We show that an open-string vertex operator algebra of central charge c [HKo1] equipped with a nondegenerate invariant bilinear form canonically gives an algebra over $\tilde{\mathbb{D}}^c$, which is a partial dioperad extension of $\mathbb{D}$.

Combining above two results, we can study the genus-zero open-closed-string interactions in which arbitrary number of in-coming open strings and closed strings fuse and split into arbitrary number of open strings and closed strings. An example of such interactions are depicted in Figure 3. Such interactions can be formulated operadically by the so-called Swiss-cheese partial dioperad (see Section 2.2) denoted as $\mathbb{S}$. When V satisfies conditions in Theorem 0.1, we show that an open-closed field algebra over V equipped with nondegenerate invariant bilinear forms for both open and closed sectors canonically gives an algebra over $\tilde{\mathbb{S}}^c$, which is an extension of $\mathbb{S}$. Moreover, we show that Ishibashi states in the closed sector can be obtained from vacuum-like states in the open sector.

Note that all surfaces of any genus with arbitrary number of boundary components, interior punctures and boundary punctures can be obtained by sewing operations among genus-zero surfaces with only one boundary component. The operadic structure of such genus-zero surfaces, are completely captured in the notion of Swiss-cheese partial dioperad. Therefore, an open-closed field algebra over V equipped with nondegenerate invariant bilinear forms for both open and closed sectors contains all the data needed for the construction of an open-closed conformal field theory of all genus. Other surfaces which are not included in the notion of Swiss-cheese partial dioperad only provide additional compatibility conditions. In the topological case, only two additional

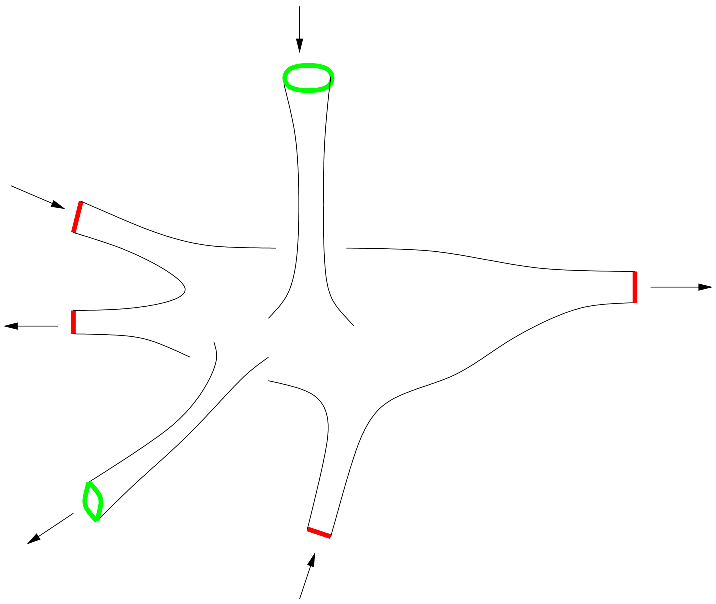


Figure 3: open strings interact with closed strings

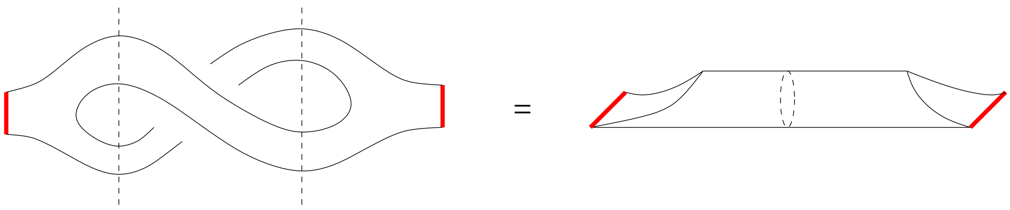


Figure 4: Cardy condition

compatibility conditions are needed to ensure the consistency of a theory of all genus [La][Mo1][MSeg][AN][LP]. One compatibility condition says that both of the open and closed sectors are finite dimensional. The other condition is the famous "Cardy condition" [C1]-[C4][La][Mo1][Mo2][MSeg], which is due to two different decompositions of a single surface as shown in Figure 4.

In open-closed conformal field theories, compatibility conditions are much more complicated. Both of the open and closed sectors in any nontrivial open-closed conformal field theories are infinite dimensional. We need require the convergence of all correlation functions of all genus. This convergence condition is highly nontrivial. So far the only known convergence results are in genus-zero [H8] and genus-one theories [Z][DLM][Mi1][Mi2][H10].

Let us recall an important Theorem by Huang [H11][H12].

Theorem 0.1. *Let $(V, Y, \mathbf{1}, \omega)$ be a simple vertex operator algebra V satisfying the following conditions:*

1. *$V_n = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V' is isomorphic to V as V-module.*

2. *Every $\mathbb{N}$-gradable weak V-module is completely reducible.*

3. V is C_2-cofinite.

Then the direct sum of all inequivalent irreducible V-modules has a natural structure of intertwining operator algebra [H5]-[H8]. Moreover, the category of V-modules, denoted as $\mathcal{C}_V$, has a natural structure of vertex tensor category [HL1]-[HL4][H3] and modular tensor category [T][BK2].

Assumption 0.2. *In this work, we fix a vertex operator algebra V, which is assumed to satisfy the conditions in Theorem 0.1 without further announcement.*

For the intertwining operator algebra given in Theorem 0.1, Huang also proved in [H8][H10] that the products of intertwining operators and q-traces of them have certain nice convergence and analytic extension properties. These properties are sufficient for the construction of genus-zero and genus-one correlation functions [HKo2][HKo3]. Since the modular tensor category $\mathcal{C}_V$ supports an action of mapping class groups of all genus, it is reasonable to believe that the conditions on V are also sufficient for the purpose of constructing correlation functions of all genus.

Another compatibility condition is that all correlation functions on surfaces without boundary (closed theory) must be invariant under the actions of mapping class groups. Sonoda argued in [So] on a physical level of rigor that for the closed theory it is sufficient to check the modular invariance of one-point correlation functions on torus. This condition on torus was studied in [HKo3]. It was formulated algebraically as a modular invariance condition on conformal full field algebra over $V^L \otimes V^R$. In [HKo3], we also showed that the conformal full field algebra over $V \otimes V$ obtained from the diagonal construction [FFFS2][FFRS][HKo2][Ko1] are modular invariant.

For the whole open-closed theory, Lewellen argued in [Le] on a physical level of rigor that the only remaining compatibility conditions one needs is the Cardy condition. Cardy condition in open-closed conformal field theory is much more complicated than that in topological theory and has never been fully written down by physicists. In Section 3.1, we derive a version of Cardy condition for an open-closed field algebra over V directly from the axioms of open-closed conformal field theory. Using results in [H10][H11], we then show how the Cardy condition is related to the modular transformation $S : \tau \mapsto -\frac{1}{\tau}$ of a q-trace of a product of intertwining operators. This relation gives us a simple formulation of Cardy condition in the framework of intertwining operator algebra (see Section 3.2).

Remark 0.3. There are still more compatibility conditions which were not discussed in [So][Le]. One also need to proved certain algebraic version of uniformization theorems (see [H1][H2] [H4] for the genus-zero case). Such results for genus large than 0 are still not available. But it seems that no additional assumption on V is needed. Such results should follows automatically from the properties of Virasoro algebra and intertwining operators. The uniformization and convergence problems are not pursued in this work.

In order to construct open-closed field algebra over V satisfying the genus-one modular invariance condition and Cardy condition, we need to obtain a tensor-categorical formulation of these conditions. This requires us to know the categorical representation of the modular transformation S. Although the action of $SL(2, \mathbb{Z})$ in a modular tensor category is explicitly known [MSei2][V][Ly][Ki][BK2], its relation to the modular transformation properties of chiral genus-one correlation functions is not completely clear. This relation was first suggested by I. Frenkel and studied by Moore and Seiberg in [MSei2] but only on a physical level of rigor. Using the tensor category theory of vertex operator algebra developed by Huang and Lepowsky [HL1]-[HL4][H3], Huang's recent results on modular tensor category [H11][H12] and results in [HKo3], we derive a graphical representation of S in $\mathcal{C}_V$. This result enables us to give a categorical formulation of Cardy condition and that of modular invariant conformal full field algebra over $V \otimes V$ [HKo3]. We incorporate them with the categorical formulation of the axioms of open-closed field algebra over V equipped with nondegenerate invariant bilinear forms into a tensor-categorical notion called *Cardy* $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$*-algebra*. As we discussed in previous paragraphs, it is reasonable to believe that the axioms of Cardy $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra is sufficient to supply an open-closed conformal field theory of all genus. However, constructing the high-genus theories is still a hard open problem which is not studied in this work. In the end, we give a categorical construction of Cardy $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra. This construction is called Cardy case in physics literature (see for example [FFRS]).

The layout of this work is as follow: in Section 1, we introduce the notion of sphere partial dioperad and disk partial dioperad and study algebras over them; in Section 2, we introduce the notions of Swiss-cheese partial dioperad $\mathbb{S}$ and its $\mathbb{C}$-extension $\tilde{\mathbb{S}}^c$, and show that an open-closed conformal field algebra over V equipped with nondegenerate invariant bilinear forms canonically gives an algebra over $\tilde{\mathbb{S}}^c$, and study Ishibashi states in such algebras; in Section 3, we give two algebraic formulations of Cardy condition; in Section 4, we derive a graphic representation of the modular transformation S; in Section 5, we give the categorical formulations of nondegenerate invariant bilinear forms, modular invariant conformal full field algebra over $V \otimes V$ and Cardy condition. Then we introduce the notion of Cardy $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra. In the end, we give a categorical construction of Cardy $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra.

Convention of notations: $\mathbb{N}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{R}, \mathbb{R}_+, \mathbb{C}$ denote the set of natural numbers, integers, positive integers, real numbers, positive real numbers, complex numbers, respectively. Let $\mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im} z > 0\}$ and $\overline{\mathbb{H}} = \{z \in \mathbb{C} | \mathrm{Im} z < 0\}$. Let $\hat{\mathbb{R}}$, $\hat{\mathbb{C}}$ and $\hat{\mathbb{H}}$ be the one point compactification of real line, complex plane and up-half plane respectively. Let $\mathbb{R}_+$ and $\mathbb{C}^\times$ be the multiplication groups of positive real and nonzero complex numbers respectively. The ground field is always chosen to be $\mathbb{C}$.

Throughout this work, we choose a branch cut for logarithm as follow:

$$\log z = \log|z| + i \mathrm{Arg}\, z, \qquad 0 \leq \mathrm{Arg}\, z < 2\pi. \tag{0.1}$$

We define power functions of two different types of complex variables as follow:

$$z^s := e^{s \log z}, \qquad \bar{z}^s := e^{s \overline{\log z}}, \qquad \forall s \in \mathbb{R}. \tag{0.2}$$

Acknowledgment The results in Section 2.4 and 3.1 are included in author's thesis. I want to thank my advisor Yi-Zhi Huang for introducing me to this interesting field and for his constant support and many important suggestions for improvement. I thank C. Schweigert for telling me the meaning of boundary states from a physical point of view. I also want to thank I. Frenkel, A. Kirillov, Jr., J. Fuchs and C. Schweigert for some inspiring conversations on the subject of Section 4.2.

1 Partial dioperads

In Section 1.1, we recall the definition of (partial) dioperad and algebra over it, and introduce sphere partial dioperad $\mathbb{K}$, disk partial dioperad $\mathbb{D}$ and their extensions $\tilde{K}^{c^L}\otimes\overline{\tilde{K}^{\overline{c^R}}}$, $\tilde{D}^c$ as examples. In Section 1.2, we discuss an algebra over $\tilde{K}^{c^L}\otimes\overline{\tilde{K}^{\overline{c^R}}}$ from a conformal full field algebra over $V^L\otimes V^R$. In Section 1.3, we disucuss an algebra over $\tilde{D}^c$ from an open-string vertex operator algebra.

1.1 Partial dioperads

Let us first recall the definition of dioperad given by Gan [G]. Let S_n be the automorphism group of the set $\{1,\dots,n\}$ for $n\in\mathbb{Z}_+$. Let $m=m_1+\cdots+m_n$ be an ordered partition and $\sigma\in S_n$. The block permutation $\sigma_{(m_1,\dots,m_n)}\in S_m$ is the permutation which permutes n intervals of lengths $m_1,\dots,m_n$ in the same way as σ permutes $1,\dots,n$. Let $\sigma_i\in S_{m_i}$, $i=1,\dots,k$. We view the element $(\sigma_1,\dots,\sigma_k)\in S_{m_1}\times\cdots\times S_{m_k}$ naturally as an element in S_m by the canonical embedding $S_{m_1}\times\cdots\times S_{m_k}\hookrightarrow S_m$. For any $\sigma\in S_n$ and $1\le i\le n$, we define a map $\hat{i}:\{1,\dots,n-1\}\to\{1,\dots,n\}$ by $\hat{i}(j)=j$ if $j<i$ and $\hat{i}(j)=j+1$ if $j\ge i$ and an element $\hat{i}(\sigma)\in S_{n-1}$ by

$$\hat{i}(\sigma)(j):=\hat{i}^{-1}\circ\sigma\circ\widehat{\sigma^{-1}(i)}(j).$$

Definition 1.1. A *dioperad* consists of a family of sets $\{\mathcal{P}(m,n)\}_{m,n\in\mathbb{N}}$ with an action of $S_m\times S_n$ on $\mathcal{P}(m,n)$ for each pair of $m,n\in\mathbb{Z}_+$, a distinguished element $I_{\mathcal{P}}\in\mathcal{P}(1,1)$ and substitution maps

$$\begin{array}{ccc}\mathcal{P}(m,n)\times\mathcal{P}(k_1,l_1)\times\cdots\times\mathcal{P}(k_n,l_n) & \xrightarrow{\gamma_{(i_1,\dots,i_n)}} & \mathcal{P}(m-n+k_1\cdots+k_n,l_1+\cdots+l_n)\\ (P,P_1,\dots,P_n) & \mapsto & \gamma_{(i_1,\dots,i_n)}(P;P_1,\dots,P_n)\end{array}\tag{1.1}$$

for $m,n,l_1,\dots,l_n\in\mathbb{N},k_1,\dots,k_n\in\mathbb{Z}_+$ and $1\le i_j\le k_j, j=1,\dots,n$, satisfying the following axioms:

1. *Unit properties*: for $P\in\mathcal{P}(m,n)$,
   (a) *left unit property*: $\gamma_{(i)}(I_{\mathcal{P}};P)=P$ for $1\le i\le m$,
   (b) *right unit property*: $\gamma_{(1,\dots,1)}(P;I_{\mathcal{P}},\dots,I_{\mathcal{P}})=P$.
2. *Associativity*: for $P\in\mathcal{P}(m,n)$, $Q_i\in\mathcal{P}(k_i,l_i), i=1,\dots,n$, $R_j\in\mathcal{P}(s_j,t_j), j=1,\dots,l=l_1+\cdots+l_n$, we have
$$\gamma_{(q_1,\dots,q_l)}\big(\gamma_{(p_1,\dots,p_n)}(P;Q_1,\dots,Q_n);R_1,\dots,R_l\big)=\gamma_{(p_1,\dots,p_n)}(P;P_1,\dots,P_n)\tag{1.2}$$
where $P_i=\gamma_{(q_{l_1+\cdots+l_{i-1}+1},\dots,q_{l_1+\cdots+l_i})}(Q_i;R_{l_1+\cdots+l_{i-1}+1},\dots,R_{l_1+\cdots+l_i})$ for $i=1,\dots,n$.
3. *Permutation property*: For $P\in\mathcal{P}(m,n)$, $Q_i\in P(k_i,l_i), i=1,\dots,n$, $(\sigma,\tau)\in S_m\times S_n$, $(\sigma_i,\tau_i)\in S_{k_i}\times S_{l_i}, i=1,\dots,n$,
$$\begin{aligned}&\gamma_{(i_1,\dots,i_n)}((\sigma,\tau)(P);Q_1,\dots,Q_n)\\ &\quad=((\sigma,\tau_{(k_1-1,\dots,k_n-1)}),\tau_{(l_1,\dots,l_n)})\gamma_{(i_{\tau(1)},\dots,i_{\tau(n)})}(P;Q_{\tau(1)},\dots,Q_{\tau(n)}),\\ &\gamma_{(i_1,\dots,i_n)}(P;(\sigma_1,\tau_1)(Q_1),\dots,(\sigma_n,\tau_n)(Q_n))\\ &\quad=((\mathrm{id},\hat{i}_1(\sigma_1),\dots,\hat{i}_n(\sigma_n)),(\tau_1,\dots,\tau_n))\gamma_{(\sigma_1^{-1}(i_1),\dots,\sigma_n^{-1}(i_n))}(P;Q_1,\dots,Q_n).\end{aligned}$$

We denote such dioperad as $(\mathcal{P}, \gamma_{\mathcal{P}}, I_{\mathcal{P}})$ or simply $\mathcal{P}$.

Remark 1.2. We define compositions ${}_i\circ_j$ as follow:

$$P\,{}_i\circ_j\, Q := \gamma_{(1,\dots,1,j,1,\dots 1)}(P; I_{\mathcal{P}}, \dots, I_{\mathcal{P}}, Q, I_{\mathcal{P}}, \dots, I_{\mathcal{P}}).$$

It is easy to see that $\gamma_{(i_1,\dots,i_n)}$ can be reobtained from ${}_i\circ_j$. In [G], the definition of dioperad is given in terms of ${}_i\circ_j$ instead of $\gamma_{(i_1,\dots,i_n)}$.

Definition 1.3. A *partial dioperad* has a similar definition as that of dioperad except the map $\gamma_{(i_1,\dots,i_n)}$ or ${}_i\circ_j$ is only partially defined and the same associativity hold whenever both sides of (1.2) exist. A *(partial) nonassociative dioperad* consists of the same data as those of (partial) dioperad satisfying all the axioms of (partial) dioperad except the associativity.

The notion of homomorphism and isomorphism of (partial pseudo-) dioperad are naturally defined.

Remark 1.4. In the case of partial dioperad, the definition using $\gamma_{(i_1,\dots,i_n)}$ or ${}_i\circ_j$ may have subtle differences in the domains on which $\gamma_{(i_1,\dots,i_n)}$ or ${}_i\circ_j$ is defined (see appendix C in [H3] for more details). These differences have no effect on those algebras over partial dioperads considered in this work. So we will simply ignore these differences.

Remark 1.5. Notice that a (partial nonassociative) dioperad $\{\mathcal{P}(m,n)\}_{m,n\in\mathbb{N}}$ naturally contains a (partial nonassociative) operad $\{\mathcal{P}(1,n)\}_{n\in\mathbb{N}}$ as a substructure.

Definition 1.6. A subset G of $\mathcal{P}(1,1)$ is called a *rescaling group of* $\mathcal{P}$ if the following conditions are satisfied:

1. For any $g, g_1, \dots, g_n \in G, Q \in P(m,n)$, $\gamma_{(i)}(g; Q)$ and $\gamma_{(1,\dots,1)}(Q; g_1, \dots, g_n)$ are always well-defined for $1 \leq i \leq m$.
2. $I_{\mathcal{P}} \in G$ and G together with the identity element $I_{\mathcal{P}}$ and multiplication map $\gamma_{(1)} : G \times G \to G$ is a group.

Definition 1.7. A *G-rescalable partial dioperad* is partial dioperad $\mathcal{P}$ such that for any $P \in \mathcal{P}(m,n)$, $Q_i \in \mathcal{P}(k_i, l_i), i = 1, \dots, n$ there exist $g_i \in G, i = 1, \dots, n$ such that $\gamma_{(i_1,\dots,i_n)}\big(P; \gamma_{(i_1)}(g_1; Q_1), \dots, \gamma_{(i_n)}(g_n; Q_n)\big)$ is well-defined.

The first example of partial dioperad in which we are interested in this work comes from $\mathbb{K} = \{\mathbb{K}(n_-, n_+)\}_{n_-,n_+\in\mathbb{N}}$ [Ko2], a natural extension of sphere partial operad K [H4]. More precisely, $\mathbb{K}(n_-, n_+)$ is the set of the conformal equivalent classes of sphere with n_- (n_+) ordered negatively (positively) oriented punctures and local coordinate map around each puncture. In particular, $\mathbb{K}(0,0)$ is an one-element set consisting of the conformal equivalent class of a sphere with no additional structure. We simply denote this element as $\hat{\mathbb{C}}$. We use

$$\begin{aligned} Q \;&=\; \big(\,(z_{-1}; a_0^{(-1)}, A^{(-1)}), \dots, (z_{-n_-}; a_0^{(-n_-)}, A^{(-n_-)})\,| \\ &\qquad\qquad (z_1; a_0^{(1)}, A^{(1)}) \dots (z_{n_+}; a_0^{(n_+)}, A^{(n_+)})\,\big)_{\mathbb{K}}, \end{aligned} \tag{1.3}$$

where $z_i \in \hat{\mathbb{C}}, a_0^{(i)} \in \mathbb{C}^\times, A^{(i)} \in \mathbb{C}^\infty$ for $i = -n_-, \dots, -1, 1, \dots, n_+$, to denote a sphere $\hat{\mathbb{C}}$ with positively (negatively) oriented punctures at $z_i \in \hat{\mathbb{C}}$ for $i = 1, \dots, n_+$ ($i = -1, \dots, -n_-$), and with local coordinate map f_i around each punctures z_i given by:

$$\begin{aligned} f_i(w) &= e^{\sum_{j=1}^\infty A_j^{(i)} x^{j+1} \frac{d}{dx}} (a_0^{(i)})^{x \frac{d}{dx}} x \bigg|_{x = w - z_i} & \text{if } z_i \in \mathbb{C}, \quad (1.4) \\ &= e^{\sum_{j=1}^\infty A_j^{(i)} x^{j+1} \frac{d}{dx}} (a_0^{(i)})^{x \frac{d}{dx}} x \bigg|_{x = \frac{-1}{w}} & \text{if } z_i = \infty. \quad (1.5) \end{aligned}$$

We introduce a useful notation $\bar{Q}$ defined as follow:

$$\begin{aligned} \bar{Q} = \; & (\,(\bar{z}_{-1}; \overline{a_0^{(-1)}}, \overline{A^{(-1)}}), \dots, (\bar{z}_{-n_-}; \overline{a_0^{(-n_-)}}, \overline{A^{(-n_-)}})| \\ & (\bar{z}_1; \overline{a_0^{(1)}}, \overline{A^{(1)}}) \dots (\bar{z}_{n_+}; \overline{a_0^{(n_+)}}, \overline{A^{(n_+)}})\,)_{\mathbb{K}}, \quad (1.6) \end{aligned}$$

where the "overline" represents complex conjugations.

We denote the set of all such Q as $\mathcal{T}_{\mathbb{K}}(n_-, n_+)$. Let $\mathcal{T}_{\mathbb{K}} := \{\mathcal{T}_{\mathbb{K}}(n_-, n_+)\}_{n_-, n_+ \in \mathbb{N}}$. There is an action of $SL(2, \mathbb{C})$ on $\mathcal{T}_{\mathbb{K}}(n_-, n_+)$ as Mobius transformations. It is clear that

$$\mathbb{K}(n_-, n_+) = \mathcal{T}_{\mathbb{K}}(n_-, n_+)/SL(2, \mathbb{C}). \quad (1.7)$$

We denote the quotient map $\mathcal{T}_{\mathbb{K}} \to \mathbb{K}$ as $\pi_{\mathbb{K}}$. The identity $I_{\mathbb{K}} \in \mathbb{K}(1, 1)$ is given by

$$I_{\mathbb{K}} = \pi_{\mathbb{K}}(\,(\infty, 1, \mathbf{0})|(0, 1, \mathbf{0})\,) \quad (1.8)$$

where $\mathbf{0} = (0, 0, \dots) \in \prod_{n=1}^\infty \mathbb{C}$. The composition ${}_i\circ_j$ is provided by the sewing operation ${}_i\infty_{-j}$ [H4]. In particular, for $n_1, m_2 \geq 1$, $P \in \mathbb{K}(m_1, n_1)$ and $Q \in \mathbb{K}(m_2, n_2)$, $P\,{}_i\infty_{-j}\,Q$ is the sphere with punctures obtained by sewing the i-th positively oriented puncture of P with the j-th negatively oriented puncture of Q. The $S_{n_-} \times S_{n_+}$-action on $\mathbb{K}(n_-, n_+)$ (or $\mathcal{T}(n_-, n_+)$) is the natural one. Moreover, the set

$$\{(\,(\infty, 1, \mathbf{0})|(0, a, \mathbf{0})\,)|a \in \mathbb{C}^\times\} \quad (1.9)$$

together with multiplication ${}_1\infty_1$ is a group which can be canonically identified with group $\mathbb{C}^\times$. It is clear that $\mathbb{K}$ is a $\mathbb{C}^\times$-rescalable partial dioperad. We call it *sphere partial dioperad*.

The $\mathbb{C}$-extensions of $\mathbb{K}$, such as $\mathbb{K}^c$ and $\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$ for $c, c^L, c^R \in \mathbb{C}$, are trivial line bundles over $\mathbb{K}$ with natural $\mathbb{C}^\times$-rescalable partial dioperad structures (see Section 6.8 in [H3]). Moreover, we denote the canonical section $K \to \tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$ as $\psi_{\mathbb{K}}$.

The next example of partial dioperad is $\mathbb{D} = \{\mathbb{D}(n_-, n_+)\}_{n_-, n_+ \in \mathbb{N}}$, which is an extension of the partial operad of disk with strips Υ introduced [HKo1]. More precisely, $\mathbb{D}(n_-, n_+)$ is the set of conformal equivalent classes of disks with ordered punctures on their boundaries and local coordinate map around each puncture. In particular, $\mathbb{D}(0, 0)$ is an one-element set consisting of the conformal equivalent class of a disk with no additional structure. We simply denote this element as $\hat{\mathbb{H}}$. We use

$$\begin{aligned} Q = \; & (\,(r_{-n_-}; b_0^{(-n_-)}, B^{(-n_-)}), \dots, (r_{-1}; b_0^{(-1)}, B^{(-1)})| \\ & (r_1; b_0^{(1)}, B^{(1)}) \dots (r_{n_+}; b_0^{(n_+)}, B^{(n_+)})\,)_{\mathbb{D}}, \quad (1.10) \end{aligned}$$

where $r_i \in \hat{\mathbb{R}}, a_0^{(i)} \in \mathbb{R}_+, B^{(i)} \in \mathbb{R}^\infty$ for $i = -n_-, \dots, -1, 1, \dots, n_+$, to denote a disk $\hat{\mathbb{H}}$ with positively (negatively) oriented punctures at $r_i \in \hat{\mathbb{R}}$ for $i = 1, \dots, n_+$ ($i = -1, \dots, -n_-$), and with local coordinate map g_i around each punctures r_i given by:

$$g_i(w) = e^{\sum_{j=1}^\infty B_j^{(i)} x^{j+1} \frac{d}{dx}} (b_0^{(i)})^{x \frac{d}{dx}} x \Big|_{x=w-r_i} \qquad \text{if } r_i \in \mathbb{R}, \tag{1.11}$$

$$= e^{\sum_{j=1}^\infty B_j^{(i)} x^{j+1} \frac{d}{dx}} (b_0^{(i)})^{x \frac{d}{dx}} x \Big|_{x=\frac{-1}{w}} \qquad \text{if } r_i = \infty. \tag{1.12}$$

We denote the set of all such Q as $\mathcal{T}_{\mathbb{D}}(n_-, n_+)$. Let $\mathcal{T}_{\mathbb{D}} := \{\mathcal{T}_{\mathbb{D}}(n_-, n_+)\}_{n_-, n_+ \in \mathbb{N}}$. The automorphism group of $\mathbb{H}$, $SL(2, \mathbb{R})$, naturally acts on $\mathcal{T}_{\mathbb{D}}(n_-, n_+)$. It is clear that

$$\mathbb{D}(n_-, n_+) = \mathcal{T}_{\mathbb{K}}(n_-, n_+)/SL(2, \mathbb{R}). \tag{1.13}$$

We denote the quotient map as $\pi_{\mathbb{D}}$. The identity $I_{\mathbb{D}} \in \mathbb{D}(1,1)$ is given by

$$I_{\mathbb{D}} = \pi_{\mathbb{D}}(\,(\infty, 1, \mathbf{0})|(0, 1, \mathbf{0})\,). \tag{1.14}$$

The composition ${}_i\circ_j$ is provided by the sewing operation ${}_i\infty_{-j}$ [HKo1]. In particular, for $n_1, m_2 \geq 1$, $P \in \mathbb{D}(m_-, m_+)$ and $Q \in \mathbb{D}(n_-, n_+)$, $P {}_i\infty_{-j} Q$ is the disk with strips obtained by sewing the i-th positively oriented puncture of P with the j-th negatively oriented puncture of Q. The $S_{n_-} \times S_{n_+}$-action on $\mathbb{D}(n_-, n_+)$ (or $\mathcal{T}(n_-, n_+)$) is the natural one. The set

$$\{(\,(\infty, 1, \mathbf{0})|(0, a, \mathbf{0})\,)|a \in \mathbb{R}_+\} \tag{1.15}$$

together with multiplication ${}_1\infty_1$ is a group which can be canonically identified with group $\mathbb{R}_+$. It is clear that $\mathbb{D}$ is a $\mathbb{R}_+$-rescalable partial dioperad. We call it *disk partial dioperad.*

$\mathbb{D}$ can be naturally embedded to $\mathbb{K}$ as a sub-dioperad. The $\mathbb{C}$-extension $\mathbb{D}^c$ of $\mathbb{D}$ for $c \in \mathbb{C}$ is just the restriction of the line bundle $\mathbb{K}^c$ on $\mathbb{D}$. $\mathbb{D}^c$ is also a $\mathbb{R}_+$-rescalable partial dioperad and a partial sub-dioperad of $\mathbb{K}^c$. We denote the canonical section on $\mathbb{D} \to \mathbb{D}^c$ as $\psi_{\mathbb{D}}$.

Now we discuss an example of partial nonassociative dioperad which is important for us. Let $U = \oplus_{n \in J} U_{(n)}$ be a graded vector space and J an index set. We denote the projection $U \to U_{(n)}$ as P_n. Now we consider a family of spaces of multilinear maps $\mathbb{E}_U = \{\mathbb{E}_U(m,n)\}_{m,n \in \mathbb{N}}$, where

$$\mathbb{E}_U(m,n) := \mathrm{Hom}_{\mathbb{C}}(U^{\otimes m}, \overline{U^{\otimes n}}). \tag{1.16}$$

For $f \in \mathbb{E}_U(m,n)$, $g_j \in \mathbb{E}_U(k_j, l_j)$ and $u_{p_j}^{(j)} \in U$, $1 \leq p_j \leq l_j$, $j = 1, \dots, n$, we say that

$$\Gamma_{(i_1, \dots, i_n)}(f; g_1, \dots, g_n)(u_1^{(1)} \otimes \cdots \otimes u_{l_n}^{(n)})$$
$$:= \sum_{s_1, \dots, s_n \in J} f\big(P_{s_1} g_1(u_1^{(1)} \otimes \cdots \otimes u_{l_1}^{(1)}) \otimes \cdots \otimes P_{s_n} g_n(u_1^{(n)} \otimes \cdots \otimes u_{l_n}^{(n)})\big)$$

is well-defined if the multiple sum converges absolutely. This give arise to a partially defined substitution map, for $1 \leq i_j \leq k_j, j = 1, \dots, n$,

$$\Gamma_{i_1, \dots, i_n} : \mathbb{E}_U(m,n) \otimes \mathbb{E}_U(k_1, l_1) \otimes \cdots \otimes \mathbb{E}_U(k_n, l_n) \to \mathbb{E}_U(m + k - n, l)$$

where $k = k_1 + \cdots + k_n$ and $l = l_1 + \cdots + l_n$. In general, the compositions of three substitution maps are not associative. The permutation groups actions on $\mathbb{E}_U$ are the usual one. Let $\Gamma = \{\Gamma_{i_1,\dots,i_n}\}$. It is clear that $(\mathbb{E}_U, \Gamma, \mathrm{id}_U)$ is a partial nonassociative dioperad. We often denote it simply by $\mathbb{E}_U$.

Definition 1.8. Let $(\mathcal{P}, \gamma_{\mathcal{P}}, I_{\mathcal{P}})$ be a partial dioperad. A *$\mathcal{P}$-algebra* (U, ν) consists of a graded vector space U and a morphism of partial nonassociative dioperad $\nu : \mathcal{P} \to \mathbb{E}_U$.

When $U = \oplus_{n\in J} U_{(n)}$ is a completely reducible module for a group G, J is the set of equivalent classes of irreducible G-modules and $U_{(n)}$ is a direct sum of irreducible G-modules of equivalent class $n \in J$, we denote $\mathbb{E}_U$ by $\mathbb{E}_U^G$.

Definition 1.9. Let $(\mathcal{P}, \gamma_{\mathcal{P}}, I_{\mathcal{P}})$ be a G-rescalable partial dioperad. A *G-rescalable $\mathcal{P}$-algebra* (U, ν) is a $\mathcal{P}$-algebra and the morphism $\nu : \mathcal{P} \to \mathbb{E}_U^G$ is so that $\nu|_G : G \to \mathrm{End}\ U$ coincides with the given G-module structure on U.

1.2 Conformal full field algebras

Let $(V^L, Y_{V^L}, \mathbf{1}^L, \omega^L)$ and $(V^R, Y_{V^R}, \mathbf{1}^R, \omega^R)$ be two vertex operator algebras with central charge c^L and c^R respectively, satisfying the conditions in Theorem 0.1. Let $(V_{cl}, m_{cl}, \iota_{cl})$ be a conformal full field algebra over $V^L \otimes V^R$. A bilinear form $(\cdot, \cdot)_{cl}$ on V_{cl} is invariant [Ko1] if, for any $u, w_1, w_2 \in V_{cl}$,

$$\begin{aligned}&(w_2, \mathbb{Y}_f(u; x, \bar{x}) w_1)_{cl} \\ &\quad = (\mathbb{Y}_f(e^{-xL^L(1)} x^{-2L^L(0)} \otimes e^{-\bar{x}L^R(1)} \bar{x}^{-2L^R(0)}\, u; e^{\pi i} x^{-1}, e^{-\pi i} \bar{x}^{-1}) w_2, w_1)_{cl}.\end{aligned} \tag{1.17}$$

or equivalently,

$$\begin{aligned}&(\mathbb{Y}_f(u; e^{\pi i} x, e^{-\pi i} \bar{x}) w_2, w_1)_{cl} \\ &\quad = (w_2, \mathbb{Y}_f(e^{xL^L(1)} x^{-2L^L(0)} \otimes e^{\bar{x}L^R(1)} \bar{x}^{-2L^R(0)}\, u; x^{-1}, \bar{x}^{-1}) w_1)_{cl}.\end{aligned} \tag{1.18}$$

We showed in [Ko1] that an invariant bilinear form on V_{cl} is automatically symmetric. Namely, for $u_1, u_2 \in V_{cl}$, we have

$$(u_1, u_2)_{cl} = (u_2, u_1)_{cl}. \tag{1.19}$$

V_{cl} has a countable basis. We choose it to be $\{e_i\}_{i\in\mathbb{N}}$. Assume that $(\cdot, \cdot)_{cl}$ is also nondegenerate, we also have the dual basis $\{e^i\}_{i\in\mathbb{N}}$. Then we define a linear map $\Delta_{cl} : \mathbb{C} \to \overline{V_{cl} \otimes V_{cl}}$ as follow:

$$\Delta_{cl} : 1 \mapsto \sum_{i\in\mathbb{N}} e_i \otimes e^i. \tag{1.20}$$

The correlation functions maps $m_{cl}^{(n)}, n \in \mathbb{N}$ of V_{cl} are canonically determined by $\mathbb{Y}$ and the identity $\mathbf{1}_{cl} := \iota_{cl}(\mathbf{1}^L \otimes \mathbf{1}^R)$ [Ko2].

For $Q \in \mathcal{T}_{\mathbb{K}}(n_-, n_+)$ given in (1.3), we define, for $\lambda \in \mathbb{C}$,

$$\nu_{cl}\big(\lambda \psi_{\mathbb{K}}(\pi_{\mathbb{K}}(Q))\big)(u_1 \otimes \cdots \otimes u_{n_+}) \tag{1.21}$$

in the following three cases:

1. If $z_k \neq \infty$ for $k = -n_-, \dots, -1, 1, \dots, n_+$, (1.21) is given by

$$\lambda \sum_{i_1,\dots,i_{n_-}\in\mathbb{N}} \Big(\mathbf{1}_{cl},\, m_{cl}^{(n_-+n_+)}(e^{-L_+^L(A^{(-1)})-L_+^R(\overline{A^{(-1)}})}(a_0^{(-1)})^{-L^L(0)}\overline{a_0^{(-1)}}^{-L^R(0)}e_{i_1},$$
$$\dots, e^{-L_+^L(A^{(-n_-)})-L_+^R(\overline{A^{(-n_-)}})}(a_0^{(-n_-)})^{-L^L(0)}\overline{a_0^{(-n_-)}}^{-L^R(0)}e_{i_{n_-}},$$
$$e^{-L_+^L(A^{(1)})-L_+^R(\overline{A^{(1)}})}(a_0^{(1)})^{-L^L(0)}\overline{a_0^{(1)}}^{-L^R(0)}u_1,$$
$$\dots, e^{-L_+^L(A^{(n_+)})-L_+^R(\overline{A^{(n_+)}})}(a_0^{(n_+)})^{-L^L(0)}\overline{a_0^{(n_+)}}^{-L^R(0)}u_{n_+};$$
$$z_{-1},\bar{z}_{-1},\dots,z_{-n_-},\bar{z}_{-n_-},z_1,\bar{z}_1,\dots,z_{n_+},\bar{z}_{n_+})\Big)_{cl} e^{i_1}\otimes\cdots\otimes e^{i_{n_-}} \quad (1.22)$$

   where $L_+^L(A) = \sum_{j=1}^{\infty} A_j L_+^L(j)$ and $L_+^R(A) = \sum_{j=1}^{\infty} A_j L_+^R(j)$ for $A = (A_1, A_2, \dots) \in \prod_{j=1}^{\infty} \mathbb{C}$;

2. If $\exists\, k \in \{-n_-, \dots, -1\}$ such that $z_k = \infty$ (recall (1.5)), (1.21) is given by the formula obtained from (1.22) by exchanging $\mathbf{1}_{cl}$ in (1.22) with

$$e^{-L_+^L(A^{(k)})-L_+^R(\overline{A^{(k)}})}(a_0^{(k)})^{-L^L(0)}\overline{a_0^{(k)}}^{-L^R(0)}e_{i_{-k}}; \quad (1.23)$$

3. If $\exists\, k \in \{1, \dots, n_+\}$ such that $z_k = \infty$ (recall (1.5)), (1.21) is given by the formula obtained from (1.22) by exchanging $\mathbf{1}_{cl}$ in (1.22) with

$$e^{-L_+^L(A^{(k)})-L_+^R(\overline{A^{(k)}})}(a_0^{(k)})^{-L^L(0)}\overline{a_0^{(k)}}^{-L^R(0)}u_k. \quad (1.24)$$

The following result is proved in [Ko1].

Proposition 1.10. *The map ν_{cl} is $SL(2,\mathbb{C})$-invariant,*

Hence ν_{cl} induces a map $\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}} \to \mathbb{E}_{V_{cl}}^{\mathbb{C}^\times}$, which is still denoted as ν_{cl}. Some interesting special cases are listed below:

$$\begin{aligned}
\nu_{cl}(\psi_{\mathbb{K}}(\hat{\mathbb{C}})) &= (\mathbf{1}_{cl}, \mathbf{1}_{cl})_{cl}\mathrm{id}_{\mathbb{C}},\\
\nu_{cl}(\psi_{\mathbb{K}}(\pi_{\mathbb{K}}((\,(\infty,1,\mathbf{0})|\,)_{\mathbb{K}})) &= \mathbf{1}_{cl},\\
\nu_{cl}(\psi_{\mathbb{K}}(\pi_{\mathbb{K}}((\,|(\infty,1,\mathbf{0}),(0,1,\mathbf{0})\,)_{\mathbb{K}})))(u\otimes v) &= (u,v)_{cl},\\
\nu_{cl}(\psi_{\mathbb{K}}(\pi_{\mathbb{K}}((\,(\infty,1,\mathbf{0}),(0,1,\mathbf{0})|\,)_{\mathbb{K}}))) &= \Delta_{cl},\\
\nu_{cl}(\psi_{\mathbb{K}}(\pi_{\mathbb{K}}((\,(\infty;1,\mathbf{0})|(z;1,\mathbf{0}),(0;1,\mathbf{0})\,)_{\mathbb{K}})))(u\otimes v) &= \mathbb{Y}(u;z,\bar{z})v,
\end{aligned}$$

where $\hat{\mathbb{C}}$ is the single element in $\mathbb{K}(0,0)$.

Definition 1.11. A $\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$-algebra (U,ν) is called *smooth* if

1. $U = \oplus_{m,n\in\mathbb{R}} U_{(m,n)}$ is a completely reducible $\mathbb{C}^\times$-module, where $z\cdot u = z^m\bar{z}^n u, \forall z \in \mathbb{C}^\times, u \in U_{(m,n)}$.
2. $\dim U_{(m,n)} < \infty, \forall m,n \in \mathbb{R}$ and $\dim U_{(m,n)} = 0$ for m or n sufficiently small.
3. ν is linear on fiber and smooth on the base space $\mathbb{K}$.

The results in [Ko1] on a conformal full field algebra over $V^L \otimes V^R$ equipped with a nondegenerate invariant bilinear form can be restated as the following theorem.

Theorem 1.12. (V_{cl}, ν_{cl}) *is a smooth* $\tilde{\mathbb{K}}^{c^L} \otimes \overline{\tilde{\mathbb{K}}^{\overline{c^R}}}$*-algebra.*

1.3 Open-string vertex operator algebras

Let $(V_{op}, Y_{op}, \mathbf{1}_{op}, \omega_{op})$ be an open-string vertex operator algebra. For $r > 0$ and $v_1, v_2 \in V_{op}$, we define $Y_{op}(v_1, -r)v_2$ by

$$Y_{op}(v_1, -r)v_2 := e^{-rL(-1)}Y_{op}(v_2, r)v_1. \tag{1.25}$$

Remark 1.13. Taking the analogy between open-string vertex operator algebra and associative algebra, $Y_{op}(\cdot, -r)\cdot$ corresponds to the opposite product [HKo1].

An invariant bilinear form on an open-string vertex operator algebra V_{op} is a bilinear form $(\cdot, \cdot)_{op}$ on V_{op} satisfying the following properties:

$$(v_3, Y_{op}(v_1, r)v_2)_{op} = (Y_{op}(e^{-rL(1)}r^{-2L(0)}\, v_1, -r^{-1})v_3, v_2)_{op} \tag{1.26}$$
$$(Y_{op}(v_1, r)v_3, v_2)_{op} = (v_3, Y_{op}(e^{-rL(1)}r^{-2L(0)}v_1, -r^{-1})v_2)_{op} \tag{1.27}$$

for $r > 0$ and $v_1, v_2, v_3 \in V_{op}$.

Lemma 1.14.
$$(v_1, v_2)_{op} = (v_2, v_1)_{op} \tag{1.28}$$

Proof. The proof is exactly same as that of Proposition 2.3 in [Ko1]. ∎

We further assume that $(\cdot, \cdot)_{op}$ is nondegenerate. Let $\{f_i\}_{i\in\mathbb{R}}$ be a basis of V_{op} and $\{f^i\}_{i\in\mathbb{R}}$ its dual basis. We define linear map $\Delta_{op} : \mathbb{C} \to \overline{V_{cl} \otimes V_{cl}}$ as follow:

$$\Delta_{op} : 1 \mapsto \sum_{i\in\mathbb{R}} f_i \otimes f^i. \tag{1.29}$$

The open-string vertex operator algebra $(V_{op}, Y_{op}, \mathbf{1}_{op}, \omega_{op})$ naturally gives a boundary field algebra $(V_{op}, m_{op}, \mathbf{d}_{op}, D_{op})$ in which the correlation-function maps $m_{op}^{(n)}, n \in \mathbb{N}$ are completely determined by Y_{op} and $\mathbf{1}_{op}$ [Ko2].

For any $Q \in \mathcal{T}_{\mathbb{D}}(n_-, n_+)$ given in (1.10). Let α be a bijective map

$$\{-n_-, \dots, -1, 1, \dots, n_+\} \xrightarrow{\alpha} \{1, \dots, n_- + n_+\} \tag{1.30}$$

so that $s_1, \dots, s_{n_-+n_+}$, defined by $s_i := r_{\alpha^{-1}(i)}$, satisfy $\infty \geq s_1 > \cdots > s_{n_-+n_+} \geq 0$. Then we define, for $\lambda \in \mathbb{C}$,

$$\nu_{op}(\lambda\psi_{\mathbb{D}}(\pi_{\mathbb{D}}(Q)))(v_1 \otimes \cdots \otimes v_{n_+}) \tag{1.31}$$

as follow:

1. If $r_k \neq \infty, \forall k = -n_-, \dots, -1, 1, \dots, n_+$, (1.31) is given by
$$\lambda \sum_{i_1,\dots,i_{n_-}\in\mathbb{R}} \big(\mathbf{1}_{op},\, m_{op}^{(n_-+n_+)}(w_1, \dots, w_{n_-+n_+}; s_1, \dots, s_{n_-+n_+})\big)_{op} f^{i_1} \otimes \cdots \otimes f^{i_{n_-}} \tag{1.32}$$
where $w_{\alpha(p)} = e^{-L_+(B^{(p)})}(b_0^{(p)})^{-L(0)}f_{i_{-p}}$ and $w_{\alpha(q)} = e^{-L_+(B^{(q)})}(b_0^{(q)})^{-L(0)}v_q$ for $p = -1, \cdots - n_-$ and $q = 1, \dots, n_+$;

2. If $\exists\, k \in \{-n_-, \dots, -1, 1, \dots, n_+\}$ such that $r_k = \infty$, (1.31) is given by the formula obtained from (1.32) by exchanging the $\mathbf{1}_{op}$ with $w_{\alpha(k)}$.

Proposition 1.15. *ν_{op} is $SL(2, \mathbb{R})$ invariant.*

Proof. The proof is same as that of Proposition 1.10. ∎

Hence ν_{op} induces a map $\tilde{\mathbb{D}}^c \to \mathbb{E}^{\mathbb{R}_+}_{V_{op}}$, which is still denoted as ν_{op}. Some interesting special cases are listed explicitly below:

$$\begin{aligned}
\nu_{op}(\psi_{\mathbb{D}}(\hat{\mathbb{H}})) &= (\mathbf{1}_{op}, \mathbf{1}_{op})_{op}\mathrm{id}_{\mathbb{C}}, \\
\nu_{op}(\psi_{\mathbb{D}}(\pi_{\mathbb{D}}((\,(\infty, 1, \mathbf{0})|\,)))) &= \mathbf{1}_{op}, \\
\nu_{op}(\psi_{\mathbb{D}}(\pi_{\mathbb{D}}((\,|(\infty, 1, \mathbf{0}), (0, 1, \mathbf{0})\,))))(u \otimes v) &= (u, v)_{op}, \\
\nu_{op}(\psi_{\mathbb{D}}(\pi_{\mathbb{D}}((\,(\infty, 1, \mathbf{0}), (0, 1, \mathbf{0})|\,)))) &= \Delta_{op}, \\
\nu_{op}(\psi_{\mathbb{D}}(\pi_{\mathbb{D}}((\,(\infty; 1, \mathbf{0})|(r; 1, \mathbf{0}), (0; 1, \mathbf{0})\,))))(u \otimes v) &= Y_{op}(u, r)v.
\end{aligned}$$

where $\hat{\mathbb{H}}$ is the single element in $\mathbb{D}(0, 0)$ and $r > 0$.

Definition 1.16. A $\tilde{\mathbb{D}}^c$-algebra (U, ν) is called *smooth* if

1. $U = \oplus_{n \in \mathbb{R}} U_{(n)}$ is a completely reducible $\mathbb{R}_+$-module, where $r \cdot u = r^n u, \forall r \in \mathbb{R}_+, u \in U_{(m,n)}$.

2. $\dim U_{(n)} < \infty, \forall n \in \mathbb{R}$ and $\dim U_{(n)} = 0$ for n sufficiently small.

3. ν is linear on fiber and smooth on the base space $\mathbb{D}$.

Theorem 1.17. *(V_{op}, ν_{op}) is a smooth $\tilde{\mathbb{D}}^c$-algebra.*

Proof. The proof is same as that of Theorem 1.12. ∎

2 Swiss-cheese partial dioperad

In Section 3.1, we introduce the notion of 2-colored partial dioperad and algebra over it. In Section 3.2, we study a special example of 2-colored partial dioperad called Swiss-cheese partial dioperad $\mathbb{S}$ and its $\mathbb{C}$-extension $\tilde{\mathbb{S}}^c$. In Section 3.3, we show that an open-closed field algebra over V equipped with nondegenerate invariant bilinear forms canonically gives an algebra over $\tilde{\mathbb{S}}^c$. In Section 3.4, we define boundary states in such algebra and show that some of the boundary states are Ishibashi states.

2.1 2-colored (partial) dioperads

Definition 2.1. A *right module over a dioperad* $(\mathcal{Q}, \gamma_{\mathcal{Q}}, I_{\mathcal{Q}})$, or a right $\mathcal{Q}$-module, is a family of sets $\{\mathcal{P}(m, n)\}_{m,n \in \mathbb{N}}$ with an $S_m \times S_n$-action on each set $\mathcal{P}(m, n)$ and substitution maps:

$$\begin{array}{ccc}
\mathcal{P}(m, n) \times \mathcal{Q}(k_1, l_1) \times \cdots \times \mathcal{Q}(k_n, l_n) & \xrightarrow{\gamma_{(i_1, \dots, i_n)}} & \mathcal{P}(m + k_1 \cdots + k_n - n, l_1 + \cdots + l_n) \\
(P, Q_1, \dots, Q_n) & \mapsto & \gamma_{(i_1, \dots, i_n)}(P; Q_1, \dots, Q_n)
\end{array} \tag{2.1}$$

for $m, n, l_1, \dots, l_n \in \mathbb{N}, k_1, \dots, k_n \in \mathbb{Z}_+$ and $1 \leq i_j \leq k_j, j = 1, \dots, n$, satisfying the right unit property, the associativity and the permutation axioms of dioperad but with the right action of $\mathcal{P}$ on itself in the definition of dioperad replaced by that of $\mathcal{Q}$.

Homomorphism and isomorphism between two right $\mathcal{Q}$-modules can be naturally defined. The right module over a partial dioperad can also be defined in the usual way.

Definition 2.2. Let $\mathcal{Q}$ be a G-rescalable partial dioperad. A right $\mathcal{Q}$-module is called G-rescalable if for any $P \in \mathcal{P}(m_-, m_+)$, $Q_i \in \mathcal{Q}(n_-^{(i)}, n_+^{(i)})$ and $1 \le j_i \le n_-^{(i)}, i = 1, \dots, m_+$, there exist $g_j \in G, j = 1, \dots, m_+$ such that

$$\gamma_{(j_1,\dots,j_{m_+})}\big(P; \gamma_{(j_1)}(g_1; Q_1), \dots, \gamma_{(j_{m_+})}(g_{m_+}; Q_{m_+})\big) \tag{2.2}$$

is well-defined.

Definition 2.3. A 2*-colored dioperad* consists of a dioperad $(\mathcal{Q}, \gamma_\mathcal{Q}, I_\mathcal{Q})$ and a family of sets $\mathcal{P}(n_-^B, n_+^B | n_-^I, n_+^I)$ equipped with a $S_{n_-^B} \times S_{n_+^B} \times S_{n_-^I} \times S_{n_+^I}$-action for $n_\pm^B, n_\pm^I \in \mathbb{N}$, a distinguished element $I_\mathcal{P} \in \mathcal{P}(1, 1|0, 0)$, and maps

$$\begin{aligned}
&\mathcal{P}(m_-, m_+|n_-, n_+) \times \mathcal{P}(k_-^{(1)}, k_+^{(1)}|l_-^{(1)}, l_+^{(1)}) \times \cdots \times \mathcal{P}(k_-^{(m_+)}, k_+^{(m_+)}|l_-^{(m_+)}, l_+^{(m_+)}) \\
&\qquad \xrightarrow{\gamma^B_{(i_1,\dots,i_{m_+})}} \mathcal{P}(m_- - m_+ + k_-, k_+ | n_- + l_-, n_+ + l_+), \\
&\mathcal{P}(m_-, m_+|n_-, n_+) \times \mathcal{Q}(p_-^{(1)}, p_+^{(1)}) \times \cdots \times \mathcal{Q}(p_-^{(n_+)}, p_+^{(n)}) \\
&\qquad \xrightarrow{\gamma^I_{(j_1,\dots,j_{n_+})}} \mathcal{P}(m_-, m_+|n_- - n_+ + p_-, p_+)
\end{aligned} \tag{2.3}$$

where $k_\pm = k_\pm^{(1)} + \cdots + k_\pm^{(m_+)}$, $l_\pm = l_\pm^{(1)} + \cdots + l_\pm^{(m_+)}$ and $p_\pm = p_\pm^{(1)} + \cdots + p_\pm^{(n_+)}$, for $1 \le i_r \le k_-^{(r)}, r = 1, \dots, m_+$, $1 \le j_s \le p_-^{(s)}, s = 1, \dots, n_+$, satisfying the following axioms:

1. The family of sets $\mathcal{P} := \{\mathcal{P}(m_-, m_+)\}_{m_-,m_+\in\mathbb{N}}$, where

$$\mathcal{P}(m_-, m_+) := \cup_{n_-,n_+\in\mathbb{N}} \mathcal{P}(m_-, m_+|n_-, n_+),$$

   equipped with the natural $S_{m_-} \times S_{m_+}$-action on $\mathcal{P}(m_-, m_+)$, together with identity element $I_\mathcal{P}$ and the family of maps $\gamma^B := \{\gamma^B_{(i_1,\dots,i_{m_+})}\}$ gives an dioperad.

2. The family of maps $\gamma^I := \{\gamma^I_{(j_1,\dots,j_{n_+})}\}$ gives each $\mathcal{P}(m_-, m_+)$ a right $\mathcal{Q}$-module structure for $m_-, m_+ \in \mathbb{N}$.

We denote such 2-colored dioperad as $(\mathcal{P}|\mathcal{Q}, (\gamma^B, \gamma^I))$. 2*-colored partial dioperad* can be naturally defined. If the associativities of $\gamma_\mathcal{Q}, \gamma^B, \gamma^I$ do not hold, then it is called 2*-colored (partial) nonassociative dioperad*.

Remark 2.4. If we restrict to $\{\mathcal{P}(1, m|0, n)\}_{m,n\in\mathbb{N}}$ and $\{\mathcal{Q}(1, n)\}_{n\in\mathbb{N}}$, they simply gives a structure of 2-colored (partial) operad [V][Kont][Ko2].

Now we discuss an important example of 2-colored partial nonassociative dioperad. Let J_1, J_2 be two index sets. $U_i = \oplus_{n\in J_i}(U_i)_{(n)}, i = 1, 2$ be two graded vector spaces. Consider two families of sets,

$$\begin{aligned}
\mathbb{E}_{U_2}(m, n) &= \mathrm{Hom}_\mathbb{C}(U_2^{\otimes m}, \overline{U_2^{\otimes n}}), \\
\mathbb{E}_{U_1|U_2}(m_-, m_+|n_-, n_+) &= \mathrm{Hom}_\mathbb{C}(U_1^{\otimes m_+} \otimes U_2^{\otimes n_+}, \overline{U_1^{\otimes m_-} \otimes U_2^{\otimes n_-}}),
\end{aligned}$$

for $m, n, m_\pm, n_\pm \in \mathbb{N}$. We denote both of the projection operators $U_1 \to (U_1)_{(n)}$, $U_2 \to (U_2)_{(n)}$ as P_n for $n \in J_1$ or J_2. For $f \in \mathbb{E}_{U_1|U_2}(k_-, k_+|l_-, l_+)$, $g_i \in \mathbb{E}_{U_1|U_2}(m_-^{(i)}, m_+^{(i)}|n_-^{(i)}, n_+^{(i)})$, $i = 1, \dots, k_+$, and $h_j \in \mathbb{E}_{U_2}(p_-^{(j)}, p_+^{(j)})$, $j = 1, \dots, l_+$, we say that

$$\begin{aligned}
&\Gamma^B_{(i_1,\dots,i_{k_+})}(f; g_1, \dots, g_{k_+})(u_1^{(1)} \otimes \cdots \otimes v^{(k_+)}_{n_+^{(k_+)}} \otimes v_1 \otimes \cdots \otimes v_{l_+}) \\
&\quad := \sum_{s_1,\dots,s_k \in J_1} f\big(P_{s_1} g_1(u_1^{(1)} \otimes \cdots \otimes u^{(1)}_{m_+^{(1)}} \otimes v_1^{(1)} \otimes \cdots \otimes v^{(1)}_{n_+^{(1)}}) \otimes \\
&\quad \cdots \otimes P_{s_{k_+}} g_{k_+}(u_1^{(k_+)} \otimes \cdots \otimes u^{(k_+)}_{m_+^{(k_+)}} \otimes v_1^{(k_+)} \otimes \cdots \otimes v^{(k_+)}_{n_+^{(k_+)}}) \otimes v_1 \otimes \cdots \otimes v_{l_+}\big) \\
&\Gamma^I_{(j_1,\dots,j_{l_+})}(f; h_1, \dots, h_{l_+})(u_1 \otimes \cdots \otimes u_{k_+} \otimes w_1^{(1)} \otimes \cdots \otimes w^{(l_+)}_{p_+^{(l_+)}}) \\
&\quad := \sum_{t_1,\dots,t_l \in J_2} f\big(u_1 \otimes \cdots \otimes u_{k_+} \otimes P_{t_1} h_1(w_1^{(1)} \otimes \cdots \otimes w^{(1)}_{p_+^{(1)}}) \\
&\qquad\qquad \otimes \cdots \otimes P_{t_{l_+}} h_{l_+}(w_1^{(l_+)} \otimes \cdots \otimes w^{(l_+)}_{p_+^{(l_+)}})\big)
\end{aligned}$$

for $u_j^{(i)} \in U_1, v_j^{(i)}, w_j^{(i)} \in U_2$, are well-defined if each multiple sum is absolutely convergent. These give arise to partially defined substitution maps:

$$\begin{aligned}
&\mathbb{E}_{U_1|U_2}(k_-, k_+|l_-, l_+) \otimes \mathbb{E}_{U_1|U_2}(m_-^{(1)}, m_+^{(1)}|n_-^{(1)}, n_+^{(1)}) \otimes \cdots \otimes \mathbb{E}_{U_1|U_2}(m_-^{(k_+)}, m_+^{(k_+)}|n_-^{(k_+)}, n_+^{(k_+)}) \\
&\qquad \xrightarrow{\Gamma^B_{(i_1,\dots,i_{k_+})}} \mathbb{E}_{U_1|U_2}(k_- - k_+ + m_-, m_+|l_- + n_-, n_+). \\
&\mathbb{E}_{U_1|U_2}(k_-, k_+|l_-, l_+) \otimes \mathbb{E}_{U_2}(p_-^{(1)}, p_+^{(1)}) \otimes \cdots \otimes \mathbb{E}_{U_2}(p_-^{(l_+)}, p_+^{(l_-)}) \\
&\qquad \xrightarrow{\Gamma^I_{(j_1+\cdots+j_{l_+})}} \mathbb{E}_{U_1|U_2}(k_-, k_+|l_- - l_+ + p_-^{(1)} + \cdots + p_-^{(l_+)}, p_+^{(1)} + \cdots + p_+^{(l_+)}).
\end{aligned}$$

where $m_\pm = m_\pm^{(1)} + \cdots + m_\pm^{(k_+)}$ and $n_\pm = n_\pm^{(1)} + \cdots + n_\pm^{(k_+)}$. In general, $\Gamma^B_{(i_1,\dots,i_{k_+})}$ and $\Gamma^I_{(j_1+\cdots+j_{l_+})}$ do not satisfy the associativity. Let

$$\mathbb{E}_{U_1|U_2} = \{\mathbb{E}_{U_1|U_2}(m_-, m_+|n_-, n_+)\}_{m_\pm, n_\pm \in \mathbb{N}},$$

$\mathbb{E}_{U_2} = \{\mathbb{E}_{U_2}(n)\}_{n \in \mathbb{N}}$, $\Gamma^B := \{\Gamma^B_{(i_1,\dots,i_n)}\}$ and $\Gamma^I := \{\Gamma^I_{(j_1,\dots,j_n)}\}$. It is obvious that $(\mathbb{E}_{U_1|U_2}|\mathbb{E}_{U_2}, (\Gamma^B, \Gamma^I))$ is a 2-colored partial nonassociative operad.

Let U_1 be a completely reducible G_1-modules and U_2 a completely reducible G_2-modules. Namely, $U_1 = \oplus_{n_1 \in J_1} (U_1)_{(n_1)}, U_2 = \oplus_{n_2 \in J_2} (U_2)_{(n_2)}$ where J_i is the set of equivalent classes of irreducible G_i-modules and $(U_i)_{(n_i)}$ is a direct sum of irreducible G_i-modules of equivalent class n_i for $i = 1, 2$. In this case, we denote $\mathbb{E}_{U_1|U_2}$ by $\mathbb{E}^{G_1|G_2}_{U_1|U_2}$.

Definition 2.5. A homomorphism between two 2-colored (partial) dioperads

$$(\mathcal{P}_i|\mathcal{Q}_i, (\gamma_i^B, \gamma_i^I)), i = 1, 2$$

consists of two (partial) dioperad homomorphisms:

$$\nu_{\mathcal{P}_1|\mathcal{Q}_1} : \mathcal{P}_1 \to \mathcal{P}_2, \qquad \text{and} \qquad \nu_{\mathcal{Q}_1} : \mathcal{Q}_1 \to \mathcal{Q}_2$$

such that $\nu_{\mathcal{P}_1|\mathcal{Q}_1} : \mathcal{P}_1 \to \mathcal{P}_2$, where $\mathcal{P}_2$ has a right $\mathcal{Q}_1$-module structure induced by dioperad homomorphism $\nu_{\mathcal{Q}_1}$, is also a right $\mathcal{Q}_1$-module homomorphism.

Definition 2.6. An *algebra over a* 2*-colored partial dioperad* $(\mathcal{P}|\mathcal{Q}, (\gamma^B, \gamma^I))$, or a $\mathcal{P}|\mathcal{Q}$-*algebra* consists of two graded vector spaces U_1, U_2 and a 2-colored partial dioperad homomorphism $(\nu_{\mathcal{P}|\mathcal{Q}}, \nu_{\mathcal{Q}}) : (\mathcal{P}|\mathcal{Q}, (\gamma^B, \gamma^I)) \to (\mathbb{E}_{U_1|U_2}|\mathbb{E}_{U_2}, (\Gamma^B, \Gamma^I))$. We denote this algebra as $(U_1|U_2, \nu_{\mathcal{P}|\mathcal{Q}}, \nu_{\mathcal{Q}})$.

Definition 2.7. If a 2-colored partial dioperad $(\mathcal{P}|\mathcal{Q}, \gamma)$ is so that $\mathcal{P}$ is a G_1-rescalable partial operad and a G_2-rescalable right $\mathcal{Q}$-module, then it is called $G_1|G_2$*-rescalable.*

Definition 2.8. A $G_1|G_2$*-rescalable* $\mathcal{P}|\mathcal{Q}$*-algebra* $(U_1|U_2, \nu_{\mathcal{P}|\mathcal{Q}}, \nu_{\mathcal{Q}})$ is a $\mathcal{P}|\mathcal{Q}$-algebra so that $\nu_{\mathcal{P}|\mathcal{Q}} : \mathcal{P} \to \mathbb{E}^{G_1|G_2}_{U_1|U_2}$ and $\nu_{\mathcal{Q}} : \mathcal{Q} \to \mathbb{E}^{G_2}_{U_2}$ are dioperad homomorphisms such that $\nu_{\mathcal{P}|\mathcal{Q}} : G_1 \to \mathrm{End}\ U_1$ coincides with the G_1-module structure on U_1 and $\nu_{\mathcal{Q}} : G_2 \to \mathrm{End}\ U_2$ coincides with the G_2-module structure on U_2.

2.2 Swiss-cheese partial dioperads

A disk with strips and tubes of type $(m_-, m_+; n_-, n_+)$ [Ko2] is a disk consisting of m_+ (m_-) ordered positively (negatively) oriented punctures on the boundary of the disk, and n_+ (n_-) ordered positively (negatively) oriented punctures in the interior of the disk, and local coordinate map around each puncture. Two disks are conformal equivalent if there exists a biholomorphic map between them preserving order, orientation and local coordinate maps. We denote the moduli space of disks with strips and tubes of type $(m_-, m_+; n_-, n_+)$ as $\mathbb{S}(m_-, m_+|n_-, n_+)$.

We use the following notation

$$
\begin{aligned}
\big[\,(\,(r_{-m_-}, b_0^{-m_-}, B^{(-m_-)}), \ldots, (r_{-1}, b_0^{-1}, B^{(-1)})\ |\\
(r_1, b_0^1, B^{(1)}), \ldots, (r_{m_+}, b_0^{m_+}, B^{(m_+)})\,)\ \|\\
(\,(z_{-n_-}, a_0^{-n_-}, A^{(-n_-)}), \ldots, (z_{-1}, a_0^{-1}, A^{(-1)})\ |\\
(z_1, a_0^1, A^{(1)}), \ldots, (z_{n_+}, a_0^{n_+}, A^{(n_+)})\,)\big]_{\mathbb{S}}
\end{aligned} \tag{2.4}
$$

where $r_i \in \hat{\mathbb{R}}$, $b_0^{(i)} \in \mathbb{R}^\times$, $B_k^{(i)} \in \mathbb{R}$ and $z_j \in \mathbb{H}$, $a_0^{(j)} \in \mathbb{C}^\times$, $A_l^{(j)} \in \mathbb{C}$ for all $i = -m_-, \ldots, -1, 1, \ldots, m_+$, $j = -n_-, \ldots, -1, 1, \ldots, n_+$ and $k, l \in \mathbb{Z}_+$, to represent a disk with strips at r_i with local coordinate map f_i and tubes at z_j with local coordinate map g_j given as follow:

$$
\begin{aligned}
f_i(w) &= e^{\sum_k B_k^{(i)} x^{k+1} \frac{d}{dx}} (b_0^{(i)})^{x \frac{d}{dx}} x|_{x=w-r_i} \quad \text{if } r_i \in \mathbb{R}\ , && (2.5)\\
&= e^{\sum_k B_k^{(i)} x^{k+1} \frac{d}{dx}} (b_0^{(i)})^{x \frac{d}{dx}} x|_{x=\frac{-1}{w}} \quad \text{if } r_i = \infty, && (2.6)\\
g_j(w) &= e^{\sum_k A_k^{(j)} x^{k+1} \frac{d}{dx}} (a_0^{(j)})^{x \frac{d}{dx}} x|_{x=w-z_j}. && (2.7)
\end{aligned}
$$

We denote the set of all such disks given in (2.4) as $\mathcal{T}_{\mathbb{S}}(m_-, m_+|n_-, n_+)$. The automorphisms of the upper half plane, which is $SL(2,\mathbb{R})$, change the disk (2.4) to a different but conformal equivalent disk. It is clear that we have

$$
\mathbb{S}(m_-, m_+|n_-, n_+) = \mathcal{T}_{\mathbb{S}}(m_-, m_+|n_-, n_+)/SL(2,\mathbb{R}). \tag{2.8}
$$

Let $\mathbb{S} = \{\mathbb{S}(m_-, m_+|n_-, n_+)\}_{m_-, m_+, n_-, n_+ \in \mathbb{N}}$. The permutation groups

$$
(S_{m_-} \times S_{m_+}) \times (S_{n_-} \times S_{n_+})
$$

acts naturally on $\mathbb{S}(m_-, m_+|n_-, n_+)$. There are so-called *boundary sewing operations* [HKo1][Ko2] on $\mathbb{S}$, denoted as ${}_i\infty^B_{-j}$, which sews the i-th positively oriented boundary puncture of the first disk with the j-th negatively oriented boundary puncture of the second disk. Boundary sewing operations naturally induce the following maps:

$$\mathbb{S}(m_-, m_+|n_-, n_+) \times \mathbb{S}(k_-^{(1)}, k_+^{(1)}|l_-^{(1)}, l_+^{(1)}) \times \cdots \times \mathbb{S}(k_-^{(m_+)}, k_+^{(m_+)}|l_-^{(m_+)}, l_+^{(m_+)})$$
$$\xrightarrow{\gamma^B_{(i_1,\ldots,i_{m_+})}} \mathbb{S}(m_- - m_+ + k_-, k_+ | n_- + l_-, n_+ + l_+),$$

where $k_\pm = k_\pm^{(1)} + \cdots + k_\pm^{(m_+)}$, $l_\pm = l_\pm^{(1)} + \cdots + l_\pm^{(m_+)}$ for $1 \le i_r \le k_-^{(r)}, r = 1, \ldots, m_+$. It is easy to see that boundary sewing operations or $\gamma^B_{(i_1,\ldots,i_{m_+})}$, together with permutation group actions on the order of boundary punctures, provide $\mathbb{S}$ with a structure of partial dioperad.

There are also so-called *interior sewing operations* [HKo1][Ko2] on $\mathbb{S}$, denoted as ${}_i\infty^I_{-j}$, which sew the i-th positively oriented interior puncture of a disk with the j-th negatively oriented puncture of a sphere. The interior sewing operations define a right action of $\mathbb{K}$ on $\mathbb{S}$:

$$\mathbb{S}(m_-, m_+|n_-, n_+) \times \mathbb{K}(p_-^{(1)}, p_+^{(1)}) \times \cdots \times \mathbb{K}(p_-^{(n_+)}, p_+^{(n_+)})$$
$$\xrightarrow{\gamma^I_{(j_1,\ldots,j_{n_+})}} \mathbb{S}(m_-, m_+|n_- - n_+ + p_-, p_+) \tag{2.9}$$

where $p_\pm = p_\pm^{(1)} + \cdots + p_\pm^{(n_+)}$, for $1 \le j_s \le p_-^{(s)}, s = 1, \ldots, n_+$. Such action gives $\mathbb{S}$ a right $\mathbb{K}$-module structure.

Let $\gamma^B = \{\gamma^B_{(i_1,\ldots,i_n)}\}$ and $\gamma^I = \{\gamma^I_{(i_1,\ldots,i_n)}\}$. The following proposition is clear.

Proposition 2.9. $(\mathbb{S}|\mathbb{K}, (\gamma^B, \gamma^I))$ *is a* $\mathbb{R}_+|\mathbb{C}^\times$*-rescalable* 2*-colored partial dioperad.*

We call $(\mathbb{S}|\mathbb{K}, (\gamma^B, \gamma^I))$ *Swiss-cheese partial dioperad.* When it is restricted on $\mathfrak{S} = \{\mathbb{S}(1, m|0, n)\}_{m,n\in\mathbb{N}}$, it is nothing but the so-called Swiss-cheese partial operad [HKo1][Ko2].

In [HKo1][Ko2], we show that the Swiss-cheese partial operad $\mathfrak{S}$ can be naturally embedded into the sphere partial operad K via the so-called doubling map, denoted as $\delta : \mathfrak{S} \hookrightarrow K$. Such doubling map δ obviously can be extended to a doubling map $\mathbb{S} \hookrightarrow \mathbb{K}$, still denoted as δ. In particular, the general element (2.4) maps under δ to

$$\begin{aligned}
(\,&(z_{-1}, a_0^{(-1)}, A^{(-1)}), \ldots, (z_{-n_-}, a_0^{(-n_-)}, A^{(-n_-)}),\\
&\quad (\bar{z}_{-1}, \overline{a_0^{(-1)}}, \overline{A^{(-1)}}), \ldots (\bar{z}_{-n_-}, \overline{a_0^{(-n_-)}}, \overline{A^{(-n_-)}}),\\
&\qquad (r_{-1}, b_0^{(-1)}, B^{(-1)}), \ldots, (r_{-m_-}, b_0^{(-m_-)}, B^{(-m_-)})\,|\\
&(z_1, a_0^{(1)}, A^{(1)}), \ldots, (z_{n_+}, a_0^{(n_+)}, A^{(n_+)}),\\
&\quad (\bar{z}_1, \overline{a_0^{(1)}}, \overline{A^{(1)}}), \ldots, (\bar{z}_{n_+}, \overline{a_0^{(n_+)}}, \overline{A^{(n_+)}}),\\
&\qquad (r_1, b_0^{(1)}, B^{(1)}), \ldots, (r_{m_+}, b_0^{(m_+)}, B^{(m_+)})\,)_{\mathbb{K}}.
\end{aligned} \tag{2.10}$$

The following proposition is clear.

Proposition 2.10. *Let* $P_i \in \mathbb{S}(m_-^{(i)}, m_+^{(i)}|n_-^{(i)}, n_+^{(i)}), i = 1, 2$ *and* $Q \in \mathbb{K}(m_-, m_+)$. *If* $P_1\,{}_i\infty^B_{-j} P_2$ *and* $P_1\,{}_k\infty^I_{-l} Q$ *exists for* $1 \leq i \leq m_+^{(1)}, 1 \leq j \leq m_-^{(2)}$ *and* $1 \leq k \leq n_+^{(1)}, 1 \leq l \leq m_-$, *then we have*

$$\begin{aligned} \delta(P_1\ {}_i\infty^B_{-j}\, P_2) &= \delta(P_1)\ {}_{2n_+^{(1)}+i}\infty_{-(2n_-^{(2)}+j)}\ \delta(P_2), \\ \delta(P_1\ {}_k\infty^I_{-l}\, Q) &= (\delta(P_1)\ {}_k\infty_{-l}\, Q\,)\ {}_{n_+^{(1)}+m_+-1+k}\infty_{-l}\ \bar{Q}. \end{aligned} \tag{2.11}$$

By above Proposition, we can identify $\mathbb{S}$ as its image under δ in $\mathbb{K}$ with boundary sewing operations replaced by ordinary sewing operations in $\mathbb{K}$ and interior sewing operations replaced by double-sewing operations in $\mathbb{K}$ as given in (2.11).

The $\mathbb{C}$-extension $\tilde{\mathbb{S}}^c(m_-, m_+|n_-, n_+)$ of $\mathbb{S}(m_-, m_+|n_-, n_+)$ is defined to be the pull-back bundle of $\tilde{\mathbb{K}}^c(2n_- + m_-, 2n_+ + m_+)$. We denote the canonical section on $\tilde{\mathbb{S}}^c$, which is induced from that on $\tilde{\mathbb{K}}^c$, as $\psi_{\mathbb{S}}$. The boundary (interior) sewing operations can be naturally extended to $\tilde{\mathbb{S}}^c$. We denote them as $\widetilde{\infty}^B$ ($\widetilde{\infty}^I$) and corresponding substitution maps as $\tilde{\gamma}^B$ ($\tilde{\gamma}^I$). There is a natural right action of $\tilde{\mathbb{K}}^c \otimes \overline{\tilde{\mathbb{K}}^{\bar{c}}}$ on $\tilde{\mathbb{S}}^c$ defined by $\tilde{\gamma}^I$.

The following proposition is also clear.

Proposition 2.11. $(\tilde{\mathbb{S}}^c|\tilde{\mathbb{K}}^c \otimes \overline{\tilde{\mathbb{K}}^{\bar{c}}}, (\tilde{\gamma}^B, \tilde{\gamma}^I))$ *is a* $\mathbb{R}_+|\mathbb{C}^\times$*-rescalable* 2*-colored partial dioperad.*

We will call the structure $(\tilde{\mathbb{S}}^c|\tilde{\mathbb{K}}^c \otimes \overline{\tilde{\mathbb{K}}^{\bar{c}}}, (\tilde{\gamma}^B, \tilde{\gamma}^I))$ as *Swiss-cheese partial dioperad with central charge c.*

Definition 2.12. An algebra over $\tilde{\mathbb{S}}^c$ viewed as a dioperad, (U, ν), is called *smooth* if

1. $U = \oplus_{n\in\mathbb{R}} U_{(n)}$ is a completely reducible $\mathbb{R}_+$-module, where $r \cdot u = r^n u, \forall r \in \mathbb{R}_+, u \in U_{(m,n)}$.
2. $\dim U_{(n)} < \infty, \forall n \in \mathbb{R}$ and $\dim U_{(n)} = 0$ for n sufficiently small.
3. ν is linear on fiber and smooth on the base space $\mathbb{S}$.

Definition 2.13. An $\tilde{\mathbb{S}}^c|\tilde{\mathbb{K}}^c \otimes \overline{\tilde{\mathbb{K}}^{\bar{c}}}$-algebra $(U_1|U_2, \nu_{\tilde{\mathbb{S}}^c|\tilde{\mathbb{K}}^c\otimes\overline{\tilde{\mathbb{K}}^{\bar{c}}}}, \nu_{\tilde{\mathbb{K}}^c\otimes\overline{\tilde{\mathbb{K}}^{\bar{c}}}})$ is *smooth* if both $(U_1, \nu_{\tilde{\mathbb{S}}^c|\tilde{\mathbb{K}}^c\otimes\overline{\tilde{\mathbb{K}}^{\bar{c}}}})$ and $(U_2, \nu_{\tilde{\mathbb{K}}^c\otimes\overline{\tilde{\mathbb{K}}^{\bar{c}}}})$ as algebras over dioperads are smooth.

2.3 Open-closed field algebras over V

Let $(V_{op}, Y_{op}, \iota_{op})$ be an open-string vertex operator algebra over V and $(V_{cl}, \mathbb{Y}, \iota_{cl})$ a conformal full field algebra over $V \otimes V$. Let

$$(\,(V_{cl}, \mathbb{Y}, \iota_{cl}), (V_{op}, Y_{op}, \iota_{op}), \mathbb{Y}_{cl-op}\,) \tag{2.12}$$

be an open-closed field algebra over V [Ko2]. We denote the formal vertex operators associated with Y_{op} and $\mathbb{Y}$ as Y_{op}^f and $\mathbb{Y}^f$ respectively. Let $\omega^L = \iota_{cl}(\omega\otimes\mathbf{1})$, $\omega^R = \iota_{cl}(\mathbf{1}\otimes\omega)$

and $\omega_{op} = \iota_{op}(\omega)$. We have

$$\begin{aligned}
\mathbb{Y}^f(\omega^L; x, \bar{x}) &= \mathbb{Y}^f(\omega^L, x) = \sum_{n\in\mathbb{Z}} L(n) \otimes 1 x^{-n-2}, \\
\mathbb{Y}^f(\omega^R; x, \bar{x}) &= \mathbb{Y}^f(\omega^R, \bar{x}) = \sum_{n\in\mathbb{Z}} 1 \otimes L(n) \bar{x}^{-n-2}, \\
Y^f_{op}(\omega_{op}, x) &= \sum_{n\in\mathbb{Z}} L(n) x^{-n-2}. \qquad (2.13)
\end{aligned}$$

We also set $L^L(n) = L(n) \otimes 1$ and $L^R(n) = 1 \otimes L(n)$ for $n \in \mathbb{Z}$.

For $u \in V$, we showed in [Ko2] that $\mathbb{Y}_{cl-op}(\iota_{cl}(u \otimes \mathbf{1}); z, \bar{z})$ and $\mathbb{Y}_{cl-op}(\iota_{cl}(\mathbf{1} \otimes u); z, \bar{z})$ are holomorphic and antiholomorphic respectively. So we also denote them simply by $\mathbb{Y}_{cl-op}(\iota_{cl}(u \otimes \mathbf{1}), z)$ and $\mathbb{Y}_{cl-op}(\iota_{cl}(\mathbf{1} \otimes u), \bar{z})$ respectively.

By V-invariant boundary condition [Ko2], we have

$$\mathbb{Y}_{cl-op}(\omega^L, r) = Y_{op}(\omega_{op}, r) = \mathbb{Y}_{cl-op}(\omega^R, r). \qquad (2.14)$$

We assume that both V_{op} and V_{cl} are equipped with nondegenerate invariant bilinear forms $(\cdot, \cdot)_{op}$ and $(\cdot, \cdot)_{cl}$ respectively.

Lemma 2.14. *For any $u \in V_{cl}$ and $v_1, v_2 \in V_{op}$ and $z \in \mathbb{H}$, we have*

$$\begin{aligned}
&(v_2, \mathbb{Y}_{cl-op}(u; z, \bar{z})v_1)_{op} \\
&\quad = (\mathbb{Y}_{cl-op}(e^{-zL(1)}z^{-2L(0)} \otimes e^{-\bar{z}L(1)}\bar{z}^{-2L(0)}u; -z^{-1}, -\bar{z}^{-1})v_2, v_1)_{op} \qquad (2.15)
\end{aligned}$$

Proof. Using (1.26), for fixed $z \in \mathbb{H}$, we have,

$$\begin{aligned}
&(v_2, \mathbb{Y}_{cl-op}(u; z, \bar{z})v_1)_{op} \\
&\quad = (v_2, \mathbb{Y}_{cl-op}(u; z, \bar{z})Y_{op}(\mathbf{1}, r)v_1)_{op} \\
&\quad = (v_2, Y_{op}(\mathbb{Y}_{cl-op}(u; z-r, \bar{z}-r)\mathbf{1}, r)v_1)_{op} \\
&\quad = (e^{-r^{-1}L(-1)}Y_{op}(v_2, r^{-1})e^{-rL(1)}r^{-2L(0)}\mathbb{Y}_{cl-op}(u; z-r, \bar{z}-r)\mathbf{1}, v_1)_{op} \qquad (2.16)
\end{aligned}$$

for $|z| > r > |z-r| > 0$. Notice that $e^{-r^{-1}L(-1)} \in \mathrm{Aut}(\overline{V}_{op})$, $\forall r \in \mathbb{C}$. By taking $v_2 = \mathbf{1}_{op}$, it is easy to see that

$$e^{-rL(1)}r^{-2L(0)}\mathbb{Y}_{cl-op}(u; z-r, \bar{z}-r)\mathbf{1} \qquad (2.17)$$

is a well-defined element in $\overline{V}_{op}$ for $|z| > r > |z-r| > 0$. Because of the chirality splitting property of $\mathbb{Y}_{cl-op}$ (see (1.72)(1.73) in [Ko2]), it is easy to show that (2.17) equals to

$$\mathbb{Y}_{cl-op}(e^{-zL(1)}z^{-2L(0)} \otimes e^{-\bar{z}L(1)}\bar{z}^{-2L(0)}u, r^{-1}-z^{-1}, r^{-1}-\bar{z}^{-1})\mathbf{1} \qquad (2.18)$$

for $r > |z-r| > 0$. By the commutativity I of analytic open-closed field algebra proved in [Ko2], we know that for fixed $z \in \mathbb{H}$,

$$e^{-r^{-1}L(-1)}Y_{op}(v_2, r^{-1})\mathbb{Y}_{cl-op}(e^{-zL(1)}z^{-2L(0)} \otimes e^{-\bar{z}L(1)}\bar{z}^{-2L(0)}u, r^{-1}-z^{-1}, r^{-1}-\bar{z}^{-1})\mathbf{1}$$

and

$$e^{-r^{-1}L(-1)}\mathbb{Y}_{cl-op}(e^{-zL(1)}z^{-2L(0)}\otimes e^{-\bar{z}L(1)}\bar{z}^{-2L(0)}u, r^{-1}-z^{-1}, r^{-1}-\bar{z}^{-1})Y_{op}(v_2, r^{-1})\mathbf{1}$$
$$= e^{-r^{-1}L(-1)}\mathbb{Y}_{cl-op}(e^{-zL(1)}z^{-2L(0)}\otimes e^{-\bar{z}L(1)}\bar{z}^{-2L(0)}u, r^{-1}-z^{-1}, r^{-1}-\bar{z}^{-1})e^{r^{-1}L(-1)}v_2 \tag{2.19}$$

converge in different domains for r, but are analytic continuation of each other along a path in $r\in\mathbb{R}_+$. Moreover, using $L(-1)$ property of intertwining operator and chirality splitting property of $\mathbb{Y}_{cl-op}$ again, the right hand side of (2.19) equals to

$$\mathbb{Y}_{cl-op}(e^{-zL(1)}z^{-2L(0)}\otimes e^{-\bar{z}L(1)}\bar{z}^{-2L(0)}u, -z^{-1}, -\bar{z}^{-1})v_2$$

for $|r^{-1}-z^{-1}|>|r^{-1}|$. Therefore, the both sides of (2.15) as constant functions of r are analytic continuation of each other. Hence (2.15) must hold identically for all $z\in\mathbb{H}$. ∎

For $u_1,\dots,u_l\in V_{cl}, v_1,\dots,v_n\in V_{cl}$, $r_1,\dots,r_n\in\mathbb{R}, r_1>\cdots>r_n$ and $z_1,\dots,z_l\in\mathbb{H}$, we define

$$m_{cl-op}^{(l;n)}(u_1,\dots,u_l;v_1,\dots,v_n;z_1,\bar{z}_1,\dots,z_l,\bar{z}_l;r_1,\dots,r_n)$$
$$:= e^{-r_nL(-1)}m_{cl-op}^{(l;n)}(u_1,\dots,u_l;v_1,\dots,v_n;z_1-r_n,\bar{z}_1-r_n,\dots,z_l-r_n,\bar{z}_l-r_n;$$
$$r_1-r_n,\dots,r_{n-1}-r_n,0). \tag{2.20}$$

We simply extend the definition of $m_{cl-op}^{(l;n)}$ to a domain where some of r_i can be negative. Note that such definition is compatible with $L(-1)$-properties of m_{cl-op}.

Lemma 2.15. *For $u_1,\dots,u_l\in V_{cl}, v, v_1,\dots,v_n\in V_{cl}$, $r_1,\dots,r_n\in\mathbb{R}, r_1>\cdots>r_n=0$ and $z_1,\dots,z_l\in\mathbb{H}$,*

$$(v,\, m_{cl-op}^{(l;n)}(u_1,\dots,u_l;v_1,\dots,v_n;z_1,\bar{z}_1,\dots,z_l,\bar{z}_l;r_1,\dots,r_n))_{op}$$
$$=(v_n,\, m_{cl-op}^{(l;n)}(F_1u_1,\dots,F_lu_l;v,G_1v_1,\dots,G_{n-1}v_{n-1};$$
$$-z_1^{-1},-\bar{z}_1^{-1},\dots,-z_l^{-1},-\bar{z}_l^{-1};0,-r_1^{-1},\dots,-r_{n-1}^{-1}))_{op} \tag{2.21}$$

where

$$\begin{aligned} F_i &= e^{-z_iL(1)}z_i^{-2L(0)}\otimes e^{-\bar{z}_iL(1)}\bar{z}_i^{-2L(0)}, && i=1,\dots,l,\\ G_j &= e^{-r_jL(1)}r_j^{-2L(0)}, && j=1,\dots,n-1. \end{aligned} \tag{2.22}$$

Proof. By Lemma 2.14, (2.21) is clearly true for $l=0,1;n=0,1$. By (1.26) and (1.27), (2.21) is true for $l=0,n=2$. We then prove the Lemma by induction. Assume that (2.21) is true for $l=k\geq 0, n=m\geq 2$ or $l=k\geq 1, n=m\geq 1$.

Let $l=k$ and $n=m+1$. It is harmless to assume that $0<r_{n-1},|z_i|<r_{n-2}, i=$

$l_1+1,\dots,l$ for some $l_1 \le l$. Using the induction hypothesis, we obtain

$$
\begin{aligned}
&(v,\, m_{cl-op}^{(l;n)}(u_1,\dots,u_l;v_1,\dots,v_n;z_1,\bar z_1,\dots,z_l,\bar z_l;r_1,\dots,r_n))_{op}\\
&= (v, m_{cl-op}^{(l_1;n-1)}(u_1,\dots,u_{l_1};v_1,\dots,v_{n-2}, m_{cl-op}^{(l-l_1;2)}(u_{l_1+1},\dots,u_l;v_{n-1},v_n;\\
&\qquad\qquad z_{l_1+1},\bar z_{l_1+1},\dots,z_l,\bar z_l;r_{n-1},0);z_1,\bar z_1,\dots,z_{l_1},\bar z_{l_1};r_1,\dots,r_{n-2},0))_{op}\\
&= (m_{cl-op}^{(l_1;n-1)}(F_1u_1,\dots,F_{l_1}u_{l_1};v,G_1v_1,\dots,G_{n-1}v_{n-1};\\
&\qquad\qquad -z_1^{-1},-\bar z_1^{-1},\dots,-z_{l_1}^{-1},-\bar z_{l_1}^{-1};0,-r_1^{-1},\dots,-r_{n-1}^{-1}),\\
&\qquad\qquad m_{cl-op}^{(l-l_1;2)}(u_{l_1+1},\dots,u_l;v_{n-1},v_n;z_{l_1+1},\bar z_{l_1+1},\dots,z_l,\bar z_l;r_{n-1},0))_{op}\\
&= (m_{cl-op}^{(l-l_1;2)}(F_{l_1+1}u_{l_1+1},\dots,F_lu_l;m_{cl-op}^{(l_1;n-1)}(F_1u_1,\dots,F_{l_1}u_{l_1};\\
&\qquad\qquad v,G_1v_1,\dots,G_{n-1}v_{n-1};-z_1^{-1},-\bar z_1^{-1},\dots,-z_{l_1}^{-1},-\bar z_{l_1}^{-1};0,-r_1^{-1},\dots,-r_{n-2}^{-1}),\\
&\qquad\qquad G_{n-1}v_{n-1};-z_{l_1+1}^{-1},-\bar z_{l_1+1}^{-1},\dots,-z_l^{-1},-\bar z_l^{-1};0,-r_{n-1}^{-1}),v_n)_{op}\\
&= \sum_{s\in\mathbb{R}}(e^{-r_{n-1}^{-1}L(-1)}m_{cl-op}^{(l-l_1;2)}(F_{l_1+1}u_{l_1+1},\dots,F_lu_l;P_se^{-r_{n-2}^{-1}L(-1)}m_{cl-op}^{(l-l_1;n-1)}(F_1u_1,\\
&\qquad\qquad \dots,F_{l_1}u_{l_1};v,G_1v_1,\dots,G_{n-1}v_{n-1};-z_1^{-1}+r_{n-2}^{-1},-\bar z_1^{-1}+r_{n-2}^{-1},\\
&\qquad\qquad \dots,-z_{l_1}^{-1}+r_{n-2}^{-1},-\bar z_{l_1}^{-1}+r_{n-2}^{-1};r_{n-2}^{-1},-r_1^{-1}+r_{n-2}^{-1},\dots,-r_{n-3}^{-1}+r_{n-2}^{-1},0),\\
&\qquad\qquad G_{n-1}v_{n-1};-z_{l_1+1}^{-1}+r_{n-1}^{-1},-\bar z_{l_1+1}^{-1}+r_{n-1}^{-1},\\
&\qquad\qquad \dots,-z_l^{-1}+r_{n-1}^{-1},-\bar z_l^{-1}+r_{n-1}^{-1};r_{n-1}^{-1},0),v_n)_{op}
\end{aligned}
\tag{2.23}
$$

Note that the position of P_s and $e^{-r_{n-2}^{-1}L(-1)}$ can not be exchanged in general. Because if we exchange their position, the sum may not converge and then the associativity law does not hold. We want to use analytic continuation to move it to a domain such that we can freely apply associativity law. By our assumption on V, both sides of (2.23) are restrictions of analytic function of $z_{l_1+1},\zeta_{l_1+1},\dots,z_l,\zeta_l,r_{n-1}$ on $\zeta_{l_1+1}=\bar z_{l_1+1},\dots,\zeta_l=\bar z_l$. Let $\tilde z_{l_1+1},\dots,\tilde z_l,\tilde r_{n-1}$ satisfy the following conditions:

$$
|-\tilde z_p^{-1}+r_{n-2}^{-1}|,\tilde r_{n-1}^{-1}-r_{n-2}^{-1} > |-z_i^{-1}+r_{n-2}^{-1}|,r_{n-2}^{-1},-r_j^{-1}+r_{n-2}^{-1}, \tag{2.24}
$$

for all $i=1,\dots,l, j=1,\dots,n-3$ and $p=l_1+1,\dots,l$. Note that such condition define a nonempty open subset on $\mathbb{H}^l\times\mathbb{R}_+$. Choose a path γ_1 in the complement of the diagonal in $\mathbb{H}^l$ from initial point $(z_{l_1+1},\dots,z_l)$ to $(\tilde z_{l_1+1},\dots,\tilde z_l)$ and a path γ_2 in $\mathbb{R}_+$ from r_{n-1} to $\tilde r_{n-1}$. We also denote the complex conjugate of path γ_1 as $\bar\gamma_1$, which is a path in $\overline{\mathbb{H}}^l$. Combine $\gamma_1,\bar\gamma_1$ with γ_2, we obtain a path γ from $(z_{l_1+1},\bar z_{l_1+1},\dots,z_l,\bar z_l,r_{n-1})$ to $(\tilde z_{l_1+1},\bar{\tilde z}_{l_1+1},\dots,\tilde z_l,\bar{\tilde z}_l,\tilde r_{n-1})$. Analytically continuating the right hand side of (2.23) along the path γ, we obtain, by the properties of $m_{cl-op}^{(l;n)}$ [Ko2],

$$
\begin{aligned}
&(e^{-\tilde r_{n-1}^{-1}L(-1)}m_{cl-op}^{(l-l_1;2)}(F_{l_1+1}u_{l_1+1},\dots,F_lu_l;P_sm_{cl-op}^{(l-l_1;n-1)}(F_1u_1,\dots,F_{l_1}u_{l_1};\\
&\qquad\qquad v,G_1v_1,\dots,G_{n-1}v_{n-1};-z_1^{-1}+r_{n-2}^{-1},-\bar z_1^{-1}+r_{n-2}^{-1},\\
&\qquad\qquad \dots,-z_{l_1}^{-1}+r_{n-2}^{-1},-\bar z_{l_1}^{-1}+r_{n-2}^{-1};r_{n-2}^{-1},-r_1^{-1}+r_{n-2}^{-1},\dots,-r_{n-3}^{-1}+r_{n-2}^{-1},0),\\
&\qquad\qquad \tilde G_{n-1}v_{n-1};-\tilde z_{l_1+1}^{-1}+r_{n-1}^{-1},-\bar{\tilde z}_{l_1+1}^{-1}+r_{n-1}^{-1},\\
&\qquad\qquad \dots,-\tilde z_l^{-1}+r_{n-1}^{-1},-\bar{\tilde z}_l^{-1}+r_{n-1}^{-1};\tilde r_{n-1}^{-1}-r_{n-2}^{-1},0),v_n)_{op}
\end{aligned}
\tag{2.25}
$$

where $\tilde{G}_{n-1} = e^{-\tilde{r}_{n-1}L(1)}\tilde{r}_{n-1}^{-2L(0)}$. Using the associativity of open-closed field algebra and $L(-1)$-properties of m_{cl-op}, we see that (2.25) further equals to

$$(m_{cl-op}^{(l;n+1)}(F_1u_1, \dots, F_lu_l; v, G_1v_1, \dots, \tilde{G}_{n-1}v_{n-1}; -z_1^{-1}, -\bar{z}_1^{-1}, \dots, \\ -z_{l_1}^{-1}, -\bar{z}_{l_1}^{-1}, -\tilde{z}_{l_1+1}^{-1}, -\bar{\tilde{z}}_{l_1+1}^{-1}, \dots, -z_l^{-1}, -\bar{z}_l^{-1}; 0, -r_1^{-1}, \dots, -\tilde{r}_{n-1}^{-1}), v_n)_{op}. \tag{2.26}$$

By analytically continuating (2.26) along the path $-\gamma$, which is γ reversed, we obtain the right hand side of (2.21). Hence (2.23) and the right hand side of (2.21) are analytic continuation of each other along path $(-\gamma)\circ(\gamma)$ which is a constant path. Hence (2.21) holds for $l = k, n = m+1$.

Now let $l = k+1, n = m$. The proof is similar to the case $l = k, n = m+1$. We only point out the difference. Using the smoothness of m_{cl-op}, it is enough to prove the case when $|z_i| \neq |z_j|$ for $i, j = 1, \dots, l$ and $i \neq j$. Without losing generality, we assume that $|z_1| > \cdots > |z_l| > 0$. Let $n_1 \leq n$ be the smallest so that $0 < r_j < |z_l|$ for $j \geq n_1$. Then we have

$$\begin{aligned}(v,\, & m_{cl-op}^{(l;n)}(u_1, \dots, u_l; v_1, \dots, v_n; z_1, \bar{z}_1, \dots, z_l, \bar{z}_l; r_1, \dots, r_n))_{op} \\ &= (v,\, m_{cl-op}^{(l-1;n_1)}(u_1, \dots, u_{l-1}; v_1, \dots, v_{n_1-1}, m_{cl-op}^{(1;n-n_1+1)}(u_l; v_{n_1}, \dots, v_n; \\ &\qquad z_l, \bar{z}_l; r_{n_1}, \dots, r_n); z_1, \bar{z}_1, \dots, z_{l-1}, \bar{z}_{l-1}; r_1, \dots, r_{n_1-1}))_{op}\end{aligned} \tag{2.27}$$

We can then apply (2.21) as in (2.23) for the case $l \leq k, n \leq m$, which is true by our induction hypothesis. The rest of proof is entirely same as that of the case $l = k, n = m+1$. ∎

We define a map, for $z, \zeta \in \mathbb{C}$ and $z \neq \zeta$, $\iota_{cl-op}(z,\zeta) : V_{cl} \to \overline{V}_{op}$ as

$$\iota_{cl-op}(z,\zeta)(u) = \mathbb{Y}_{cl-op}(u; z, \zeta)\mathbf{1}_{op}.$$

We denote its adjoint as $\iota_{cl-op}^*(z,\zeta)$. Namely, $\iota_{cl-op}^*(z,\zeta) : V_{op} \to \overline{V_{cl}}$ is given by

$$(\iota_{cl-op}^*(z,\zeta)(w), u)_{cl} = (w, \iota_{cl-op}(z,\zeta)(u))_{op} \tag{2.28}$$

for any $u \in V_{cl}$ and $w \in V_{op}$.

Let Q be an element in $\mathcal{T}_{\mathbb{S}}(n_-, n_+|m_-, m_+)$ of form (2.4). Let α be the map (1.30) so that $s_1, \dots, s_{n_-+n_+}$, defined as $s_i := r_{\alpha^{-1}(i)}$, satisfy $\infty \geq s_1 > \cdots > s_{n_-+n_+} \geq 0$. Then we define

$$\nu_{cl-op}(\lambda\psi_{\mathbb{S}}(Q))(u_1 \otimes \cdots \otimes u_{m_+} \otimes v_1 \otimes \cdots \otimes v_{n_+}) \tag{2.29}$$

as follow:

1. If $s_1 \neq \infty$, (2.29) is given by

$$\lambda \sum_{i_1,\dots,i_{m_-};j_1,\dots,j_{n_-}} \big(\mathbf{1}_{op},\; m_{cl-op}^{(m_-+m_+;n_-+n_+)}(u_{-1}, \dots, u_{-m_-}, u_1, \dots, u_{m_+}; \\ w_1, \dots, w_{n_-+n_+}; z_{-1}, \bar{z}_{-1}, \dots, z_{-m_-}, \bar{z}_{-m_-}, z_1, \bar{z}_1, \dots, z_{m_+}, \bar{z}_{m_+}; \\ s_1, \dots, s_{n_-+n_+})\big)_{op} \cdot e^{i_1} \otimes \cdots \otimes e^{i_{n_-}} \otimes f^{j_1} \otimes \cdots \otimes f^{j_{m_-}}. \tag{2.30}$$

where

$$\begin{aligned}
u_p &= e^{-L_+^L(A^{(p)})-L_+^R(\overline{A^{(p)}})}(a_0^{(p)})^{-L^L(0)}\overline{a_0^{(p)}}^{-L^R(0)}e_{i_{-p}},\\
u_q &= e^{-L_+^L(A^{(q)})-L_+^R(\overline{A^{(q)}})}(a_0^{(q)})^{-L^L(0)}\overline{a_0^{(q)}}^{-L^R(0)}u_q,\\
w_{\alpha(k)} &= e^{-L_+(B^{(k)})}(b_0^{(k)})^{-L(0)}f_{j_{-k}},\\
w_{\alpha(l)} &= e^{-L_+(B^{(l)})}(b_0^{(l)})^{-L(0)}v_l,
\end{aligned} \tag{2.31}$$

for $p=-1,\dots,-n_-$, $q=1,\dots,n_+$, $k=-1,\dots,-m_-$ and $l=1,\dots,m_+$.

2. When $r_k=\infty$ for some $k=-m_-,\dots,-1,1,\dots,m_+$. (2.29) is given by the formula obtain from (2.30) by exchanging $\mathbf{1}_{op}$ with $w_{\alpha(k)}$.

Lemma 2.16. ν_{cl-op} is $SL(2,\mathbb{R})$-invariant.

Proof. The $SL(2,\mathbb{R})$ is generated by the following three transformations 1. $w\mapsto aw, \forall a\in\mathbb{R}^+$; 2. $w\mapsto w-b, \forall b\in\mathbb{R}$; 3. $w\mapsto\frac{-1}{w}$. That ν_{cl-op} is invariant under the first two transformations simply follows from the $L(0)$- and $L(-1)$-properties of m_{cl-op}. That ν_{cl-op} is invariant under the third transformation is proved in Lemma 2.15. ∎

Hence ν_{cl-op} induces a map $\tilde{\mathbb{S}}^c\to\mathbb{E}^{\mathbb{R}_+|\mathbb{C}^\times}_{V_{op}|V_{cl}}$, which is still denoted as ν_{cl-op}. We list a few interesting cases:

$$\begin{aligned}
\nu_{cl-op}\big(\psi_{\mathbb{S}}\big(\big[(\,(\infty,1,\mathbf{0})|\,)\big\|(\,|(z,1,\mathbf{0}))\big]_{\mathbb{S}}\big)\big) &= \iota_{cl-op}(z,\bar{z}),\\
\nu_{cl-op}\big(\psi_{\mathbb{S}}\big(\big[(\,|(\infty,1,\mathbf{0}))\big\|((z,1,\mathbf{0})|\,)\big]_{\mathbb{S}}\big)\big) &= \iota^*(z,\bar{z})
\end{aligned} \tag{2.32}$$

and for $b\in\mathbb{R}_+$, $B\in\mathbb{R}^\infty$, $a\in\mathbb{C}^\times$, $A\in\mathbb{C}^\infty$ and $v\in V_{op}$, we have

$$\begin{aligned}
&\nu_{cl-op}\big(\psi_{\mathbb{S}}\big(\big[(\,|(\infty,b,B))\big\|((z,a,A)|\,)\big]_{\mathbb{S}}\big)\big)\\
&\quad= a^{-L^L(0)}\bar{a}^{-L^R(0)}e^{-\sum_{j=1}^\infty(-1)^j[A_jL^L(-j)+\overline{A_j}L^R(-j)]}\iota^*(z,\bar{z})(e^{-L_+(B)}b^{-L(0)}v).
\end{aligned} \tag{2.33}$$

Theorem 2.17. *$(V_{op}|V_{cl},\nu_{cl-op},\nu_{cl})$ is an $\mathbb{R}_+|\mathbb{C}^\times$-rescalable smooth $\tilde{\mathbb{S}}^c|\tilde{\mathbb{K}}^c\otimes\overline{\tilde{\mathbb{K}}^{\bar{c}}}$-algebra.*

Proof. The smoothness is automatic. We showed in [Ko2] that $(V_{op}|V_{cl},\nu_{cl-op},\nu_{cl})$ is an $\mathbb{R}_+|\mathbb{C}^\times$-rescalable smooth $\tilde{\mathfrak{S}}^c|\tilde{K}^c\otimes\overline{\tilde{K}^{\bar{c}}}$-algebra. The rest of the proof is similar to that of Theorem 1.12 in [Ko1]. We omit the detail here. ∎

2.4 Ishibashi states

As we mentioned in the introduction, an open-closed field algebra over V equipped with nondegenerate invariant bilinear forms for both open theory and closed theory contains all the data needed to grow to an open-closed field field theory of all genus. Without adding more compatibility conditions, itself is already an interesting object to study. We show in this subsection that the famous "Ishibashi states" can be studied in the framework of such algebra. Throughout this subsection, we fix an open-closed algebra over V given in (2.12) and equipped with nondegenerate invariant bilinear forms $(\cdot,\cdot)_{cl}$ and $(\cdot,\cdot)_{op}$.

For $u\in V_{op}$ and $z_0\in\mathbb{H}$, we define the boundary state $B_{z_0}(u)\in\overline{V_{cl}}$ associated with u and z_0 by

$$B_{z_0}(u)=e^{L(-1)}(\bar{z}_0-z_0)^{L(0)}\otimes e^{L(-1)}\overline{\bar{z}_0-z_0}^{L(0)}\iota^*_{cl-op}(z_0,\bar{z}_0)(u). \tag{2.34}$$

Proposition 2.18. *Let $u \in V_{op}$ be a vacuum-like vector [LL], i.e. $L(-1)u = 0$. For $z_0 \in \mathbb{H}$, we have the following Ishibashi identity:*

$$(L^L(n) - L^R(-n))B_{z_0}(u) = 0, \qquad \forall n \in \mathbb{Z}. \tag{2.35}$$

Proof. For $v \in V_{cl}$, the following two functions of z

$$\begin{aligned}&(u,\ \mathbb{Y}_{cl-op}(\omega^L, z+z_0)\mathbb{Y}(v; z_0, \bar{z}_0)\mathbf{1}_{op})_{op},\\ &(u,\ \mathbb{Y}_{cl-op}(\omega^R; \overline{z+z_0})\mathbb{Y}(v; z_0, \bar{z}_0)\mathbf{1}_{op})_{op}\end{aligned} \tag{2.36}$$

can be extended to a holomorphic function and an antiholomorphic function in $\{z | z + z_0 \in \mathbb{H}, z \neq 0\}$ respectively by our assumption on V. We denote the extended functions by $g_1(\omega^L, z)$ and $g_2(\omega^R, \bar{z})$ respectively.

The following two limits

$$\begin{aligned}&\lim_{z+z_0\to r}\left(1 - \frac{z}{\bar{z}_0 - z_0}\right)^4 g_1(\omega^L, z)\\ &\lim_{z+z_0\to r}\left(1 - \frac{z}{\bar{z}_0 - z_0}\right)^4 g_2(\omega^R, \bar{z})\end{aligned} \tag{2.37}$$

exist for all $r \in \mathbb{R}$. Using (2.15), it is easy to see that above two limits also exist for $r = \infty \in \hat{\mathbb{R}}$ if and only if u is vacuum-like. Hence, by V-invariant (or conformal invariant) boundary condition, we have

$$\lim_{z+z_0\to r}\left(1 - \frac{z}{\bar{z}_0 - z_0}\right)^4 g_1(\omega^L, z) = \lim_{z+z_0\to r}\left(1 - \frac{z}{\bar{z}_0 - z_0}\right)^4 g_2(\omega^R, z) \tag{2.38}$$

for all $r \in \hat{\mathbb{R}}$ when $L(-1)u = 0$.

On the other hand, for $|z + z_0| > |z_0|$, we have

$$\begin{aligned}&(u,\ \mathbb{Y}_{cl-op}(\omega^L, z+z_0)\mathbb{Y}_{cl-op}(v; z_0, \bar{z}_0)\mathbf{1}_{op})_{op}\\ &\quad = (u,\ \mathbb{Y}_{cl-op}(\mathbb{Y}(\omega^L, z)v; z_0, \bar{z}_0)_{op}\mathbf{1}_{op})_{op}\\ &\quad = (u,\ \iota_{cl-op}(z_0, \bar{z}_0)(\mathbb{Y}(\omega^L, z)v))_{op}\\ &\quad = (\iota^*_{cl-op}(z_0, \bar{z}_0)(u),\ \mathbb{Y}(\omega^L, z)v)_{cl}\\ &\quad = (B_{z_0}(u),\ e^{L(1)}(\bar{z}_0 - z_0)^{-L(0)} \otimes e^{L(1)}\overline{\bar{z}_0 - z_0}^{-L(0)}\mathbb{Y}(\omega^L, z)v)_{cl}.\end{aligned} \tag{2.39}$$

Note that one should check the convergence property of each step in (2.39). In particular, in the last step, the convergence and equality follow from the convergence of early steps and the fact that $(\bar{z}_0 - z_0)^{-L(0)}e^{-L(-1)} \otimes \overline{\bar{z}_0 - z_0}^{-L(0)}e^{-L(-1)} \in \mathrm{Aut}\,\overline{V_{cl}}$. For $0 < |z| < |\mathrm{Im} z_0|$, it is easy to show that

$$\begin{aligned}&e^{L(1)}(\bar{z}_0 - z_0)^{-L(0)} \otimes e^{L(1)}\overline{\bar{z}_0 - z_0}^{-L(0)}\mathbb{Y}(\omega^L, z)v\\ &\quad = \mathbb{Y}(e^{(1-\frac{z}{\bar{z}_0 - z_0})L^L(1)}\left(1 - \frac{z}{\bar{z}_0 - z_0}\right)^{-2L^L(0)} \cdot\\ &\qquad\qquad \cdot(\bar{z}_0 - z_0)^{-L^L(0)}\omega^L, \frac{1}{\bar{z}_0 - z_0}\frac{z}{1 - \frac{z}{\bar{z}_0 - z_0}})v_1\\ &\quad = \left(1 - \frac{z}{\bar{z}_0 - z_0}\right)^{-4}(\bar{z}_0 - z_0)^{-2}\mathbb{Y}(\omega^L, \frac{1}{\bar{z}_0 - z_0}\frac{z}{1 - \frac{z}{\bar{z}_0 - z_0}})v_1\end{aligned} \tag{2.40}$$

where $v_1 = e^{L(1)}(\bar{z}_0 - z_0)^{-L(0)} \otimes e^{L(1)}\overline{\bar{z}_0 - z_0}^{-L(0)}v$. Hence, for all $0 < |z| < |\mathrm{Im} z_0|$ and $|z + z_0| > |z_0|$, we obtain

$$
\begin{aligned}
&(u, \mathbb{Y}_{cl-op}(\omega^L, z + z_0)\mathbb{Y}_{cl-op}(v; z_0, \bar{z}_0)\mathbf{1}_{op})_{op} \\
&= (B_{z_0}(u), \left(1 - \frac{z}{\bar{z}_0 - z_0}\right)^{-4} (\bar{z}_0 - z_0)^{-2}\mathbb{Y}(\omega^L, f(z))v_1)_{cl}
\end{aligned}
\tag{2.41}
$$

where f is the composition of the following maps:

$$
w \mapsto w + z_0 \mapsto -\frac{(w + z_0) - z_0}{(w + z_0) - \bar{z}_0} = \frac{1}{\bar{z}_0 - z_0}\frac{w}{1 - \frac{w}{\bar{z}_0 - z_0}}
$$

which maps the domain $\mathbb{H} - z_0$ to the unit disk. Since $g_1(\omega^L, z)$ is analytic and free of singularities for $z + z_0 \in \mathbb{H}\backslash\{z_0\}$, the right hand side of (2.41) can also be extended to an analytic function in $z \in \mathbb{H} - z_0\backslash\{0\}$. If we view $f(z)$ as a new variable ξ, then the right hand side of (2.41) can be extended to an analytic function on $\{\xi | 1 > |\xi| > 0\}$, which has a Laurent series expansion. By the uniqueness of Laurent expansion, the right hand side of (2.41) gives exactly such Laurent expansion and thus is absolutely convergent in $\{\xi | 1 > |\xi| > 0\}$. Moreover, $\lim_{z+z_0\to r} g_1(\omega^L, z)$ exists for all $r \in \hat{\mathbb{R}}$. By the properties of Laurent series, the right hand side of (2.41) must converge absolutely for all $f(z) \in \{\xi | |\xi| = 1\}$ to the function given by $\lim_{z+z_0\to r} g_1(\omega^L, z), r \in \hat{\mathbb{R}}$.

Similarly, for all $0 < |\bar{z}| < |\mathrm{Im} z_0|$ and $|z + z_0| > |z_0|$, we have

$$
\begin{aligned}
&(u, \mathbb{Y}_{cl-op}(\omega^R, \overline{z + z_0})\mathbb{Y}_{cl-op}(v; z_0, \bar{z}_0)\mathbf{1}_{op})_{op} \\
&= (B_{z_0}(u),\ e^{L(1)}(\bar{z}_0 - z_0)^{-L(0)} \otimes e^{L(1)}\overline{\bar{z}_0 - z_0}^{-L(0)}\mathbb{Y}(\omega^R, \bar{z})v)_{cl} \\
&= (B_{z_0}(u),\ \mathbb{Y}(e^{(1-\frac{\bar{z}}{z_0-\bar{z}_0})L^R(1)}\left(1 - \frac{z}{z_0 - \bar{z}_0}\right)^{-2L^R(0)} (z_0 - \bar{z}_0)^{-L^R(0)}\omega^R, g(\bar{z}))v_1)_{cl} \\
&= \left(1 - \frac{\bar{z}}{z_0 - \bar{z}_0}\right)^{-4} (z_0 - \bar{z}_0)^{-2}(B_{z_0}(u), \mathbb{Y}(\omega^R, g(\bar{z}))v_1)_{cl}
\end{aligned}
\tag{2.42}
$$

where g is the composition of the following maps:

$$
w \mapsto w + \bar{z}_0 \mapsto -\frac{(w + \bar{z}_0) - z_0}{(w + \bar{z}_0) - \bar{z}_0} \mapsto -\frac{(w + \bar{z}_0) - \bar{z}_0}{(w + \bar{z}_0) - z_0} = \frac{1}{z_0 - \bar{z}_0}\frac{w}{1 - \frac{w}{z_0 - \bar{z}_0}}
$$

which maps the domain $-\mathbb{H} - \bar{z}_0$ to the unit disk. Moreover, the right hand side of (2.42), as a Laurent series of $g(\bar{z})$, is absolutely convergent for all $g(\bar{z}) \in \{\xi | |\xi| = 1\}$ to $\lim_{z+z_0\to r} g_2(\omega^R, z), r \in \hat{\mathbb{R}}$.

Also notice that

$$
g(r - \bar{z}_0) = \frac{1}{f(r - z_0)} \in \{e^{i\theta} | 0 \le \theta < 2\pi\}
\tag{2.43}
$$

for all $r \in \hat{\mathbb{R}}$. Using (2.38) and by replacing z in (2.41) by $r - z_0$ and $\bar{z}$ in (2.42) by $r - \bar{z}_0$, we obtain the following identity:

$$
(B_{z_0}(u),\ \mathbb{Y}(\omega^L, e^{i\theta})v_1)_{cl} = (B_{z_0}(u),\ \mathbb{Y}(\omega^R, e^{-i\theta})v_1)_{cl}\ e^{-4i\theta}
\tag{2.44}
$$

where $e^{i\theta} = f(r - z_0)$, for all $0 \leq \theta < 2\pi$. Notice that the existence of both sides of (2.44) follows directly from (2.38), which further follows from the condition of u being vacuum-like. Then we obtain

$$\sum_{n\in\mathbb{Z}}(B_{z_0}(u),\ L^L(n)v_1)_{cl}\ e^{i\theta(-n-2)} = \sum_{n\in\mathbb{Z}}(B_{z_0}(u), L^R(-n)v_1)_{cl}\ e^{i\theta(-n-2)}$$

for all $0 \leq \theta < 2\pi$. Notice that v_1 can be arbitrary. Therefore, we must have (2.35) when $L(-1)u = 0$. ∎

In physics, boundary states are usually obtained by solving the equation (2.35). The solutions of such equation was first obtained by Ishibashi [I]. They are called Ishibashi states. The definition of boundary states we gives in (2.34) is more constructive. Boundary conditions are also called "D-branes" in string theory. If u is not a vacuum-like vector, the boundary state (2.34) associated with u is also very interesting in physics (see for example [FFFS1]). Such boundary states are associated to the geometry on D-branes.

3 Cardy condition

In this section, we discuss the famous Cardy condition. We first derive an algebraic formulation of Cardy condition directly from algebraic realizations of Figure 4 (see also Figure 5). Then we reformulate Cardy condition in the framework of intertwining operator algebra. Throughout this section, we fix an open-closed field algebra over V given in (2.12) equipped with nondegenerate bilinear forms $(\cdot,\cdot)_{op}$ and $(\cdot,\cdot)_{cl}$.

3.1 The first version

In the notion of Swiss-cheese dioperad, we exclude interior sewing operations between two disks with strips and tubes and self-sewing operations between two oppositely oriented boundary punctures on a single disk. The surface obtained after these two types of sewing operations is a cylinder or an annulus. The axioms of open-closed conformal field theory require that the algebraic realization of these two sewing operations must coincide. This gives a nontrivial condition, called Cardy condition.

Although the Cardy condition is only a requirement from genus-zero surfaces, its algebraic realization is genus-one in nature. This fact is manifest if we consider the doubling map δ. A double of a cylinder is actually a torus. Hence the Cardy condition is a condition on the equivalence of two algebraic realizations of two different decompositions of a torus. This is nothing but a condition associated to modularity.

That an annulus can be obtained by two different sewing operations is shown in Figure 5. In particular, the surface (A) in Figure 5 shows how an annulus is obtained by sewing two oppositely oriented boundary punctures on the same disk with strips and tubes in $\mathbb{S}(1,3|0,0)$, and surface (C) in Figure 5, viewed as a propagator of close string, can be obtained by sewing an element in $\mathbb{S}(0,1|1,0)$ with an element in $\mathbb{S}(0,1|0,1)$ along the interior punctures.

We only show in Figure 5 a simple case in which there are only two boundary punctures and no interior puncture. In general, the number of boundary punctures and

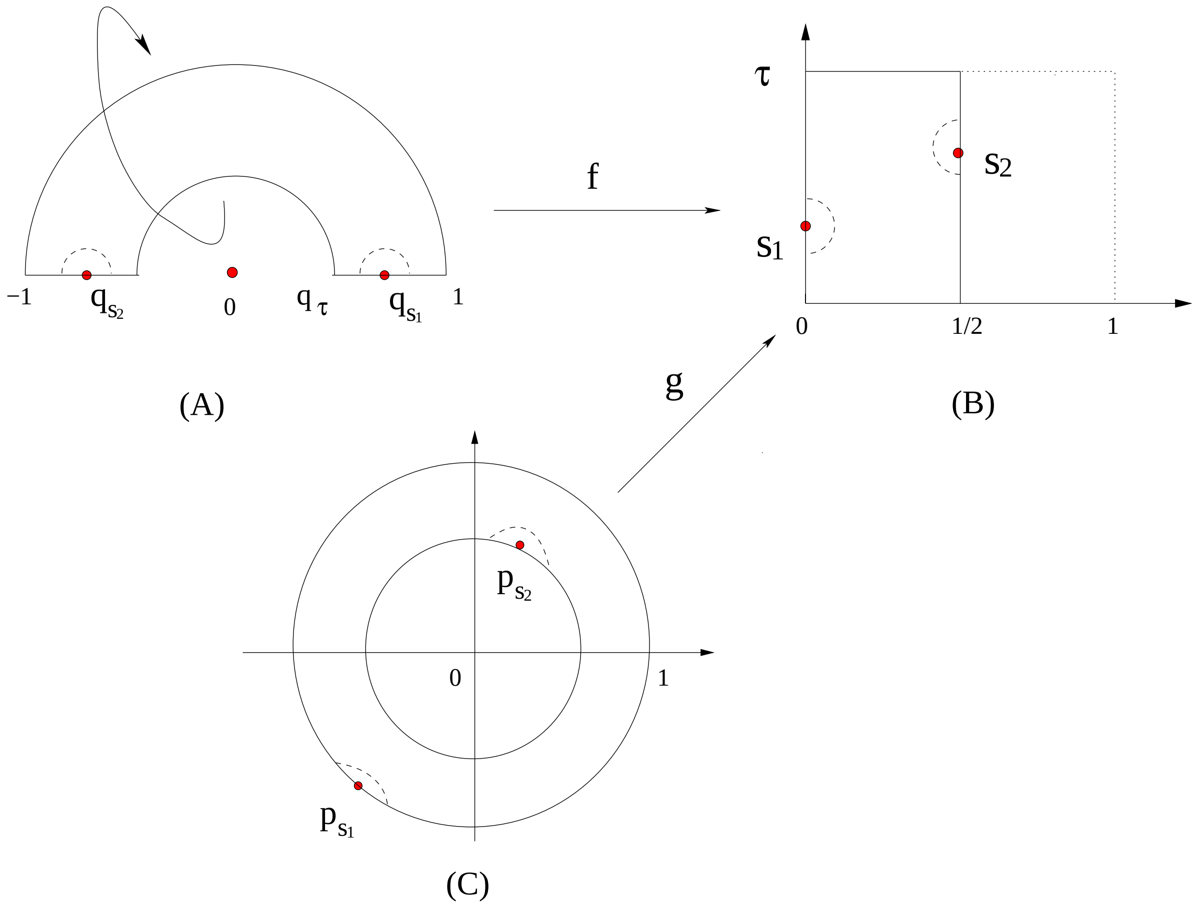


Figure 5: Cardy condition: two different sewings give same annulus.

interior punctures can be arbitrary. However, all general cases can be reduced to this simple case by applying associativities. Notice that the two boundary punctures in this simple case can not be reduced further by the associativities. We only focus on this case in this work.

The conformal map f between the surface (A) and (B) and g between the surface (C) and (B) in Figure 5 are given by

$$\begin{aligned} f(w) &= \frac{1}{2\pi i}\log w, \\ g(w) &= \frac{-\tau}{2\pi i}\log w. \end{aligned} \tag{3.1}$$

It is also useful to know their inverses $f^{-1}(w) = e^{2\pi i w}$, $g^{-1}(w) = e^{2\pi i(-\frac{1}{\tau})w}$. For any $z \in \mathbb{C}$, we set $q_z := e^{2\pi i z}$ and $p_z := e^{2\pi i(-\frac{1}{\tau})z}$. The radius of the outer circle of the surface (C) in Figure 5 is $|p_{s_1}| = 1$ and that of inner circle is

$$|p_{s_2}| = e^{\pi i(-\frac{1}{\tau})} = q_{-\frac{1}{\tau}}^{1/2}. \tag{3.2}$$

As we have argued and will show more explicitly later, Cardy condition deeply related to modularity. In [H6], Huang introduced the so-called geometrically modified intertwining operators, which is very convenient for the study of modularity. He was motivated by the fact that it is much easier to study modularity in the global coordinates. Namely, one should choose the local coordinate maps at s_1, s_2 in surface (B) in Figure 5 as simple as possible. More precisely, we choose the local coordinate maps at s_1, s_2 as

$$\begin{aligned} w &\mapsto e^{\frac{\pi i}{2}}(w-s_1) \\ w &\mapsto e^{-\frac{\pi i}{2}}(w-s_2) \end{aligned} \tag{3.3}$$

respectively. Correspondingly, the local coordinate maps at punctures q_{s_1}, q_{s_2} are

$$\begin{aligned} f_{q_{s_1}}(w) &= e^{\frac{\pi i}{2}}\frac{1}{2\pi i}\log\left(1+\frac{1}{q_{s_1}}x\right)\Big|_{x=w-q_{s_1}} \\ f_{q_{s_2}}(w) &= e^{-\frac{\pi i}{2}}\frac{1}{2\pi i}\log\left(1+\frac{1}{q_{s_2}}x\right)\Big|_{x=w-q_{s_2}} \end{aligned} \tag{3.4}$$

respectively.

Notice that both local coordinate maps $f_{q_{s_1}}(w)$ and $f_{q_{s_2}}(w)$ are real analytic. Hence $\exists B_j^{(i)} \in \mathbb{R}, b_0^{(i)} \in \mathbb{R}_+, i=1,2$ such that

$$\begin{aligned} f_{q_{s_1}}(w) &= e^{\sum_{j=1}^{\infty} B_j^{(1)} x^{j+1}\frac{d}{dx}}(b_0^{(1)})^{x\frac{d}{dx}}x\Big|_{x=w-q_{s_1}}, \\ f_{q_{s_2}}(w) &= e^{\sum_{j=1}^{\infty} B_j^{(2)} x^{j+1}\frac{d}{dx}}(b_0^{(2)})^{x\frac{d}{dx}}x\Big|_{x=w-q_{s_2}}. \end{aligned} \tag{3.5}$$

Then the algebraic realization of the surface (A) gives a map $V_{op}\otimes V_{op}\to\mathbb{C}$. We assume $1>|q_{s_1}|>|q_{s_2}|>|q_\tau|>0$. By the axiom of open-closed conformal field theory, this map must be given by (recall (1.25))

$$v_1\otimes v_2 \mapsto \mathrm{Tr}_{V_{op}}\left(Y_{op}(T_{q_{s_1}}v_1, q_{s_1})Y_{op}(T_{q_{s_2}}v_2, q_{s_2})q_\tau^{L(0)}\right), \tag{3.6}$$

where

$$\begin{aligned} T_{q_{s_1}} &= e^{-\sum_{j=1}^{\infty} B_j^{(1)} L(j)}(b_0^{(1)})^{-L(0)}, \\ T_{q_{s_2}} &= e^{-\sum_{j=1}^{\infty} B_j^{(2)} L(j)}(b_0^{(2)})^{-L(0)}. \end{aligned} \tag{3.7}$$

We need rewrite $T_{q_{s_1}}$ and $T_{q_{s_2}}$. Let $A_j, j\in\mathbb{Z}_+$, be the complex numbers defined by

$$\log(1+y) = e^{\sum_{j=1}^{\infty} A_j y^{j+1}\frac{d}{dy}}y.$$

It is clear that $A_j\in\mathbb{R}$. Hence we also obtain:

$$\frac{1}{2\pi i}\log\left(1+\frac{1}{z}x\right) = z^{-x\frac{d}{dx}}e^{\sum_{j\in\mathbb{Z}_+} A_j x^{j+1}\frac{d}{dx}}(2\pi i)^{-x\frac{d}{dx}}x. \tag{3.8}$$

Lemma 3.1.

$$\begin{aligned}
T_{q_{s_1}} &= (q_{s_1})^{L(0)} e^{-\sum_{j=1}^{\infty} A_j L(j)} (2\pi i)^{L(0)} e^{-\frac{\pi i}{2} L(0)}, \\
T_{q_{s_2}} &= (q_{s_2})^{L(0)} e^{-\sum_{j=1}^{\infty} A_j L(j)} (2\pi i)^{L(0)} e^{-2\pi i L(0)} e^{\frac{\pi i}{2} L(0)}.
\end{aligned} \tag{3.9}$$

Proof. By results in [BHL], if $e^{\sum_{j=1}^{\infty} C_j L(j)} c_0^{L(0)} = e^{\sum_{j=1}^{\infty} D_j L(j)} d_0^{L(0)}$ for any $C_j, D_j \in \mathbb{C}, c_0, d_0 \in \mathbb{C}$, we must have $C_j = D_j$ and $c_0 = d_0$. Therefore, by moving the factor $(q_{s_1})^{L(0)}$ to the right side of $e^{-\sum_{j=1}^{\infty} A_j L(j)}$ in (3.9) and similarly moving the factor $(q_{s_2})^{L(0)}$ in (3.9), we see that it is enough to show

$$\begin{aligned}
(b_0^{(1)})^{L(0)} &= (q_{s_1})^{L(0)} (2\pi i)^{L(0)} e^{-\frac{\pi i}{2} L(0)}, \\
(b_0^{(1)})^{L(0)} &= (q_{s_2})^{L(0)} (2\pi i)^{L(0)} e^{-2\pi i L(0)} e^{\frac{\pi i}{2} L(0)}.
\end{aligned} \tag{3.10}$$

Using our conventions (0.1)(0.2), it is easy to check that above identities hold. ∎

Let W be a V-module. Huang introduced the following operator in [H10].

$$\mathcal{U}(x) := x^{L(0)} e^{-\sum_{j=1}^{\infty} A_j L(j)} (2\pi i)^{L(0)} \in (\mathrm{End} W)\{x\}.$$

Thus (3.6) can be rewritten as follow:

$$v_1 \otimes v_2 \mapsto \mathrm{Tr}_{V_{op}} \left(Y_{op}(\mathcal{U}(q_{s_1}) v_1', q_{s_1}) Y_{op}(\mathcal{U}(q_{s_2}) e^{-2\pi i L(0)} v_2', q_{s_2}) q_\tau^{L(0)} \right) \tag{3.11}$$

where $v_1' = e^{-\frac{\pi i}{2} L(0)} v_1$ and $v_2' = e^{\frac{\pi i}{2} L(0)} v_2$.

Now we consider the algebraic realization of the surface (C) in Figure 5 obtained from an interior sewing operation between an element in $\mathbb{S}(0,1|1,0)$ and an element in $\mathbb{S}(0,1|0,1)$.

Lemma 3.2. $\forall r \in (0,1) \subset \mathbb{R}_+$, *the surface C in Figure 5 is conformally equivalent to a surface $Q_1 \ {}_1\infty_1^I \ Q_2$, where*

1. $Q_1 \in \mathbb{S}(0,1|0,1)$ *with punctures at $z_1 \in \mathbb{H}, \infty \in \hat{\mathbb{R}}$ and local coordinate maps:*

$$f_{z_1} : w \mapsto \left(\frac{p_{s_1}}{r}\right) \frac{w - z_1}{w - \bar{z}_1}, \tag{3.12}$$

$$f_\infty^{(1)} : w \mapsto e^{\frac{\pi i}{2}} \left(\frac{-\tau}{2\pi i}\right) \log \frac{w - z_1}{w - \bar{z}_1} \ ; \tag{3.13}$$

2. $Q_2 \in \mathbb{S}(0,1|1,0)$ *with punctures at $z_2 \in \mathbb{H}, \infty \in \hat{\mathbb{R}}$ with local coordinate maps*

$$f_{z_2} : w \mapsto \left(\frac{-r}{p_{s_2}}\right) \frac{w - z_2}{w - \bar{z}_2}, \tag{3.14}$$

$$f_\infty^{(2)} : w \mapsto e^{-\frac{\pi i}{2}} \left(\frac{-\tau}{2\pi i}\right) \log \frac{w - \bar{z}_2}{w - z_2} \ . \tag{3.15}$$

Proof. Let us first define two disks D_1 and D_2. D_1 is the unit disk, i.e. $D_1 := \{z \in \mathbb{C} | |z| \leq 1\}$, which has punctures at $p_{s_1}, 0$ and local coordinate maps:

$$g_{p_{s_1}} : w \mapsto e^{\frac{\pi i}{2}} \left(\frac{-\tau}{2\pi i} \log w - s_1\right), \tag{3.16}$$

$$g_0 : w \mapsto r^{-1} w. \tag{3.17}$$

D_2 is the disk $\{z \in \hat{\mathbb{C}} | |z| \geq |p_{s_2}|\}$ which has punctures at p_{s_2}, ∞ and local coordinate maps:

$$
\begin{aligned}
g_{p_{s_2}} : w &\mapsto e^{-\frac{\pi i}{2}} \left(\frac{-\tau}{2\pi i} \log w - s_2 \right), &(3.18)\\
g_{\infty} : w &\mapsto \frac{-r}{w}. &(3.19)
\end{aligned}
$$

It is not hard to see that the surface C in Figure 1 can be obtained by sewing the puncture $0 \in D_1$ with the puncture $\infty \in D_2$ according to the usual definition of interior sewing operation. Then it is enough to show that D_1 and D_2 are conformally equivalent to P and Q respectively. We define two maps $h_1 : Q_1 \to D_1$ and $h_2 : Q_2 \to D_2$ as follow:

$$
\begin{aligned}
h_1 : w &\mapsto p_{s_1} \frac{w - z_1}{w - \bar{z}_1},\\
h_2 : w &\mapsto p_{s_2} \frac{w - \bar{z}_2}{w - z_2}.
\end{aligned}
$$

It is clear that h_1 and h_2 are both biholomorphic. We can check directly that h_1 and h_2 map punctures to punctures and preserve local coordinate maps as well. ∎

Using (2.33), we obtain the the algebraic realization the annulus C in Figure 1 as follow:

$$
v_1 \otimes v_2 \mapsto ((T_1^L \otimes T_1^R)^* \, \iota_{cl-op}^*(z_1, \bar{z}_1)(T_2 v_1'), \ (T_3^L \otimes T_3^R)^* \, \iota_{cl-op}^*(z_2, \bar{z}_2)(T_4 v_2'))_{cl} \qquad (3.20)
$$

where $T_1^{L,R}, T_2, T_3^{L,R}, T_4$ are conformal transformations determined by local coordinate maps $f_{z_1}, f_{\infty}^{(2)}, f_{z_2}^{(1)}, f_{\infty}^{(2)}$, and $(T_i^L \otimes T_i^R)^*$ is the adjoint of $T_i^L \otimes T_i^R$ with respect to the bilinear form $(\cdot, \cdot)_{cl}$ for $i = 1, 3$, and $v_1' = e^{-\frac{\pi i}{2} L(0)} v_1$ and $v_2' = e^{\frac{\pi i}{2} L(0)} v_2$.

Lemma 3.3. *Recall the convention (0.2), we have*

$$
\begin{aligned}
T_1^L &= (z_1 - \bar{z}_1)^{L(0)} e^{L(1)} \left(\frac{p_{s_1}}{r} \right)^{-L(0)}, T_1^R = \overline{(z_1 - \bar{z}_1)}^{L(0)} e^{L(1)} \overline{\left(\frac{p_{s_1}}{r} \right)}^{-L(0)}, &(3.21)\\
T_2 &= e^{\bar{z}_1 L(1)} (z_1 - \bar{z}_1)^{-L(0)} \mathcal{U}(1) \left(-\frac{1}{\tau} \right)^{L(0)}, &(3.22)\\
T_3^L &= (\bar{z}_2 - z_2)^{L(0)} e^{-L(1)} \left(\frac{p_{s_2}}{r} \right)^{L(0)}, T_3^R = \overline{(\bar{z}_2 - z_2)}^{L(0)} e^{-L(1)} \overline{\left(\frac{p_{s_2}}{r} \right)}^{L(0)}, &(3.23)\\
T_4 &= e^{z_2 L(1)} (\bar{z}_2 - z_2)^{-L(0)} \mathcal{U}(1) \left(-\frac{1}{\tau} \right)^{L(0)}. &(3.24)
\end{aligned}
$$

Proof. From (3.12) and (3.14), we obtain

$$
\begin{aligned}
f_{z_1} : w &\mapsto (z_1 - \bar{z}_1)^{-x\frac{d}{dx}} e^{-x^2 \frac{d}{dx}} \left(\frac{p_{s_1}}{r} \right)^{x\frac{d}{dx}} x \Big|_{x = w - z_1}\\
f_{z_2} : w &\mapsto (\bar{z}_2 - z_2)^{-x\frac{d}{dx}} e^{x^2 \frac{d}{dx}} \left(\frac{p_{s_2}}{r} \right)^{-x\frac{d}{dx}} x \Big|_{x = w - z_2} &(3.25)
\end{aligned}
$$

Then (3.21) and (3.23) is obvious. Notice that the expression (3.21) is independent of our choice of branch cut as long as we keep the convention (0.2).

From (3.13) and (3.15), we obtain

$$f_\infty^{(1)}(w) = e^{-\bar{z}_1 x^2 \frac{d}{dx}} (z_1 - \bar{z}_1)^{x\frac{d}{dx}} e^{\sum_{j=1}^\infty A_j x^{j+1}\frac{d}{dx}} (2\pi i)^{-x\frac{d}{dx}} \left(-\frac{1}{\tau}\right)^{-x\frac{d}{dx}} e^{\frac{\pi i}{2}x\frac{d}{dx}} x \Big|_{x=\frac{-1}{w}} \tag{3.26}$$

$$f_\infty^{(2)}(w) = e^{-z_2 x^2 \frac{d}{dx}} (\bar{z}_2 - z_2)^{x\frac{d}{dx}} e^{\sum_{j=1}^\infty A_j x^{j+1}\frac{d}{dx}} (2\pi i)^{-x\frac{d}{dx}} \left(-\frac{1}{\tau}\right)^{-x\frac{d}{dx}} e^{-\frac{\pi i}{2}x\frac{d}{dx}} x \Big|_{x=\frac{-1}{w}} \tag{3.27}$$

Recall that $f_\infty^{(1)}, f_\infty^{(2)}$ are both real analytic. Similar to the proof of Lemma 3.1, to show (3.22) and (3.24) is enough to show that

$$(b_\infty^{(1)})^{L(0)} = (z_1 - \bar{z}_1)^{-L(0)} (2\pi i)^{L(0)} \left(-\frac{1}{\tau}\right)^{L(0)} e^{-\frac{\pi i}{2}L(0)} \tag{3.28}$$

$$(b_\infty^{(2)})^{L(0)} = (\bar{z}_2 - z_2)^{-L(0)} (2\pi i)^{L(0)} \left(-\frac{1}{\tau}\right)^{L(0)} e^{\frac{\pi i}{2}L(0)} \tag{3.29}$$

for some $b_\infty^{(1)}, b_\infty^{(2)} \in \mathbb{R}_+$. Using our convention (0.1) and (0.2), it is a direct check that (3.28) and (3.29) holds. ∎

The last piece of ingredients needed is from the determinant line bundle on torus. It says that two different ways of obtain the same torus differ by a factor which depends on the moduli associated with two sewing operations [Se1][Kr].

Combining (3.11), (3.20) and additional factor from the determinant line bundle on torus, we obtain the following formulation of Cardy condition:

Definition 3.4. The open-closed field algebra over V given in (2.12) and equipped with nondegenerate bilinear forms $(\cdot,\cdot)_{op}$ and $(\cdot,\cdot)_{cl}$ is said to satisfy *Cardy condition* if the left hand sides of the following formula, $\forall z_1, z_2 \in \mathbb{H}, v_1, v_2 \in V_{op}$,

$$\begin{aligned}&\mathrm{Tr}_{V_{op}} \left(Y_{op}(\mathcal{U}(q_{s_1})v_1, q_{s_1}) Y_{op}(\mathcal{U}(q_{s_2})e^{-2\pi i L(0)}v_2, q_{s_2}) q_\tau^{L(0)-c/24}\right) \\ &= \Big((T_1^L \otimes T_1^R)^* \iota^*_{cl-op}(z_1,\bar{z}_1)(T_2 v_1),\ q_{-\frac{1}{\tau}}^{-c/24} (T_3^L \otimes T_3^R)^* \iota^*_{cl-op}(z_2,\bar{z}_2)(T_4 v_2)\Big)_{cl}\end{aligned} \tag{3.30}$$

converge absolutely when $1 > |q_{s_1}| > |q_{s_2}| > |q_\tau| > 0$, and the right hand side of (3.30) converge absolutely for all $s_1, s_2 \in \mathbb{H}$ satisfying $\mathrm{Re}\, s_1 = 0, \mathrm{Re}\, s_2 = \frac{1}{2}$. Moreover, the equality (3.30) holds when $1 > |q_{s_1}| > |q_{s_2}| > |q_\tau| > 0$.

Remark 3.5. The dependence of z_1, z_2, r of the right hand side of (3.30) is superficial as required by the independence of z_1, z_2, r of the left hand side of (3.30). We will see it more explicitly later.

Using the definition of boundary states (2.34), (3.30) can also be written as follow:

$$\begin{aligned}&\mathrm{Tr}_{V_{op}} \left(Y_{op}(\mathcal{U}(q_{s_1})v_1, q_{s_1}) Y_{op}(\mathcal{U}(q_{s_2})e^{-2\pi i L(0)}v_2, q_{s_2}) q_\tau^{L(0)-c/24}\right) \\ &= \left(B_{z_1}(T_2 v_1),\ \left(\frac{p_{s_2}}{-p_{s_1}}\right)^{L(0)} \otimes \left(\overline{\frac{p_{s_2}}{-p_{s_1}}}\right)^{L(0)} q_{-\frac{1}{\tau}}^{-c/24} B_{z_2}(T_4 v_2)\right)_{cl}.\end{aligned} \tag{3.31}$$

3.2 The second version

In this subsection, we rewrite the Cardy condition (3.30) in the framework of intertwining operator algebra [H5]-[H8].

Since V satisfies the conditions in Theorem 0.1, it has only finite number of inequivalent irreducible modules. Let $\mathcal{I}$ be the set of equivalence classes of irreducible V-modules. We denote the equivalence class of the adjoint module V as e, i.e. $e \in \mathcal{I}$. Let W_a be a chosen representative of $a \in \mathcal{I}$.

For any V-module (W, Y_W), we denote the graded dual space of W as W', i.e. $W' = \oplus_{n\in\mathbb{C}}(W_{(n)})^*$. There is a contragredient module structure on W' [FHL] given by a vertex operator Y'_W, which is defined as follow

$$\langle Y'_W(u,x)w', w\rangle := \langle w', Y_W(e^{-xL(1)}x^{-2L(0)}u, -x^{-1})w\rangle \tag{3.32}$$

for $u \in V, w \in W, w' \in W'$. (W', Y'_W) (or simply W) is the only module structure on W' we use in this work. So we can set $Y_{W'} := Y'_W$. We denote the equivalent class of W'_a as a'. It is harmless to set $W'_a = W_{a'}$. Moreover, W'' is canonically identified with W. Hence $a'' = a$ for $a \in \mathcal{I}$.

By assumption on V, $V' \cong V$, i.e. $e' = e$. From [FHL], there is a nondegenerate invariant bilinear form $(\cdot,\cdot)$ on V such that $(\mathbf{1},\mathbf{1}) = 1$. This bilinear form specifies a unique isomorphism from V to V'. In the rest of this work, we identify V' with V using this isomorphism without mentioning it explicitly.

For any triple V-modules W_1, W_2, W_3, we have isomorphisms

$$\Omega_r : \mathcal{V}_{W_1W_2}^{W_3} \to \mathcal{V}_{W_2W_1}^{W_3}, \qquad \forall r \in \mathbb{Z}$$

given as follow:

$$\Omega_r(\mathcal{Y})(w_2, z)w_1 = e^{zL(-1)}\mathcal{Y}(w_1, e^{(2r+1)\pi i}z)w_2, \tag{3.33}$$

for $\mathcal{Y} \in \mathcal{V}_{W_1W_2}^{W_3}$ and $w_i \in W_i, i = 1, 2$. The following identity

$$\Omega_r \circ \Omega_{-r-1} = \Omega_{-r-1} \circ \Omega_r = \mathrm{id} \tag{3.34}$$

is proved in [HL3].

For $\mathcal{Y} \in \mathcal{V}_{W_1W_2}^{W_3}$ and $r \in \mathbb{Z}$, a so-called *r-contragredient operator* $A_r(\mathcal{Y})$ was introduced in [HL3]. Here, we use two slightly different operators $\tilde{A}_r(\mathcal{Y})$ and $\hat{A}_r(\mathcal{Y})$ introduced in [Ko1] and defined as follow:

$$\begin{aligned} \langle \tilde{A}_r(\mathcal{Y})(w_1, e^{(2r+1)\pi i}x)w'_3, w_2\rangle &= \langle w'_3, \mathcal{Y}(e^{xL(1)}x^{-2L(0)}w_1, x^{-1})w_2\rangle, \\ \langle \hat{A}_r(\mathcal{Y})(w_1, x)w'_3, w_2\rangle &= \langle w'_3, \mathcal{Y}(e^{-xL(1)}x^{-2L(0)}w_1, e^{(2r+1)\pi i}x^{-1})w_2\rangle, \end{aligned} \tag{3.35}$$

for $\mathcal{Y} \in \mathcal{V}_{W_1W_2}^{W_3}$ and $w_1 \in W_1, w_2 \in W_2$, $w'_3 \in W'_3$. In particular, when $W_1 = V$ and $W_2 = W_3 = W$, we have $\tilde{A}_r(Y_W) = \hat{A}_r(Y_W) = Y'_W = Y_{W'}, \forall r \in \mathbb{Z}$. If $\mathcal{Y} \in \mathcal{V}_{W_1W_2}^{W_3}$, then $\tilde{A}_r(\mathcal{Y})$, $\hat{A}_r(\mathcal{Y}) \in \mathcal{V}_{W_1W'_3}^{W'_2}$ for $r \in \mathbb{Z}$ and

$$\tilde{A}_r \circ \hat{A}_r(\mathcal{Y}) = \hat{A}_r \circ \tilde{A}_r(\mathcal{Y}) = \mathcal{Y}. \tag{3.36}$$

Let $\mathcal{Y} \in \mathcal{V}_{a_1 a_2}^{a_3}$. We define $\sigma_{123} := \Omega_r \circ \tilde{A}_r$. It is easy to see that

$$\begin{aligned} &\langle w'_{a_3}, \mathcal{Y}(w_{a_1}, x) w_{a_2} \rangle \\ &\quad = \langle e^{-x^{-1}L(-1)} \sigma_{123}(\mathcal{Y})(w'_{a_3}, x^{-1}) e^{-xL(1)} x^{-2L(0)} w_{a_1}, w_{a_2} \rangle \end{aligned} \tag{3.37}$$

for $w_{a_1} \in W_{a_1}, w_{a_2} \in W_{a_2}, w'_{a_3} \in W'_{a_3}$. It is also clear that σ_{123} is independent of $r \in \mathbb{Z}$. It is proved in [Ko1] that $\sigma_{123}^3 = \mathrm{id}_{\mathcal{V}_{a_1 a_2}^{a_3}}$. We also denote σ_{123}^{-1} as σ_{132}. Clearly, we have $\sigma_{132} = \hat{A}_r \circ \Omega_{-r-1}$ and

$$\begin{aligned} &\langle \sigma_{132}(\mathcal{Y})(w_1, x) w'_3, w_2 \rangle \\ &\quad = \langle w'_3, e^{-x^{-1}L(-1)} \mathcal{Y}(w_2, x^{-1}) e^{-xL(1)} x^{-2L(0)} w_1 \rangle \end{aligned} \tag{3.38}$$

for $w_{a_1} \in W_{a_1}, w_{a_2} \in W_{a_2}, w'_{a_3} \in W'_{a_3}$.

For any V-module W, we define a V-module map $\theta_W : W \to W$ by

$$\theta_W : w \mapsto e^{-2\pi i L(0)} w. \tag{3.39}$$

For W_a, we have $\theta_{W_a} = e^{-2\pi i h_a} \mathrm{id}_{W_a}$ where $h_a \in \mathbb{C}$ is the lowest conformal weight of W_a.

We denote the graded dual space of V_{cl} and V_{op} by V'_{cl} and V'_{op} respectively. Let $\varphi_{cl} : V_{cl} \to V'_{cl}$ and $\varphi_{op} : V_{op} \to V'_{op}$ be the isomorphisms induced from $(\cdot, \cdot)_{cl}$ and $(\cdot, \cdot)_{op}$ respectively. Namely, we have

$$\begin{aligned} (u_1, u_2)_{cl} &= \langle \varphi_{cl}(u_1), u_2 \rangle \\ (v_1, v_2)_{op} &= \langle \varphi_{op}(v_1), v_2 \rangle \end{aligned} \tag{3.40}$$

for $u_1, u_2 \in V_{cl}$ and $v_1, v_2 \in V_{op}$.

V_{cl} as a conformal full field algebra over $V \otimes V$ can be expanded as follow:

$$V_{cl} = \oplus_{i=1}^{N_{cl}} W_{r_L(i)} \otimes W_{r_R(i)} \tag{3.41}$$

where $r_L, r_R : \{1, \dots, N_{cl}\} \to \mathcal{I}$. For $a \in \mathcal{I}$, we choose a basis $\{e_{a;\alpha}\}_{\alpha \in \mathbb{N}}$ of W_a and a dual basis $\{e'_{a;\alpha}\}_{\alpha \in \mathbb{N}}$ of W'_a. Then

$$\{e_{r_L(i),\alpha} \otimes e_{r_R(i),\beta}\}_{i=1,\dots,N_{cl},\alpha,\beta \in \mathbb{N}} \tag{3.42}$$

is a basis of V_{cl} and

$$\{\varphi_{cl}^{-1}(e'_{r_L(i);\alpha} \otimes e'_{r_R(i),\beta})\}_{i=1,\dots,N_{cl},\alpha,\beta \in \mathbb{N}} \tag{3.43}$$

is its dual basis with respect to the nondegenerate bilinear form $(\cdot, \cdot)_{cl}$.

Let $T : \mathcal{C}_{V \otimes V} \to \mathcal{C}_V$ be the tensor bifunctor. We showed in [Ko2] that there is a morphism $\iota_{cl-op} : T(V_{cl}) \to V_{op}$ in $\mathcal{C}_V$ (see (3.81),(3.82) in [Ko2] for definition). We define a morphism $\iota'_{cl-op} : T(V'_{cl}) \to V_{op}$ as a composition of maps as follow:

$$\iota'_{cl-op} : T(V'_{cl}) \xrightarrow{T(\varphi_{cl}^{-1})} T(V_{cl}) \xrightarrow{\iota_{cl-op}} V_{op}\, . \tag{3.44}$$

By the universal property of tensor product $\boxtimes$ ([HL1]-[HL4]), $\mathcal{V}_{W_1 W_2}^{W_3}$ and $\mathrm{Hom}_V(W_1 \boxtimes W_2, W_3)$ for any three V-modules W_1, W_2, W_3 are canonically isomorphic. Given a morphism $m \in \mathrm{Hom}_V(W_1 \boxtimes W_2, W_3)$, we denote the corresponding intertwining operator as $\mathcal{Y}_m$. Conversely, given an intertwining operator $\mathcal{Y}$, we denote its corresponding morphism as $m_{\mathcal{Y}}$. Therefore, we have two intertwining operators $\mathcal{Y}_{\iota_{cl-op}}$ and $\mathcal{Y}_{\iota'_{cl-op}}$ corresponding to morphisms ι_{cl-op} and ι'_{cl-op} respectively.

Lemma 3.6. *For $z \in \mathbb{H}$, we have*

$$\iota_{cl-op}(z, \bar{z})(e_{r_L(i),\alpha} \otimes e_{r_R(i),\beta}) = e^{\bar{z}L(-1)} \mathcal{Y}_{\iota_{cl-op}}(e_{r_L(i),\alpha}, z - \bar{z}) e_{r_R(i),\alpha}. \tag{3.45}$$

Proof. It is proved in [Ko2] that

$$m_{\mathbb{Y}_{cl-op}} = m_{Y_{op}^f} \circ (\iota_{cl-op} \boxtimes \mathrm{id}_{V_{op}}). \tag{3.46}$$

Using (3.46), when $z \in \mathbb{H}, \zeta \in \overline{\mathbb{H}}$ and $|\zeta| > |z - \zeta| > 0$, we have

$$\begin{aligned} \iota_{cl-op}(z, \zeta)(e_{r_L(i),\alpha} \otimes e_{r_R(i),\beta}) &= \mathbb{Y}_{cl-op}(e_{r_L(i),\alpha} \otimes e_{r_R(i),\beta}; z, \zeta)\mathbf{1}_{op} \\ &= Y_{op}^f(\mathcal{Y}_{\iota_{cl-op}}(e_{r_L(i),\alpha}, z - \zeta)e_{r_R(i),\beta}, \zeta)\mathbf{1} \\ &= e^{\zeta L(-1)} \mathcal{Y}_{\iota_{cl-op}}(e_{r_L(i),\alpha}, z - \zeta)e_{r_R(i),\beta}. \end{aligned} \tag{3.47}$$

By the convergence property of the iterate of two intertwining operators, the right hand side of (3.47) is a power series of ζ absolutely convergent for $|\zeta| > |z - \zeta| > 0$. By the property of power series, the right hand side of (3.47) must converge absolutely for all $z \in \mathbb{H}, \zeta \in \overline{\mathbb{H}}$. Because analytic extension in a simply connected domain is unique, we obtain that the equality (3.47) holds for all $z \in \mathbb{H}, \zeta \in \overline{\mathbb{H}}$. When $\zeta = \bar{z}$, we obtain (3.45). ∎

Now we consider the both sides of Cardy condition (3.30) for an open-closed field algebra over V. On the left hand side of (3.30), we have $q_{s_2} < 0$. Using (1.25) and (3.33), we obtain, $\forall v_3 \in V_{op}$,

$$\begin{aligned} Y_{op}(\mathcal{U}(q_{s_2})v_2, q_{s_2})v_3 &= e^{-|q_{s_2}|L(-1)} Y_{op}(v_3, |q_{s_2}|)\, \mathcal{U}(q_{s_2})v_2 \\ &= \Omega_{-1}(Y_{op}^f)(\mathcal{U}(q_{s_2})v_2, e^{\pi i}|q_{s_2}|)v_3. \end{aligned} \tag{3.48}$$

Hence we can rewrite the left hand side of (3.30) as follow:

$$\mathrm{Tr}_{V_{op}} \left(Y_{op}^f(\mathcal{U}(q_{s_1})v_1, q_{s_1})\Omega_{-1}(Y_{op}^f)(\mathcal{U}(q_{s_2})e^{-2\pi i L(0)}v_2, e^{\pi i}|q_{s_2}|) q_\tau^{L(0)-c/24} \right) \tag{3.49}$$

for $q_{s_1} > |q_{s_2}| > |q_\tau| > 0$.

We have the following result for the right hand side of (3.30).

Proposition 3.7. *For $s_1, s_2 \in \mathbb{H}, \mathrm{Re}\, s_1 = 0, \mathrm{Re}\, s_2 = 0$,*

$$\begin{aligned} &((T_1^L \otimes T_1^R)^* \, \iota_{cl-op}^*(z_1, \bar{z}_1)(T_2 v_1),\ q_{-\frac{1}{\tau}}^{-\frac{c}{24}} (T_3^L \otimes T_3^R)^* \, \iota_{cl-op}^*(z_2, \bar{z}_2)(T_4 v_2))_{cl} \\ &= \sum_{i=1}^{N_{cl}} \mathrm{Tr}_{W_{r_R(i)}} \ e^{-2\pi i L(0)} \ \mathcal{Y}_1 \left(\mathcal{U}(q_{-\frac{1}{\tau}s_1}) \left(-\frac{1}{\tau} \right)^{L(0)} v_1, q_{-\frac{1}{\tau}s_1} \right) \\ &\qquad \mathcal{Y}_2 \left(\mathcal{U}(q_{-\frac{1}{\tau}s_2}) \left(-\frac{1}{\tau} \right)^{L(0)} v_2, q_{-\frac{1}{\tau}s_2} \right) q_{-\frac{1}{\tau}}^{L(0)-\frac{c}{24}} \end{aligned} \tag{3.50}$$

where $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are intertwining operators of types $\binom{W_{r_R(i)}}{V_{op} W'_{r_L(i)}}$ and $\binom{W'_{r_L(i)}}{V_{op} W_{r_R(i)}}$ respectively and are given by

$$\mathcal{Y}_1 = \sigma_{123}(\mathcal{Y}_{\iota'_{cl-op}}) \circ (\varphi_{op} \otimes \mathrm{id}_{W'_{r_L(i)}}), \tag{3.51}$$

$$\mathcal{Y}_2 = \Omega_0(\sigma_{132}(\mathcal{Y}_{\iota_{cl-op}})) \circ (\varphi_{op} \otimes \mathrm{id}_{W_{r_R(i)}}). \tag{3.52}$$

Proof. Let $z_3 := z_1 - \bar{z}_1$ and $z_4 := z_2 - \bar{z}_2$. By (2.28) and (3.45), the left hand side of (3.50) equals to

$$
\begin{aligned}
&((T_1^L \otimes T_1^R)^* \, \iota_{cl-op}^*(z_1, \bar{z}_1)(T_2 v_1), \ (T_3^L \otimes T_3^R)^* \, \iota_{cl-op}^*(z_2, \bar{z}_2)(T_4 v_2))_{cl} \\
&= \sum_{i=1}^{N_{cl}} \sum_{\alpha,\beta} \big((T_3^L \otimes T_3^R)^* \, \iota_{cl-op}^*(z_2, \bar{z}_2)(T_4 v_2), e_{r_L(i),\alpha} \otimes e_{r_R(i),\beta}\big)_{cl} \\
&\qquad\qquad \big(\varphi_{cl}^{-1}(e'_{r_L(i),\alpha} \otimes e'_{r_R(i),\beta}), (T_1^L \otimes T_1^R)^* \, \iota_{cl-op}^*(z_1, \bar{z}_1)(T_2 v_1)\big)_{cl} \\
&= \sum_{i=1}^{N_{cl}} \sum_{\alpha,\beta} \big(T_4 v_2, \iota_{cl-op}(z_2, \bar{z}_2)(T_3^L e_{r_L(i),\alpha} \otimes T_3^R e_{r_R(i),\beta})\big)_{op} \\
&\qquad\qquad \big(\iota_{cl-op}(z_1, \bar{z}_1)(\varphi_{cl}^{-1}(T_1^L e'_{r_L(i),\alpha} \otimes T_1^R e'_{r_R(i),\beta})), T_2 v_1\big)_{op} \\
&= \sum_{i=1}^{N_{cl}} \sum_{\alpha,\beta} \big(T_4 v_2 , \ e^{\bar{z}_2 L(-1)} \mathcal{Y}_{\iota_{cl-op}}(T_3^L e_{r_L(i),\alpha}, z_2 - \bar{z}_2) T_3^R e_{r_R(i),\beta}\big)_{op} \\
&\qquad\qquad \Big(e^{\bar{z}_1 L(-1)} \mathcal{Y}_{\iota'_{cl-op}}(T_1^L e'_{r_L(i),\alpha}, z_1 - \bar{z}_1) T_1^R e'_{r_R(i),\beta}, \ T_2 v_1\Big)_{op} \\
&= \sum_{i=1}^{N_{cl}} \sum_{\alpha,\beta} \langle\, \varphi_{op}(T_4 v_2), \ e^{\bar{z}_2 L(-1)} \mathcal{Y}_{\iota_{cl-op}}(T_3^L e_{r_L(i),\alpha}, z_2 - \bar{z}_2) T_3^R e_{r_R(i),\beta} \,\rangle \\
&\qquad\qquad \langle\, e^{\bar{z}_1 L(-1)} \mathcal{Y}_{\iota'_{cl-op}}(T_1^L e'_{r_L(i),\alpha}, \ z_1 - \bar{z}_1) T_1^R e'_{r_R(i),\beta}, \ \varphi_{op}(T_2 v_1) \,\rangle \\
&= \sum_{i=1}^{N_{cl}} \sum_{\alpha,\beta} \langle\, e^{-z_4^{-1} L(-1)} \sigma_{123}(\mathcal{Y}_{\iota_{cl-op}})(\varphi_{op}(e^{-\bar{z}_2 L(1)} T_4 v_2), z_4^{-1}) \\
&\qquad\qquad\qquad z_4^{-2L(0)} e^{-z_4^{-1} L(1)} T_3^L e_{r_L(i),\alpha}, \ T_3^R e_{r_R(i),\beta} \,\rangle \\
&\qquad\qquad \langle\, T_1^R e'_{r_R(i),\beta}, \ e^{-z_3^{-1} L(-1)} \sigma_{123}(\mathcal{Y}_{\iota'_{cl-op}})(\varphi_{op}(e^{-\bar{z}_1 L(1)} T_2 v_1), z_3^{-1}) \\
&\qquad\qquad\qquad z_3^{-2L(0)} e^{-z_3^{-1} L(1)} T_1^L e'_{r_L(i),\alpha} \rangle \\
&= \sum_{i=1}^{N_{cl}} \sum_{\alpha,\beta} \langle\, \sigma_{123}(\mathcal{Y}_{\iota_{cl-op}})(\varphi_{op}(e^{-\bar{z}_2 L(1)} T_4 v_2), z_4^{-1}) z_4^{-2L(0)} e^{-z_4^{-1} L(1)} T_3^L e_{r_L(i),\alpha}, \\
&\qquad\qquad\qquad e^{z_4^{-1} L(1)} T_3^R e_{r_R(i),\beta} \rangle \, \langle e'_{r_R(i),\beta}, \ (T_1^R)^* e^{-z_3^{-1} L(-1)} \cdot \\
&\qquad\quad \sigma_{123}(\mathcal{Y}_{\iota'_{cl-op}})(\varphi_{op}(e^{-\bar{z}_1 L(1)} T_2 v_1), z_3^{-1}) z_3^{-2L(0)} e^{-z_3^{-1} L(1)} T_1^L e'_{r_L(i),\alpha} \rangle \\
&= \sum_{i=1}^{N_{cl}} \sum_{\alpha,\beta} \langle\, e_{r_L(i),\alpha}, (T_3^L)^* e^{z_4^{-1} L(-1)} z_4^{-2L(0)} \tilde{A}_0 \circ \sigma_{123}(\mathcal{Y}_{\iota_{cl-op}})(e^{-z_4^{-1} L(1)} z_4^{2L(0)} \cdot \\
&\qquad\quad \varphi_{op}(e^{-\bar{z}_2 L(1)} T_4 v_2), e^{\pi i} z_4) e^{z_4^{-1} L(1)} T_3^R e_{r_R(i),\beta} \rangle \, \langle e'_{r_R(i),\beta}, \ (T_1^R)^* e^{-z_3^{-1} L(-1)} \cdot \\
&\qquad\quad \sigma_{123}(\mathcal{Y}_{\iota'_{cl-op}})(\varphi_{op}(e^{-\bar{z}_1 L(1)} T_2 v_1), z_3^{-1}) z_3^{-2L(0)} e^{-z_3^{-1} L(1)} T_1^L e'_{r_L(i),\alpha} \rangle
\end{aligned}
\tag{3.53}
$$

We define two intertwining operators as follow:

$$
\begin{aligned}
\mathcal{Y}_1^{(0)} &= \sigma_{123}(\mathcal{Y}_{\iota'_{cl-op}}) \circ (\varphi_{op} \otimes \mathrm{id}_{W'_{r_L(i)}}), \\
\mathcal{Y}_2^{(0)} &= \tilde{A}_0(\sigma_{123}(\mathcal{Y}_{\iota_{cl-op}})) \circ (\varphi_{op} \otimes \mathrm{id}_{W'_{r_R(i)}}).
\end{aligned}
\tag{3.54}
$$

Using (3.21),(3.22),(3.23) and (3.24), we further obtain that the left hand side of (3.53) equals to

$$\begin{aligned}
&\sum_{i=1}^{N_{cl}}\sum_{\alpha,\beta}\langle\, e_{r_L(i),\alpha},\left(\frac{p_{s_2}}{r}\right)^{L(0)}(\bar{z}_2-z_2)^{L(0)}z_4^{-2L(0)}\cdot\\
&\qquad\mathcal{Y}_2^{(0)}(z_4^{2L(0)}(\bar{z}_2-z_2)^{-L(0)}\mathcal{U}(1)\left(-\frac{1}{\tau}\right)^{L(0)}v_2,e^{\pi i}z_4)\\
&\qquad\overline{\bar{z}_2-z_2}^{L(0)}\overline{\left(\frac{p_{s_2}}{r}\right)}^{L(0)}e_{r_R(i),\beta}\rangle\langle e'_{r_R(i),\beta},\overline{\left(\frac{p_{s_1}}{r}\right)}^{-L(0)}\overline{(z_1-\bar{z}_1)}^{L(0)}\\
&\qquad\mathcal{Y}_2^{(0)}(z_3^{-L(0)}\mathcal{U}(1)\left(-\frac{1}{\tau}\right)^{L(0)}v_1,z_3^{-1})z_3^{-L(0)}\left(\frac{p_{s_1}}{r}\right)^{-L(0)}e'_{r_L(i),\alpha}\rangle\\
&=\sum_{i=1}^{N_{cl}}\mathrm{Tr}_{W'_{r_L(i)}}\mathcal{Y}_2^{(0)}(\mathcal{U}(p_{s_2})(-1/\tau)^{L(0)}v_2,E_1)E_2\,\mathcal{Y}_1^{(0)}(\mathcal{U}(p_{s_1})(-1/\tau)^{L(0)}v_1,p_{s_1})
\end{aligned}\tag{3.55}$$

where $E_1=p_{s_2}(\bar{z}_2-z_2)z_4^{-2}e^{\pi i}z_4$ and

$$E_2=p_{s_2}^{L(0)}(\bar{z}_2-z_2)^{L(0)}z_4^{-2L(0)}\overline{(\bar{z}_2-z_2)}^{L(0)}\overline{p_{s_2}}^{L(0)}\overline{p_{s_1}}^{-L(0)}\overline{z_1-\bar{z}_1}^{L(0)}z_3^{-L(0)}p_{s_1}^{-L(0)}.$$

For E_1, since z_4^{-1} is obtained by operations on intertwining operator where z_4 is treated formally, z_4^{-1} really means $|z_4|^{-1}e^{-i\frac{\pi i}{2}}$. Therefore, we have

$$E_1=p_{s_2}e^{\frac{3\pi i}{2}}e^{-\pi i}e^{\pi i}e^{\frac{\pi i}{2}}=e^{2\pi i}p_{s_2}.\tag{3.56}$$

For E_2, keep in mind (0.1) and (0.2), we have

$$\begin{aligned}
E_2\;&=\;\left|\frac{p_{s_2}}{p_{s_1}}\right|^{2L(0)}(\bar{z}_2-z_2)^{L(0)}(z_2-\bar{z}_2)^{-2L(0)}\overline{(\bar{z}_2-z_2)}^{L(0)}\overline{(z_1-\bar{z}_1)}^{L(0)}(z_1-\bar{z}_1)^{-L(0)}\\
&=\;q_{-\frac{1}{\tau}}^{L(0)}\;e^{\frac{3\pi i}{2}L(0)}e^{-2\frac{\pi i}{2}L(0)}e^{-\frac{3\pi i}{2}L(0)}e^{-\frac{\pi i}{2}L(0)}e^{-\frac{\pi i}{2}L(0)}\\
&=\;q_{-\frac{1}{\tau}}^{L(0)}\;e^{-2\pi iL(0)}.
\end{aligned}\tag{3.57}$$

Therefore, we obtain that the left hand side of (3.53) further equals to

$$\begin{aligned}
&\sum_{i=1}^{N_{cl}}\mathrm{Tr}_{W'_{r_L(i)}}\Omega_0^2(\mathcal{Y}_2^{(0)})\left(\mathcal{U}(p_{s_2})\left(-\frac{1}{\tau}\right)^{L(0)}v_2,p_{s_2}\right)\\
&\qquad\qquad q_{-\frac{1}{\tau}}^{L(0)}\,e^{-2\pi iL(0)}\;\mathcal{Y}_1^{(0)}\left(\mathcal{U}(p_{s_1})\left(-\frac{1}{\tau}\right)^{L(0)}v_1,p_{s_1}\right),
\end{aligned}\tag{3.58}$$

where we have used the fact that $\mathcal{Y}(\cdot,e^{2\pi i}x)\cdot=\Omega_0^2(\mathcal{Y})(\cdot,x)\cdot$ for any intertwining operator $\mathcal{Y}$. By using the property of trace, it is easy to see that (3.58) multiplying $q_{-\frac{1}{\tau}}^{-\frac{c}{24}}$ is nothing but the right hand side of (3.50). ∎

Remark 3.8. It is easy to check that the absolute convergence of the left hand side of (3.53) by our assumption easily implies the absolute convergence of each step in (3.53). Notice that the absolute convergence of the right side of (3.50) is automatic because V is assumed to satisfy the conditions in Theorem 0.1. Hence, by tracing back the steps in above proof, we see that the absolute convergence of the left hand side of (3.50) is also automatic.

Now we recall some results in [H10][H11]. We follow the notations in [HKo3]. We denote the unique analytic extension of

$$\mathrm{Tr}_{W_{a_1}}\mathcal{Y}_{aa_1;i}^{a_1;(1)}(\mathcal{U}(e^{2\pi iz})w_a, e^{2\pi iz})q_\tau^{L(0)-\frac{c}{24}}$$

in the universal covering space of $1 > |q_\tau| > 0$ as

$$E(\mathrm{Tr}_{W_{a_1}}\mathcal{Y}_{aa_1;i}^{a_1;(1)}(\mathcal{U}(e^{2\pi iz})w_a, e^{2\pi iz})q_\tau^{L(0)-\frac{c}{24}}).$$

By [Mi2][H10], above formula is independent of z. Consider the map: for $w_a \in W_a$,

$$\Psi_1(\mathcal{Y}_{aa_1;i}^{a_1;(1)}) : w_a \to E(\mathrm{Tr}_{W_{a_1}}\mathcal{Y}_{aa_1;i}^{a_1;(1)}(\mathcal{U}(e^{2\pi iz})w_a, e^{2\pi iz})q_\tau^{L(0)-\frac{c}{24}}). \tag{3.59}$$

We denote the right hand side of (3.59) as $\Psi_1(\mathcal{Y}_{aa_1;i}^{a_1;(1)})(w_a; z, \tau)$. Notice that we choose to add z in the notation even though it is independent of z. We define an action of $SL(2,\mathbb{Z})$ on the map (3.59) as follow:

$$\begin{aligned}&\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}(\Psi_1(\mathcal{Y}_{aa_1;i}^{a_1;(1)}))\right)(w_a; z, \tau) \qquad (3.60)\\ &\quad = E\left(\mathrm{Tr}_{W_{a_1}}\mathcal{Y}_{aa_1;i}^{a_1;(1)}\left(\mathcal{U}(e^{2\pi iz'})\left(\frac{1}{c\tau+d}\right)^{L(0)} w_a, e^{2\pi iz'}\right)q_{\tau'}^{L(0)-\frac{c}{24}}\right)\end{aligned}$$

where $\tau' = \frac{a\tau+b}{c\tau+d}$ and $z' = \frac{z}{c\tau+d}$, for $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{Z})$ and $w_a \in W_a$. The following Theorem is proved in [Mi2][H8].

Theorem 3.9. *There exists a unique $A_{a_2a_3}^{ij} \in \mathbb{C}$ for $a_2, a_3 \in \mathcal{I}$ such that, for $w_a \in W_a$,*

$$\begin{aligned}E&\left(\mathrm{Tr}_{W_{a_1}}\mathcal{Y}_{aa_1;i}^{a_1;(1)}\left(\mathcal{U}(e^{2\pi iz'})\left(\frac{1}{c\tau+d}\right)^{L(0)} w_a, e^{2\pi iz'}\right)q_{\tau'}^{L(0)-\frac{c}{24}}\right)\\ &= \sum_{a_3\in\mathcal{I}} A_{a_1a_2}^{ij}E(\mathrm{Tr}_{W_{a_1}}\mathcal{Y}_{aa_2;j}^{a_2;(2)}(\mathcal{U}(e^{2\pi iz})w_a, e^{2\pi iz})q_\tau^{L(0)-\frac{c}{24}}) \qquad (3.61)\end{aligned}$$

where $\tau' = \frac{a\tau+b}{c\tau+d}$ and $z' = \frac{z}{c\tau+d}$.

In particular, the action of $S = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$ on (3.59) induces, for each $a \in \mathcal{I}$, an automorphism on $\oplus_{a_1\in\mathcal{I}}\mathcal{V}_{aa_1}^{a_1}$, denoted as $S(a)$. Namely, we have

$$S(\Psi_1(\mathcal{Y}_{aa_1;i}^{a_1;(1)})) = \Psi_1(S(a)(\mathcal{Y}_{aa_1;i}^{a_1;(1)})) \tag{3.62}$$

Combining all such $S(a)$, we obtain an automorphism on $\oplus_{a,a_1\in\mathcal{I}}\mathcal{V}_{aa_1}^{a_1}$. We still denote it as S, i.e. $S=\oplus_{a\in\mathcal{I}}S(a)$. Then S can be further extended to a map on $\oplus_{a,a_3\in\mathcal{I}}\mathcal{V}_{aa_3}^{a_3}\otimes\mathcal{V}_{a_1a_2}^{a}$ given as follow:

$$S(\mathcal{Y}_{aa_3;i}^{a_3;(1)}\otimes\mathcal{Y}_{a_1a_2;j}^{a;(2)}):=S(\mathcal{Y}_{aa_3;i}^{a_3;(1)})\otimes\mathcal{Y}_{a_1a_2;j}^{a;(2)}. \tag{3.63}$$

There is a fusing isomorphism map [H7]:

$$\mathcal{F}:\oplus_{a\in\mathcal{I}}\mathcal{V}_{a_1a}^{a_4}\otimes\mathcal{V}_{a_2a_3}^{a}\xrightarrow{\cong}\oplus_{b\in\mathcal{I}}\mathcal{V}_{ba_3}^{a_4}\otimes\mathcal{V}_{a_1a_2}^{b} \tag{3.64}$$

for $a_1,a_2,a_3,a_4\in\mathcal{I}$. Using the isomorphism $\mathcal{F}$, we obtain a natural action of S on $\oplus_{b,a_3}\mathcal{V}_{a_1b}^{a_3}\otimes\mathcal{V}_{a_2a_3}^{b}$.

It is shown in [H8] that the following 2-points genus-one correlation function, for $a_1,a_2,a_3,a_4\in\mathcal{I}$, $i=1,\dots,N_{aa_1}^{a_1}$, $j=1,\dots,N_{a_2a_3}^{a}$ and $w_{a_k}\in W_{a_k}, k=2,3$,

$$\mathrm{Tr}_{W_{a_1}}\mathcal{Y}_{aa_1;i}^{a_1;(1)}(\mathcal{U}(q_{z_2})\mathcal{Y}_{a_2a_3;j}^{a;(2)}(w_{a_2},z_1-z_2)w_{a_3},q_{z_2})q_\tau^{L(0)-\frac{c}{24}} \tag{3.65}$$

is absolutely convergent when $1>|e^{2\pi iz_2}|>|q_\tau|>0$ and $1>|e^{2\pi i(z_1-z_2)}|>0$ and single-valued in the chosen branch. It can be extended uniquely to a single-valued analytic function on the universal covering space of

$$M_1^2=\{(z_1,z_2,\tau)\in\mathbb{C}^3|z_1\neq z_2+p\tau+q,\forall p,q\in Z,\tau\in\mathbb{H}\}.$$

This universal covering space is denoted by $\tilde{M}_1^2$. We denote this single-valued analytic function on $\tilde{M}_1^2$ as

$$E(\mathrm{Tr}_{W_{a_1}}\mathcal{Y}_{aa_1;i}^{a_1;(1)}(\mathcal{U}(q_{z_2})\mathcal{Y}_{a_2a_3;j}^{a;(2)}(w_{a_2},z_1-z_2)w_{a_3},q_{z_2})q_\tau^{L(0)-\frac{c}{24}}).$$

We denote the space spanned by such functions on $\tilde{M}_1^2$ by $\mathbb{G}_{1;2}$.

For $\mathcal{Y}_{aa_1;i}^{a_1;(1)}\in\mathcal{V}_{aa_1}^{a_1}$ and $\mathcal{Y}_{a_2a_3;j}^{a;(2)}\in\mathcal{V}_{a_2a_3}^{a}$, we now define the following linear map:

$$\Psi_2(\mathcal{Y}_{aa_1;i}^{a_1;(1)}\otimes\mathcal{Y}_{a_2a_3;j}^{a;(2)}):\quad \oplus_{b_2,b_3\in\mathcal{I}}W_{b_2}\otimes W_{b_3}\quad\to\quad\mathbb{G}_{1;2}$$

as follow: the map restricted on $W_{b_2}\otimes W_{b_3}$ is define by 0 for $b_2\neq a_2$ or $b_3\neq a_3$, and by

$$E(\mathrm{Tr}_{W_{a_1}}\mathcal{Y}_{aa_1;i}^{a_1;(1)}(\mathcal{U}(q_{z_2})\mathcal{Y}_{a_2a_3;j}^{a;(2)}(w_{a_2},z_1-z_2)w_{a_3},q_{z_2})q_\tau^{L(0)-\frac{c}{24}}), \tag{3.66}$$

for all $w_{a_k}\in W_{a_k}, k=2,3$. The following identity was proved in [H10].

$$\begin{aligned}\left(\Psi_2(\mathcal{Y}_{aa_1;i}^{a_1;(1)}\otimes\mathcal{Y}_{a_2a_3;j}^{a;(2)})\left(\left(-\frac{1}{\tau}\right)^{L(0)}w_{a_2}\otimes\left(-\frac{1}{\tau}\right)^{L(0)}w_{a_3}\right)\right)\left(-\frac{1}{\tau}z_1,-\frac{1}{\tau}z_2;-\frac{1}{\tau}\right)\\ =\big(\Psi_2(S(\mathcal{Y}_{aa_1;i}^{a_1;(1)}\otimes\mathcal{Y}_{a_2a_3;j}^{a;(2)}))(w_{a_2}\otimes w_{a_3})\big)(z_1,z_2,\tau).\end{aligned} \tag{3.67}$$

One can also produce 2-point genus-one correlation functions from a product of two intertwining operators. It is proved in [H8] that $\forall w_{a_k}\in W_{a_k}, k=1,2$,

$$\mathrm{Tr}_{W_{a_4}}\mathcal{Y}_{a_1a_3;i}^{a_4;(1)}(\mathcal{U}(q_{z_1})w_{a_1},q_{z_1})\mathcal{Y}_{a_2a_4;j}^{a_3;(2)}(\mathcal{U}(q_{z_2})w_{a_2},q_{z_2})q_\tau^{L(0)-\frac{c}{24}}, \tag{3.68}$$

is absolutely convergent when $1 > |q_{z_1}| > |q_{z_2}| > |q_\tau| > 0$. (3.68) has a unique extension to the universal covering space $\tilde{M}_1^2$, denoted as

$$E(\mathrm{Tr}_{W_{a_4}} \mathcal{Y}_{a_1 a_3;i}^{a_4;(1)}(\mathcal{U}(q_{z_1})w_{a_1}, q_{z_1}) \mathcal{Y}_{a_2 a_4;j}^{a_3;(2)}(\mathcal{U}(q_{z_2})w_{a_2}, q_{z_2}) q_\tau^{L(0)-\frac{c}{24}}). \tag{3.69}$$

Such functions on $\tilde{M}_1^2$ also span $\mathbb{G}_{1;2}$. We define a map

$$\Psi_2(\mathcal{Y}_{a_1 a_3;i}^{a_4;(1)} \otimes \mathcal{Y}_{a_2 a_4;j}^{a_3;(2)}) : \oplus_{b_1, b_2 \in \mathcal{I}} W_{b_1} \otimes W_{b_2} \to \mathbb{G}_{1;2} \tag{3.70}$$

as follow: the map restrict on $W_{b_1} \otimes W_{b_2}$ is defined by 0 for $b_1 \neq a_1, b_2 \neq a_2$, and by (3.69) for $w_{a_1} \in W_{a_1}, w_{a_2} \in W_{a_2}$.

It was proved by Huang in [H8] that the fusing isomorphism (3.64) gives the following associativity:

$$\begin{aligned}
&E(\mathrm{Tr}_{W_{a_4}} \mathcal{Y}_{a_1 a_3;i}^{a_4;(1)}(\mathcal{U}(q_{z_1})w_{a_1}, q_{z_1}) \mathcal{Y}_{a_2 a_4;j}^{a_3;(2)}(\mathcal{U}(q_{z_2})w_{a_2}, q_{z_2}) q_\tau^{L(0)-\frac{c}{24}}) \\
&\quad = \sum_{a_5 \in \mathcal{I}} \sum_{k,l} F(\mathcal{Y}_{a_1 a_3;i}^{a_4;(1)} \otimes \mathcal{Y}_{a_2 a_4;j}^{a_3;(2)}, \mathcal{Y}_{a_5 a_4;k}^{a_4;(3)} \otimes \mathcal{Y}_{a_1 a_2;l}^{a_5;(4)}) \\
&\qquad E(\mathrm{Tr}_{W_{a_4}} \mathcal{Y}_{a_5 a_4;k}^{a_4;(3)}(\mathcal{U}(q_{z_2}) \mathcal{Y}_{a_1 a_2;l}^{a_5;(4)}(w_{a_1}, z_1 - z_2) w_{a_2}, q_{z_2}) q_\tau^{L(0)-\frac{c}{24}}),
\end{aligned} \tag{3.71}$$

where $F(\mathcal{Y}_{a_1 a_3;i}^{a_4;(1)} \otimes \mathcal{Y}_{a_2 a_4;j}^{a_3;(2)}, \mathcal{Y}_{a_5 a_4;k}^{a_4;(3)} \otimes \mathcal{Y}_{a_1 a_2;l}^{a_5;(4)})$ is the matrix representation of $\mathcal{F}$ in the basis $\{\mathcal{Y}_{a_1 a_3;i}^{a_4;(1)} \otimes \mathcal{Y}_{a_2 a_4;j}^{a_3;(2)}\}_{i,j}, \{\mathcal{Y}_{a_5 a_4;k}^{a_4;(3)} \otimes \mathcal{Y}_{a_1 a_2;l}^{a_5;(4)}\}_{k,l}$. Therefore, $\forall w_{a_k} \in W_{a_k}, k = 1, 2$, we obtain

$$\begin{aligned}
&\left(\Psi_2(\mathcal{Y}_{a_1 a;i}^{a_3;(1)} \otimes \mathcal{Y}_{a_2 a_3;j}^{a;(2)}) \left(\left(-\frac{1}{\tau}\right)^{L(0)} w_{a_1} \otimes \left(-\frac{1}{\tau}\right)^{L(0)} w_{a_2} \right) \right) \left(-\frac{1}{\tau} z_1, -\frac{1}{\tau} z_2; -\frac{1}{\tau} \right) \\
&\quad = \big(\Psi_2(S(\mathcal{Y}_{a_1 a;i}^{a_3;(1)} \otimes \mathcal{Y}_{a_2 a_3;j}^{a;(2)}))(w_{a_1} \otimes w_{a_2}) \big)(z_1, z_2, \tau).
\end{aligned} \tag{3.72}$$

Combining (3.49), (3.50), (3.51), (3.52) and (3.72) and Remark 3.8, we obtain a simpler version of Cardy condition.

Theorem 3.10. *The Cardy condition can be rewritten as follow:*

$$\begin{aligned}
&\big(\theta_{W_{r_{R(i)}}} \circ \sigma_{123}(\mathcal{Y}_{\iota'_{cl-op}}) \circ (\varphi_{op} \otimes \mathrm{id}_{W'_{r_{L(i)}}})\big) \otimes \big(\Omega_0(\sigma_{132}(\mathcal{Y}_{\iota_{cl-op}})) \circ (\varphi_{op} \otimes \mathrm{id}_{W_{r_{R(i)}}})\big) \\
&\quad = S^{-1}\left(Y_{op}^f \otimes \left(\Omega_{-1}(Y_{op}^f) \circ (\theta_{V_{op}} \otimes \mathrm{id}_{V_{op}})\right)\right).
\end{aligned} \tag{3.73}$$

4 Modular tensor categories

This section is independent of the rest of this work. The tensor product theory of modules over a vertex operator algebra is developed by Huang and Lepowsky [HL1]-[HL4][H3]. In particular, the notion of vertex tensor category is introduced in [HL1]. Huang later proved in [H12] that $\mathcal{C}_V$ is a modular tensor category for V satisfying conditions in Theorem 0.1. In Section 4.1, we review some basic ingredients of modular tensor category $\mathcal{C}_V$. In Section 4.2, we show how to find in $\mathcal{C}_V$ a graphical representation of the modular transformation $S : \tau \mapsto -\frac{1}{\tau}$ discussed in Section 3.2.

4.1 Preliminaries

We recall some ingredients of vertex tensor category $\mathcal{C}_V$ and those of modular tensor category structure on $\mathcal{C}_V$ constructed in [H12].

There is an associativity isomorphism $\mathcal{A}$

$$\mathcal{A} : W_1 \boxtimes (W_2 \boxtimes W_3) \to (W_1 \boxtimes W_2) \boxtimes W_3$$

for each triple of V-modules W_1, W_2, W_3. The relation between the fusing isomorphism $\mathcal{F}$ (recall (3.64)) and the associativity isomorphism $\mathcal{A}$ in $\mathcal{C}_V$ is described by the following commutative diagram:

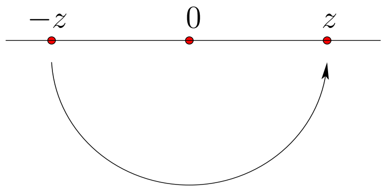
$$\begin{tikzcd}
\oplus_{a_5 \in \mathcal{I}} \mathcal{V}^{a_4}_{a_1 a_5} \otimes \mathcal{V}^{a_5}_{a_2 a_3} \arrow[r, "\cong"] \arrow[d, "\mathcal{F}"] & \mathrm{Hom}_V(W_{a_1} \boxtimes (W_{a_2} \boxtimes W_{a_3}), W_{a_4}) \arrow[d, "(\mathcal{A}^{-1})^*"] \\
\oplus_{a_6 \in \mathcal{I}} \mathcal{V}^{a_4}_{a_6 a_3} \otimes \mathcal{V}^{a_6}_{a_1 a_2} \arrow[r, "\cong"] & \mathrm{Hom}_V((W_{a_1} \boxtimes W_{a_2}) \boxtimes W_{a_3}, W_{a_4})
\end{tikzcd} \tag{4.1}$$

where the two horizontal maps are canonical isomorphisms induced from the universal property of $\boxtimes$.

We recall the braiding structure on $\mathcal{C}_V$. For each pair of V-modules W_1, W_2, there is also a natural isomorphism, for $z > 0$, $\mathcal{R}_+^{P(z)} : W_1 \boxtimes_{P(z)} W_2 \to W_2 \boxtimes_{P(z)} W_1$, defined by

$$\overline{\mathcal{R}_+^{P(z)}}(w_1 \boxtimes_{P(z)} w_2) = e^{L(-1)} \overline{\mathcal{T}}_{\gamma_+}(w_2 \boxtimes_{P(-z)} w_1), \tag{4.2}$$

where γ_+ is a path from $-z$ to z inside the lower half plane as shown in the following graph

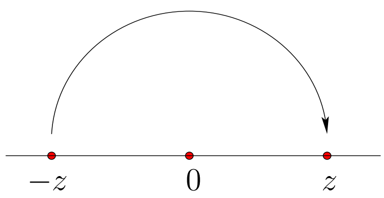


. (4.3)

The inverse of $\mathcal{R}_+^{P(z)}$ is denoted by $\mathcal{R}_-^{P(z)}$, which is characterized by

$$\overline{\mathcal{R}_-^{P(z)}}(w_2 \boxtimes_{P(z)} w_1) = e^{L(-1)} \overline{\mathcal{T}}_{\gamma_-}(w_1 \boxtimes_{P(-z)} w_2), \tag{4.4}$$

where γ_- is a path in the upper half plane as shown in the following graph



. (4.5)

We denote $\mathcal{R}_\pm^{P(1)}$ simply as $\mathcal{R}_\pm$. The natural isomorphisms $\mathcal{R}_\pm$ give $\mathcal{C}_V$ two different braiding structures. We choose $\mathcal{R}_+$ as the default braiding structure on $\mathcal{C}_V$. Sometimes, we will denote it by $(\mathcal{C}_V, \mathcal{R}_+)$ to emphasis our choice of braiding isomorphisms.

Notice that our choice of $\mathcal{R}_\pm$ follows that in [Ko2], which is different from that in [H9][H12][Ko1]. For each V-module W, (3.39) defines an automorphism $\theta_W : W \to W$ called a *twist*. A V-module W is said to have a trivial twist if $\theta_W = \mathrm{id}_W$. The twist θ and braiding $\mathcal{R}_+$ satisfy the following three balancing axioms

$$\begin{aligned} \theta_{W_1 \boxtimes W_2} &= \mathcal{R}_+ \circ \mathcal{R}_+ \circ (\theta_{W_1} \boxtimes \theta_{W_2}) && (4.6)\\ \theta_V &= \mathrm{id}_V && (4.7)\\ \theta_{W'} &= (\theta_W)^*. && (4.8) \end{aligned}$$

for any pair of V-modules W_1, W_2.

Let $\{\mathcal{Y}_{ea}^a\}$ be a basis of $\mathcal{V}_{ea}^a$ for all $a \in \mathcal{I}$ such that it coincides with the vertex operator Y_{W_a}, which defines the V-module structure on W_a, i.e. $\mathcal{Y}_{ea}^a = Y_{W_a}$. We choose a basis $\{\mathcal{Y}_{ae}^a\}$ of $\mathcal{V}_{ae}^a$ as follow:

$$\mathcal{Y}_{ae}^a = \Omega_{-1}(\mathcal{Y}_{ea}^a). \tag{4.9}$$

We also choose a basis $\{\mathcal{Y}_{aa'}^e\}$ of $\mathcal{V}_{aa'}^e$ as

$$\mathcal{Y}_{aa'}^e = \mathcal{Y}_{aa'}^{e'} = \hat{A}_0(\mathcal{Y}_{ae}^a) = \sigma_{132}(\mathcal{Y}_{ea}^a). \tag{4.10}$$

Notice that these choices are made for all $a \in \mathcal{I}$. In particular, we have

$$\mathcal{Y}_{a'e}^{a'} = \Omega_{-1}(\mathcal{Y}_{ea'}^{a'}), \qquad \mathcal{Y}_{a'a}^e = \mathcal{Y}_{a'a}^{e'} = \hat{A}_0(\mathcal{Y}_{a'e}^{a'}).$$

The following relation was proved in [Ko1].

$$\mathcal{Y}_{a'a}^e = e^{2\pi i h_a}\Omega_0(\mathcal{Y}_{aa'}^e) = e^{-2\pi i h_a}\Omega_{-1}(\mathcal{Y}_{aa'}^e). \tag{4.11}$$

For any V-modules W_1, W_2, W_3 and $\mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3}$, we denote by $m_{\mathcal{Y}}$ the morphism in $\mathrm{Hom}_V(W_1 \boxtimes W_2, W_3)$ associated to $\mathcal{Y}$ under the identification of two spaces induced by the universal property of $\boxtimes$.

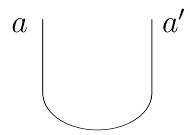
Now we recall the construction of duality maps [H12]. We will follow the convention in [Ko1]. Since $\mathcal{C}_V$ is semisimple, we only need to discuss irreducible modules. For $a \in \mathcal{I}$, the right duality maps $e_a : W_a' \boxtimes W_a \to V$ and $i_a : V \to W_a \boxtimes W_a'$ for $a \in \mathcal{I}$ are given by

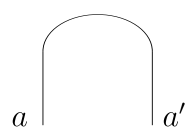
$$e_a = m_{\mathcal{Y}_{a'a}^e}, \qquad m_{\mathcal{Y}_{aa'}^e} \circ i_a = \dim a \ \mathrm{id}_V$$

where $\dim a \neq 0$ for $a \in \mathcal{I}$ (proved by Huang in [H11]). The left duality maps $e_a' : W^a \boxtimes W_a' \to V$ and $i_a' : V \to W_a' \boxtimes W^a$ are given by

$$e_a' = m_{\mathcal{Y}_{aa'}^e}, \qquad m_{\mathcal{Y}_{a'a}^e} \circ i_a' = \dim a \ \mathrm{id}_V.$$

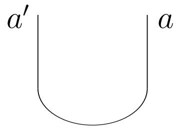
In a ribbon category, there is a powerful tool called graphic calculus. One can express various morphisms in terms of graphs. In particular, the right duality maps i_a and e_a are denoted by the following graphs:

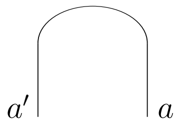
$i_a =$ 

, $\qquad e_a =$ 

,

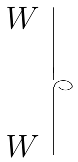
the left duality maps are denoted by

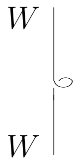
$i'_a =$ 

, $e'_a =$ 

,

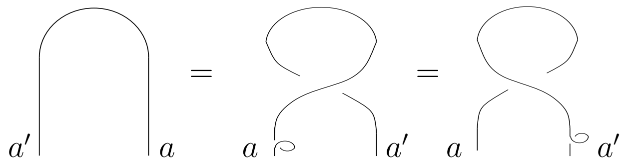
and the twist and its inverse, for any object W, are denoted by

$\theta_W =$ 

, $\theta_W^{-1} =$ 

.

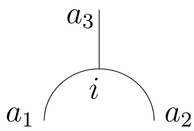
The identity (4.11) proved in [Ko2] is nothing but the following identity:



. (4.12)

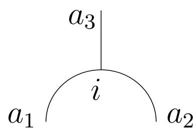
This formula (4.12) is implicitly used in many graphic calculations in this work.

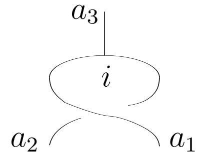
A basis $\{\mathcal{Y}_{a_1a_2;i}^{a_3;(1)}\}_{i=1}^{N_{a_1a_2}^{a_3}}$ of $\mathcal{V}_{a_1a_2}^{a_3}$ for $a_1, a_2, a_3 \in \mathcal{I}$ induces a basis $\{e_{a_1a_2;i}^{a_3}\}$ of $\mathrm{Hom}(W_{a_1} \boxtimes W_{a_2}, W_{a_3})$. One can also denote $e_{a_1a_2;i}^{a_3}$ as the following graph:



. (4.13)

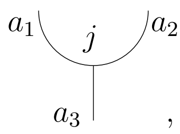
Note that we will always use a to represent W_a and a' to represent W'_a in graph for simplicity. By the universal property of $\boxtimes_{P(z)}$, the map $\Omega_0 : \mathcal{V}_{a_1a_2}^{a_3} \to \mathcal{V}_{a_2a_1}^{a_3}$ induces a linear map $\Omega_0 : \mathrm{Hom}_V(W_{a_1} \boxtimes W_{a_2}, W_{a_3}) \to \mathrm{Hom}_V(W_{a_2} \boxtimes W_{a_1}, W_{a_3})$ given as follow:

$\Omega_0 :$ 

$\mapsto$ 

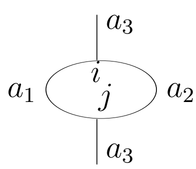
. (4.14)

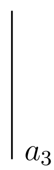
Let us choose a basis $\{f_{a_3;j}^{a_1a_2}\}_{j=1}^{N_{a_1a_2}^{a_3}}$ of $\mathrm{Hom}_V(W_{a_3}, W_{a_1} \boxtimes W_{a_2})$, denoted by

$f_{a_3;j}^{a_1a_2} =$ 

, (4.15)

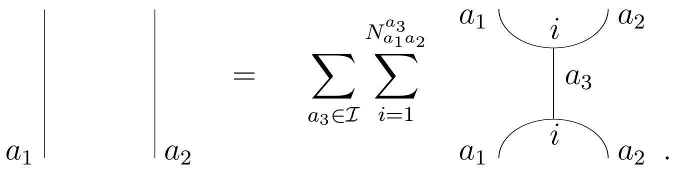
such that



$= \delta_{ij}$ 

. (4.16)

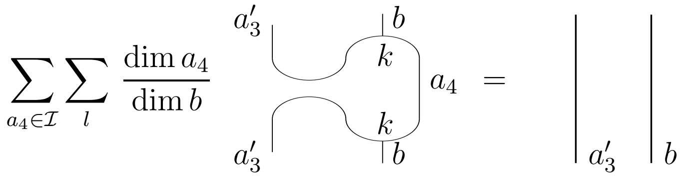
The following identity is proved in [Ko1]



(4.17)

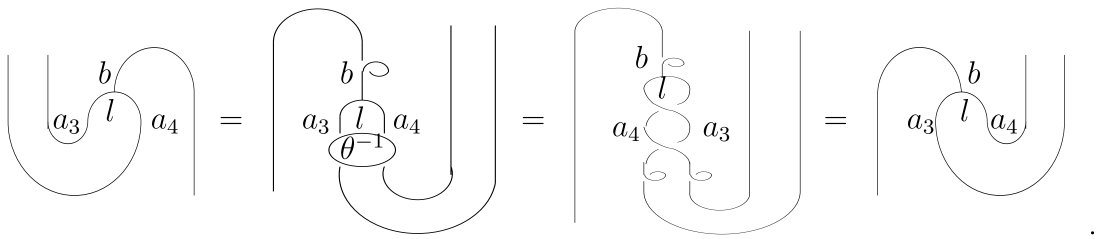
We prove a similar identity below.

Lemma 4.1.



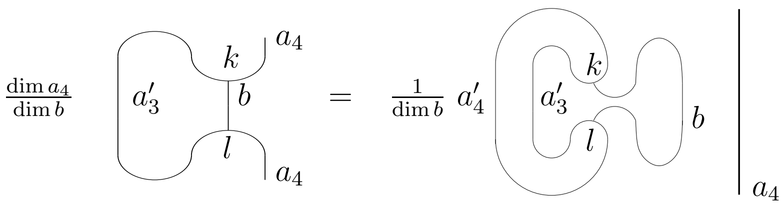
(4.18)

Proof. Using the first balancing axiom (4.6), we have

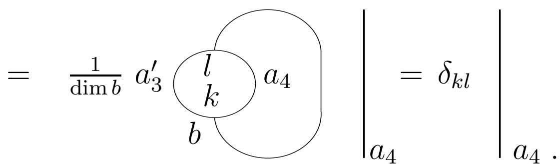


(4.19)

Then the Lemma follows from the following relations:



(4.20)

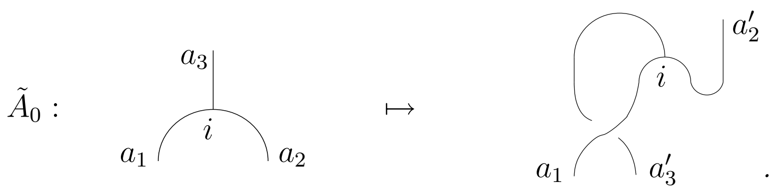


(4.21)

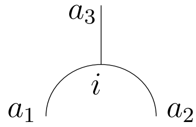
∎

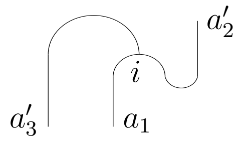
Similar to Ω_0, $\tilde{A}_0$, σ_{123} and σ_{132} can also be described graphically as proved in [Ko1]. We recall these results below.

Proposition 4.2.

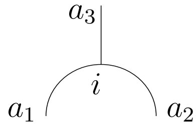


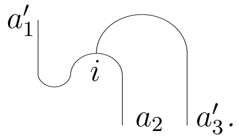
(4.22)

$\sigma_{123}:$ 

$\mapsto$ 

, (4.23)

$\sigma_{132}:$ 

$\mapsto$ 

. (4.24)

4.2 Graphical representation of $S : \tau \mapsto -\frac{1}{\tau}$

In [HKo3], we defined an action of α, β on $\Psi_2(\mathcal{Y}_{aa_1;p}^{a_1;(1)} \otimes \mathcal{Y}_{a_2a_3;q}^{a;(2)})$. More precisely,

$$\alpha(\Psi_2(\mathcal{Y}_{aa_1;p}^{a_1;(1)} \otimes \mathcal{Y}_{a_2a_3;q}^{a;(2)})) : \oplus_{a_2,a_3\in\mathcal{I}} W_{a_2} \otimes W_{a_3} \;\; \rightarrow \;\; \mathbb{G}_{1;2} \tag{4.25}$$

$$\beta(\Psi_2(\mathcal{Y}_{aa_1;p}^{a_1;(1)} \otimes \mathcal{Y}_{a_2a_3;q}^{a;(2)})) : \oplus_{a_2,a_3\in\mathcal{I}} W_{a_2} \otimes W_{a_3} \;\; \rightarrow \;\; \mathbb{G}_{1;2} \tag{4.26}$$

are defined by

$$\begin{aligned}
&(\alpha(\Psi_2(\mathcal{Y}_{aa_1;p}^{a_1;(1)} \otimes \mathcal{Y}_{a_2a_3;q}^{a;(2)})))(w_2 \otimes w_3) \\
&\qquad = (\Psi_2(\mathcal{Y}_{aa_1;p}^{a_1;(1)} \otimes \mathcal{Y}_{a_2a_2';q}^{a;(2)}))(w_2 \otimes w_3; z_1, z_2 - 1; \tau) \\
&(\beta(\Psi(\mathcal{Y}_{aa_1;p}^{a_1;(1)} \otimes \mathcal{Y}_{a_2a_3;q}^{a;(2)})))(w_2 \otimes w_3) \\
&\qquad = (\Psi_2(\mathcal{Y}_{aa_1;p}^{a_1;(1)} \otimes \mathcal{Y}_{a_2a_3;q}^{a;(2)}))(w_2 \otimes w_3; z_1, z_2 + \tau; \tau)
\end{aligned} \tag{4.27}$$

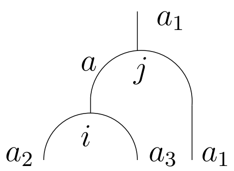
if $w_2 \otimes w_3 \in W_{a_2} \otimes W_{a_3}$, and by 0 if otherwise.

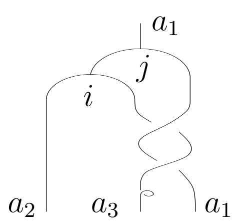
We also showed in [HKo3] that α induces an automorphism on $\oplus_{a\in\mathcal{I}} \mathcal{V}_{aa_1}^{a_1} \otimes \mathcal{V}_{a_2a_3}^{a}$ given as follow:

$$\begin{aligned}
\mathcal{Y}_{aa_1;i}^{a_1;(1)} \otimes \mathcal{Y}_{a_2a_3;j}^{a;(2)} \mapsto \sum_{b,c\in\mathcal{A}} \sum_{k,l,p,q} & e^{-2\pi i h_{a_3}} F^{-1}(\mathcal{Y}_{aa_1;i}^{a_1;(1)} \otimes \mathcal{Y}_{a_2a_3;j}^{a;(2)}; \mathcal{Y}_{a_2b;k}^{a_1;(3)} \otimes \mathcal{Y}_{a_3a_1;l}^{b;(4)}) \\
& F(\mathcal{Y}_{a_2b;k}^{a_1;(3)} \otimes \Omega_{-1}^2(\mathcal{Y}_{a_3a_1;l}^{b;(4)}); \mathcal{Y}_{ca_1;p}^{a_1;(5)} \otimes \mathcal{Y}_{a_2a_3;q}^{c;(6)}) \\
& \mathcal{Y}_{ca_1;p}^{a_1;(5)} \otimes \mathcal{Y}_{a_2a_3;q}^{c;(6)}.
\end{aligned} \tag{4.28}$$

We still denoted this automorphism and its natural extension on $\oplus_{a,a_1\in\mathcal{I}} \mathcal{V}_{aa_1}^{a_1} \otimes \mathcal{V}_{a_2a_3}^{a}$ by α. The following Lemma follows immediately from (4.28).

Lemma 4.3. *α can also be expressed graphically as follow:*

$\alpha:$ 

$\mapsto$ 

. (4.29)

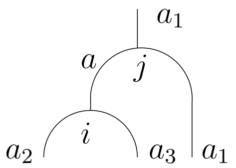
For β, we prefer to use maps $\tilde{A}_0$ and $\hat{A}_0$ defined in (3.35) instead of the map A_0 used in [HKo3]. We obtain the following Lemma, which is proved in appendix.

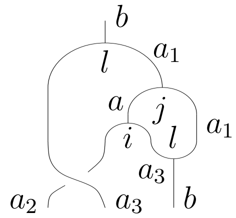
Lemma 4.4. *β also induces an automorphism on $\oplus_{a,a_1\in\mathcal{I}}\mathcal{V}_{aa_1}^{a_1}\otimes\mathcal{V}_{a_2a_3}^{a}$ given by*

$$\begin{aligned}\mathcal{Y}_{aa_1;i}^{a_1;(1)}\otimes\mathcal{Y}_{a_2a_3;j}^{a;(2)} \quad\mapsto\quad & \sum_{b\in\mathcal{I}}\sum_{k,l}\sum_{c\in\mathcal{I}}\sum_{p,q}F^{-1}(\mathcal{Y}_{aa_1;i}^{a_1;(1)}\otimes\mathcal{Y}_{a_2a_3;j}^{a;(2)};\mathcal{Y}_{a_2b;k}^{a_1;(3)}\otimes\mathcal{Y}_{a_3a_1;l}^{b;(4)})\\ & F(\tilde{A}_0(\mathcal{Y}_{a_2b;k}^{a_1;(3)})\otimes\tilde{A}_0(\mathcal{Y}_{a_3a_1;l}^{b;(4)}),\mathcal{Y}_{cb';p}^{b';(5)}\otimes\mathcal{Y}_{a_2a_3;q}^{c;(6)})\\ & \hat{A}_0(\mathcal{Y}_{cb';p}^{b';(5)})\otimes\Omega_0^2(\mathcal{Y}_{a_2a_3;q}^{c;(6)}).\end{aligned}\tag{4.30}$$

We still denote this automorphism by β.

Lemma 4.5. *β can be expressed graphically as follow:*

β : 

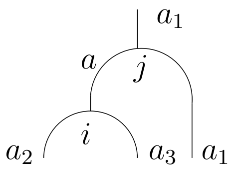
$\mapsto \quad \sum_{b\in\mathcal{I}}\sum_l$ 

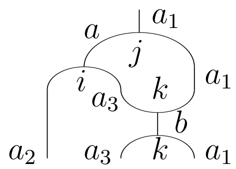
. (4.31)

Proof. Using (4.30), we can see that β is the composition of following maps

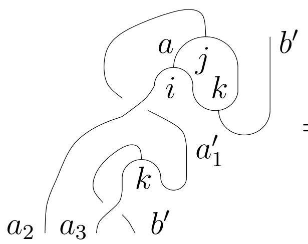
$$\begin{tikzcd}\oplus_{a,a_1\in\mathcal{I}}\mathcal{V}_{aa_1}^{a_1}\otimes\mathcal{V}_{a_2a_3}^{a} \arrow[r, "\mathcal{F}^{-1}"] & \oplus_{b,a_1\in\mathcal{I}}\mathcal{V}_{a_2b}^{a_1}\otimes\mathcal{V}_{a_3a_1}^{b} \arrow[r, "\tilde{A}_0\otimes\tilde{A}_0"] & \oplus_{b,a_1\in\mathcal{I}}\mathcal{V}_{a_2a_1'}^{b'}\otimes\mathcal{V}_{a_3b'}^{a_1'} \arrow[d, "\mathcal{F}"]\\ & \oplus_{b,c\in\mathcal{I}}\mathcal{V}_{cb}^{b}\otimes\mathcal{V}_{a_2a_3}^{c} & \oplus_{b,c\in\mathcal{I}}\mathcal{V}_{cb'}^{b'}\otimes\mathcal{V}_{a_2a_3}^{c}. \arrow[l, "\hat{A}_0\otimes\Omega_0^2"']\end{tikzcd}\tag{4.32}$$

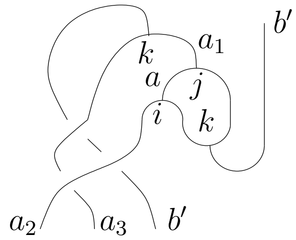
By the commutative diagram (4.1), (4.32) can be rewritten graphically as follow:

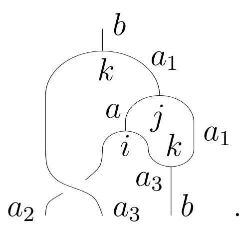
β : 

$= \quad \sum_{b\in\mathcal{I}}\sum_k$ 

(4.33)

$\mapsto \sum_{b\in\mathcal{I}}\sum_k$ 

$= \sum_{b\in\mathcal{I}}\sum_k$ 

$\mapsto \sum_{b\in\mathcal{I}}\sum_k$ 

.

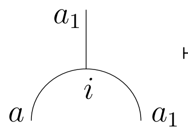
∎

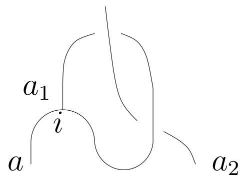
We have introduced S, α, β all as isomorphisms on $\oplus_{a,a_1\in\mathcal{I}}\mathcal{V}_{aa_1}^{a_1}\otimes\mathcal{V}_{a_2a_3}^{a}$. They satisfy the following well-known equation [MSei1][MSei2][H11][HKo3]:

$$S\alpha = \beta S. \tag{4.34}$$

We proved in [HKo3] that S is determined by the identity (4.34) up to a constant S_e^e. We will solve the equation (4.34) for S graphically below.

Proposition 4.6.

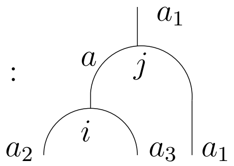
$S(a)$: 

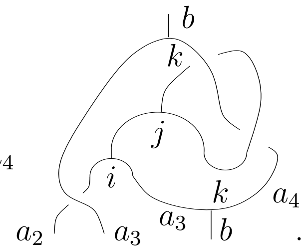
$\mapsto \sum_{a_2\in\mathcal{I}} S_e^e \dim a_2$ 

(4.35)

Proof. Since we know that the equation (4.34) determine S up to an overall constant S_e^e. Hence we only need to check that (4.35) gives a solution to (4.34).

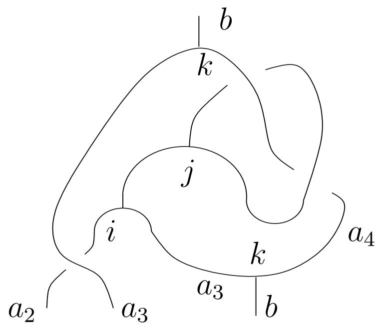
Combining (4.31) with (4.35), we obtain that

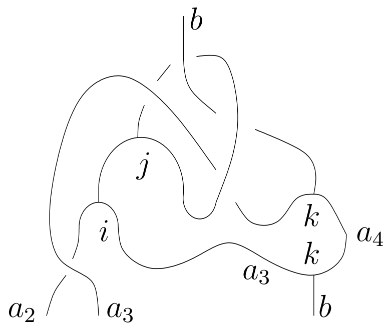
$\beta\circ S$: 

$\mapsto \sum_{b,a_4\in\mathcal{I}}\sum_k S_e^e \dim a_4$ 

. (4.36)

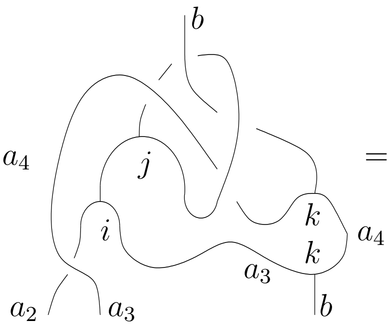
The diagram in the right hand side of (4.36) can be deformed as follow:

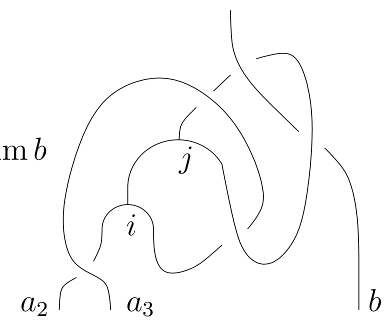


=


(4.37)

By (4.18), we have

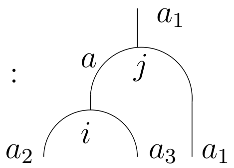
$\sum_{a_4\in\mathcal{I}}\sum_k S_e^e \dim a_4$ 

$= S_e^e \dim b$ 

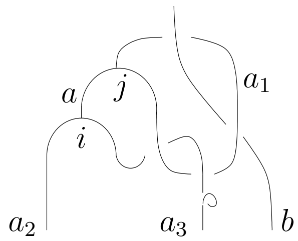
(4.38)

On the other hand, combining (4.29) with (4.35), we obtain

$S \circ \alpha :$



$\mapsto \sum_{b \in \mathcal{I}} S_e^e \dim b$



. (4.39)

Notice the diagram on the right hand side of (4.39) is exactly a deformation of the diagram on the right hand side of (4.38). Hence we obtain that the map defined as (4.35) give a solution to the equation $S\alpha = \beta S$. ∎

To determine S completely, we need determine S_e^e. This can be done by using other identities satisfied by S.

Proposition 4.7.

$$S^2(a)(\mathcal{Y}_{aa_1;i}^{a_1;(1)}) = \tilde{A}_0(\mathcal{Y}_{aa_1;i}^{a_1;(1)}) \tag{4.40}$$

Proof. By the definition of S-cation on $\mathcal{Y}_{aa_1;i}^{a_1;(1)} \in \mathcal{V}_{aa_1}^{a_1}$, we have

$$\Psi_1(S^2(a)(\mathcal{Y}_{aa_1;i}^{a_1;(1)}))(w_a; z, \tau) = \Psi_1(\mathcal{Y}_{aa_1;i}^{a_1;(1)}) \left(\tau^{L(0)} \left(\frac{-1}{\tau} \right)^{L(0)} w_a; -z, \tau \right) \tag{4.41}$$

for $w_a \in W_a$. Keep in mind of our convention on branch cut for logarithm. We have

$$\tau^{L(0)} \left(\frac{-1}{\tau} \right)^{L(0)} w_a = e^{\pi i L(0)} w_a.$$

Hence we obtain

$$\Psi_1(S^2(a)(\mathcal{Y}_{aa_1;i}^{a_1;(1)}))(w_a; z, \tau) = \Psi_1(\mathcal{Y}_{aa_1;i}^{a_1;(1)})(e^{\pi i L(0)} w_a; -z, \tau) \tag{4.42}$$

By (A.62), we also have

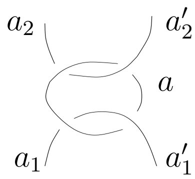
$$\begin{aligned}
&\Psi_1(\mathcal{Y}_{aa_1;i}^{a_1;(1)})(e^{\pi i L(0)} w_a; -z, \tau) \\
&\quad = E\big(\mathrm{Tr}_{W_{a_1}} \mathcal{Y}_{aa_1;i}^{a_1;(1)}(\mathcal{U}(e^{-2\pi i z}) e^{\pi i L(0)} w_a, e^{-2\pi i z}) q_\tau^{L(0)-\frac{c}{24}}\big) \\
&\quad = E\big(\mathrm{Tr}_{(W_{a_1})'} \tilde{A}_0(\mathcal{Y}_{aa_1;i}^{a_1;(1)})(e^{\pi i L(0)} \mathcal{U}(e^{2\pi i z}) w_a, e^{\pi i} e^{2\pi i z}) q_\tau^{L(0)-\frac{c}{24}}\big) \\
&\quad = E\big(\mathrm{Tr}_{(W_{a_1})'} \tilde{A}_0(\mathcal{Y}_{aa_1;i}^{a_1;(1)})(\mathcal{U}(e^{2\pi i z}) w_a, e^{2\pi i z}) q_\tau^{L(0)-\frac{c}{24}}\big) \\
&\quad = \Psi_1(\tilde{A}_0(\mathcal{Y}_{aa_1;i}^{a_1;(1)}))(w_a; z, \tau)
\end{aligned} \tag{4.43}$$

Therefore, combining (4.42) and (4.43), we obtain (4.40). ∎

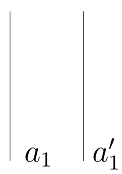
The following lemma is proved in [BK2].

Lemma 4.8. Let $D^2 = \sum_{a \in \mathcal{I}} \dim^2 a$. Then $D \neq 0$ and we have

$\sum_{a \in \mathcal{I}} \frac{\dim a_2 \dim a}{D^2}$



$= \delta_{a_1 a_2}$



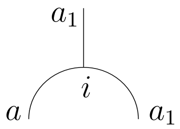
. (4.44)

Proposition 4.9.

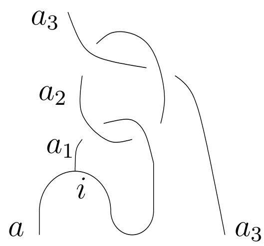
$$(S_e^e)^2 = \frac{1}{D^2}. \tag{4.45}$$

Proof. By (4.35), we have

$S^2(a):$



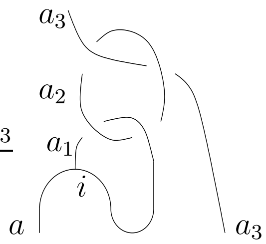
$$\mapsto \sum_{a_2,a_3\in\mathcal{I}} (S_e^e)^2 \dim a_2 \dim a_3$$



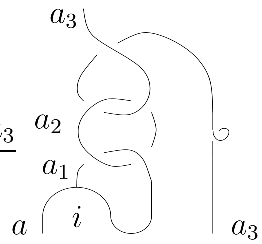
$$. \tag{4.46}$$

Apply (4.12) to the graph in the right hand side of (4.46), we obtain

$$\sum_{a_2\in\mathcal{I}} \frac{\dim a_2 \dim a_3}{D^2}$$



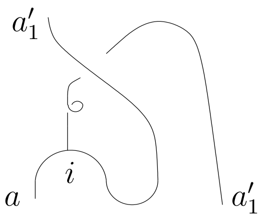
$$= \sum_{a_2\in\mathcal{I}} \frac{\dim a_2 \dim a_3}{D^2}$$



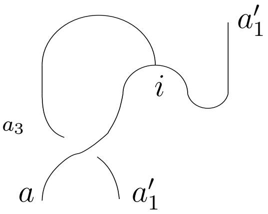
$$. \tag{4.47}$$

By (4.44), the right hands side of (4.47) equals to

$\delta_{a_1a_3}$



$= \delta_{a_1a_3}$



$$. \tag{4.48}$$

By (4.40) and (4.22), we obtain (4.45). ∎

So far, we have determined S_e^e up to a sign. Now we consider the relation between S and another generator of modular group $T : \tau \mapsto \tau + 1$. We define a T-action on $\Psi_1(\mathcal{Y}_{aa_1;i}^{a_1})$ as follow:

$$T(\Psi_1(\mathcal{Y}_{aa_1;i}^{a_1}))(w_a; z, \tau) := \Psi_1(\mathcal{Y}_{aa_1;i}^{a_1})(w_a; z, \tau + 1). \tag{4.49}$$

It is clear that this action induces an action of T on $\mathcal{V}_{aa_1}^{a_1}$ for all $a, a_1 \in \mathcal{I}$, given by

$$T|_{\mathcal{V}_{aa_1}^{a_1}} = e^{2\pi i(h_a - \frac{c}{24})} \tag{4.50}$$

where h_a is the lowest conformal weight of W_a.

Lemma 4.10. *S and T satisfy the following relation:*

$$(T^{-1}S)^3 = S^2 = T^{-1}S^2T. \tag{4.51}$$

Proof. Let $w_a \in W_a$. We have

$$\begin{aligned}
&(T^{-1}S)^3((\Psi_1(\mathcal{Y}^{a_1;(1)}_{aa_1;i})))(w_a; z, \tau)\\
= \quad & \Psi_1(\mathcal{Y}^{a_1;(1)}_{aa_1;i})\left(\tau^{L(0)}\left(\frac{-1}{\frac{-1}{\tau-1}-1}\right)^{L(0)}\left(\frac{-1}{\tau-1}\right)^{L(0)} w_a, -z, \tau\right) \qquad (4.52)
\end{aligned}$$

Keeping in mind our choice of branch cut. Then it is easy to show that

$$\tau^{L(0)}\left(\frac{-1}{\frac{-1}{\tau-1}-1}\right)^{L(0)}\left(\frac{-1}{\tau-1}\right)^{L(0)} w_a = e^{\pi i L(0)} w_a.$$

By (4.42), we obtain the first equality of (4.51). The proof of the second equality (4.51) is similar. ∎

Proposition 4.11. Let $p_\pm = \sum_{a\in\mathcal{I}} e^{\pm 2\pi i h_a} \dim^2 a$. Then we have

$$S^e_e = \frac{1}{p_-} e^{2\pi i c/8} = \frac{1}{p_+} e^{-2\pi i c/8}. \qquad (4.53)$$

Proof. In the proof of the Theorem 3.1.16 in [BK2], Bakalov and Kirillov proved an identity, which, in our own notation, can be written as follow:

$$\frac{1}{(S^e_e)^2 D^2} e^{-2\pi i \frac{c}{24}} ST^{-1}S = \frac{1}{S^e_e}\frac{p_+}{D^2} e^{2\pi i \frac{2c}{24}} TST.$$

By (4.51) and the fact $p_-p_+ = D^2$ which is proved in [BK2], we simply obtain that

$$S^e_e = \frac{1}{p_-} e^{2\pi i c/8}.$$

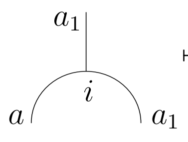
Using (4.45) and $p_+p_- = D^2$, we also obtain the second equality. ∎

We thus define

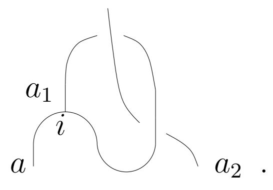
$$D := p_- e^{-2\pi i c/8} = p_+ e^{2\pi i c/8}. \qquad (4.54)$$

Notice that this notation is compatible with the definition of D^2. Then the action of modular transformation $S(a)$ on $\oplus_{a_1\in\mathcal{I}} \mathrm{Hom}_V(W_a \boxtimes W_{a_1}, W_{a_1})$ can be expressed graphically as follow:

$S(a)$:



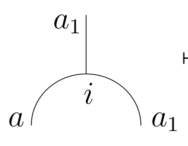
$\mapsto \quad \sum_{a_2\in\mathcal{I}} \frac{\dim a_2}{D}$



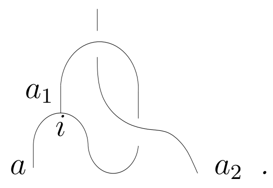
. (4.55)

Proposition 4.12.

$S^{-1}(a)$:

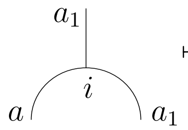


$\mapsto \quad \sum_{a_2\in\mathcal{I}} \frac{\dim a_2}{D}$

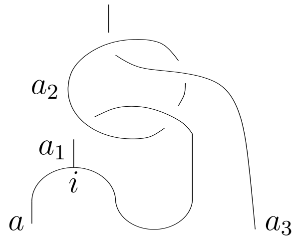


. (4.56)

Proof. Composing the map (4.55) with (4.56), we obtain a map given as follow:



$$\mapsto \quad \sum_{a_2,a_3\in\mathcal{I}} \frac{\dim a_2 \dim a_3}{D^2}$$



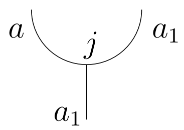
$$. \qquad (4.57)$$

Apply (4.44) to the graph in (4.57), it is easy to see that above map is the identity map. ∎

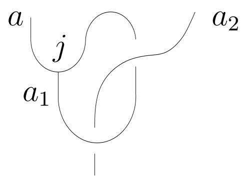
Remark 4.13. Bakalov and Kirillov obtained the same formula (4.55) in [BK2] by directly working with modular tensor category and solving equations obtained in the so-called Lego-Teichmüller game [BK1]. In our approach, we see the direct link between the modular transformations of q-traces of the product (or iterate) of intertwining operators and their graphic representations in a modular tensor category.

Proposition 4.14.

$$(S(a))^* :$$

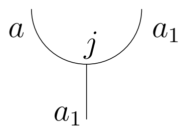


$$\mapsto \quad \sum_{a_2\in\mathcal{I}} \frac{\dim a_2}{D}$$

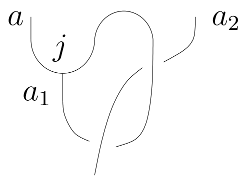


$$, \qquad (4.58)$$

$$(S^{-1}(a))^* :$$



$$\mapsto \quad \sum_{a_2\in\mathcal{I}} \frac{\dim a_2}{D}$$

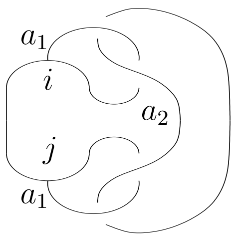


$$. \qquad (4.59)$$

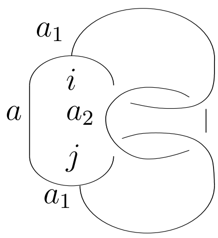
Proof. We only prove (4.58). The proof of (4.59) is analogous to that of (4.58).

It is enough to show that the pairing between the image of (4.35) and that of (4.58) still gives δ_{ij}. This can be proved as follow:

$$\sum_{a_2} \frac{\dim a_2}{D^2}$$



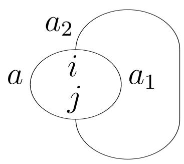
$$= \sum_{a_2} \frac{\dim a_1 \dim a_2}{\dim a_1 D^2}$$



$$(4.60)$$

By (4.44), the right hand side of (4.60) equals to

$$\frac{1}{\dim a_1}$$



$$= \quad \delta_{ij}.$$

∎

5 Categorical formulations and constructions

In this section, we give a categorical formulation of modular invariant conformal full field algebra over $V^L \otimes V^L$, open-string vertex operator algebra over V equipped with invariant bilinear form and Cardy condition. Then we introduce a notion called Cardy $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra. In the end, we give a categorical construction of such algebra in the Cardy case [FFFS2].

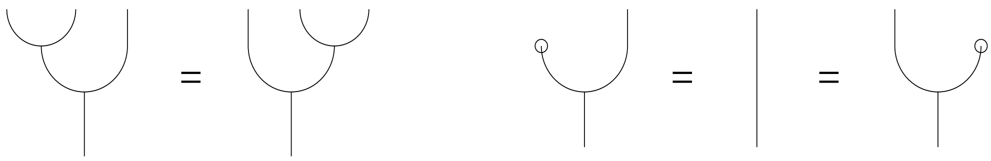
5.1 Modular invariant $\mathcal{C}_{V^L\otimes V^R}$-algebras

We first recall the notion of coalgebra and Frobenius algebra ([FS]) in a tensor category.

Definition 5.1. A coalgebra A in a tensor category $\mathcal{C}$ is an object with a coproduct $\Delta \in \mathrm{Mor}(A, A \otimes A)$ and a counit $\epsilon \in \mathrm{Mor}(A, \mathbf{1}_{\mathcal{C}})$ such that

$$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta, \qquad (\epsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A = (\mathrm{id}_A \otimes \epsilon) \circ \Delta, \tag{5.1}$$

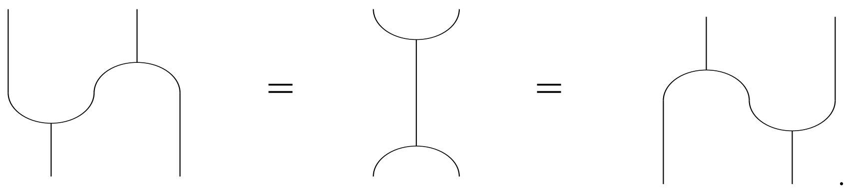
which can also be expressed in term of the following graphic equations:



Definition 5.2. Frobenius algebra in $\mathcal{C}$ is an object that is both an algebra and a coalgebra and for which the product and coproduct are related by

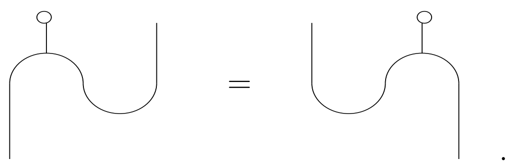
$$(\mathrm{id}_A \otimes m) \circ (\Delta \otimes \mathrm{id}_A) = \Delta \circ m = (m \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \Delta), \tag{5.2}$$

or as the following graphic equations,



(5.3)

A Frobenius algebra is called *symmetric* if the following condition is satisfied.



(5.4)

Let V^L and V^R be vertex operator algebras satisfying the conditions in Theorem 0.1. Then the vertex operator algebra $V^L \otimes V^R$ also satisfies the conditions in Theorem 0.1 [HKo1]. Thus $\mathcal{C}_{V^L\otimes V^R}$ also has a structure of modular tensor category. In particular, we choose the braiding structure on $\mathcal{C}_{V\otimes V}$ to be $\mathcal{R}_{+-}$ which is defined in [Ko2]. The twist θ_{+-} for each $V^L\otimes V^R$-module is defined by

$$\theta_{+-} = e^{-2\pi i L^L(0)} \otimes e^{2\pi i L^R(0)}. \tag{5.5}$$

Duality maps are naturally induced from those of $\mathcal{C}_{V^L}$ and $\mathcal{C}_{V^R}$.

The following theorem is proved in [Ko1].

Theorem 5.3. *The category of conformal full field algebras over $V^L\otimes V^R$ equipped with nondegenerate invariant bilinear form is isomorphic to the category of commutative Frobenius algebra in $\mathcal{C}_{V^L\otimes V^R}$ with a trivial twist.*

Remark 5.4. In a ribbon category, it was proved in [FFRS] that a commutative Frobenius algebra with a trivial twist is equivalent to a commutative symmetric Frobenius algebra.

Let $\mathcal{I}^L$ and $\mathcal{I}^R$ denote the set of equivalent class of irreducible V^L-modules and V^R-modules respectively. We use a and a_i for $i\in\mathbb{N}$ to denote elements in $\mathcal{I}^L$ and we use e to denote the equivalent class of V^L. We use $\bar{a}$ and $\bar{a}_i$ for $i\in\mathbb{N}$ to denote elements in $\mathcal{I}^R$, and $\bar{e}$ to denote the equivalent class of V^R. For each $a\in\mathcal{I}^L$ ($\bar{a}\in\mathcal{I}^R$), we choose a representative W_a ($W_{\bar{a}}$). We denote the vector space of intertwining operators of type $\binom{W_{a_3}}{W_{a_1}W_{a_2}}$ and $\binom{W_{\bar{a}_3}}{W_{\bar{a}_1}W_{\bar{a}_2}}$ as $\mathcal{V}_{a_1a_2}^{a_3}$ and $\bar{\mathcal{V}}_{\bar{a}_1\bar{a}_2}^{\bar{a}_3}$ respectively, the fusion rule as $N_{a_1a_2}^{a_3}$ and $N_{\bar{a}_1\bar{a}_2}^{\bar{a}_3}$ respectively.

A conformal full field algebra over $V^L\otimes V^R$, denoted as F, is a direct sum of irreducible modules of $V^L\otimes V^R$, i.e.

$$F = \oplus_{\alpha=1}^N W_{r(\alpha)} \otimes W_{\bar{r}(\alpha)}, \tag{5.6}$$

where $r:\{1,\dots,N\}\to\mathcal{I}^L$ and $\bar{r}:\{1,\dots,N\}\to\mathcal{I}^R$, for some $N\in\mathbb{Z}_+$. Let $\{e_{ab;i}^c\}$ and $\{\bar{e}_{\bar{a}\bar{b};j}^{\bar{c}}\}$ be basis for $\mathcal{V}_{ab}^c$ and $\bar{\mathcal{V}}_{\bar{a}\bar{b}}^{\bar{c}}$, and $\{f_{c;i}^{ab}\}$ and $\{\bar{f}_{\bar{c};j}^{\bar{a}\bar{b}}\}$ be the dual basis respectively. Then the vertex operator $\mathbb{Y}$ can also be expanded as follow:

$$\mathbb{Y} = \sum_{\alpha,\beta,\gamma}\sum_{i,j} d_{\alpha\beta}^{\gamma}(f_{r(\gamma);i}^{r(\alpha)r(\beta)}, \bar{f}_{\bar{r}(\gamma);j}^{\bar{r}(\alpha)\bar{r}(\beta)})\, e_{r(\alpha)r(\beta)}^{r(\gamma);i} \otimes \bar{e}_{\bar{r}(\alpha)\bar{r}(\beta)}^{\bar{r}(\gamma);j}, \tag{5.7}$$

where $d_{\alpha\beta}^{\gamma}$ defines a bilinear map

$$(\mathcal{V}_{r(\alpha)r(\beta)}^{r(\gamma)})^* \otimes (\bar{\mathcal{V}}_{\bar{r}(\alpha)\bar{r}(\beta)}^{\bar{r}(\gamma)})^* \to \mathbb{C}$$

for all $\alpha,\beta,\gamma = 1,\dots,N$ for some $N\in\mathbb{N}$.

Since the trace function pick out $\gamma=\beta$ terms, we define $\mathbb{Y}_{\alpha}^{diag}$ by

$$\mathbb{Y}_{\alpha}^{diag} := \sum_{\beta}\sum_{i,j} d_{\alpha\beta}^{\beta}(f_{r(\beta);i}^{r(\alpha)r(\beta)}, \bar{f}_{\bar{r}(\beta);j}^{\bar{r}(\alpha)\bar{r}(\beta)})\, e_{r(\alpha)r(\beta)}^{r(\beta);i} \otimes \bar{e}_{\bar{r}(\alpha)\bar{r}(\beta)}^{\bar{r}(\beta);j}. \tag{5.8}$$

Let $\mathbb{Y}^{diag} := \sum_{\alpha}\mathbb{Y}_{\alpha}^{diag}$. Of course, it is obvious to see that such defined $\mathbb{Y}^{diag}$ is independent of the choice of basis. We denote the representation of the modular transformation $S:\tau\mapsto\frac{-1}{\tau}$ on $\oplus_{b\in\mathcal{I}^L}\mathcal{V}_{ab}^b$ and $\oplus_{\bar{b}\in\mathcal{I}^R}\mathcal{V}_{\bar{a}\bar{b}}^{\bar{b}}$, by $S^L(a)$ and $S^R(\bar{a})$ respectively.

In [HKo3], we defined the notion of modular invariant conformal full field algebra over $V^L \otimes V^R$ (see [HKo3] for the precise definition). It basically means that n-point genus-one correlation functions built out of q-$\bar{q}$-traces are invariant under the action of modular group $SL(2, \mathbb{Z})$ for all $n \in \mathbb{N}$. Moreover, we proved the following results in [HKo3].

Proposition 5.5. *F, a conformal full field algebra over $V^L \otimes V^R$, is modular invariant if it satisfies $c^L - c^R = 0 \mod 24$ and*

$$S^L(r(\alpha)) \otimes (S^R(\bar{r}(\alpha)))^{-1} \mathbb{Y}_\alpha^{diag} = \mathbb{Y}_\alpha^{diag} \tag{5.9}$$

for all $\alpha = 1, \dots, N$.

We first introduce an object H in $\mathcal{C}_{V^L \otimes V^R}$ as follow:

$$H = \oplus_{a \in \mathcal{I}^L, \bar{a} \in \mathcal{I}^R} (W_a \otimes W_{\bar{a}}) \boxtimes (W_a' \otimes W_{\bar{a}}'). \tag{5.10}$$

For any object A in $\mathcal{C}_{V^L \otimes V^R}$, we choose an arbitrary decomposition of A. It is equivalent to specify the following maps:

$$\pi_{a \otimes \bar{a};p} : A \to W_a \otimes W_{\bar{a}}, \quad \iota_{a \otimes \bar{a};q} : W_a \otimes W_{\bar{a}} \to A, \quad p, q = 1, \dots, m_{a \otimes \bar{a}},$$

where $m_{a \otimes \bar{a}}$ denotes the multiplicity of $W_a \otimes W_{\bar{a}}$ in A, such that

$$\pi_{a \otimes \bar{a};p} \circ \iota_{b \otimes \bar{b};q} = \delta_{ab} \delta_{\bar{a}\bar{b}} \delta_{pq} I_{W_a \otimes W_{\bar{a}}}; \qquad \sum_{a \in \mathcal{I}^L, \bar{a} \in \mathcal{I}^R; p} \iota_{a \otimes \bar{a};p} \circ \pi_{a \otimes \bar{a};p} = I_A. \tag{5.11}$$

It is clear that $\pi_{a \otimes \bar{a},p}^*$ and $\iota_{a \otimes \bar{a};p}^*$ also gives a decomposition of A'. We define a morphism $\phi_A : A \otimes A' \to H$ as follow:

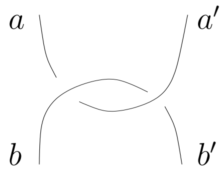
$$\phi_A := \sum_{a \in \mathcal{I}^L, \bar{a} \in \mathcal{I}^R} \sum_{p=1}^{m_{a \otimes \bar{a}}} \pi_{a \otimes \bar{a};p} \otimes \iota_{a \otimes \bar{a};p}^*. \tag{5.12}$$

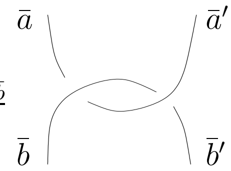
Such defined ϕ_A is independent of the decomposition of A. Indeed, let $f_1 \in W_a \otimes W_{\bar{a}}$, $f^1 \in W_a' \otimes W_{\bar{a}}'$ be so that $\langle f^1, f_1 \rangle = 1$. Let $\{e_i\}_{i \in \mathbb{N}}$ be a basis of A and $\{e^i\}_{i \in \mathbb{N}}$ be the dual basis of A'. Then we have the pairing

$$\sum_{i \in \mathbb{N}} \langle f^1, \pi_{a \otimes \bar{a};p} e_i \rangle \langle \iota_{a \otimes \bar{a};q}^* e^i, f_1 \rangle = \delta_{pq} \tag{5.13}$$

by (5.11). Namely, $\{\pi_{a \otimes \bar{a};p}\}_{p=1}^{m_{a \otimes \bar{a}}}$ and $\{\iota_{a \otimes \bar{a};p}^*\}_{p=1}^{m_{a \otimes \bar{a}}}$, as basis in $\mathrm{Hom}(A, a \otimes \bar{a})$ and $\mathrm{Hom}(A', a' \otimes \bar{a}')$ respectively, are mutually dual to each other with respect to the pairing (5.13). Hence ϕ_A is independent of the decomposition of A.

Let D^L in $\mathcal{C}_{V^L}$ and D^R in $\mathcal{C}_{V^R}$ be defined same as D in $\mathcal{C}_V$ (recall (4.54)). There are two morphisms $\check{S}_{ab}^L : W_a \otimes W_a' \to W_b \otimes W_b'$ in $(\mathcal{C}_{V^L}, \mathcal{R}_+)$ and $\check{S}_{ab}^R : W_{\bar{a}} \otimes W_{\bar{a}}' \to W_{\bar{b}} \otimes W_{\bar{b}}'$ in $(\mathcal{C}_{V^R}, \mathcal{R}_-)$, defined as follow

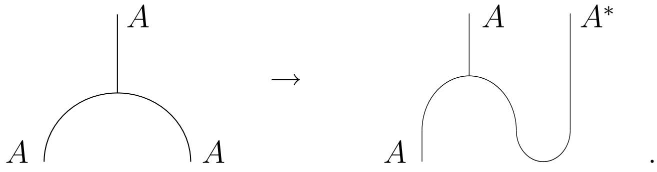
$\check{S}_{ab}^L := \frac{\dim b}{D^L}$ 

$\check{S}_{\bar{a}\bar{b}}^R := \frac{\dim \bar{b}}{D^R}$ 

. (5.14)

Let $\breve{S}_{\mathcal{C}_{V^L}} = \oplus_{a,b\in\mathcal{I}^L}\breve{S}^R_{ab}$ and $\breve{S}_{\mathcal{C}_{V^R}} = \oplus_{\bar{a},\bar{b}\in\mathcal{I}^R}\breve{S}^R_{\bar{a}\bar{b}}$. By (4.55) and (4.56), it is also clear that $\breve{S}^R$ in $(\mathcal{C}_{V^R}, \mathcal{R}_-)$ is exactly the map $(\breve{S}^R)^{-1}$ in $(\mathcal{C}_{V^R}, \mathcal{R}_+)$. Since $(\mathcal{C}_{V^L\otimes V^R}, \mathcal{R}_{+-})$ is also a modular tensor category, the automorphism $\breve{S}_{\mathcal{C}_{V^L\otimes V^R}}$, which is canonically defined on H as (5.14), is nothing but $\oplus_{a,b\in\mathcal{I}^L;\bar{a},\bar{b}\in\mathcal{I}^R}\breve{S}^L_{ab}\otimes\breve{S}^R_{\bar{a}\bar{b}}$.

If (A, ν_A, ι_A) is an associative algebra in $\mathcal{C}_{V\otimes V}$, then using duality, the map $\mu_A : A\otimes A \to A$ induces a map $A \to A\otimes A^*$ as follow:



. (5.15)

We denote the induced map as μ_A^*.

Definition 5.6. Let V^L and V^R be so that $c^L - c^R = 0 \mod 24$. A modular invariant $\mathcal{C}_{V^L\otimes V^R}$-algebra is an associative algebra (A, μ_A, ι_A) satisfying

$$\breve{S}_{\mathcal{C}_{V^L\otimes V^R}} \circ \phi_A \circ \mu_A^* = \phi_A \circ \mu_A^*. \tag{5.16}$$

Theorem 5.7. *Let V^L and V^R be so that $c^L - c^R = 0 \mod 24$. The following two notions are equivalent:*

1. *Modular invariant conformal full field algebra over $V^L\otimes V^R$ with a nondegenerate invariant bilinear form.*

2. *Modular invariant commutative Frobenius algebra with a trivial twist.*

Proof. The Theorem follows from Theorem 5.3 and the equivalence between (5.9) and (5.16) immediately. ∎

5.2 Cardy $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebras

For an open-string vertex operator algebra V_{op} over V equipped with a nondegenerate invariant bilinear form $(\cdot,\cdot)_{op}$, there is an isomorphism $\varphi_{op} : V_{op} \to V_{op}$ induce from $(\cdot,\cdot)_{op}$ (recall (3.40)).

In this case, V_{op} is a V-module and Y^f_{op} is an intertwining operator. By comparing (1.26) with (3.37), and (1.27) with (3.38), we see that the conditions (1.26) and (1.27) can be rewritten as

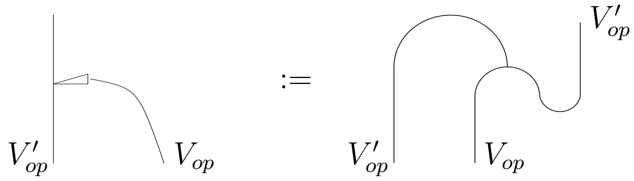
$$\begin{aligned} Y^f_{op} &= \varphi_{op}^{-1}\circ\sigma_{123}(Y^f_{op})\circ(\varphi_{op}\otimes \mathrm{id}_{V_{op}}) && (5.17)\\ &= \varphi_{op}^{-1}\circ\sigma_{132}(Y^f_{op})\circ(\mathrm{id}_{op}\otimes\varphi_{op}). && (5.18)\end{aligned}$$

Remark 5.8. The representation theory of open-string vertex operator algebra was briefly developed in the chapter 4 of [Ko1]. In that context, $\sigma_{123}(Y^f_{op})$ gives V'_{op} a right V_{op}-module structure and the equation (5.17) is equivalent to the statement that φ_{op} is an isomorphism between two right V_{op}-modules. Similarly, $\sigma_{132}(Y^f_{op})$ gives V'_{op} a left V_{op}-module structure and the equation (5.18) is equivalent to the statement that φ_{op} is an isomorphism between two left V_{op}-modules.

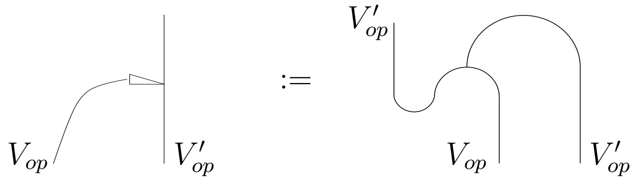
Theorem 5.9. *The category of open-string vertex operator algebras over V equipped with a nondegenerate invariant bilinear form is isomorphic to the category of symmetric Frobenius algebras in $\mathcal{C}_V$.*

Proof. We have already shown in [HKo1] that an open-string vertex operator algebra over V is equivalent to an associative algebra in $\mathcal{C}_V$.

Let V_{op} be an open-string vertex operator algebra over V. Giving a nondegenerate invariant bilinear form (recall (1.26) and (1.27)) on V_{op} is equivalent to give an isomorphism $\varphi_{op} : V_{op} \to V'_{op}$ satisfying the conditions (5.17) and (5.18). If we define

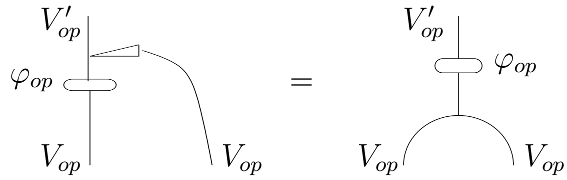


, (5.19)

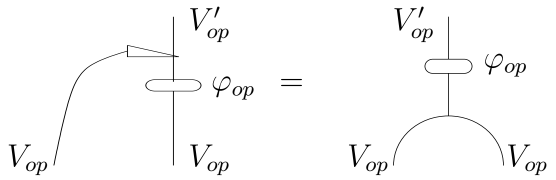


, (5.20)

then (5.17) and (5.18) can be rewritten as

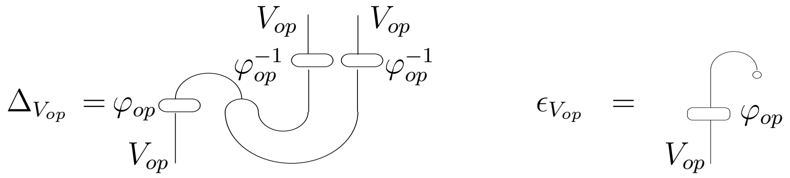


. (5.21)



, (5.22)

Using the map φ_{op} and its inverse, we can obtain a natural coalgebra structure on F defined as follow:



. (5.23)

We showed in [Ko1] that (5.21) implies that such defined $\Delta_{V_{op}}$ and $\epsilon_{V_{op}}$ give V_{op} a Frobenius algebra structure. Moreover, we also showed in [Ko1] that (5.21) implies the equality between φ_{op} and the left hand side of (5.4). Similarly, using (5.22), we can show that the right hand side of (5.4) also equals to φ_{op}. Thus V_{op} has a structure of symmetric Frobenius algebra.

We thus obtain a functor from the first category to the second category.

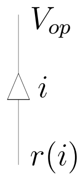
Conversely, given a symmetric Frobenius algebra in $\mathcal{C}_V$. In [HKo1], we showed that it gives an open-string vertex operator algebra over V. It is shown in [FRS] that either side of (5.4) is an isomorphism. Take φ_{op} to be either side of (5.4). Then (5.21) and (5.22) follow automatically from the definition (5.19) and (5.20). They are nothing but the invariance properties (recall (5.17)(5.18)) of the bilinear form associated with φ_{op}. Thus we obtain a functor from the second category to the first category.

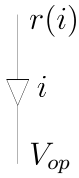
It is routine to check that these two functors are inverse to each other. ∎

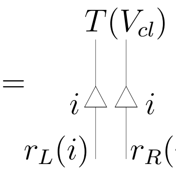
Now we consider an open-closed field algebra over V given in (2.12) equipped with nondegenerate invariant bilinear forms $(\cdot,\cdot)_{op}$ and $(\cdot,\cdot)_{cl}$. We assume that V_{cl} and V_{op} have the following decompositions:

$$V_{cl} = \oplus_{i=1}^{N_{cl}} W_{r_L(i)} \otimes W_{r_R(i)}, \qquad V_{op} = \oplus_{i=1}^{N_{op}} W_{r(i)}$$

where $r_L, r_R : \{1, \dots, N_{cl}\} \to \mathcal{I}$ and $r : \{1, \dots, N_{op}\} \to \mathcal{I}$. We denote the embedding $b^{V_{op}}_{(i)} : W_{r(i)} \hookrightarrow V_{op}$, the projection $b^{(i)}_{V_{op}} : V_{op} \to W_{r(i)}$ and $b^{V_{cl}}_{(i)} : W_{r_L(i)} \boxtimes W_{r_R(i)} \hookrightarrow T(V_{cl})$ by the following graphs:

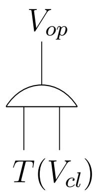
$b^{V_{op}}_{(i)} =$ 

, $b^{(i)}_{V_{op}} =$ 

, $b^{V_{cl}}_{(i)} =$ 

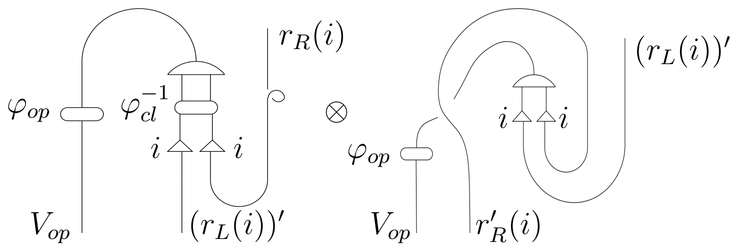
. (5.24)

We denote the map $\iota_{cl-op} : T(V_{cl}) \to V_{op}$ [Ko2] by the following graph

$\iota_{cl-op} :=$ 

. (5.25)

Now we can express the Cardy condition (3.73) in graphs. The left hand side of (3.73) can be expressed by:

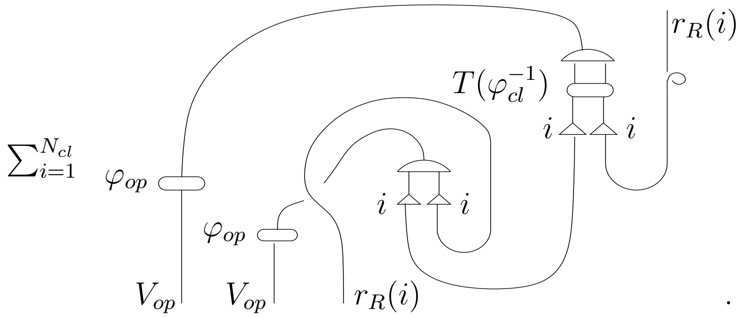
$\sum_{i=1}^{N_{cl}}$ 

. (5.26)

By the universal property of tensor product, for $a_i \in \mathcal{I}, i = 1, \dots, 6$, we have a canonical isomorphism:

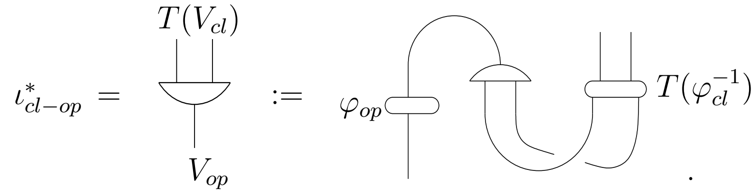
$$\begin{array}{rcl} \oplus_{a\in\mathcal{I}} \mathcal{V}^{a_4}_{a_1 a} \otimes \mathcal{V}^{a}_{a_2 a_3} & \xrightarrow{\cong} & \mathrm{Hom}_V(W_{a_1} \boxtimes (W_{a_2} \boxtimes W_{a_3}), W_{a_4}) \\ \mathcal{Y}_1 \otimes \mathcal{Y}_2 & \mapsto & m_{\mathcal{Y}_1} \circ (\mathrm{id}_{W_{a_1}} \boxtimes m_{\mathcal{Y}_2}). \end{array} \qquad (5.27)$$

Under this canonical isomorphism, the Cardy condition (3.73) can be viewed as a condition on two morphisms in $\mathrm{Hom}_V(V_{op} \boxtimes (V_{op} \boxtimes W_{r_R(i)}), W_{r_R(i)})$. In particular, the left hand side of (3.73) viewed as a morphism in $\mathrm{Hom}_V(V_{op} \boxtimes (V_{op} \boxtimes W_{r_R(i)}), W_{r_R(i)})$ can be expressed as follow:



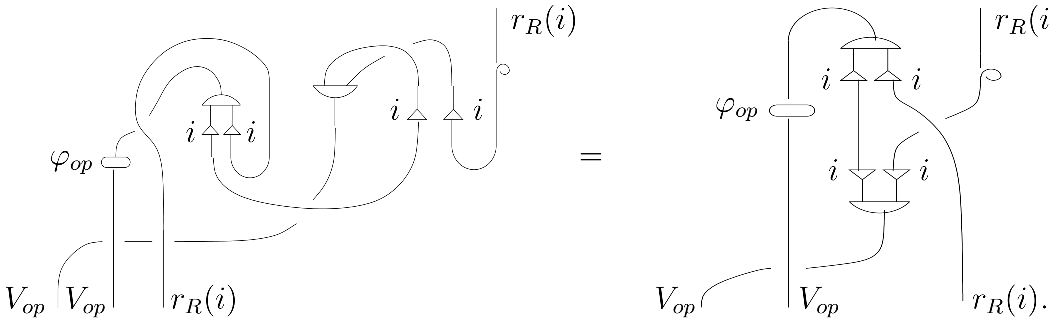
(5.28)

We define a morphism $\iota^*_{cl-op} : V_{op} \to T(V_{cl})$ by



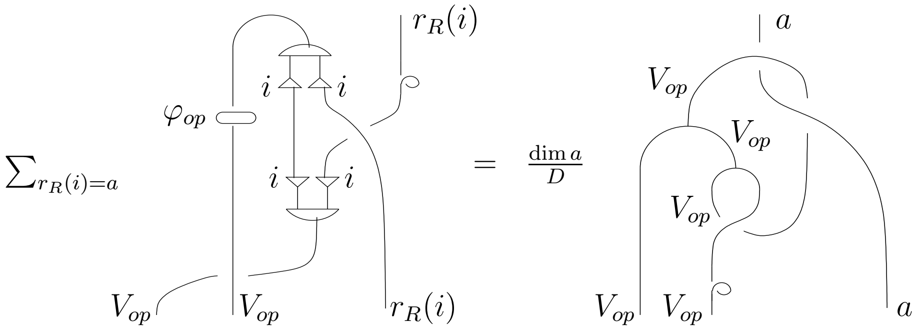
(5.29)

Then using the morphism ι^*_{cl-op}, we can rewrite the graph in (5.28) as follow:



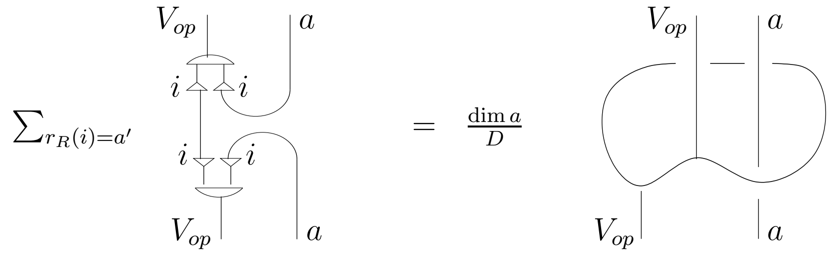
(5.30)

Using (4.55), (5.28) and (5.30), we obtain a graphic version of Cardy condition (3.73) as follow:



(5.31)

Remark 5.10. (5.31) is not the most efficient way to describe Cardy condition. For example, using Frobenius properties of V_{op}, one can easily obtain the following simpler version of Cardy condition:

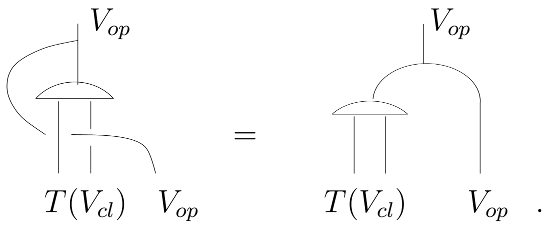


. (5.32)

The asymmetry between chiral and antichiral parts in (5.31)(5.32) is also superficial. One can find some other versions of Cardy conditions in which such asymmetry disappear. We will not pursue further in this direction. We leave it to the future publications.

We recall a definition in [Ko2].

Definition 5.11. An *open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra*, denoted as $(A_{op}|A_{cl}, \iota_{cl-op})$, consists of a commutative associative algebra with a trivial twist A_{cl} in $\mathcal{C}_{V\otimes V}$, an associative algebra A_{op} in $\mathcal{C}_V$ and an associative algebra morphism $\iota_{cl-op} : T(V_{cl}) \to V_{op}$, satisfying the following condition:



. (5.33)

The following Theorem is proved in [Ko2].

Theorem 5.12. *The category of open-closed field algebras over V is isomorphic to the category of open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebras.*

Definition 5.13. A *Cardy $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra* is an open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra $(A_{op}|A_{cl}, \iota_{cl-op})$ such that A_{cl} is a modular invariant commutative Frobenius algebra in $\mathcal{C}_{V\otimes V}$ with a trivial twist and A_{op} a symmetric Frobenius algebra in $\mathcal{C}_V$ and Cardy condition (5.31) hold.

Remark 5.14. As we discussed in the introduction, we believe that the axioms of Cardy $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra should be sufficient to supply an open-closed conformal field theory of all genus. However, constructing the high-genus theories is still a hard open problem.

5.3 Constructions

In this section, we give a categorical construction of Cardy $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra. This construction is called Cardy case in physics literature [FFFS2].

Let us first recall the diagonal construction [FFRS][HKo2][Ko1][HKo3]. We will follow the categorical construction given in [Ko1].

Let V_{cl} is be the object in $\mathcal{C}_{V\otimes V}$ given as follow

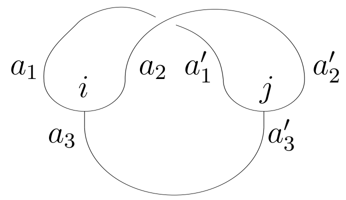
$$V_{cl} = \oplus_{a\in\mathcal{I}} W_a \otimes W'_a. \tag{5.34}$$

The decomposition of V_{cl} as a direct sum gives a natural embedding $V \otimes V \hookrightarrow V_{cl}$. We denote this embedding as ι_{cl}. We define a morphism $\mu_{cl} \in \mathrm{Hom}_{V\otimes V}(V_{cl} \boxtimes V_{cl}, V_{cl})$ by

$$\mu_{cl} = \sum_{a_1,a_2,a_3\in\mathcal{A}} \sum_{i,j=1}^{N_{a_1a_2}^{a_3}} \langle f_{a_3;i}^{a_1a_2}, f_{a_3';j}^{a_1'a_2'} \rangle \, e_{a_1a_2;i}^{a_3} \otimes e_{a_1'a_2';j}^{a_3'}, \tag{5.35}$$

where $e_{a_1a_2;i}^{a_3}$ and $f_{a_3}^{a_1a_2;j}$ are basis vectors given in (4.13) and (4.15) and $\langle \cdot, \cdot \rangle$ is a bilinear pairing given by

$$\frac{1}{\dim a_3}$$



. (5.36)

Notice that V_{cl}' has the same decomposition as V_{cl} in (5.34). They are isomorphic as $V \otimes V$-modules. There is, however, no canonical isomorphism. Now we choose a particular isomorphism $\varphi_{cl} : V_{cl} \to V_{cl}'$ given by

$$\varphi_{cl} = \oplus_{a\in\mathcal{I}} \frac{1}{\dim a} e^{-2\pi i h_a} \,\mathrm{id}_{W_a\otimes W_a'}. \tag{5.37}$$

The isomorphism φ_{cl} induces a nondegenerate invariant bilinear form on V_{cl} viewed as $V \otimes V$-module.

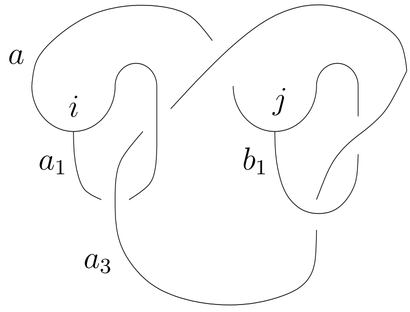
The following Theorem is a categorical version of Theorem 5.1 in [HKo3]. We give a categorical proof here.

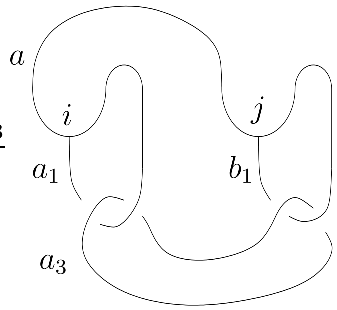
Theorem 5.15. *$(V_{cl}, \mu_{cl}, \iota_{cl})$ together with isomorphism φ_{cl} gives a modular invariant commutative Frobenius algebra in $\mathcal{C}_{V\otimes V}$ with a trivial twist.*

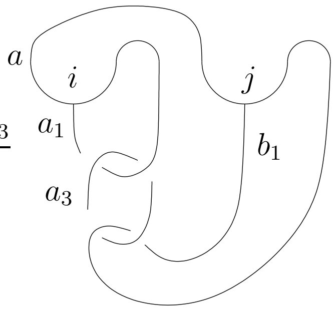
Proof. It was proved in [Ko1] that $(V_{cl}, \mu_{cl}, \iota_{cl})$ together with isomorphism φ_{cl} gives a commutative Frobenius algebra with a trivial twist.

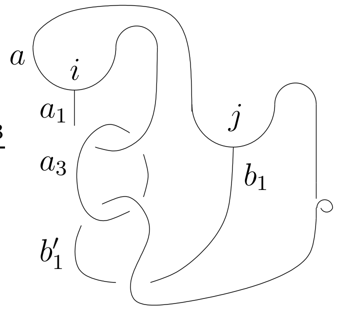
It remains to show the modular invariance. The bilinear pairing $\langle \cdot, \cdot \rangle$ given in (5.36) can be naturally extended to a bilinear form on $\oplus_{a,a_1\in\mathcal{I}}(W_{a_1}, W_a \boxtimes W_{a_1})$. We still denote this bilinear form as $\langle \cdot, \cdot \rangle$. Then to prove the modular invariance of V_{cl} amounts to show that the bilinear form $\langle \cdot, \cdot \rangle$ is invariant under the action $(S^{-1})^* \otimes S^*$. Clearly, when $b \neq a'$, $\langle (S^{-1}(a))^* f_{a_1;i}^{aa_1}, (S(b))^* f_{b_1;j}^{bb_1} \rangle = 0$. When $b = a'$, we have (using (4.58), (4.59) and (4.44))

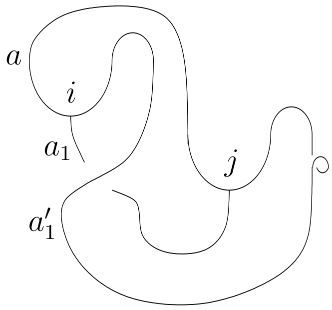
$$\langle (S^{-1}(a))^* f_{a_1;i}^{aa_1}, (S(a'))^* f_{b_1;j}^{a'b_1} \rangle = \sum_{a_3\in\mathcal{I}} \frac{\dim a_3}{D^2}$$

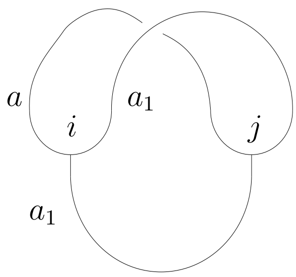


$$= \sum_{a_3 \in \mathcal{I}} \frac{\dim a_3}{D^2}$$ 

$$= \sum_{a_3 \in \mathcal{I}} \frac{\dim a_3}{D^2}$$ 

$$= \sum_{a_3 \in \mathcal{I}} \frac{\dim a_3}{D^2}$$ 

$$= \frac{\delta_{a_1 b_1'}}{\dim a_1}$$ 

$$= \frac{\delta_{a_1 b_1'}}{\dim a_1}$$ 

.

∎

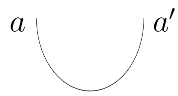
Now we give an associative algebra in $\mathcal{C}_V$. Let

$$V_{op} := \oplus_{a,b \in \mathcal{I}} \, W_a \boxtimes W_b'. \tag{5.38}$$

Let $e_a : W_a' \boxtimes W_a \to V$ and $i_a : V \to W_a \boxtimes W_a'$ be the duality maps defined in [Ko1]. Let $\sqrt{D} := e^{\frac{1}{2}\log D}$. We define ι_{op} as the following composition of maps:

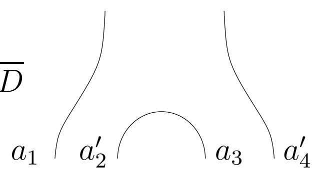
$$\iota_{op} : V \xrightarrow{\frac{1}{\sqrt{D}} \oplus_{a \in \mathcal{I}} i_a} \oplus_{a \in \mathcal{I}} W_a \boxtimes W_a' \hookrightarrow V_{op}. \tag{5.39}$$

or equivalently

$$\iota_{op} := \sum_{a \in \mathcal{I}} \frac{1}{\sqrt{D}}$$ 

. (5.40)

We define the multiplication morphism μ_{op} by

$$\mu_{op} := \sum_{a_1, a_2, a_3, a_4 \in \mathcal{I}} \delta_{a_2 a_3} \sqrt{D}$$ 

. (5.41)

V_{op} is clearly isomorphic to V'_{op} as V-modules. We define an isomorphism $\varphi_{op} : V_{op} \to V'_{op}$ by specify its restriction on $W_a \otimes W'_b$ as follow:

$$\varphi_{op}|_{W_a \boxtimes W'_b} := \mathrm{id}_{a \boxtimes b'} \tag{5.42}$$

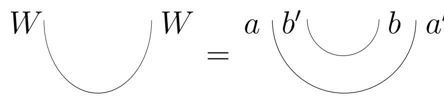
for $a, b' \in \mathcal{I}$.

Proposition 5.16. *$(V_{op}, \mu_{op}, \iota_{op})$ together with φ_{op} gives a symmetric Frobenius algebra in $\mathcal{C}_V$.*

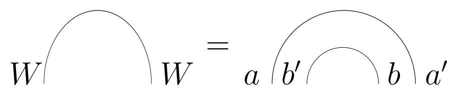
Proof. It is trivial to see that $(V_{op}, \mu_{op}, \iota_{op})$ is an associative algebra in $\mathcal{C}_V$. It remains to prove that its symmetric and Frobenius properties, This amounts to prove the following two identities (recall (5.17) and (5.18))

$$\begin{aligned} \mu_{op} &= \varphi_{op}^{-1} \circ \sigma_{123}(\mu_{op}) \circ (\varphi_{op} \boxtimes \mathrm{id}_{V_{op}}), && (5.43)\\ \mu_{op} &= \varphi_{op}^{-1} \circ \sigma_{132}(\mu_{op}) \circ (\mathrm{id}_{V_{op}} \boxtimes \varphi_{op}). && (5.44) \end{aligned}$$

For any $W = W_a \boxtimes W'_b$, the duality maps can be defined as follow:

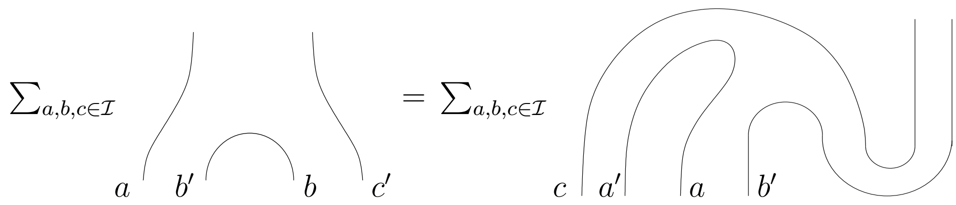


(5.45)



(5.46)

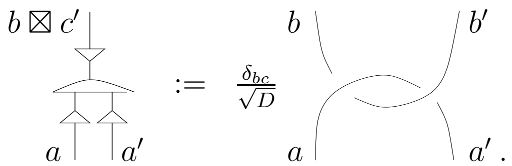
because the rigidity axioms are obviously true by above definition. Then (5.43) can be proved as follow:



. (5.47)

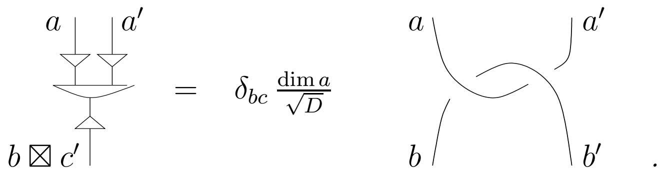
The proof of (5.44) is similar. ∎

Now we define a map $\iota_{cl-op} : T(V_{cl}) \to V_{op}$ by



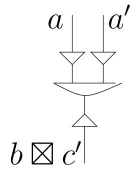
(5.48)

Lemma 5.17.

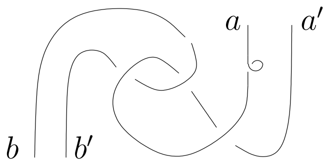


(5.49)

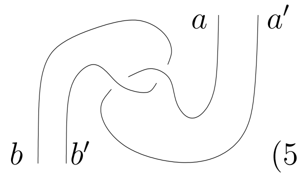
Proof. By (5.29) and (5.48), we have



$= \delta_{bc} \frac{\dim a}{\sqrt{D}}$



$= \delta_{bc} \frac{\dim a}{\sqrt{D}}$



(5.50)

It is easy to see that the last figure in (5.50) can be deformed to that on the right hand side of (5.49). ∎

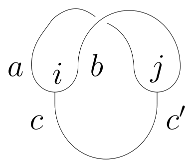
Theorem 5.18. $(V_{cl}, V_{op}, \iota_{cl-op})$ *is a Cardy* $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$*-algebra.*

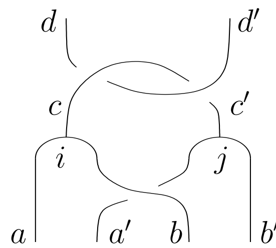
Proof. Recall that $T(V_{cl})$ together with multiplication morphism $\mu_{T(V_{cl})} = T(\mu_{cl}) \circ \varphi_2$ and morphism $\iota_{T(V_{cl})} = T(\iota_{cl}) \circ \varphi_0$ is an associative algebra. We first prove that ι_{cl-op} is an algebra morphism. It is clear that $\iota_{cl-op} \circ \iota_{T(V_{cl})} = \iota_{op}$. It remains to show the following identity

$$\iota_{cl-op} \circ \mu_{T(V_{cl})} = \mu_{op} \circ (\iota_{cl-op} \boxtimes \iota_{cl-op}). \tag{5.51}$$

By the definition of φ_2, that of μ_{cl} and (5.48), we obtain

$\iota_{cl-op} \circ \mu_{T(V_{cl})} = \sum_{a,b,c,d \in \mathcal{I}} \sum_{i,j} \frac{1}{\sqrt{D} \dim c}$

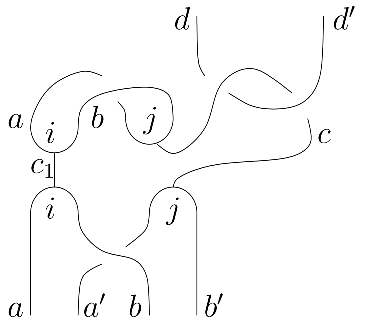




(5.52)

It is easy to see that the right hand side of (5.52) equals to

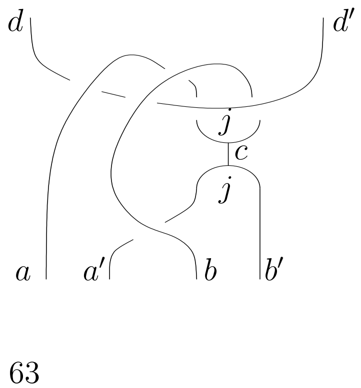
$\sum_{a,b,c,d \in \mathcal{I}} \sum_{i,j} \sum_{c_1 \in \mathcal{I}} \frac{1}{\sqrt{D}}$



(5.53)

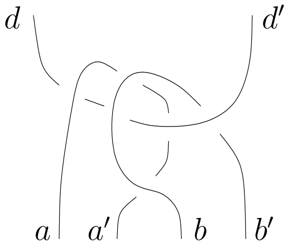
Using (4.17) to sum up the indices c_1 and i, we obtain that (5.53) further equals to

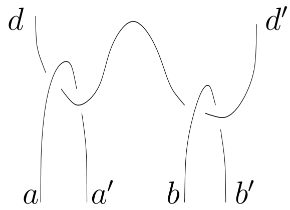
$\sum_{a,b,c,d \in \mathcal{I}} \sum_{j} \frac{1}{\sqrt{D}}$



(5.54)

Using (4.17) again, we obtain

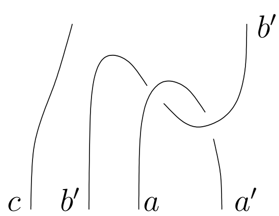
$$\sum_{a,b,d\in\mathcal{I}} \frac{1}{\sqrt{D}}$$ 

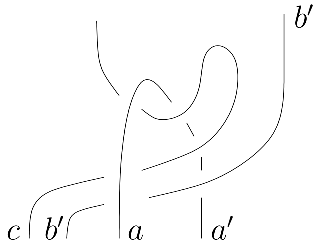
$$= \sum_{a,b,d\in\mathcal{I}} \frac{\sqrt{D}}{\sqrt{D}\sqrt{D}}$$ 

, (5.55)

the right hand side of which is nothing but $\mu_{op} \circ (\iota_{cl-op} \boxtimes \iota_{cl-op})$.

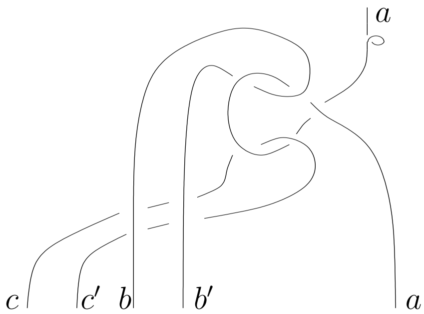
The commutativity (5.33) follows from the following identity:



$=$ 

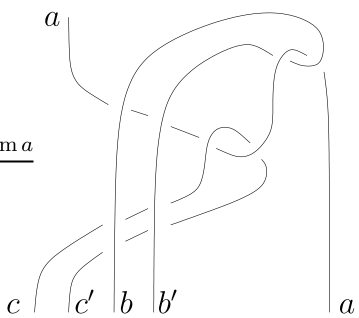
. (5.56)

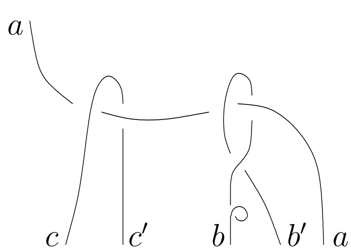
In summary, we have proved that the triple $(V_{cl}, V_{op}, \iota_{cl-op})$ is an open-closed $\mathcal{C}_V|\mathcal{C}_{V\otimes V}$-algebra. It remains to show that it also satisfies the Cardy condition (5.31). The left hand side of (5.31) restricted on $c \boxtimes c_1' \subset V_{op}$ and $b \boxtimes b_1' \subset V_{op}$ equals to

$$\delta_{cc_1} \frac{\dim a}{\sqrt{D}} \frac{\delta_{bb_1}}{\sqrt{D}}$$ 

, (5.57)

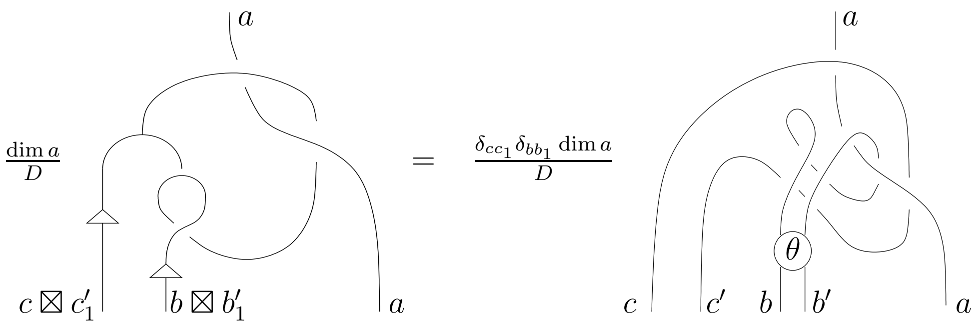
which can be simplified as follow:

$$\frac{\delta_{cc_1}\delta_{bb_1} \dim a}{D}$$ 

$$= \frac{\delta_{cc_1}\delta_{bb_1} \dim a}{D}$$ 

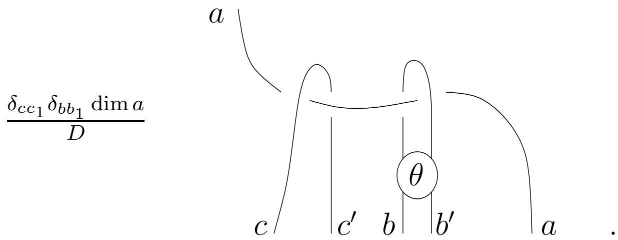
. (5.58)

The right hand side of (5.31) restricted on $c \boxtimes c_1' \subset V_{op}$ and $b \boxtimes b_1' \subset V_{op}$ equals to



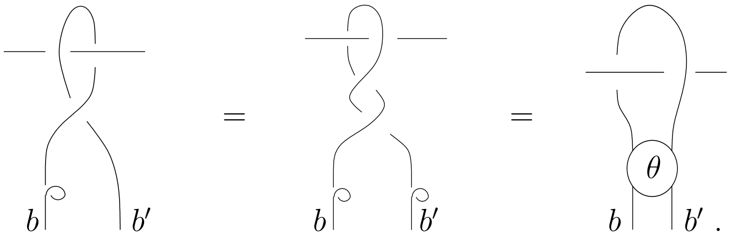
, (5.59)

which can be simplified as follow:



. (5.60)

The right hand side of (5.58) equals to (5.60) because of the following identities:



. (5.61)

∎

One can see from (5.38), the vacuum-like states in V_{op} forms a $|\mathcal{I}|$-dimensional vector spaces with a natural choice of basis vectors. By Proposition 2.18, they give $|\mathcal{I}|$ fundamental D-branes in $\overline{V_{cl}}$.

A The Proof of Lemma 4.30

Lemma A.1. For $w_1 \in W_{a_1}, w_a \in W_{a_2}$ and $w'_{a_3} \in (W_{a_3})'$ and $\mathcal{Y}_{a_1a_2}^{a_3} \in \mathcal{V}_{a_1a_2}^{a_3}$, we have

$$\begin{aligned}&\langle w'_{a_3}, \mathcal{Y}(\mathcal{U}(x)w_{a_1}, x)w_{a_2}\rangle \\ &\quad = \langle \tilde{A}_r(\mathcal{Y})(\mathcal{U}(e^{(2r+1)\pi i}x^{-1})e^{-(2r+1)\pi i L(0)}w_{a_1}, e^{(2r+1)\pi i}x^{-1})w'_{a_3}, w_{a_2}\rangle. \end{aligned} \tag{A.62}$$

Proof. Using the definition of $\hat{A}_r$, we see that the left hand side of (A.62) equals to

$$\langle \tilde{A}_r(\mathcal{Y})(e^{-xL(1)}x^{-2L(0)}\mathcal{U}(x)w_{a_1}, e^{(2r+1)\pi i}x^{-1})w'_{a_3}, w_{a_2}\rangle. \tag{A.63}$$

In [H9], the following formula

$$e^{xL(1)}x^{-2L(0)}e^{(2r+1)\pi i L(0)}\mathcal{U}(x)e^{-(2r+1)\pi i L(0)} = \mathcal{U}(x^{-1}) \tag{A.64}$$

is proved. Applying (A.64) to (A.63), we obtain (A.62) immediately. ∎

Now we are ready to give a proof of Lemma 4.4.

Proof. We have, for $w_{a_2} \in W_{a_2}, w_{a_3} \in W_{a_3}$,

$$\begin{aligned}
&((\Psi_2(\mathcal{Y}^{a_1;(1)}_{aa_1;i} \otimes \mathcal{Y}^{a;(2)}_{a_2a_3;j}))(w_{a_2} \otimes w_{a_3}))(z_1, z_2+\tau, \tau)\\
&= E\Bigg(\mathrm{Tr}_{W_{a_1}} \mathcal{Y}^{a_1;(1)}_{aa_1;i}(\mathcal{U}(e^{2\pi i(z_2+\tau)}) \cdot\\
&\qquad\qquad \cdot \mathcal{Y}^{a;(2)}_{a_2a_3;j}(w_{a_2}, z_1-(z_2+\tau))w_{a_3}, e^{2\pi i(z_2+\tau)})q_\tau^{L(0)-\frac{c}{24}}\Bigg)\\
&= \sum_{b\in\mathcal{I}}\sum_{k,l} F^{-1}(\mathcal{Y}^{a_1;(1)}_{aa_1;i} \otimes \mathcal{Y}^{a;(2)}_{a_2a_3;j}; \mathcal{Y}^{a_1;(3)}_{a_2b;k} \otimes \mathcal{Y}^{b;(4)}_{a_3a_1;l})\\
&\qquad E\Bigg(\mathrm{Tr}_{W_{a_1}} \mathcal{Y}^{a_1;(3)}_{a_2b;k}(\mathcal{U}(e^{2\pi i z_1})w_{a_2}, e^{2\pi i z_1}) \cdot\\
&\qquad\qquad \cdot \mathcal{Y}^{b;(4)}_{a_3a_1;l}(\mathcal{U}(e^{2\pi i(z_2+\tau)})w_{a_3}, e^{2\pi i(z_2+\tau)})q_\tau^{L(0)-\frac{c}{24}}\Bigg).
\end{aligned} \tag{A.65}$$

Using the $L(0)$-conjugation formula, we can move q_τ from the right side of $\mathcal{Y}^{b;(4)}_{a_3a_1;l}$ to the left side of $\mathcal{Y}^{b;(4)}_{a_3a_1;l}$. Then using the following property of trace:

$$\mathrm{Tr}_{W_{a_1}}(AB) = \mathrm{Tr}_{W_b}(BA), \tag{A.66}$$

for all $A : W_b \to \overline{W_{a_1}}, B : W_{a_1} \to \overline{W_b}$ whenever the multiple sums in either side of (A.66) converge absolutely, we obtain that the left hand side of (A.65) equals to

$$\begin{aligned}
&\sum_{b\in\mathcal{I}}\sum_{k,l} F^{-1}(\mathcal{Y}^{a_1;(1)}_{aa_1;i} \otimes \mathcal{Y}^{a;(2)}_{a_2a_3;j}; \mathcal{Y}^{a_1;(3)}_{a_2b;k} \otimes \mathcal{Y}^{b;(4)}_{a_3a_1;l})\\
&\quad E\Bigg(\mathrm{Tr}_{W_b} \mathcal{Y}^{b;(4)}_{a_3a_1;l}(\mathcal{U}(e^{2\pi i z_2})w_{a_3}, e^{2\pi i z_2})\mathcal{Y}^{a_1;(3)}_{a_2b;k}(\mathcal{U}(e^{2\pi i z_1})w_{a_2}, e^{2\pi i z_1})q_\tau^{L(0)-\frac{c}{24}}\Bigg).
\end{aligned} \tag{A.67}$$

Now apply (A.62) to (A.67). We then obtain that (A.67) equals to

$$\begin{aligned}
&\sum_{b\in\mathcal{I}}\sum_{k,l} F^{-1}(\mathcal{Y}^{a_1;(1)}_{aa_1;i} \otimes \mathcal{Y}^{a;(2)}_{a_2a_3;j}; \mathcal{Y}^{a_1;(3)}_{a_2b;k} \otimes \mathcal{Y}^{b;(4)}_{a_3a_1;l})\\
&\quad E\Bigg(\mathrm{Tr}_{(W_b)'} \tilde{A}_r(\mathcal{Y}^{a_1;(3)}_{a_2b;k})(\mathcal{U}(e^{(2r+1)\pi i}e^{-2\pi i z_1})e^{-(2r+1)\pi i L(0)}w_{a_2}, e^{(2r+1)\pi i}e^{-2\pi i z_1})\\
&\quad \tilde{A}_r(\mathcal{Y}^{b;(4)}_{a_3a_1;l})(\mathcal{U}(e^{(2r+1)\pi i}e^{-2\pi i z_2})e^{-(2r+1)\pi i L(0)}w_{a_3}, e^{(2r+1)\pi i}e^{-2\pi i z_2})q_\tau^{L(0)-\frac{c}{24}}\Bigg).
\end{aligned}$$

Now apply the associativity again and be careful about the branch cut as in [H9], then the left hand side of (A.65) further equals to

$$\sum_{b\in\mathcal{I}}\sum_{k,l}\sum_{c\in\mathcal{I}}\sum_{p,q} F^{-1}(\mathcal{Y}^{a_1;(1)}_{aa_1;i}\otimes\mathcal{Y}^{a;(2)}_{a_2a_3;j};\mathcal{Y}^{a_1;(3)}_{a_2b;k}\otimes\mathcal{Y}^{b;(4)}_{a_3a_1;l})$$
$$F(\tilde{A}_r(\mathcal{Y}^{a_1;(3)}_{a_2b;k})\otimes\tilde{A}_r(\mathcal{Y}^{b;(4)}_{a_3a_1;l}),\mathcal{Y}^{b';(5)}_{cb';p}\otimes\mathcal{Y}^{c;(6)}_{a_2a_3;q})$$
$$E\Bigg(\mathrm{Tr}_{(W_b)'}\, q_\tau^{L(0)-\frac{c}{24}}\,\mathcal{Y}^{b';(5)}_{cb';p}(\mathcal{U}(e^{(2r+1)\pi i}e^{-2\pi i z_2})\cdot$$
$$\cdot\mathcal{Y}^{c;(6)}_{a_2a_3;q}(e^{-(2r+1)\pi i L(0)}w_{a_2},e^{\pi i}(z_1-z_2))e^{-(2r+1)\pi i L(0)}w_{a_3},e^{(2r+1)\pi i}e^{-2\pi i z_2})\Bigg).$$
$$=\sum_{b\in\mathcal{I}}\sum_{k,l}\sum_{c\in\mathcal{I}}\sum_{p,q} F^{-1}(\mathcal{Y}^{a_1;(1)}_{aa_1;i}\otimes\mathcal{Y}^{a;(2)}_{a_2a_3;j};\mathcal{Y}^{a_1;(3)}_{a_2b;k}\otimes\mathcal{Y}^{b;(4)}_{a_3a_1;l})$$
$$F(\tilde{A}_r(\mathcal{Y}^{a_1;(3)}_{a_2b;k})\otimes\tilde{A}_r(\mathcal{Y}^{b;(4)}_{a_3a_1;l}),\mathcal{Y}^{b';(5)}_{cb';p}\otimes\mathcal{Y}^{c;(6)}_{a_2a_3;q})$$
$$E\Bigg(\mathrm{Tr}_{W_b}\, q_\tau^{L(0)-\frac{c}{24}}\,\hat{A}_r(\mathcal{Y}^{b';(5)}_{cb';p})(e^{(2r+1)\pi i L(0)}$$
$$\mathcal{Y}^{c;(6)}_{a_2a_3;q}(e^{-(2r+1)\pi i L(0)}w_{a_2},e^{\pi i}(z_1-z_2))e^{-(2r+1)\pi i L(0)}w_{a_3},e^{2\pi i z_2})\Bigg) \tag{A.68}$$

Choosing $r=0$ and using $\mathcal{Y}(\cdot,e^{2\pi i}x)\cdot=\Omega_0^2(\mathcal{Y})(\cdot,x)\cdot$, we obtain

$$((\Psi_2(\mathcal{Y}^{a_1;(1)}_{aa_1;i}\otimes\mathcal{Y}^{a;(2)}_{a_2a_3;j}))(w_{a_2}\otimes w_{a_3}))(z_1,z_2+\tau,\tau)$$
$$=\sum_{b\in\mathcal{I}}\sum_{k,l}\sum_{c\in\mathcal{I}}\sum_{p,q} F^{-1}(\mathcal{Y}^{a_1;(1)}_{aa_1;i}\otimes\mathcal{Y}^{a;(2)}_{a_2a_3;j};\mathcal{Y}^{a_1;(3)}_{a_2b;k}\otimes\mathcal{Y}^{b;(4)}_{a_3a_1;l})$$
$$F(\tilde{A}_0(\mathcal{Y}^{a_1;(3)}_{a_2b;k})\otimes\tilde{A}_0(\mathcal{Y}^{b;(4)}_{a_3a_1;l}),\mathcal{Y}^{b';(5)}_{cb';p}\otimes\mathcal{Y}^{c;(6)}_{a_2a_3;q})$$
$$E\Bigg(\mathrm{Tr}_{W_b}\hat{A}_0(\mathcal{Y}^{b';(5)}_{cb';p})(\Omega_0^2(\mathcal{Y}^{c;(6)}_{a_2a_3;q})(w_{a_2},z_1-z_2)w_{a_3},e^{2\pi i z_2})q_\tau^{L(0)-\frac{c}{24}}\Bigg). \tag{A.69}$$

By the linear independency proved in [H11] of the last factor in each term of above sum, it is clear that β induces a map given by (4.30). ∎

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