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AN A_∞-STRUCTURE FOR LINES IN A PLANE

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Dedicated to Professor Jim Stasheff in honor of his 70th birthday

ABSTRACT. As an explicit example of an A_∞-structure associated to geometry, we construct an A_∞-structure for a Fukaya category of finitely many lines (Lagrangians) in $\mathbb{R}^2$, *i.e.*, we define also *non-transversal* A_∞-products. This construction is motivated by homological mirror symmetry of (two-)tori, where $\mathbb{R}^2$ is the covering space of a two-torus. The strategy is based on an algebraic reformulation of Morse homotopy theory through homological perturbation theory (HPT) as discussed by Kontsevich and Soibelman in [21], where we introduce a special DG category which is a key idea of our construction.

CONTENTS

1. INTRODUCTION

For a graded vector space A, a strong homotopy associative structure (or an A_∞-structure) on A is a family of multilinear maps $m_k : A^{\otimes k} \to A$ for $k \geq 1$ satisfying certain constraints, first introduced by Jim Stasheff [26, 27] in the study of H-spaces such as based loop spaces. In particular, $m_1 = d$ forms a differential on A, m_2 is a product which is associative up to homotopy, where m_3 defines the homotopy and $m_4, m_5, \ldots$ define higher homotopies. An A_∞-algebra with higher products $m_3, m_4, \ldots$ all zero is a differential graded (DG) algebra, which appears as the structure DeRham complexes have in general. A category version of an A_∞-algebra $(A, \{m_k\}_{k\geq 1})$ is called an A_∞-category introduced by Fukaya [2] to formulate Morse homotopy theory and Floer

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theory of Lagrangian submanifolds (shortly, Lagrangians) in a symplectic manifold. In particular, a category for the latter theory, called a Fukaya category, can give an interesting example of A_∞-structures associated to geometry. In this paper, we shall construct an A_∞-structure for a Fukaya category $Fuk(\mathbb{R}^2)$ consisting of lines in $\mathbb{R}^2$. For each two lines L_a, L_b which intersect with each other at one point $v_{ab} \in \mathbb{R}^2$, the space of morphisms $\mathrm{Hom}(a,b)$ is a one-dimensional vector space (over $\mathbb{R}$) spanned by a base $[v_{ab}]$ associated to the intersection point v_{ab}. Then, the (higher) A_∞-product $m_k : \mathrm{Hom}(a_1,a_2)\otimes\cdots\otimes\mathrm{Hom}(a_k,a_{k+1}) \to \mathrm{Hom}(a_1,a_{k+1})$, $a_1,\ldots,a_{k+1}\in\mathrm{Ob}(\mathcal{C})$, is defined by polygons surrounded by lines in $\mathbb{R}^2$

$$m_k([v_{a_1a_2}],\ldots,[v_{a_ka_{k+1}}]) = \pm e^{-Area(\vec{v})}[v_{a_1a_{k+1}}]$$

if the sequence $\vec{v} := (v_{a_1a_2},\ldots,v_{a_ka_{k+1}},v_{a_{k+1}a_1})$, $v_{a_{k+1}a_1} = v_{a_1a_{k+1}}$, of the intersection points forms a clockwise convex (CC-) polygon (Figure 1). From the viewpoint of Lagrangian intersection Floer

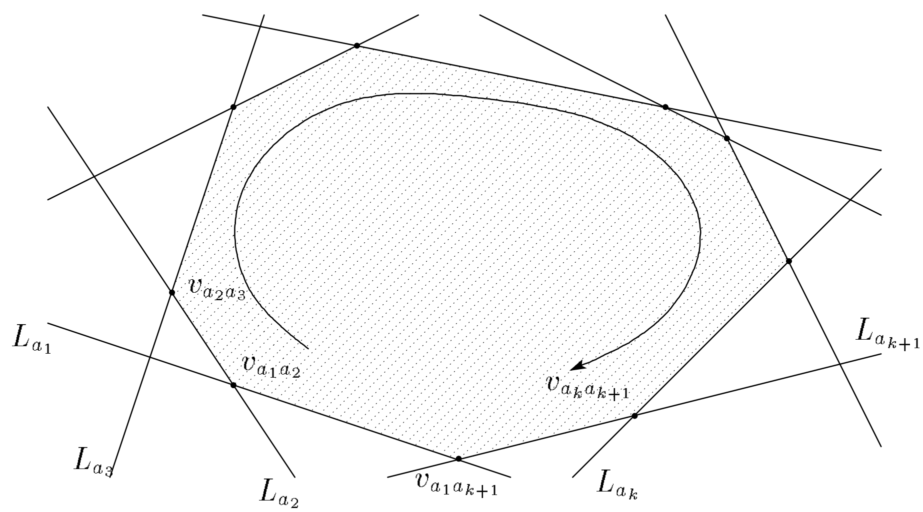


FIGURE 1. A clockwise convex polygon (CC-polygon) defined by lines $L_{a_1},\ldots,L_{a_{k+1}}$.

theory, these lines are thought of as special Lagrangian submanifolds in a symplectic manifold $\mathbb{R}^2 \simeq T^*\mathbb{R}$, the cotangent bundle over $\mathbb{R}$. We in particular construct such an A_∞-category $Fuk(\mathbb{R}^2)$ with finitely many objects, in which $\oplus_{a,b\in\mathrm{Ob}(Fuk(\mathbb{R}^2))}\mathrm{Hom}(a,b)$ is an example of an A_∞-algebra.

Although the definition of an A_∞-structure of a Fukaya category is clear for the multilinear maps m_k on morphisms $\mathrm{Hom}(a,b)$ with L_a and L_b transversal to each other, even in this $\mathbb{R}^2$ case, it is technically not easy to define multilinear maps m_k on morphisms including $\mathrm{Hom}(a,a)$ for some line L_a because of non-transversality of Lagrangians. (See FOOO [6] for the problem of transversality in a more general setup, where, I have heard, another way of resolution is discussed.) However, we can not define an A_∞-category without defining all those non-transversal multilinear maps. The aim of this paper is to define explicitly all the A_∞-products of the Fukaya category $Fuk(\mathbb{R}^2)$ including these non-transversal ones. To derive those A_∞-products, the rough direction of our strategy is first to define a DG category $\mathcal{C}_{DR}$ with the same objects, and then to apply to $\mathcal{C}_{DR}$ homological perturbation theory (HPT) developed by Gugenheim, Lambe, Stasheff, Huebschmann, Kadeishvili, etc., [10, 8, 9, 12] (see also the decomposition theorem in [18, 19]). For a DG algebra or an A_∞-algebra A, HPT starts with what is called *strong deformation retract (SDR)* data

$$(\, B \underset{\pi}{\overset{\iota}{\rightleftarrows}} A \,, h),$$

where B is a complex, ι and π are chain maps so that $\pi \circ \iota = \mathrm{Id}_B$ and $h : A \to A$ is the contracting homotopy defined by

$$d_A h + h d_A = \mathrm{Id}_A - P, \qquad P := \iota \circ \pi,$$

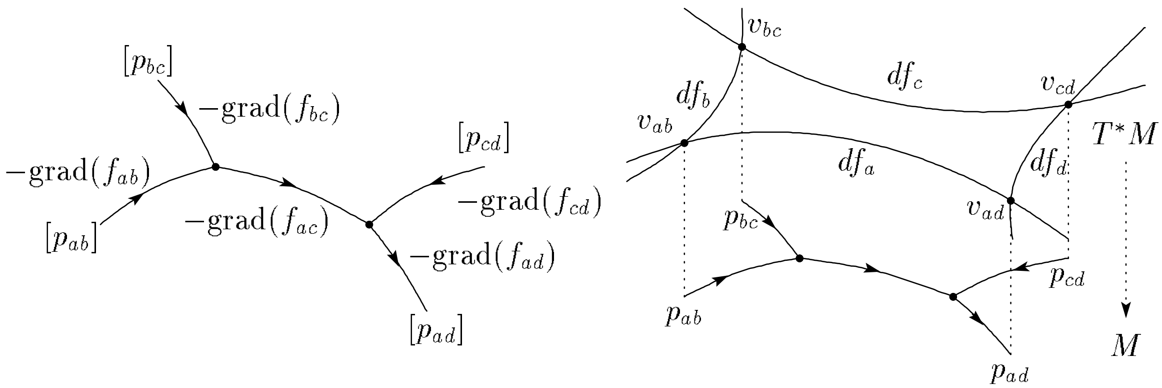
together with some additional conditions. By definition, P will be an idempotent in A. Given a contracting homotopy h, one obtains SDR data as above, and then the HPT machinery gives a way to produce an A_∞-structure on B which is homotopy equivalent to the original A_∞-algebra A. In particular, the induced A_∞-structure on B can be described in terms of planar rooted trees (Feynman graphs). The HPT for A_∞-algebras is extended straightforwardly to A_∞-categories [21]. Since the A_∞-structure is manifest on $\mathcal{C}_{DR}$ as its DG-category structure and there are not subtleties about transversality there, one can expect that HPT yields an A_∞-structure on $Fuk(\mathbb{R}^2)$ if we can find a suitable contracting homotopy h.

Physically, this DG category $\mathcal{C}_{DR}$ is related to a kind of Chern-Simons field theory (on one-dimensional space $\mathbb{R}$). Applying HPT to $\mathcal{C}_{DR}$ then corresponds to considering perturbation theory of the Chern-Simons theory at tree level. This kind of Chern-Simons theory is thought of as a topological open string field theory (SFT) [29], where the choice of a homotopy operator in applying HPT corresponds to the choice of a gauge fixing for the open SFT (see [13, 18] for open SFT and [22, 1] for topological open SFT). From such a physical viewpoint, it is interesting that the result of this paper indicates the (disk) instantons, which are nonperturbative effects in string theory, are also derived by perturbation theory of string *field* theory.

The homotopy equivalence $Fuk(\mathbb{R}^2) \simeq \mathcal{C}_{DR}$ obtained via the HPT plays the key role in discussing homological mirror symmetry [20], since the DG category $\mathcal{C}_{DR}$ is related to a category of holomorphic vector bundles on a complex manifold. Since $\mathbb{R}^2$ is the covering space of a two-torus, the arguments in this paper are directly applied to homological mirror symmetry for two-tori, and higher dimensional generalization of the torus analog of the DG-category $\mathcal{C}_{DR}$ is also straightforward (for instance see [17]). Homological mirror symmetry is discussed positively for two-tori [25, 23], for abelian varieties [3], and for (complex) noncommutative tori [14, 24, 15, 16, 17]; in particular, for two tori, transversal A_∞-products are defined explicitly and the homological mirror is also shown for the transversal A_∞-products by Polishchuk [23]. However, the reason why such equivalence holds has still been unclear even for the transversal A_∞-products.

Kontsevich-Soibelman [21] then proposed a strategy to show the homological mirror symmetry based on the viewpoint of Strominger-Yau-Zaslow torus fibrations (see also [4] for a related approach). The strategy is to reformulate Fukaya-Oh Morse homotopy theory [2, 5] algebraically in terms of a DG category $DR(M)$ consisting of DeRham complexes and to apply HPT to $DR(M)$ together with Harvey-Lawson's Morse theory [11]. For a compact manifold M with a given metric (which is used to define the gradient grad, see below), the objects of the category $Ms(M)$ [1] of Fukaya-Oh Morse homotopy [2, 5] are smooth functions $f \in C^\infty(M)$ on M. If the difference $f_{ab} := f_a - f_b$ of two functions $f_a, f_b \in C^\infty(M)$ is a Morse function, the space $\mathrm{Hom}_{Ms(M)}(a, b)$ of morphisms is defined as the vector space spanned by bases $[p_{ab}]$ associated to the critical points p_{ab} of f_{ab}. The A_∞-structure on $Ms(M)$ is defined by trivalent planar trees so that each edge is associated to the gradient flow of the difference of the corresponding two functions (see Figure 2 (a)). The equivalence of the Morse A_∞-category $Ms(M)$ with the Fukaya A_∞-category $Fuk(T^*M)$ on T^*M is discussed in [5], where an object of $Fuk(T^*M)$ is a Lagrangian $L_a \subset T^*M$ defined by the graph of $df \in \Gamma(T^*M)$ of a Morse function f_a (Figure 2 (b)), and the space $\mathrm{Hom}_{Fuk(T^*M)}(a, b)$ of morphisms from L_a to L_b is spanned by the bases $[v_{ab}]$ associated to the intersection points v_{ab} of L_a with L_b, whose images by the projection $x : T^*M \to M$ are the critical points $p_{ab} = x(v_{ab})$ of $f_{ab} = f_a - f_b$. Kontsevich-Soibelman [21] discussed obtaining the Morse A_∞-category $Ms(M)$ by applying HPT to the DG-category $DR(M)$. The key idea there is to identify the contracting homotopy of the SDR in HPT with Harvey-Lawson's chain homotopy in [11], which will allow us to identify the planar trees in HPT with the trees of gradient flows defining the A_∞-structure of

[1] This category $Ms(M)$ is denoted by $\mathrm{M}(Y)$ in [21] where Y is the compact smooth manifold M here.



(a) (b)

FIGURE 2. (a): A tree of gradient flows in M, where $f_{ab} := f_a - f_b$, while p_{ab} is a critical point of f_{ab} and $[p_{ab}]$ the associated base of $\mathrm{Hom}_{Ms(M)}(a,b)$. An A_∞-product $m_3([p_{ab}],[p_{bc}],[p_{cd}])$ is defined by counting all such trees of gradient flows. (b): The Lagrangians in T^*M defined by df_a, df_b, df_c, df_d corresponding to the Morse functions f_a, f_b, f_c, f_d. By definition, the dimension of the Lagrangians is the same as the dimension of M. By the projection $x : T^*M \to M$, an intersection point v_{ab} of L_a with L_b corresponds to a critical point $p_{ab} = x(v_{ab})$.

$Ms(M)$. Let $\varphi_t : M \to M$, $t \in [0,\infty)$ be the flow generated by the gradient $\mathrm{grad}(f)$ of a given Morse function f. Harvey-Lawson [11] showed the existence of the limit $\mathbf{P} := \lim_{t\to\infty} \varphi_t^*$ of the pullback φ_t^* together with the chain homotopy

$$d_{D'}h + hd_{DR} = \mathbf{I} - \mathbf{P}, \tag{1.1}$$

where $\mathbf{I} : (\Omega(M), d_{DR}) \to (D'(M), d_{DR'})$ is the inclusion of smooth differential forms on M to the space $D'(M)$ of distribution forms and $\mathbf{P} : (\Omega(M), d_{DR}) \to (D'(M), d_{D'})$ turns out to be a linear map such that $\mathbf{P}(\Omega(M)) \subset D'(M)$ forms a subcomplex spanned by DeRham currents $[U_p]$ with support the unstable manifolds U_p of critical points p of f. In [21], the SDR data for the complex $\mathrm{Hom}_{DR(M)}(a,b) := \Omega(M)$ was identified with the Harvey-Lawson's chain homotopy (1.1) with $f = f_{ab} := f_a - f_b$. All these tools for T^*M were then extended to torus fibrations over M to discuss homological mirror symmetry for torus fibrations.

Strongly motivated by this story, we define a DG-category $\mathcal{C}_{DR}$ which is similar to $DR(M)$ in [21] with $M = \mathbb{R}$. Here, for two Morse functions $f_a, f_b \in C^\infty(\mathbb{R})$, we set the differential $d_{ab} : \mathrm{Hom}_{\mathcal{C}_{DR}}(a,b) \to \mathrm{Hom}_{\mathcal{C}_{DR}}(a,b)$ as the twisted differential

$$d_{ab} = d - d(f_{ab})\wedge \;= e^{f_{ab}}\, d\, e^{-f_{ab}}$$

of Witten's Morse complex [28]. This leads to the correct structure constant of the transversal A_∞-products, *i.e.*, the area of the corresponding CC-polygons, via the HPT. Though the case $M = \mathbb{R}$ looks too simple, because $\mathbb{R}$ is noncompact, this case can include more nontrivial phenomena than the traditional setting where M is a *compact* smooth manifold. We consider a set $\mathfrak{F}_N := \{f_a, f_b, \dots\} = \{a, b, \dots\}$ of N lines, and denote by $\mathcal{C}_{DR}(\mathfrak{F}_N)$ the DG-category $\mathcal{C}_{DR}$ with $\mathrm{Ob}(\mathcal{C}_{DR}) = \mathfrak{F}_N$. As mentioned above, this theorem is motivated by the case where $\mathbb{R}^2 \simeq T^*\mathbb{R}$ is replaced by a two-torus and its higher dimensional generalizations. Our choice of this DG category $\mathcal{C}_{DR}(\mathfrak{F}_N)$ then comes from the DG category of holomorphic vector bundles on a noncommutative torus with the noncommutativity set to be zero, which gives (an equivalent but) different description from the usual commutative torus setting (cf. [25, 23]). For our purposes, this noncommutative tori setting fits better even in discussing commutative tori. In particular, we identify $\Omega(M)$, $M = \mathbb{R}$, with the space of *rapidly decreasing* smooth differential forms; for instance, $\Omega^0(\mathbb{R})$ is the space $\mathcal{S}(\mathbb{R})$ of Schwartz functions instead of $C^\infty(M)$.

The main theorem (Theorem 3.2) of this paper is to show the existence of an A_∞-structure on the category $Fuk(\mathbb{R}^2, \mathfrak{F}_N)$ of lines in $\mathbb{R}^2$ whose transversal A_∞-products are those associated with CC-polygons as in Figure 1 and which is homotopy equivalent to the DG-category $\mathcal{C}_{DR}(\mathfrak{F}_N)$. We also present explicitly one such A_∞-category, which we denote $\mathcal{C}(\mathfrak{F}_N)$. As stated previously, the rough idea to obtain $\mathcal{C}(\mathfrak{F}_N)$ is to apply HPT to $\mathcal{C}_{DR}(\mathfrak{F}_N)$. In order to reproduce the area of the CC-polygon as a structure constant of an A_∞-product of $\mathcal{C}(\mathfrak{F}_N)$, the contracting homotopy h of SDR in HPT should be of the type in the identity (1.1). However, unfortunately, the h in the identity (1.1) is the chain homotopy between $\mathbf{I}$ and $\mathbf{P}$, which map $\Omega(M)$ to not $\Omega(M)$ itself but to $D'(M)$, since the DeRham currents $[U_p]$ are *not smooth* differential forms. Thus, we need some modification of the story. One natural way may be to modify h as h_ϵ with a parameter ϵ such that $d_{DR}h_\epsilon + h_\epsilon d_{DR} = \mathrm{Id} - P_\epsilon$ holds on $\Omega(M)$ if $\epsilon \neq 0$ and $\lim_{\epsilon\to 0} h_\epsilon = h$. Then, we may apply HPT with contracting homotopy h_ϵ, $\epsilon \neq 0$, construct the induced A_∞-products, and finally take the limit $\epsilon \to 0$.

One such modification h_ϵ is discussed in [21], but the strategy in the present paper is instead to define a suitable subcomplex of $D'(\mathbb{R})$. Though $D'(\mathbb{R})$ can not be equipped with a product structure, we can introduce a product structure [2] in the subcomplex and apply HPT directly to the subcomplex. More precisely, we introduce a DG-category $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ as the smallest DG-category with the same objects $\mathrm{Ob}(\mathcal{C}'_{DR}(\mathfrak{F}_N)) = \mathrm{Ob}(\mathcal{C}_{DR}(\mathfrak{F}_N)) = \mathfrak{F}_N$ so that, for any $a \neq b \in \mathfrak{F}_N$, $\mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a,b)$ includes $[U_{p_{ab}}] \subset D'(\mathbb{R})$ for any critical point $p_{ab} = x(v_{ab})$ of $f_{ab} = f_a - f_b$ and is closed with respect to the operation h. Note that the latter requirement enables us to apply HPT directly to $\mathcal{C}'_{DR}(\mathfrak{F}_N)$. This $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ is in fact homotopy equivalent to the original DG-category $\mathcal{C}_{DR}(\mathfrak{F}_N)$ as shown in subsetion 5.1.

In our case, $M = \mathbb{R}$ and the graph df_a of f_a is a line L_a in $\mathbb{R}^2 \simeq T^*\mathbb{R}$ for any $a \in \mathfrak{F}_N$. Thus, for any $a \neq b \in \mathfrak{F}_N$, the intersection point v_{ab} of L_a and L_b is only one and so is the critical point $p_{ab} = x(v_{ab})$ of f_{ab}. Then, $[U_{p_{ab}}]$ will be the gaussian $e^{f_{ab}} \in \Omega^0(\mathbb{R}) \subset D'^0(\mathbb{R})$ whose support is $\mathbb{R}$ itself (but multiplied by $e^{f_{ab}}$ due to our choice of the differential d_{ab}) if the Hessian of $-f_{ab}$ is positive and the delta function one-form $\delta_{v_{ab}} \in D'^1(\mathbb{R})$ with support p_{ab} if the Hessian of $-f_{ab}$ is negative. In order for $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ to be closed with respect to the composition of morphisms, for any $a \neq b \in \mathfrak{F}_N$ we need to include $\delta_{v_{cd}} \in \mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a,b)$ for any $c \neq d \in \mathfrak{F}_N$. The operation of h on $\delta_{v_{cd}}$ will then produce step functions $\vartheta_{v_{cd}}$. Consequently, it turns out that the DG category $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ is generated by step functions and delta function one forms. The contracting homotopy h_{ab} of the type in (1.1) gives a desirable idempotent $P_{ab} : \mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a,b) \to \mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a,b)$ such that $P_{ab}\mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a,b) = \mathbb{R}\cdot[U_{p_{ab}}] \simeq \mathbb{R}\cdot[v_{ab}]$ for $a \neq b \in \mathfrak{F}_N$. Here, the corresponding SDR gives a Hodge decomposition of $\mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a,b)$. However, as we will mention also in the final section, there is no natural choice of the Hodge decomposition, *i.e.*, contracting homotopy h_{ab} if $a = b$. Therefore, we set $h_{aa} = 0$. Consequently, the space $\mathrm{Hom}_{\mathcal{C}(\mathfrak{F}_N)}(a,a)$ will be a commutative DG algebra (denoted by $A_S(\mathbb{R})$) which is also generated by the step functions and the delta function one-forms.

This paper is organized as follows. After recalling terminologies for A_∞-categories and HPT in section 2, we present the main theorem (Theorem 3.2) in subsection 3.1. Before proving it in section 5, we present the A_∞-category $\mathcal{C}(\mathfrak{F}_N)$ explicitly in subsection 3.3. To define the A_∞-category $\mathcal{C}(\mathfrak{F}_N)$, we introduce the commutative DG algebra $A_S(\mathbb{R})$, which is prepared in subsection 3.2. Section 4 is devoted to presenting geometric interpretations of some basic properties of the transversal part of the Fukaya A_∞-category $\mathcal{C}(\mathfrak{F}_N)$ in some examples. Thus, the contents in section 4 may essentially be known to experts. In subsection 4.1, we observe that a transversal A_∞-product can be nonzero if and only if the corresponding lines form a CC-polygon as in Figure 1. In subsection 4.2, we see an A_∞-constraint for transversal A_∞-products consists of only two terms which correspond to the ways to divide a clockwise polygon with one nonconvex

[2] The product structure we shall introduce is also motivated by one such modification h_ϵ which is however different from the one discussed in [21]. We hope to discuss the limit $\epsilon \to 0$ in this approach elswhere.

vertex into two. We also include a reason why we can not avoid non-transversal A_∞-products in subsection 4.3. Then, in section 5 we prove the main theorem (Theorem 3.2). In subsection 5.1, we introduce the DG-category $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ and prove Theorem 3.2 assuming a proposition (Proposition 5.4). Then, in subsection 5.2 we prove Proposition 5.4, where we derive the A_∞-category $\mathcal{C}(\mathfrak{F}_N)$ by applying HPT to $\mathcal{C}'_{DR}(\mathfrak{F}_N)$. Several examples of the explicit calculations of the derived A_∞-products are also given there. Since we consider the case $M=\mathbb{R}$, the trees of gradient flows in M in the sense of $Ms(M)$ are degenerate to be intervals and points on them. On the other hand, the HPT suggests the use of planar trees which are useful to determine the signs of the A_∞-products, too. Thus, in those calculations, we introduce planar trees associated to CC-polygons which are lifts of the trees of gradient flows in $M=\mathbb{R}$ to $T^*\mathbb{R}$. Finally, we end with mentioning applications of the main theorem to the case of tori, etc., in section 6.

Throughout this paper, by (graded) vector spaces we indicate those over fields $k=\mathbb{R}$. Though motivated strongly by the background stated above, the body of this paper can be read independently.

Acknowledgments : First of all, I would like to thank Jim Stasheff for his continuous encouragement and valuable discussions. I remember communications with him started when I discussed some application of HPT to open string field theory; the present work is thought of a topological string analog. As for discussions related to the background of this work, I would also like to thank M. Akaho, T. Kondo and Y. Terashima. K. Saito and A. Takahashi pointed out many faults in the formulation of an earlier stage, which helped to arrive at the present formulation. I am grateful to K. Fukaya who called my attention to various issues about transversality. It is needless to say that the present work is motivated greatly by what I have learnt from him. This work was completed during my visit in IHES where the environment for research is excellent and I would like to thank all researchers and staff there.

2. A_∞-categories and their homotopical properties

2.1. A_∞-algebras.

Definition 2.1 (A_∞-algebra (strong homotopy associative algebra) [26, 27]). An A_∞-algebra $(V,\mathfrak{m})$ consists of a $\mathbb{Z}$-graded vector space V with a collection of multilinear maps $\mathfrak{m}:=\{m_n : V^{\otimes n}\to V\}_{n\ge 1}$ of degree $(2-n)$ satisfying

$$0=\sum_{k+l=n+1}\sum_{j=0}^{k-1}(-1)^{\sigma}\ m_k(w_1,\dots,w_j,m_l(w_{j+1},\dots,w_{j+l}),w_{j+l+1},\dots,w_n)\ ,\qquad n\ge 1 \qquad (2.1)$$

for homogeneous elements $w_i\in V$, $i=1,\dots,n$, with degree $|w_i|\in\mathbb{Z}$, where $\sigma=(j+1)(l+1)+l(|w_1|+\cdots+|w_j|)$.

That the multilinear map m_k has degree $(2-k)$ indicates the degree of $m_k(w_1,\dots,w_k)$ is $|w_1|+\cdots+|w_k|+(2-k)$.

For $m_1=d$, $m_2=\cdot$, the first three relations of the above A_∞-condition are:

$$\begin{aligned}&d^2=0\ ,\\ &d(w\cdot w')=d(w)\cdot w'+(-1)^{|w|}w\cdot d(w')\ ,\\ &(w\cdot w')\cdot w''-w\cdot(w'\cdot w'')=d(m_3)(w,w',w''),\\ &\qquad d(m_3):=dm_3+m_3(d\otimes\mathbf{1}\otimes\mathbf{1}+\mathbf{1}\otimes d\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{1}\otimes d)\end{aligned}$$

for homogeneous elements $w,w',w''\in V$. The first identity implies that (V,d) defines a complex. The second identity implies that the differential d satisfies the Leibniz rule with respect to the product $\cdot$.

The third identity implies that the product $\cdot$ is associative up to homotopy. In particular, the product $\cdot$ is strictly associative if m_3=0.

Definition 2.2. An A_∞-algebra $(V, \mathfrak{m})$ with vanishing higher products $m_3 = m_4 = \cdots = 0$ is called a *differential graded algebra (DGA)*.

There exists a different definition of A_∞-algebras via a shift in degree.

Definition 2.3. An A_∞-algebra $(\mathcal{H}, \mathfrak{m})$ consists of a $\mathbb{Z}$-graded vector space $\mathcal{H}$ with a collection of *degree one* multilinear maps $\mathfrak{m} := \{m_n : \mathcal{H}^{\otimes n} \to \mathcal{H}\}_{n\geq 1}$ satisfying

$$0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^{|o_1|+\cdots+|o_j|}\, m_k(o_1, \ldots, o_j, m_l(o_{j+1}, \ldots, o_{j+l}), o_{j+l+1}, \ldots, o_n)\ .$$

These two definitions of A_∞-algebras are in fact equivalent. They are related by a degree shifting operator

$$s : V^r \to (V[1])^{r-1} =: \mathcal{H}^{r-1}$$

called the *suspension*. The direct relation between multilinear maps in these two definitions is given [7] by

$$m_n^{\mathcal{H}} = (-1)^{\sum_{i=1}^{n-1}(n-i)} s m_n^V((s^{-1})^{\otimes n})$$

or more explicitly:

$$m_n^{\mathcal{H}}(o_1, \ldots, o_n) = (-1)^{\sum_{i=1}^{n-1}(n-i)|o_i|}\ s m_n^V(s^{-1}(o_1), \ldots, s^{-1}(o_n)), \tag{2.2}$$

where we denoted the multilinear maps of $(V, \mathfrak{m})$ and that of $(\mathcal{H}, \mathfrak{m})$ by $\mathfrak{m}^V$ and $\mathfrak{m}^{\mathcal{H}}$, respectively.

The original definition in Definition 2.1 is natural in the sense that the differential m_1 has degree one, the product m_2 preserves the degree and then m_n, $n \geq 3$, are the higher homotopies. However, one can see that Definition 2.3 is simpler in sign.

Definition 2.4 (A_∞-morphism)**.** Given two A_∞-algebras $(\mathcal{H}, \mathfrak{m})$ and $(\mathcal{H}', \mathfrak{m}')$, a collection of degree preserving (= degree zero) multilinear maps $\mathcal{G} := \{g_k : \mathcal{H}^{\otimes k} \to \mathcal{H}'\}_{k\geq 1}$, is called an A_∞-morphism $\mathcal{G} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}', \mathfrak{m}')$ if and only if the following relations hold:

$$\sum_i \sum_{k_1+\cdots+k_n=n} m_i(g_{k_1} \otimes \cdots \otimes g_{k_i}) = \sum_{i+1+j=k} \sum_{i+l+j=n} g_k(\mathbf{1}^{\otimes i} \otimes m_l \otimes \mathbf{1}^{\otimes j}) \tag{2.3}$$

for $n = 1, 2, \ldots$.

The above relation for $n = 1$ implies that $g_1 : \mathcal{H} \to \mathcal{H}'$ forms a chain map $g_1 : (\mathcal{H}, m_1) \to (\mathcal{H}', m_1')$.

Definition 2.5. An A_∞-morphism $\mathcal{G} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}', \mathfrak{m}')$ is called an A_∞*-quasi-isomorphism* if and only if $g_1 : (\mathcal{H}, m_1) \to (\mathcal{H}', m_1')$ induces an isomorphism between the cohomologies of these two complexes. In this situation, we say $(\mathcal{H}, \mathfrak{m})$ *is homotopy equivalent to* $(\mathcal{H}', \mathfrak{m}')$ and call the A_∞-quasi-isomorphism $\mathcal{G} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}', \mathfrak{m}')$ *homotopy equivalence.*

It is known that there exists an inverse A_∞-quasi-isomorphism $\mathcal{G}' : (\mathcal{H}', \mathfrak{m}') \to (\mathcal{H}, \mathfrak{m})$ for a given A_∞-quasi-isomorphism $\mathcal{G} : (\mathcal{H}, \mathfrak{m}) \to (\mathcal{H}', \mathfrak{m}')$ and the notion of A_∞-quasi-isomorphisms in fact defines a homotopy equivalence relation between A_∞-algebras (see [18] and reference therein).

2.2. **Homological perturbation theory for A_∞-structures.** A version of homological perturbation theory we shall employ is as follows.

Theorem 2.6. *For an A_∞-algebra $(\mathcal{H}, \mathfrak{m})$, suppose given linear maps $h : \mathcal{H}^r \to \mathcal{H}^{r-1}$ and $P : \mathcal{H}^r \to \mathcal{H}^r$ satisfying*

$$dh + hd = \mathrm{Id}_{\mathcal{H}} - P, \quad P^2 = P, \qquad d := m_1 \tag{2.4}$$

on $\mathcal{H}$. Then, there exists a canonical way to construct an A_∞-structure $\mathfrak{m}'$ on $P\mathcal{H}$ such that $(P\mathcal{H}, \mathfrak{m}')$ is homotopy equivalent to the original A_∞-algebra $(\mathcal{H}, \mathfrak{m})$.

Note that if $dP = 0$, then eq.(2.4) gives a Hodge decomposition of the complex $(\mathcal{H}, d)$, where $P(\mathcal{H}) = H(\mathcal{H})$ gives the cohomology.

Proof. Let $\iota : \mathcal{H}' \to \mathcal{H}$ be the embedding and $\pi : \mathcal{H} \to \mathcal{H}'$ the projection, respectively, such that $\pi \circ \iota = \mathrm{Id}_{\mathcal{H}'}$ and $\iota \circ \pi = P$. Namely, ι is the embedding $\mathcal{H}' \simeq P(\mathcal{H}) \subset \mathcal{H}$. A collection of degree zero maps $\mathcal{G} = \{g_l : (\mathcal{H}')^{\otimes l} \to \mathcal{H}\}_{l \geq 1}$ is defined recursively with respect to k as

$$g_k = -h \sum_{i \geq 2} \sum_{1 \leq k_1 < k_2 \cdots < k_i = k} m_i(g_{k_1} \otimes g_{k_2 - k_1} \otimes \cdots \otimes g_{k - k_{i-1}}) \tag{2.5}$$

with $g_1 := \iota : \mathcal{H}' \to \mathcal{H}$ the inclusion. Then, $\mathfrak{m}' = \{m'_k : (\mathcal{H}')^{\otimes k} \to \mathcal{H}\}_{k \geq 1}$ is given recursively by

$$m'_k = \pi \sum_{i \geq 2} \sum_{1 \leq k_1 < k_2 \cdots < k_i = k} m'_i(g_{k_1} \otimes g_{k_2 - k_1} \otimes \cdots \otimes g_{k - k_{i-1}}) \ . \tag{2.6}$$

Note that $(m_1)^2 = \pi \circ d \circ \iota \circ \pi \circ d \circ \iota = \pi \circ d \circ P \circ d \circ \iota = 0$ since d commutes with P due to the condition (2.4). One can check that these actually give an A_∞-structure and an A_∞-quasi-isomorphism (see [18]). $\square$

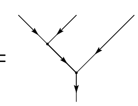
Equivalently, $\mathfrak{m}'$ are described in terms of rooted planar trees as follows.

A *planar tree* (a simply connected planar graph without loops) consists of vertices, internal edges and external edges. An internal edge has two distinct vertices at its ends. An external edge has one end on a vertex and another end is free. The number of incident edges at a vertex is greater than two. The term 'planar' means the cyclic order of edges at each vertex is distinguished. A *rooted planar tree* is a planar tree graph with one of its external edges distinguished from others as a root edge. The remaining external edges are called the *leaves*. Each edge of a planar rooted tree has a unique orientation so that the orientations form a flow from the leaves to the root edge. We sometimes describe the orientation as an arrow. We call a vertex at which the number of incident edges is $(k+1)$ a *k-vertex*.

We call a rooted planar tree having k leaves a *k-tree*. The set of (the isomorphism classes of) k-trees is denoted by G_k, $k \geq 2$.

For any element $\Gamma_n \in G_n$, $n \geq 2$, let us define $m'_{\Gamma_n} : (\mathcal{H}')^{\otimes n} \to \mathcal{H}'$ by attaching $\iota : \mathcal{H}' \to \mathcal{H}$ to each leaf, $m_k : \mathcal{H}^{\otimes k} \to \mathcal{H}$ to each k-vertex, $-h : \mathcal{H} \to \mathcal{H}$ to each internal edge, $\pi : \mathcal{H} \to \mathcal{H}'$ to the root edge and then composing them. For example,

$$\begin{aligned} &m'_{\Gamma_3}(o'_1, o'_2, o'_3) \\ &\quad = \pi m_2(-h m_2(\iota(o'_1), \iota(o'_2)), \iota(o'_3)) \end{aligned} \quad , \qquad \Gamma_3 =$$



for $o'_1, o'_2, o'_3 \in \mathcal{H}'$. Then, $\{m'_n\}_{n \geq 1}$ is given by $m'_1 = \pi \circ m_1 \circ \iota$ and

$$m'_n = \sum_{\Gamma_n \in G_n} m'_{\Gamma_n} \tag{2.7}$$

for $n \geq 2$. Thus, m'_n is described as the sum of the value m'_{Γ_n} over all the n-trees $\Gamma_n \in G_n$. Similarly, $\{g_n\}_{n \geq 1}$ is given by $g_1 = \iota$ and $g_n = \sum_{\Gamma_n \in G_n} g_{\Gamma_n}$ for $n \geq 2$, where $g_{\Gamma_n} : (\mathcal{H}')^{\otimes n} \to \mathcal{H}$ is obtained by replacing π by $-h$ in the definition of m'_{Γ_n}.

Remark 2.7. The data $(\, P\mathcal{H} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \mathcal{H}\,, h)$ used in the proof above is often called a *strong deformation retract (SDR)* of the complex $(\mathcal{H}, d)$, the starting point of the traditional HPT (for instance [10, 8, 9, 12]). There, it is discussed that the A_∞-quasi-isomorphism (2.5) induces homotopy equivalence of the induced A_∞-products (2.6) with the original one $\mathfrak{m}$, mainly in the case $(\mathcal{H}, \mathfrak{m})$ is a DGA. The extension to the case when $(\mathcal{H}, \mathfrak{m})$ is a general A_∞-algebra is not difficult. The present form of HPT (Theorem 2.6) is due to [21], where the above planar tree expression of the recursive formula eq.(2.5) and (2.6) is also presented.

2.3. A_∞**-categories.** We need the categorical version of these terminologies.

Definition 2.8 (A_∞-category [2]). An A_∞-*category* $\mathcal{C}$ consists of the set of objects $\mathrm{Ob}(\mathcal{C}) = \{a, b, \dots\}$, $\mathbb{Z}$-graded vector space $V_{ab} := \mathrm{Hom}_{\mathcal{C}}(a, b)$ for each two objects $a, b \in \mathrm{Ob}(\mathcal{C})$ and a collection of multilinear maps

$$\mathfrak{m} := \{m_n : V_{a_1a_2} \otimes \cdots \otimes V_{a_na_{n+1}} \to V_{a_1a_{n+1}}\}_{n\geq 1}$$

of degree $(2-n)$ defining the A_∞-structure, that is, $\mathfrak{m}$ satisfies the A_∞-relations (2.1).

In particular, an A_∞-category $\mathcal{C}$ with vanishing higher products $m_3 = m_4 = \cdots = 0$ is called a *DG category.*

The suspension $s(\mathcal{C})$ of an A_∞-category $\mathcal{C}$ is defined by the shift

$$s : \mathrm{Hom}_{\mathcal{C}}(a, b) \to s(\mathrm{Hom}_{\mathcal{C}}(a, b)) =: \mathrm{Hom}_{s(\mathcal{C})}(a, b)$$

for any $a, b \in \mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(s(\mathcal{C}))$, where the degree $|m_n|$ of the A_∞-products becomes one for all $n \geq 1$ as in the case of A_∞-algebras. We sometimes denote $\mathrm{Hom}_{s(\mathcal{C})}(a, b) = \mathcal{H}_{ab}$ as we do $\mathrm{Hom}_{\mathcal{C}}(a, b) = V_{ab}$.

Definition 2.9 (A_∞-functor). Given two A_∞-categories $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{G} := \{g, g_1, g_2, \dots\} : s(\mathcal{C}) \to s(\mathcal{C}')$ is called an A_∞-*functor* if and only if $g : \mathrm{Ob}(s(\mathcal{C})) \to \mathrm{Ob}(s(\mathcal{C}'))$ is a map of object and

$$g_k : \mathrm{Hom}_{s(\mathcal{C})}(a_1, a_2) \otimes \cdots \otimes \mathrm{Hom}_{s(\mathcal{C})}(a_k, a_{k+1}) \to \mathrm{Hom}_{s(\mathcal{C}')}(g(a_1), g(a_{k+1})), \quad k \geq 1$$

are degree preserving multilinear maps satisfying the defining relations of an A_∞-morphism (2.3).

In particular, if $g : \mathrm{Ob}(s(\mathcal{C})) \to \mathrm{Ob}(s(\mathcal{C}'))$ and $g_1 : \mathrm{Hom}_{s(\mathcal{C})}(a, b) \to \mathrm{Hom}_{s(\mathcal{C}')}(f(a), f(b))$ induces an isomorphism between the cohomologies for any $a, b \in \mathrm{Ob}(s(\mathcal{C}))$, we call the A_∞-functor *homotopy equivalence.*

The generalization of HPT for A_∞-algebras to A_∞-categories is straightforward [21].

Theorem 2.10. *For an A_∞-category $\mathcal{C}$, suppose given linear maps $h_{ab} : \mathcal{H}^r_{ab} \to \mathcal{H}^{r-1}_{ab}$ and $P_{ab} : \mathcal{H}^r_{ab} \to \mathcal{H}^r_{ab}$ satisfying*

$$d_{ab}h_{ab} + h_{ab}d_{ab} = \mathrm{Id}_{\mathcal{H}_{ab}} - P_{ab}, \quad (P_{ab})^2 = P_{ab}, \qquad d_{ab} := m_1 : \mathcal{H}_{ab} \to \mathcal{H}_{ab} \tag{2.8}$$

on $\mathcal{H}_{ab}$ for any $a, b \in \mathrm{Ob}(\mathcal{C})$. Then, there exists a canonical way to construct an A_∞-category $\mathcal{C}'$ which is homotopy equivalent to the original A_∞-category $\mathcal{C}$ and in particular the space of morphisms is defined by $\mathrm{Hom}_{s(\mathcal{C}')}(a, b) = \mathcal{H}'_{ab} = P_{ab}\mathcal{H}_{ab}$.

Proof. Let $\iota_{ab} : \mathcal{H}'_{ab} \to \mathcal{H}_{ab}$ be the embedding and $\pi_{ab} : \mathcal{H}_{ab} \to \mathcal{H}'_{ab}$ the projection such that $\pi_{ab} \circ \iota_{ab} = \mathrm{Id}_{\mathcal{H}'_{ab}}$ and $\iota_{ab} \circ \pi_{ab} = P_{ab}$. Then, for $a_1, \dots, a_{n+1} \in \mathrm{Ob}(\mathcal{C}')$, the A_∞-product $m'_n : \mathcal{H}'_{a_1a_2} \otimes \cdots \otimes \mathcal{H}'_{a_na_{n+1}} \to \mathcal{H}'_{a_1a_{n+1}}$ is given by $m'_n = \sum_{\Gamma_n \in G_n} m'_{\Gamma_n}$, where m'_{Γ_n} is defined in the same way as the one for an A_∞-algebra, but we attach $\iota_{a_ia_{i+1}} : \mathcal{H}'_{a_ia_{i+1}} \to \mathcal{H}_{a_ia_{i+1}}$, $i = 1, \dots, n$, for each leaf (instead of ι), m_k to each k-vertex, h_{ab} to each internal edge, where $a, b \in \{a_1, \dots, a_{n+1}\}$ is uniquely determined by the graph Γ_n, and finally $\pi_{a_1a_{n+1}}$ to the root edge of Γ_n (instead of π). The construction of homotopy equivalence is also parallel to the case of A_∞-algebras, though we do not use it in the present paper. □

3. A_∞-CATEGORY OF LINES IN A PLANE

3.1. **The main theorem.** For a fixed integer $N \geq 2$, let $\{f_1, \dots, f_N\}$ be a set of polynomial functions on $\mathbb{R}$ of degree equal or less than two. For each $a \in \{1, \dots, N\}$, $y = df_a/dx$ is a line L_a in $\mathbb{R}^2$ with coordinates (x, y) described as

$$L_a : y = t_ax + s_a, \qquad t_a, s_a \in \mathbb{R}.$$

Let us consider such a collection $\{f_1, \dots, f_N\}$ satisfying the following two conditions:

(i) For any $a \neq b = 1, \dots, N$, the slopes of the lines L_a and L_b are not the same: $t_a \neq t_b$.

(ii) More than two lines do not intersect at the same point in $\mathbb{R}^2$.

We identify the set $\{f_a \mid a = 1, \dots, N\}$ with the label set $\{a|a = 1, \dots, N\}$. Then, denote by $\mathfrak{F}_N := \{a|a = 1, \dots, N\}$ such a set satisfying the above conditions (i) and (ii).

We shall construct a Fukaya A_∞-category $\mathcal{C}(\mathfrak{F}_N)$ with $\mathrm{Ob}(\mathcal{C}(\mathfrak{F}_N)) = \mathfrak{F}_N$ from another A_∞-category, in particular, a DG category $\mathcal{C}_{DR}(\mathfrak{F}_N)$. Let $\Omega(\mathbb{R}) := \Omega^0(\mathbb{R}) \oplus \Omega^1(\mathbb{R})$ be the graded vector space defined by $\Omega^0(\mathbb{R}) := \mathcal{S}(\mathbb{R})$, the space of Schwartz functions, and $\Omega^1(\mathbb{R}) := \mathcal{S}(\mathbb{R}) \cdot dx$, where dx is the base of one-form on $\mathbb{R}$.

Definition 3.1 ($\mathcal{C}_{DR}(\mathfrak{F}_N)$)**.** The DG category $\mathcal{C}_{DR}(\mathfrak{F}_N)$ consists of the set of objects $\mathrm{Ob}(\mathcal{C}_{DR}(\mathbb{R})) = \mathfrak{F}_N$ and the space of morphisms $\Omega_{ab} := \mathrm{Hom}_{\mathcal{C}_{DR}(\mathfrak{F}_N)}(a, b) = \Omega(\mathbb{R})$ for each $a, b \in \mathfrak{F}_N$, where we set

- the differential $d_{ab} : \Omega^0_{ab} \to \Omega^1_{ab}$ by $d_{ab} := d - df_{ab}\wedge$, where $d = dx \cdot (d/dx)$ is the exterior derivative and $f_{ab} := f_a - f_b$;
- the product $m : \Omega^{r_{ab}}_{ab} \otimes \Omega^{r_{bc}}_{bc} \to \Omega^{r_{ab}+r_{bc}}_{ac}$ by the usual wedge product.

It is clear that $\mathcal{C}_{DR}(\mathfrak{F})$ forms a DG category.

The following is the main theorem of this paper.

Theorem 3.2. *There exists an A_∞-category $\mathcal{C}(\mathfrak{F}_N)$ with $\mathrm{Ob}(\mathfrak{F}_N) = \mathfrak{F}_N$ such that*

(i) *For two objects $a \neq b \in \mathfrak{F}_N$, the space $\mathrm{Hom}_{\mathcal{C}(\mathfrak{F}_N)}(a, b) =: V_{ab}$ of morphisms is the following graded vector space of degrees zero and one*:

$$\begin{array}{lll} V^0_{ab} = \mathbb{R} \cdot [v_{ab}], & V^1_{ab} = 0, & t_a < t_b, \\ V^0_{ab} = 0, & V^1_{ab} = \mathbb{R} \cdot [v_{ab}], & t_a > t_b. \end{array}$$

Here, $[v_{ab}]$ are the bases of the vector spaces attached to the intersection points $v_{ab} (= v_{ba})$ of L_a and L_b.

(ii) *Let $a_1, \dots, a_{k+1} \in \mathfrak{F}_N$, $k \geq 1$, be objects such that $a_i \neq a_j$ for any $i \neq j \in \{1, \dots, k+1\}$ and $\vec{v} := (v_{a_1}, \dots, v_{a_k a_{k+1}}, v_{a_{k+1} a_1})$. Then, for $k = 1$, the differential $m_1 : V_{a_1 a_2} \to V_{a_1 a_2}$ is zero, $m_1 = 0$. For $k \geq 2$, the structure constant $c(\vec{v}) \in \mathbb{R}$ for the higher A_∞-product*

$$m_k([v_{a_1 a_2}], \dots, [v_{a_k a_{k+1}}]) = c(\vec{v}) \cdot [v_{a_1 a_{k+1}}]$$

is zero if $\vec{v}$ does not form a clockwise convex polygon (see also Definition 3.6 for the definition of clockwise convex polygon), and if $\vec{v}$ forms a clockwise convex polygon, it is given by $c(\vec{v}) = \pm e^{-Area(\vec{v})}$, with an appropriate sign $\pm$, where $Area(\vec{v})$ is the area of the clockwise convex polygon.

(iii) *$\mathcal{C}(\mathfrak{F}_N)$ is homotopy equivalent to $\mathcal{C}_{DR}(\mathfrak{F}_N)$.*

Conditions (i) and (ii) are the ones for $\mathcal{C}(\mathfrak{F}_N)$ to be a Fukaya category. We call a multilinear map m_k, $k \geq 2$, of the type in Condition (ii) a *transversal (higher) A_∞-product.* Multilinear maps m_k of the other type are then called *non-transversal A_∞-products.* Condition (iii) is motivated by homological mirror symmetry (HMS)[20] of (non)commutative complex tori. As discussed in [21], this homotopy equivalence should be the key idea of HMS for tori or more general cases, where both $\mathcal{C}(\mathfrak{F}_N)$ and $\mathcal{C}_{DR}(\mathfrak{F}_N)$ are A_∞-categories associated to a symplectic structure, but $\mathcal{C}_{DR}(\mathfrak{F}_N)$ is canonically isomorphic to a DG category associated to the mirror dual complex structure. In fact, the relation of this DG category $\mathcal{C}_{DR}(\mathfrak{F}_N)$ with the DG category of holomorphic vector bundles on a noncommutative complex torus [24, 15] with noncommutativity set to be zero is clear. For the precise relation of the *noncommutative* complex torus description and the usual complex torus description, for instance see [17].

3.2. **Commutative DG algebra $A_S(F)$.** We shall give a sketch of the proof of Theorem 3.2 in section 5. Before that, we present an A_∞-category, which hereafter we denote by $\mathcal{C}(\mathfrak{F}_N)$, shown to satisfy Conditions (i), (ii) and (iii) in the next subsection.

In order to construct an A_∞-structure including non-transversal A_∞-products, we introduce a (commutative) DG algebra $A_S(F)$ over $k=\mathbb{R}$. This notion is motivated by an extension of a subalgebra F of the commutative DG algebra of smooth differential forms on $\mathbb{R}$ by including step functions and delta-function one forms.

Definition 3.3 (Commutative DG algebra $A_S(F)$)**.** Let $F=F^0\oplus F^1$ be a commutative DG subalgebra of the commutative DG algebra of smooth differential forms on $\mathbb{R}$, and S be a finite set with a map $x:S\to\mathbb{R}$. For each $v\in S$, we introduce degree zero base ϑ_v and degree one base $\delta_v:=d(\vartheta_v)$ with (degree zero) unit $\mathbf{1}$: $\mathbf{1}\cdot\vartheta_v=\vartheta_v\cdot\mathbf{1}=\vartheta_v$, $\mathbf{1}\cdot\delta_v=\delta_v\cdot\mathbf{1}=\delta_v$. Consider the commutative algebra $\tilde{A}_S(F):=F\otimes\langle\mathbf{1},\vartheta_v,\delta_v\mid v\in S\rangle$ of degrees zero and one, and relations defined as follows:

$$\vartheta_v\vartheta_{v'}=\vartheta_{v'},\quad \delta_v\vartheta_{v'}=0$$

for any $v,v'\in S$ such that $x(v)<x(v')$,

$$\vartheta_v=\vartheta_{v'},\quad \delta_v=\delta_{v'}$$

for any $v,v'\in S$ such that $x(v)=x(v')$,

$$\alpha\cdot\delta_v=\alpha(x(v))\cdot\delta_v,\qquad \alpha(x(v))\in k=\mathbb{R},$$

for any $v\in S$,

$$\alpha\cdot\vartheta_v=0,\quad \beta\cdot\vartheta_v=0,\qquad \alpha\in F^0,\ \ \beta\in F^1$$

for any $v\in S$ if $\alpha(x)=0$ or $\beta(x)$=0 for any $x\geq x(v)$,

$$\alpha\cdot(\mathbf{1}-\vartheta_v)=0,\quad \beta\cdot(\mathbf{1}-\vartheta_v)=0,\qquad \alpha\in F^0,\ \ \beta\in F^1$$

for any $v\in S$ if $\alpha(x)=0$ or $\beta(x)=0$ for any $x\leq x(v)$, $F^1\cdot\delta_v=0$ for any $v\in S$ and $\delta_v\cdot\delta_{v'}=0$ for any $v,v'\in S$. More explicitly, the graded vector space $\tilde{A}^r_S(F)$, $r=0,1$, is

$$\tilde{A}^0_S(F):=F^0\otimes\langle\mathbf{1},\vartheta_v|v\in S\rangle,\qquad \tilde{A}^1_S(F):=F^1\otimes\langle\mathbf{1},\vartheta_v|v\in S\rangle\oplus\oplus_{v\in S}\tilde{A}^0_S(F)\otimes\delta_v.$$

By the commutativity and the relations above, any element is described as

$$\alpha=\alpha_0+\sum_{v\in S,n\in\mathbb{Z}_{>0}}\alpha_{v,n}(\vartheta_v)^n,\qquad \alpha_0,\ \alpha_{v,n}\in F^0$$

for $\alpha\in\tilde{A}^0_S(F)$ and

$$\beta=\beta_0+\sum_{v\in S,n\in\mathbb{Z}_{>0}}\beta_{v,n}(\vartheta_v)^n+\sum_{v\in S,n\in\mathbb{Z}_{>0}}c_{v,n}(\vartheta_v)^{n-1}\cdot\delta_v,\qquad \beta_0,\ \beta_{v,n}\in F^1,\quad c_{v,n}\in k=\mathbb{R}$$

for $\beta\in\tilde{A}^1_S(F)$. The differential $d:\tilde{A}^0_S(F)\to\tilde{A}^1_S(F)$ is defined by extending the differential $d:F^0\to F^1$ with $d(\vartheta_v)=\delta_v$, $v\in S$, so that they satisfy the Leibniz rule with respect to the commutative product.

In this paper, we shall consider the two cases $F=\Omega(\mathbb{R})$ and $F=F^0=\mathbb{R}$. For $F=\Omega(\mathbb{R})$, we set $A_S(\Omega(\mathbb{R})):=\tilde{A}_S(\Omega(\mathbb{R}))$. For $F=\mathbb{R}$ (note that the differential on F is trivial), we set $A_S(\mathbb{R})$ as a commutative DG subalgebra of $\tilde{A}_S(F)$ as follows:

$$A^0_S(\mathbb{R}):=\left\{\alpha_0+\sum_{v\in S,n\in\mathbb{Z}_{>0}}\alpha_{v,n}(\vartheta_v)^n\in\tilde{A}_S(\mathbb{R})\,\middle|\,\alpha_0=0,\sum_{v\in S,n\in\mathbb{Z}_{>0}}\alpha_{v,n}=0\right\},$$

$$A^1_S(\mathbb{R}):=\tilde{A}^1_S(\mathbb{R})=\left\{\sum_{v\in S,n\in\mathbb{Z}_{>0}}c_{v,n}(\vartheta_v)^{n-1}\cdot\delta_v,\quad c_{v,n}\in k=\mathbb{R}\right\}.$$

By the map $x : S \to \mathbb{R}$, S is identified with the set of finitely many points on $\mathbb{R}$. Then, δ_v can be regarded as the delta function on $\mathbb{R}$ with support at $x(v)$ and ϑ_v is the step function whose value is zero at $x \to -\infty$, one at $x \to \infty$ and which is discontinuous at $x(v)$. In the case $F = \mathbb{R}$, taking the subspace $A^0_S(\mathbb{R}) \subset \tilde{A}^0_S(\mathbb{R})$ implies that we concentrate on constant functions α which are discontinuous at some points in $x(S)$ and further $\alpha = 0$ at $x \to \pm\infty$. Note that we do not impose the relation $(\vartheta_v)^2 = \vartheta_v$. Thus, for any element in $A^0_S(\mathbb{R})$, any discontinuous point $x(v)$ is associated with $\mathbb{Z}_{>0}$-valued weight corresponding to the power of ϑ_v. The commutative DGA $A_S(\mathbb{R})$ is useful in the sense that it is defined only in terms of finitely many points on $\mathbb{R}$, though $A_S(\mathbb{R})$ is infinite dimensional as a vector space.

For the construction of the A_∞-category $\mathcal{C}(\mathfrak{F}_N)$ we need only $A_S(\mathbb{R})$. The cohomology of $A_S(\mathbb{R})$ is $H^0(A_S(\mathbb{R})) = 0$ and $H^1(A_S(\mathbb{R})) \simeq \mathbb{R}$ (one dimensional); a base of H^1 is δ_v for an element $v \in S$, but one has $\delta_{v'} - \delta_v = d(\vartheta_{v'} - \vartheta_v)$ for $v, v' \in S$.

At a first look the reader can skip the following lemma, which shall be employed as a key step of the proof of Theorem 3.2 in subsection 5.1.

Lemma 3.4. *There exist inclusions*

$$\iota : A_S(\mathbb{R}) \to A_S(\Omega(\mathbb{R})), \qquad \iota : \Omega(\mathbb{R}) \to A_S(\Omega(\mathbb{R})),$$

both of which induce homotopy equivalences as A_∞-algebras.

Proof. The existence of the inclusions ι is clear. Also, for each case, the ι forms a chain map with respect to the differentials on both sides, and also defines an algebra homomorphism. Then, for each case, by setting $g_1 := \iota$ and $g_2 = g_3 = \cdots = 0$, $\mathcal{G} := \{g_1, g_2, \dots\}$ forms an A_∞-morphism.

For $A_S(\mathbb{R})$, $\Omega(\mathbb{R})$ and $A_S(\Omega(\mathbb{R}))$, their cohomologies are isomorphic to each other: $H^0 = 0$ and $H^1 = \mathbb{R}$. In order to show that $g_1 := \iota$ induces isomorphism on the cohomologies, we need only see that the image of a representative of H^1 of $A_S(\mathbb{R})$ or $\Omega(\mathbb{R})$ is not exact in $A_S(\Omega(\mathbb{R}))$. It is clear that the image of $\delta_v \in A_S(\mathbb{R})$ and the image of $\beta_0 \in \Omega^1(\mathbb{R})$ such that $\int_{-\infty}^{\infty} \beta_0 \neq 0$ are not exact. □

3.3. **The A_∞-category $\mathcal{C}(\mathfrak{F}_N)$.** Let us define the A_∞-category $\mathcal{C}(\mathfrak{F}_N)$. First of all, for each $a \in \mathfrak{F}_N$ the graded vector space $V_{aa} = V^0_{aa} \oplus V^1_{aa}$ is set to be

$$V^r_{aa} = A^r_{S_a}(\mathbb{R}), \qquad r = 0, 1,$$

on which we set the differential $m_1 = d : V^0_{aa} \to V^1_{aa}$ and the product $m_2 : V_{aa} \otimes V_{aa} \to V_{aa}$ as those in $A_{S_a}(\mathbb{R})$.

For $a \neq b \in \mathfrak{F}_N$, the graded vector space V_{ab} is taken to be the one given in Theorem 3.2 (i), on which the differential $m_1 : V_{ab} \to V_{ab}$ is set to be zero. Then, next, let us define multilinear maps

$$m_k : V_{a_1 a_2} \otimes \cdots \otimes V_{a_k a_{k+1}} \to V_{a_1 a_{k+1}}$$

of degree $(2 - k)$ for $k \geq 2$. By degree counting, the following holds.

Lemma 3.5. *Any multilinear maps $m_k(w_1, \dots, w_k)$ can be nonzero only if there exists a nonzero element $w_{k+1} \in V_{a_{k+1} a_1}$ such that the number of degree zero elements in $\{w_1, \dots, w_{k+1}\}$ is two.* □

We first define multilinear maps $m_k(w_1, \dots, w_k)$ on $\tilde{V}_{**}$, where $\tilde{V}_{ab} = V_{ab}$ for $a \neq b \in \mathfrak{F}_N$ and $\tilde{V}_{aa} = \tilde{A}_{S_a}(\mathbb{R})$ for $a \in \mathfrak{F}_N$. The multilinear maps on the $\mathbb{Z}$-graded vector spaces $\tilde{V}_{**}$ given below are closed in the $\mathbb{Z}$-graded subvector spaces V_{**} and thus the restriction of them onto V_{**} gives the multilinear maps on V_{**}.

We determine those multilinear maps on $\tilde{V}_{ab}$ separately in each case $\sharp(\vartheta) := \sharp\{1 \leq i \leq k | w_i \in V^0_{aa},\ a \in \mathfrak{F}_N\}$ is two, one, or zero.

• **The case $\sharp(\vartheta) = 2$:** By degree counting (Lemma 3.5), the multilinear map $m_k(w_1, \dots, w_k)$ can be nonzero only if $w_i \in \tilde{V}_{aa}$ for all $i = 1, \dots, k$ with some $a \in \mathfrak{F}_N$. We set $m_k(w_1, \dots, w_k)$ is nonzero only if it is of the form $m_2(w_1, w_2)$, $w_1, w_2 \in \tilde{V}^0_{aa}$, for some $a \in \mathfrak{F}_N$. This is the product m_2 in $\tilde{A}_{S_a}(\mathbb{R})$.

• **The case** $\sharp(\vartheta) = 1$: By degree counting (Lemma 3.5), all such A_∞-products to be nonzero are only of the following types: for any $a \neq b \in \mathfrak{F}_N$,

$$
\begin{array}{ll}
\mathrm{A} & m_* : (V^1_{aa})^{\otimes k_1} \otimes \tilde{V}^0_{aa} \otimes (V^1_{aa})^{\otimes k_2} \otimes V^r_{ab} \otimes (V^1_{bb})^{\otimes k_3} \to V^r_{ab}, \\
\mathrm{B} & m_* : (V^1_{aa})^{\otimes k_1} \otimes V^r_{ab} \otimes (V^1_{bb})^{\otimes k_2} \otimes \tilde{V}^0_{bb} \otimes (V^1_{bb})^{\otimes k_3} \to V^r_{ab}, \\
\mathrm{C}_1 & m_* : (V^1_{aa})^{\otimes k_1} \otimes V^r_{ab} \otimes (V^1_{bb})^{\otimes k_2} \otimes \tilde{V}^0_{bb} \otimes (V^1_{bb})^{\otimes k_3} \otimes V^{1-r}_{ba} \otimes (V^1_{aa})^{\otimes k_4} \to \tilde{V}^0_{aa}, \\
\mathrm{C}_2 & m_* : (V^1_{aa})^{\otimes k_1} \otimes \tilde{V}^0_{aa} \otimes (V^1_{aa})^{\otimes k_2} \otimes V^1_{ab} \otimes (V^1_{bb})^{\otimes k_3} \otimes V^0_{ba} \otimes (V^1_{aa})^{\otimes k_4} \to \tilde{V}^0_{aa}, \\
\mathrm{C}_3 & m_* : (V^1_{aa})^{\otimes k_1} \otimes V^0_{ab} \otimes (V^1_{bb})^{\otimes k_2} \otimes V^1_{ba} \otimes (V^1_{aa})^{\otimes k_3} \otimes \tilde{V}^0_{aa} \otimes (V^1_{aa})^{\otimes k_4} \to \tilde{V}^0_{aa},
\end{array}
$$

where $*$'s are the appropriate numbers. We set the A_∞-products of the following types to be zero; type A with $r = 0$ if $k_1 \neq 0$, type A with $r = 1$ if $k_2 \neq 0$, type B with $r = 0$ if $k_3 \neq 0$, type B with $r = 1$ if $k_2 \neq 0$, type C_1 with $r = 0$ if $k_3 \neq 0$, type C_1 with $r = 1$ if $k_2 \neq 0$, type C_2 and type C_3 if $k_2 \neq 0$.

We set the multilinear maps which do not include degree one elements in V^1_{aa} for any $a \in \mathfrak{F}_N$ as follows.

For type A, the product $m_2 : \tilde{V}^0_{aa} \otimes V^r_{ab} \to V^r_{ab}$, $a \neq b$, $r = 0, 1$, is given by

$$
m_2((\vartheta_{v_a})^n, [v_{ab}]) \begin{cases} [v_{ab}] & x(v_a) < x(v_{ab}) \\ \frac{1}{2^n}[v_{ab}] & v_a = v_{ab}, \quad t_a < t_b \\ \frac{1}{n+1}[v_{ab}] & v_a = v_{ab}, \quad t_a > t_b \\ 0 & x(v_{ab}) < x(v_a) \end{cases} \tag{3.1}
$$

for $n \geq 1$, where recall that the degree of $[v_{ab}]$ is zero for $t_a < t_b$ and one for $t_a > t_b$. In the same way, for type B, the product $V^r_{ab} \otimes \tilde{V}^0_{bb} \to V^r_{ab}$, $a \neq b$, $r = 0, 1$, is given by

$$
m_2([v_{ab}], (\vartheta_{v_b})^n) = \begin{cases} [v_{ab}] & x(v_b) < x(v_{ab}) \\ \frac{1}{2^n}[v_{ab}] & v_b = v_{ab}, \quad t_a < t_b \\ \frac{1}{n+1}[v_{ab}] & v_b = v_{ab}, \quad t_a > t_b \\ 0 & x(v_{ab}) < x(v_b) \end{cases} \tag{3.2}
$$

for $n \geq 1$. In addition, we set $m_2(\mathbf{1}_a, [v_{ab}]) = [v_{ab}]$ and $m_2([v_{ab}], \mathbf{1}_b) = [v_{ab}]$ for the identities $\mathbf{1}_a \in \tilde{V}^0_{aa}$ and $\mathbf{1}_b \in \tilde{V}^0_{bb}$.

For type C_1, C_2, C_3, $a \neq b \in \mathfrak{F}_N$ such that $t_a < t_b$,

$$
\begin{aligned}
m_3([v_{ba}], (\vartheta_{v_a})^n, [v_{ab}]) &= \frac{1}{n+1} \vartheta_{v_{ab}} (1 - (\vartheta_{v_{ab}})^n) \ \in \tilde{V}^0_{bb}, \\
m_3((\vartheta_{v_b})^n, [v_{ba}], [v_{ab}]) &= \frac{1}{n+1} \vartheta_{v_{ab}} (1 - (\vartheta_{v_{ab}})^n) \ \in \tilde{V}^0_{bb}, \\
m_3([v_{ab}], (\vartheta_{v_b})^n, [v_{ba}]) &= -\frac{1}{n+1} \vartheta_{v_{ab}} (1 - (\vartheta_{v_{ab}})^n) \ \in \tilde{V}^0_{aa}, \\
m_3([v_{ab}], [v_{ba}], (\vartheta_{v_a})^n) &= -\frac{1}{n+1} \vartheta_{v_{ab}} (1 - (\vartheta_{v_{ab}})^n) \ \in \tilde{V}^0_{aa}
\end{aligned}
$$

for $n \geq 1$ if $v_a = v_{ab}$ or $v_b = v_{ab}$, and they are equal to zero if $v_a \neq v_{ab}$ or $v_b \neq v_{ab}$. In addition, we set $m_3([v_{ba}], \mathbf{1}_a, [v_{ab}]) = m_3(\mathbf{1}_b, [v_{ba}], [v_{ab}]) = 0$ and $m_3([v_{ab}], \mathbf{1}_b, [v_{ba}]) = m_3([v_{ab}], [v_{ba}], \mathbf{1}_a) = 0$.

• **The case** $\sharp(\vartheta) = 0$: We first prepare some terminology for polygons.

Definition 3.6 (CC-polygon, Degree of points, Sign of the CC-polygon)**.** Let $\vec{v}$ be a sequence of points v_{ab}, $a, b \in \mathfrak{F}_N$, in $\mathbb{R}^2$ with coordinates (x, y). Any $\vec{v}$ is described in the form

$$
\vec{v} = (v_1, \ldots, v_1, v_2, \ldots, v_2, \ldots\ldots, v_n, \ldots, v_n),
$$

where $\{v_1, \ldots, v_n\}$, $n \in \mathbb{Z}_{>0}$, are points in $\mathbb{R}^2$ such that $v_i \neq v_{i+1}$ for $i = 1, \ldots, n-1$. In this expression, we call $\vec{v}$ a *point* if $n = 1$. On the other hand, we call $\vec{v}$ a *clockwise convex polygon (CC-polygon)* if and only if $0 < Angle(v_{i-1}v_iv_{i+1}) \leq \pi$ for any $i \in \mathbb{Z}$, where, we identify $v_i = v_{i+n}$ if $v_1 \neq v_n$ and $v_i = v_{i+(n-1)}$ if $v_1 = v_n$. By definition $n \geq 3$ if $\vec{v}$ is a CC-polygon.

For a CC-polygon $\vec{v} = (v_1, \ldots, v_1, v_2, \ldots, v_2, \ldots\ldots, v_n, \ldots, v_n)$, we attach a *degree* $|v_i|$ for each point v_i, $i = 1, \ldots, n$, as follows. Consider the map $x : \{v_1, \ldots, v_n\} \to \mathbb{R}$, where the image $x(v_i)$ is the x-coordinate of the point v_i. Let $\{x_L < \cdots < x_R\} \subset \mathbb{R}$ be the ordered subset consisting of the image $x(\{v_1, \ldots, v_n\})$, where x_L and x_R indicate the left/right extrema. We fix $i \neq j \in \{1, \ldots, n\}$ such that $x(v_i) = x_L$ and $x(v_j) = x_R$ and assign the degree as $|v_i| = |v_j| = 0$. The degree of the remaining points is set to be one. The choice of such i, j is not unique only if $v_1 = v_n$ and further $x(v_1) = x(v_n) = x_L$ or $x(v_1) = x(v_n) = x_R$. Hereafter, by a CC-polygon $\vec{v}$, we mean that with a degree attached in the sense above. The *sign* $\sigma(\vec{v})$ *of the CC-polygon* $\tilde{v}$ is then defined by

$$\sigma(\vec{v}) := \begin{cases} -1 & i < j \\ +1 & j < i. \end{cases}$$

(See Figure 3.)

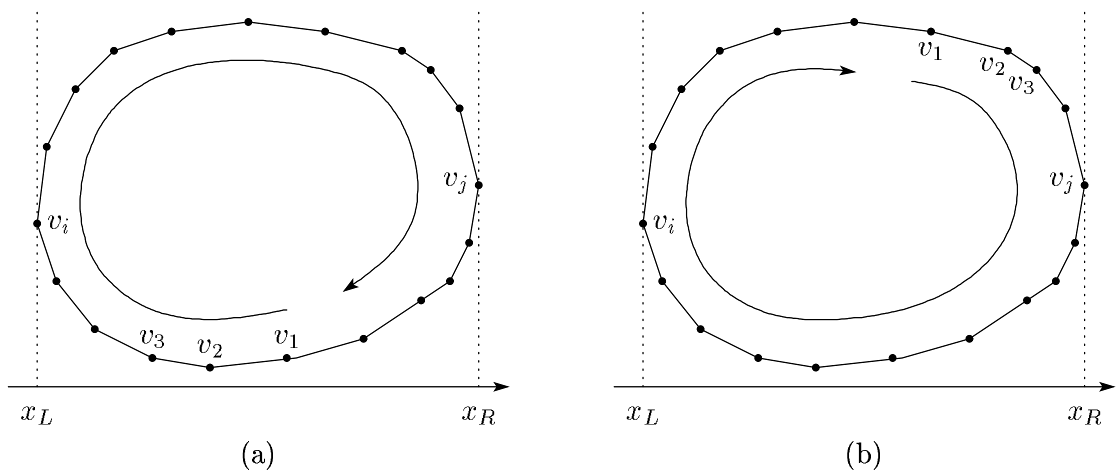


FIGURE 3. CC-polygons $\vec{v}$ with (a): $\sigma(\vec{v}) = -1$ and (b): $\sigma(\vec{v}) = +1$.

For $k \geq 2$, let us define degree $(2-k)$ multilinear maps $m_k : V_{a_1a_2} \otimes \cdots \otimes V_{a_ka_{k+1}} \to V_{a_1a_{k+1}}$,

$$m_k(w_{a_1a_2}, \ldots, w_{a_ka_{k+1}}) \in V_{a_1a_{k+1}}$$

which do not include degree one element in V_{aa} for any $a \in \mathfrak{F}_N$. Namely, we consider the case $a_i \neq a_{i+1}$ for any $i = 1, \ldots, k$, which implies that $w_{a_ia_{i+1}}$, $i = 1, \ldots, k$, is spanned by the base $[v_{a_ia_{i+1}}]$. Thus, it is enough to determine the multilinear maps

$$m_k([v_{a_1a_2}], \ldots, [v_{a_ka_{k+1}}]). \tag{3.3}$$

In the case $a_1 \neq a_{k+1}$ $(k \geq 2)$: Let us set $\vec{v} := (v_{a_1a_2}, \ldots, v_{a_ka_{k+1}}, v_{a_{k+1}a_1})$, where $v_{a_{i-1}a_i} \neq v_{a_ia_{i+1}}$ for any $i \in \mathbb{Z}$, $a_i = a_{i+(k+1)}$. By degree counting (Lemma 3.5), the multilinear maps (3.3) can be nonzero only if $\vec{v}$ forms a CC-polygon (we shall check this fact in subsection 4.1), where the degree for the points is attached uniquely as $|v_{a_ia_{i+1}}| = |[v_{a_ia_{i+1}}]|$, $i \in \mathbb{Z}$, $a_i = a_{i+(k+1)}$. We set the structure constant $c_{a_1\cdots a_{k+1}} \in \mathbb{R}$ of

$$m_k([v_{a_1a_2}], \ldots, [v_{a_ka_{k+1}}]) = c_{a_1\cdots a_{k+1}}[v_{a_1a_{k+1}}]$$

by

$$c_{a_1\cdots a_{k+1}} := (\sigma(\vec{v}))^k \; e^{-Area(\vec{v})}, \tag{3.4}$$

where $Area(\vec{v})$ is the area of the CC-polygon $\vec{v}$.

In the case $a_1 = a_{k+1} =: a$ ($k \geq 2$): Let us set $\vec{v} := (v_{a_1a_2}, \ldots, v_{a_ka_{k+1}})$. By degree counting (Lemma 3.5), the multilinear maps (3.3) can be nonzero only if $\vec{v}$ forms a CC-polygon or a point. If $\vec{v}$ forms a CC-polygon, with the degree attached uniquely as $|v_{a_ia_{i+1}}| = |[v_{a_ia_{i+1}}]|$, $i = 1, \ldots, k$, we set

$$\begin{aligned} &m_k([v_{aa_2}], \ldots, [v_{a_ka}]) = c_{aa_2\cdots a_ka} \cdot \ \vartheta_{v_{aa_2}}^{-\sigma(\vec{v})} \vartheta_{v_{a_ka}}^{\sigma(\vec{v})} \in V_{aa}^0 \\ &c_{a_1\cdots a_{k+1}} := (\sigma(\vec{v}))^k \ e^{-Area(\vec{v})}, \end{aligned} \tag{3.5}$$

where $\vartheta_v^{\pm 1}$ denotes

$$\vartheta_v^{+1} = \vartheta_v, \qquad \vartheta_v^{-1} = 1 - \vartheta_v.$$

If $\vec{v}$ forms a point, the corresponding multilinear maps (3.3) become bilinear one $m_2 : V_{ab} \otimes V_{ba} \to V_{aa}$ for some $a \neq b \in \mathfrak{F}_N$, which we set as

$$m_2([v_{ab}], [v_{ba}]) = \delta_{v_{ab}} \in V_{aa}^1. \tag{3.6}$$

Theorem 3.7. *These multilinear maps m_k define a unique A_∞-structure in $\mathcal{C}(\mathfrak{F}_N)$.*

Proof. The multilinear maps m_k given above (those which do not include elements in V_{aa}^1 for some $a \in \mathfrak{F}_N$) is in fact compatible with the A_∞-constraint (2.1); this fact can be checked directly. On the other hand, for any $a \in \mathfrak{F}_N$, elements in $V_{aa}^1 = \tilde{V}_{aa}^1$ is m_1-exact in $\tilde{V}_{aa}$ and then all the A_∞-products including those are determined uniquely by the A_∞-constraint (2.1). The A_∞-structure on $\tilde{V}_{**}$ obtained so is in fact closed in V_{**}.

Alternatively, all the compatibility, existence, and the uniqueness of the A_∞-structure stated above can also be obtained as a corollary of the proof of Proposition (5.4) in subsection 5.2 where the A_∞-structure on $\mathcal{C}(\mathfrak{F}_N)$ is derived in the framework of HPT. □

Note that the A_∞-product (3.4) is just the transversal A_∞-product in Condition (ii) in Theorem 3.2. This definition of transversal A_∞-products (3.4) *agrees in the sign* with that given by Polishchuk [23] in the two-tori case.

As for the A_∞-products of $\sharp(\vartheta) = 1$, as a consequence, the A_∞-products can be nonzero only if $k_1 = k_2 = k_3 = 0$ for type A and B, $k_1 = k_3 = 0$ for type C$_1$ with $r = 0$ and type C$_3$, and $k_2 = k_4 = 0$ for type C$_1$ with $r = 1$ and type C$_2$.

For applications in the future, it should be worth giving the formula for the A_∞-products $m_k : V_{a_1a_2} \otimes \cdots \otimes V_{a_ka_{k+1}} \to V_{a_1a_{k+1}}$ which do not include elements in V_{aa}^0 for any a and which may include degree one elements in V_{aa}^1 for any a of the form $\delta_{v_a} \in V_{aa}^1$ only. Namely, for the A_∞-products

$$m_k(w_{a_1a_2}, \ldots, w_{a_ka_{k+1}}) \in V_{a_1a_{k+1}}, \qquad c_{a_1\cdots a_{k+1}} \in \mathbb{R}, \quad (k \geq 2), \tag{3.7}$$

we let $w_{a_ia_{i+1}} = [v_{a_ia_{i+1}}] \in V_{a_ia_{i+1}}^0$ if $t_{a_i} < t_{a_{i+1}}$, $w_{a_ia_{i+1}} = [v_{a_ia_{i+1}}] \in V_{a_ia_{i+1}}^1$ if $t_{a_i} > t_{a_{i+1}}$ and $w_{a_ia_{i+1}} = \delta_{v_{a_i}} \in V_{a_ia_{i+1}}^1$ for some $v_{a_i} \in S_{a_i}$ if $a_i = a_{i+1}$, where $i = 1, \ldots, k$. Recall that, in this case, any $w_{a_ia_{i+1}}$ is associated with a point in $\mathbb{R}^2$. If $a_i \neq a_{i+1}$, the associated point is $v_{a_ia_{i+1}}$. If $a_i = a_{i+1}$, we denote the associated point again by $v_{a_ia_{i+1}}$, where $v_{a_ia_{i+1}} \in S_{a_i}$. Then, for the case $a_1 \neq a_{a_{k+1}}$, we set $\vec{v} := (v_{a_1a_2}, \ldots, v_{v_{a_ka_{k+1}}}, v_{a_{k+1}a_1})$, the points associated to elements $(w_{a_1a_2}, \ldots, w_{a_ka_{k+1}}, [v_{a_{k+1}a_1}])$. For the case $a_1 = a_{a_{k+1}}$, we set $\vec{v} := (v_{a_1a_2}, \ldots, v_{v_{a_ka_{k+1}}})$, the points associated to elements $(w_{a_1a_2}, \ldots, w_{a_ka_{k+1}})$. In both cases, the A_∞-product $m_k(w_{a_1a_2}, \ldots, w_{a_ka_{k+1}})$ in eq.(3.7) can be zero only if $\vec{v}$ forms a CC-polygon or a point. Let us describe $\vec{v}$ as the form

$$\vec{v} = (v_1, \ldots, v_1, v_2, \ldots, v_2, \ldots\ldots, v_n, \ldots, v_n).$$

Suppose that $\vec{v}$ is a CC-polygon, where the degree for each point is given as follows. For any $i = 1, \ldots, n$, if $v_i, \ldots, v_i$ includes the point associated to a degree zero element, we set $|v_i| = 0$

and denote the copy of v_i by

$$(v_i)^{\otimes(d_{i_-},d_{i_+})} := \overbrace{v_i,\dots,v_i}^{d_{i_-}}, \overset{\circ}{v}_i, \overbrace{v_i,\dots,v_i}^{d_{i_+}},$$

where we attach $\circ$ to the point associated to the degree zero element. We put $d_i := d_{i-} + d_{i_+}$. If $v_i,\dots,v_i$ does not include the point associated to a degree zero element, we set $|v_i| = 1$ and denote the d_i copy of v_i by

$$(v_i)^{\otimes d_i} := \overbrace{v_i,\dots,v_i}^{d_i}.$$

For any point v_i of degree zero or one, we call the integer d_i the *multiplicity* of v_i. For any $i = 1,\dots,n$, we define

$$D_i = \begin{cases} 2^{d_i}(d_{i_-})!(d_{i_+})! & |v_i| = 0 \\ (d_i)! & |v_i| = 1. \end{cases}$$

In the setting above, the A_∞-products (3.7) is determined as follows.

In the case $a_1 \neq a_{k+1}$ ($k \geq 2$): The A_∞-product (3.7) is nonzero only if $\vec{v}$ forms a CC-polygon (*i.e.*, is zero if $\vec{v}$ is a point). Then, the structure constant $c_{a_1\cdots a_{k+1}} \in \mathbb{R}$ of

$$m_k(w_{a_1a_2},\dots,w_{a_ka_{k+1}}) = c_{a_1\cdots a_{k+1}}[v_{a_1a_{k+1}}]$$

is given by

$$c_{a_1\cdots a_{k+1}} := \frac{(\sigma(\vec{v}))^k}{D_1\cdots D_n}\, e^{-Area(\vec{v})}. \tag{3.8}$$

In the case $a_1 = a_{k+1} =: a$ ($k \geq 3$): The A_∞-product (3.7) can be nonzero only if $\vec{v}$ forms a CC-polygon or a point. If $\vec{v}$ forms a CC-polygon, it is given by

$$\begin{aligned} m_k(w_{aa_2},\dots,w_{a_ka}) &= c_{aa_2\cdots a_ka}\cdot\alpha_1(\vec{v})\alpha_n(\vec{v}) \in V^0_{aa} \\ c_{a_1\cdots a_{k+1}} &:= \frac{(\sigma(\vec{v}))^k}{D_1\cdots D_n}\, e^{-Area(\vec{v})}, \end{aligned} \tag{3.9}$$

where $\alpha_1(\vec{v}), \alpha_n(\vec{v}) \in V^0_{aa}$ are defined by

$$\alpha_1(\vec{v}) = \begin{cases} \vartheta_{v_1}^{-\sigma(\vec{v})}(-\sigma(\vec{v})2(\vartheta_{v_1} - \frac{1}{2}))^{d_1} & |v_1| = 0, \\ (\vartheta_{v_1}^{-\sigma(\vec{v})})^{d_1} & |v_1| = 1, \end{cases}$$

$$\alpha_n(\vec{v}) = \begin{cases} \vartheta_{v_n}^{\sigma(\vec{v})}(\sigma(\vec{v})2(\vartheta_{v_n} - \frac{1}{2}))^{d_n} & |v_n| = 0, \\ (\vartheta_{v_n}^{\sigma(\vec{v})})^{d_n} & |v_n| = 1. \end{cases}$$

If $\vec{v}$ forms a point, nonzero A_∞-products are only the followings:

$$m_{2+d_-+d_+}(\overbrace{\delta_{v_{ab}},\dots,\delta_{v_{ab}}}^{d_-},[v_{ab}],\overbrace{\delta_{v_{ab}},\dots,\delta_{v_{ab}}}^{d_+},[v_{ba}]) = \frac{\left(\frac{1}{2}-\vartheta_{v_{ab}}\right)^{d_-+d_+}}{(d_-)!(d_+)!}\cdot\delta_{v_{ab}} \in V^1_{aa},$$

$$m_{2+d_-+d_+}([v_{ba}],\overbrace{\delta_{v_{ab}},\dots,\delta_{v_{ab}}}^{d_-},[v_{ab}],\overbrace{\delta_{v_{ab}},\dots,\delta_{v_{ab}}}^{d_+}) = \frac{\left(\frac{1}{2}-\vartheta_{v_{ab}}\right)^{d_-+d_+}}{(d_-)!(d_+)!}\cdot\delta_{v_{ab}} \in V^1_{bb}$$

for any $a, b \in \mathfrak{F}_N$ such that $t_a < t_b$ and $d_-, d_+ \in \mathbb{Z}_{\geq 0}$.

Theorem 3.8. *For any given $\mathfrak{F}_N$ and $\mathfrak{F}'_N$, the two A_∞-categories $\mathcal{C}(\mathfrak{F}_N)$ and $\mathcal{C}'(\mathfrak{F}'_N)$ are homotopy equivalent.*

We shall prove this after the proof of the main theorem (Theorem 3.2) in subsection 5.1. This result seems reasonable from the viewpoint of symplectic geometry, since the second cohomology of $\mathbb{R}^2$ is trivial, that is, any symplectic form on $\mathbb{R}^2$ is exact. Extending this construction of an A_∞-structure to tori, the A_∞-categories with different configurations of lines are not homotopy equivalent in general even if the number of the lines (objects) is the same. For instance, for a two-torus case, a geodesic cycle in a torus T^2 is described by a $\mathbb{Z}$-copy of lines in the covering space $\mathbb{R}^2$ and the dimension of the space $\mathrm{Hom}_{Fuk(T^2)}(a,b)$ of morphisms for two transversal geodesic cycles a and b in T^2 is the number of the intersection points of a and b in T^2, which will change if we change the slops of a and b even if we keep the ordering (cf. [25, 14, 24, 15]).

4. Interpretations and Examples

In this section, we explain more on the relation of polygons and the A_∞-products in the Fukaya A_∞-category $\mathcal{C}(\mathfrak{F}_N)$ mainly for the transversal part. The relation between polygons and the degrees of the the intersection points is explained in subsection 4.1. The realization of the A_∞-constraints in terms of polygons is given in subsection 4.2. Also, the necessity of nontrivial non-transversal products is observed in an example in subsection 4.3.

4.1. **CC-polygon and the degree.** In the previous subsection we stated that by degree counting (Lemma 3.5) the transversal A_∞-products $m_n([v_{a_1a_2}],\dots,[v_{a_na_{n+1}}])$ in eq. (3.4) can be nonzero only if the corresponding sequence $\vec{v}=(v_{a_1a_2},\dots,v_{a_na_{n+1}})$ of points forms a CC-polygon. Let us check this fact. If we go around the CC $(n+1)$-gon $\vec{v}$ in the clockwise direction and count

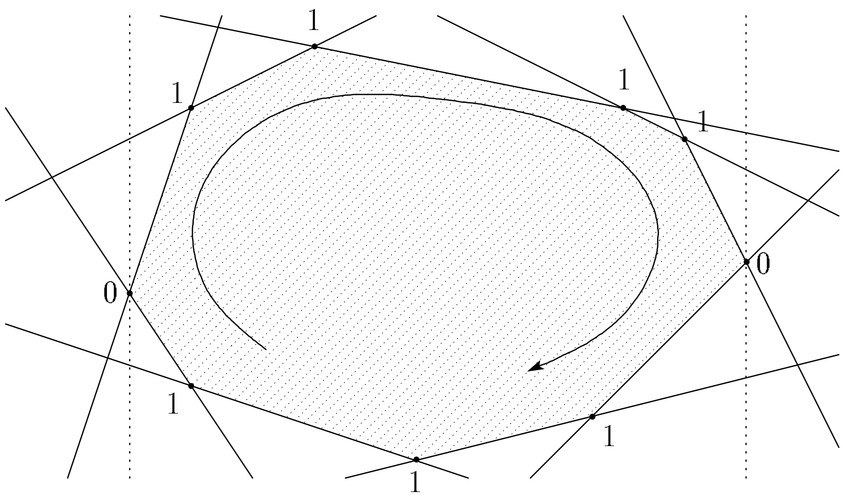


FIGURE 4. A CC-polygon $\vec{v}$ with degree (0 or 1) of the intersection points.

the degree r (zero or one) of each point $v_{a_ia_{i+1}}$, we always have two points of degree zero and $(n+1)-2$ points of degree one as in Figure 4. Thus, we have

$$\sum_{i=1}^{n+1}|v_{a_ia_{i+1}}|=\sum_{i=1}^{n+1}|[v_{a_ia_{i+1}}]|=(n+1)-2 \tag{4.1}$$

for the CC $(n+1)$-gon $\vec{v}$, where $v_{a_{n+1}a_{n+2}}:=v_{a_{n+1}a_1}$. One can also see that the equation above holds true only if $\vec{v}$ forms a CC-polygon.

On the other hand, for the transversal A_∞-product $m_n([v_{a_1a_2}],\dots,[v_{a_na_{n+1}}])$, Lemma 3.5 implies that one has only two elements of degree zero in $\{[v_{a_1a_2}],\dots,[v_{a_na_{n+1}}],[v_{a_{n+1}a_1}]\}$. This exactly implies the identity (4.1) since the degree of the remaining elements is one.

To make sure, let us check Lemma 3.5 in this transversal situation. Let us assume that $|[v_{a_{n+1}a_1}]|=r$, where r is equal to zero or one. Then, one has

$$|m_n([v_{a_1a_2}],\dots,[v_{a_na_{n+1}}])|=\sum_{i=1}^{n}|[v_{a_ia_{i+1}}]|-r+(2-n)$$

since the degree of m_n is $(2-n)$. Here, $\vec{v}$ is a CC-polygon if and only if the identity (4.1) holds, where the right hand side of the equation above turns out to be $(n+1)-2+(2-n)-r=1-r=|[v_{a_1 a_n}]|$.

Thus, one can indeed define nonzero transversal A_∞-product m_n only when the corresponding $\vec{v}$ forms a CC $(n+1)$-gon. This fact was obtained by counting the degrees of the corresponding points, which are related to their Maslov indices (see [2].)

4.2. **A_∞-constraint and polygons.** In the rest of this section, we denote $v_{ab} := [v_{ab}]$ since it does not cause any confusion.

The A_∞-constraints for transversal A_∞-products has a geometric interpretation in terms of a clockwise polygon which has one nonconvex point (=vertex of the polygon). There exist two ways to divide the polygon into two convex polygons. The corresponding terms then appear with opposite signs and cancel each other in the A_∞-constraint. For example, in Figure 5, we have the

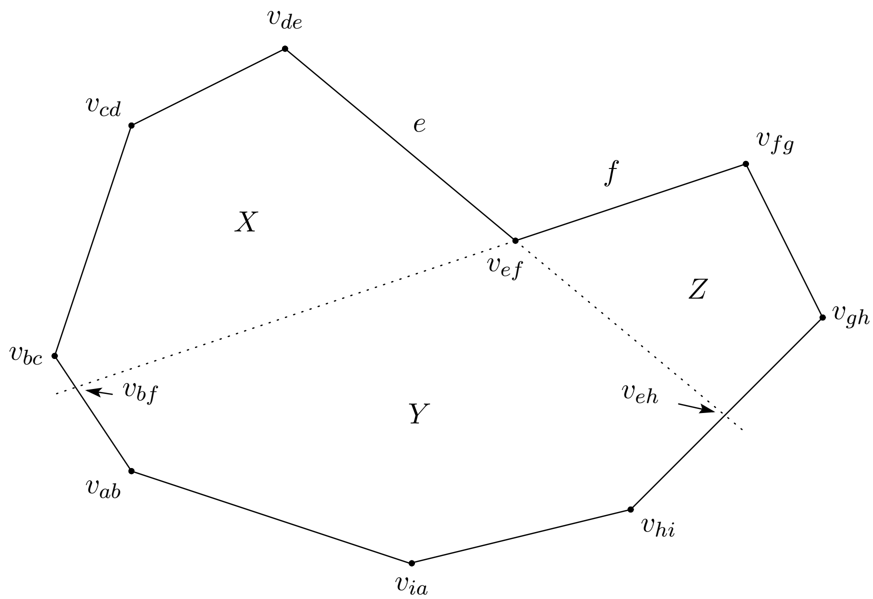


FIGURE 5. A clockwise polygon which has one nonconvex point.

following (intersection) points with their degrees assigned:

$$\begin{array}{cccccccc} v_{ab} & v_{bc} & v_{cd} & v_{de} & v_{ef} & v_{fg} & v_{gh} & v_{hi} \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array} .$$

Corresponding to the way of dividing the area $X+Y+Z$ into (i) $X+(Y+Z)$ or (ii) $(X+Y)+Z$, we have the following composition of transversal A_∞-products:

$$\text{(i)} \pm (v_{ab}(v_{bc}v_{cd}v_{de}v_{ef})v_{fg}v_{gh}v_{hi}),$$
$$\text{(ii)} \pm (v_{ab}v_{bc}v_{cd}v_{de}(v_{ef}v_{fg}v_{gh})v_{hi}),$$

where $(v_{bc}v_{cd}v_{de}v_{ef})$ indicates $m_4(v_{bc}, v_{cd}, v_{de}, v_{ef})$ and so on. There does not exist any other composition of A_∞-products since a transversal A_∞-product can be nonzero only if the corresponding polygon forms a CC-polygon. According to the definition, one obtains

$$m_4(v_{bc}, v_{cd}, v_{de}, v_{ef}) = e^{-X} v_{bf} ,$$
$$m_5(v_{ab}, v_{bf}, v_{fg}, v_{gh}, v_{hi}) = -e^{-(Y+Z)} v_{ai} ,$$
$$m_3(v_{ef}, v_{fg}, v_{gh}) = -e^{-Z} v_{eh} ,$$
$$m_6(v_{ab}, v_{bc}, v_{cd}, v_{de}, v_{eh}, v_{hi}) = e^{-(X+Y)} v_{ai} .$$

Combining the first two equations leads to

$$m_5(v_{ab}, m_4(v_{bc}, v_{cd}, v_{de}, v_{ef}), v_{fg}, v_{gh}, v_{hi}) = -e^{-X-(Y+Z)} v_{ai} \ ,$$

and combining the last two gives

$$m_6(v_{ab}, v_{bc}, v_{cd}, v_{de}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi}) = -e^{-(X+Y)-Z} v_{ai} \ .$$

Thus, we obtain

$$0 = m_5(v_{ab}, m_4(v_{bc}, v_{cd}, v_{de}, v_{ef}), v_{fg}, v_{gh}, v_{hi}) - m_6(v_{ab}, v_{bc}, v_{cd}, v_{de}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi}) \ ,$$

which is just the A_∞-constraints (2.1) on $(v_{ab}, v_{bc}, v_{cd}, v_{de}, v_{ef}, v_{fg}, v_{gh}, v_{hi})$.

4.3. **Why can we not avoid non-transversal products ?** Using Figure 5, we show that we can not avoid non-transversal A_∞-products, *i.e.*, we can not define an A_∞-structure for the Fukaya category such that all non-transversal A_∞-products are zero.

Consider the sequence $(v_{ab}, v_{bf}, v_{fe}, v_{ef}, v_{fg}, v_{gh}, v_{hi})$ of elements and the corresponding A_∞-products. There exists a composition $m_5(v_{ab}, v_{bf}, v_{fe}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi}) = e^{-(Y+Z)} v_{ai}$ of two transversal A_∞-products. The A_∞-constraint (2.1) then implies that this composition cancels with other terms. However, there does not exist any more composition of two non-zero transversal A_∞-products on the sequence $(v_{ab}, v_{bf}, v_{fe}, v_{ef}, v_{fg}, v_{gh}, v_{hi})$. This shows the necessity of nonzero non-transversal A_∞-products. For the case of the A_∞-category $\mathcal{C}(\mathfrak{F}_N)$, one has $m_2(v_{fe}, v_{ef}) = \delta_{v_{fe}} \in V^1_{ff}$ and the A_∞-constraint on the sequence $(v_{ab}, v_{bf}, v_{fe}, v_{ef}, v_{fg}, v_{gh}, v_{hi})$ is

$$0 = m_5(v_{ab}, v_{bf}, v_{fe}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi}) - m_6(v_{ab}, v_{bf}, m_2(v_{fe}, v_{ef}), v_{fg}, v_{gh}, v_{hi}),$$

where $m_6(v_{ab}, v_{bf}, m_2(v_{fe}, v_{ef}), v_{fg}, v_{gh}, v_{hi}) = m_6(v_{ab}, v_{bf}, \delta_{v_{fe}}, v_{fg}, v_{gh}, v_{hi}) = e^{-(Y+Z)} v_{ai}$.

5. Proof of Theorem 3.2

5.1. **The outline of the proof.** We define two DG categories $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ and $\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)$, which are DG-categorical extensions of DG-algebras $A_S(\mathbb{R})$ and $A_S(\Omega(\mathbb{R}))$, respectively.

Let S_{all} be the set of all intersection points v_{ab} for all $a \neq b \in \mathfrak{F}_N$ equipped with a map $x : S_{all} \to \mathbb{R}$. Here, note that $v_{ab} = v_{ba} \in S_{all}$, $v_{ab} = v_{ac}$ if and only if $b = c \in \mathfrak{F}_N$, and in general $x(v) = x(v')$ possibly holds for $v \neq v' \in S_{all}$.

Definition 5.1 ($\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)$)**.** The set of objects is taken to be the same as that of $\mathcal{C}_{DR}(\mathfrak{F}_N)$:

$$\mathrm{Ob}(\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)) := \mathrm{Ob}(\mathcal{C}_{DR}(\mathfrak{F}_N)) = \mathfrak{F}_N.$$

For any $a, b \in \mathfrak{F}_N$, we set the space of morphisms by

$$\mathrm{Hom}_{\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)}(a,b) = \tilde{\Omega}_{ab} := A_{S_{all}}(\Omega(\mathbb{R}))$$

as a graded vector space of degree zero and one. For $a, b, c \in \mathfrak{F}$, the composition $m : \tilde{\Omega}_{ab} \otimes \tilde{\Omega}_{bc} \to \tilde{\Omega}_{ac}$ is defined as the product in $A_{S_{all}}(\Omega(\mathbb{R}))$. For $a, b \in \mathfrak{F}_N$, the differential $d_{ab} : \tilde{\Omega}_{ab} \to \tilde{\Omega}_{ab}$ is given by

$$d_{ab} = d - df_{ab} \wedge,$$

where $d : A_{S_{all}}(\Omega(\mathbb{R})) \to A_{S_{all}}(\Omega(\mathbb{R}))$ is the differential of $A_{S_{all}}(\Omega(\mathbb{R}))$, and $\wedge$ is the graded commutative product in $A_{S_{all}}(\Omega(\mathbb{R}))$.

Definition 5.2 ($\mathcal{C}'_{DR}(\mathfrak{F}_N)$)**.** The set of objects is the same as that of $\mathcal{C}_{DR}(\mathfrak{F}_N)$:

$$\mathrm{Ob}(\mathcal{C}'_{DR}(\mathfrak{F}_N)) := \mathfrak{F}_N.$$

For any $a, b \in \mathfrak{F}_N$, we set the space of morphisms by

$$\mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a,b) = \Omega'_{ab} := \{ e^{f_{ab}} \cdot \alpha \in A_{S_{all}}(\Omega(\mathbb{R})) \mid \alpha \in \tilde{A}_{S_{all}}(\mathbb{R}) \}$$

as a graded vector space of degree zero and one. For $a,b,c\in\mathfrak{F}$, the composition $m:\Omega'_{ab}\otimes\Omega'_{bc}\to\Omega'_{ac}$ is defined as the product in $A_{S_{all}}(\Omega(\mathbb{R}))$. For $a,b\in\mathfrak{F}_N$, the differential $d_{ab}:\Omega'_{ab}\to\Omega'_{ab}$ is the same as that in $\mathcal{C}_{DR}(\mathfrak{F}_N)$ or $\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)$:

$$d_{ab}=d-df_{ab}\wedge .$$

Clearly, $\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)$ forms a DG category, and the DG subcategory $\mathcal{C}'_{DR}(\mathfrak{F}_N)\subset\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)$ is well-defined.

The following can be showed just in the same way as Lemma 3.4:

Lemma 5.3. *The inclusions*

$$\iota:\mathcal{C}'_{DR}(\mathfrak{F}_N)\to\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N),\qquad \iota:\mathcal{C}_{DR}(\mathfrak{F}_N)\to\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N),$$

induce homotopy equivalences $\mathcal{C}_{DR}(\mathfrak{F}_N)\simeq\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)$ and $\mathcal{C}'_{DR}(\mathfrak{F}_N)\simeq\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)$ as A_∞-categories. □

On the other hand, one has the following:

Proposition 5.4. *There exists an A_∞-functor $\mathcal{G}:\mathcal{C}(\mathfrak{F}_N)\to\mathcal{C}'_{DR}(\mathfrak{F}_N)$ which induces a homotopy equivalence*

$$\mathcal{C}(\mathfrak{F}_N)\simeq\mathcal{C}'_{DR}(\mathfrak{F}_N).$$

We shall show this in the next subsection, where HPT is applied to derive the A_∞-structure of $\mathcal{C}(\mathfrak{F}_N)$.

Then, we obtain the following homotopy equivalences

$$\mathcal{C}(\mathfrak{F}_N)\overset{\mathcal{G}}{\to}\mathcal{C}'_{DR}(\mathfrak{F}_N)\overset{\iota}{\to}\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)\overset{\iota}{\leftarrow}\mathcal{C}_{DR}(\mathfrak{F}_N),$$

which give a proof of Theorem 3.2. □

Proof. of Theorem 3.8. By Theorem 3.2 $\mathcal{C}(\mathfrak{F}_N)\simeq\mathcal{C}_{DR}(\mathfrak{F}_N)$ and $\mathcal{C}(\mathfrak{F}'_N)\simeq\mathcal{C}_{DR}(\mathfrak{F}'_N)$. Also, by Lemma 5.3 we obtained the equivalence $\mathcal{C}_{DR}(\mathfrak{F}_N)\simeq\mathcal{C}'_{DR}(\mathfrak{F}_N)$ and $\mathcal{C}_{DR}(\mathfrak{F}'_N)\simeq\mathcal{C}'_{DR}(\mathfrak{F}'_N)$.

Thus, one may show the homotopy equivalence $\mathcal{C}'_{DR}(\mathfrak{F}_N)\simeq\mathcal{C}'_{DR}(\mathfrak{F}'_N)$. These two categories are in fact isomorphic to each other. Let us denote the objects by $\mathfrak{F}_N=\{a,b,\dots\}$ and $\mathfrak{F}'_N=\{a',b',\dots\}$, where we can assume $t_a<t_b\cdots$ and $t_{a'}<t_{b'}\cdots$ without lose of generality. The functor between the objects is given by $a\mapsto a'$ for any $a\in\mathfrak{F}_N$, and the functor between the space of morphisms is given by

$$\begin{array}{ccc}\mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a,b) & \to & \mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}'_N)}(a',b')\\ \omega & \mapsto & e^{f_{a'b'}-f_{ab}}\omega\end{array}$$

for any $a,b\in\mathfrak{F}_N$. □

5.2. **Deriving the A_∞-category $\mathcal{C}(\mathfrak{F}_N)$.** Now, we shall show Proposition 5.4 stating the homotopy equivalence $\mathcal{C}(\mathfrak{F}_N)\simeq\mathcal{C}'_{DR}(\mathfrak{F}_N)$. We apply HPT (Theorem 2.10) to $\mathcal{C}'_{DR}(\mathfrak{F}_N)$. In order to do so, for any $a,b\in\mathfrak{F}_N$, we first define homotopy $h_{ab}:\Omega'^1_{ab}\to\Omega'^0_{ab}$, $\Omega'_{ab}:=\mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a,b)$, so that $P_{ab}:\Omega'^r_{ab}\to\Omega'^r_{ab}$, $r=0,1$, defined by the $d_{ab}h_{ab}+h_{ab}d_{ab}=\mathrm{Id}_{\Omega'_{ab}}-P_{ab}$ gives a projection on Ω'_{ab}.

For any $a\in\mathfrak{F}_N$, we set $h_{aa}=0$ and then $P_{aa}=\mathrm{Id}$.

For all $a\neq b\in\mathfrak{F}_N$, define the homotopy $h_{ab}:\Omega'^1_{ab}\to\Omega'^0_{ab}$ and the projection $P_{ab}:\Omega'^r_{ab}\to\Omega'^r_{ab}$ as follows. For the base $d(\vartheta_v)^n\in\Omega'^1_{ab}$,

$$h_{ab}(d(\vartheta_v)^n):=e^{f_{ab}-f_{ab}(x(v))}((\vartheta_v)^n-c)\tag{5.1}$$

for the case $t_a<t_b$, where $c=0$ if $x(v_{ab})<x(v)$, $c=1/2^n$ if $x(v_{ab})=x(v)$, and $c=1$ if $x(v)<x(v_{ab})$. On the other hand, if $t_a>t_b$,

$$h_{ab}(d(\vartheta_v)^n)=e^{f_{ab}-f_{ab}(x(v))}((\vartheta_v)^n-\vartheta_{v_{ab}})\tag{5.2}$$

for any $n \geq 1$ and $v \in S_{all}$.

For $t_a < t_b$, the projection $P_{ab} : {\Omega'}^0_{ab} \to {\Omega'}^0_{ab}$ is

$$P_{ab}(e^{f_{ab}} \cdot (\vartheta_v)^n) = \begin{cases} e^{f_{ab}} & x(v) < x(v_{ab}) \\ \frac{1}{2^n} e^{f_{ab}} & x(v) = x(v_{ab}) \\ 0 & x(v_{ab}) < x(v) \end{cases}$$

for any $n \geq 1$ and $v \in S_{all}$, $P_{ab}(e^{f_{ab}}) = e^{f_{ab}}$, and $P_{ab} = 0$ for $P_{ab} : {\Omega'}^1_{ab} \to {\Omega'}^1_{ab}$.

For $t_a > t_b$, the projection $P_{ab} : {\Omega'}^1_{ab} \to {\Omega'}^1_{ab}$ is

$$P_{ab}(d(\vartheta_v)^n) = \delta_{v_{ab}}$$

for any $n \geq 1$ and $v \in S_{all}$ and $P_{ab} = 0$ for $P_{ab} : {\Omega'}^0_{ab} \to {\Omega'}^0_{ab}$.

Then, for any $a, b \in \mathfrak{F}_N$, one has the identity

$$d_{ab}h_{ab} + h_{ab}d_{ab} = \mathrm{Id} - P_{ab} \tag{5.3}$$

on Ω'_{ab}. We denote the base of $P_{ab}{\Omega'}^r_{ab}$ by $\mathbf{e}_{ab} := e^{f_{ab} - f_{ab}(x(v_{ab}))}$ for $r = 0$ and $\mathbf{e}_{ab} := \delta_{v_{ab}}$ for $r = 1$.

Now, applying HPT (Theorem 2.10) to $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ with the identity (5.3) leads to an A_∞-category $\mathcal{C}'(\mathfrak{F}_N)$ with homotopy equivalence $\mathcal{C}'(\mathfrak{F}_N) \stackrel{\sim}{\to} \mathcal{C}'_{DR}(\mathfrak{F}_N)$. Here, the set of the objects is $\mathrm{Ob}(\mathcal{C}'(\mathfrak{F}_N)) = \mathfrak{F}_N$. The space of morphisms is defined so that

$$\iota_{ab} : \mathrm{Hom}_{\mathcal{C}'(\mathfrak{F}_N)}(a, b) \to \mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a, b) \tag{5.4}$$

gives the embedding to $P_{ab}\mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a, b) \subseteq \mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a, b)$ for any $a, b \in \mathfrak{F}_N$, which turns out to be $\mathrm{Hom}_{\mathcal{C}'(\mathfrak{F}_N)}(a, b) = \mathrm{Hom}_{\mathcal{C}(\mathfrak{F}_N)}(a, b) = V_{ab}$ for any $a \neq b \in \mathfrak{F}_N$ but $\mathrm{Hom}_{\mathcal{C}'(\mathfrak{F}_N)}(a, a) = \mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a, a) \supset \mathrm{Hom}_{\mathcal{C}(\mathfrak{F}_N)}(a, a) = V_{aa}$ for any $a \in \mathfrak{F}_N$. For $a \neq b \in \mathfrak{F}_N$, we identify the base $\mathbf{e}_{ab} \in P_{ab}\mathrm{Hom}_{\mathcal{C}'_{DR}(\mathfrak{F}_N)}(a, b)$ with the base $[v_{ab}] \in \mathrm{Hom}_{\mathcal{C}'(\mathfrak{F}_N)}(a, b) = V_{ab}$ by the embedding ι_{ab} in eq.(5.4) as $\mathbf{e}_{ab} = \iota_{ab}[v_{ab}]$. Then, the A_∞-structure on $\mathcal{C}'(\mathfrak{F}_N)$ is closed in the subspace $\mathrm{Hom}_{\mathcal{C}(\mathfrak{F}_N)}(a, b) \subseteq \mathrm{Hom}_{\mathcal{C}'(\mathfrak{F}_N)}(a, b)$, which gives the A_∞-structure on $\mathcal{C}(\mathfrak{F}_N)$. Clearly, the inclusion $\mathcal{C}(\mathfrak{F}_N) \to \mathcal{C}'(\mathfrak{F}_N)$ gives an A_∞-functor, and in fact gives homotopy equivalence since the inclusion is a quasi-isomorphism of chain complexes $\mathrm{Hom}_{\mathcal{C}(\mathfrak{F}_N)}(a, b) \to \mathrm{Hom}_{\mathcal{C}'(\mathfrak{F}_N)}(a, b)$. $\square$

Lastly, we end with deriving some examples of the A_∞-products (3.8) (3.9) of $\mathcal{C}(\mathfrak{F}_N)$ associated with CC-polygons.

Consider an A_∞-product

$$m_k(w_{a_1a_2}, \ldots, w_{a_ka_{k+1}})$$

which is associated with a CC-polygon $\vec{v}$ in the sense in eq.(3.7) and the descriptions below. We describe the CC-polygon as

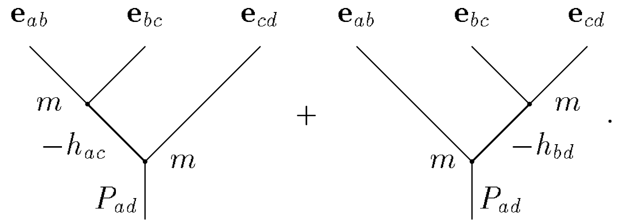
$$\vec{v} = ((v_1)^{\otimes d_1}, (v_2)^{\otimes d_2}, \ldots, (\mathring{v}_i)^{\otimes (d_{i-}, d_{i+})}, \ldots, (\mathring{v}_j)^{\otimes (d_{j-}, d_{j+})}, \ldots, (v_n)^{\otimes d_n})$$

with the map $x : \{v_1, \ldots, v_n\} \to \mathbb{R}$ together. As above, we attach $\circ$ on degree zero points v_i and v_j to distinguish them from other points. If $d = 1$ for $(v)^{\otimes d}$, we simply denote it as $(v)^{\otimes 1} = v$. Similarly, if $(d_-, d_+) = (0, 0)$ for $(\mathring{v})^{\otimes (d_-, d_+)}$, we denote it as $(\mathring{v})^{\otimes (0,0)} = \mathring{v}$. We shall derive these A_∞-products by applying HPT (Theorem 2.10) to the DG category $\mathcal{C}'_{DR}(\mathfrak{F}_N)$. We denote the product in $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ by m. To simplify the formula, in these examples, we identify $[v_{ab}] \in V_{ab}$ with $\mathbf{e}_{ab} \in P_{ab}\Omega'_{ab}$ for $a \neq b \in \mathfrak{F}_N$, and then the surjection $\pi_{ab} : \Omega'_{ab} \to V_{ab}$ with P_{ab}.

Let us start from deriving a transversal A_∞-product with an example. One can see how the area of the corresponding CC-polygon appears, where the correspondence of the CC-polygon and a planar tree (a Feynman graph) is a key point. The way of determining the sign shall be explained in the end of this subsection. Therefore, in the examples below, we do not care about the sign and denote it simply by $\pm$.

Example 1. $\vec{v} = (\overset{\circ}{v}_{ab}, v_{bc}, \overset{\circ}{v}_{cd}, v_{da})$, $x(v_{ab}) < x(v_{bc}) < x(v_{da}) < x(v_{cd})$. The HPT implies

$$m_3(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd}) \quad = \qquad\qquad\qquad + \qquad\qquad\qquad . \tag{5.5}$$



On the other hand, one can associate a planar tree graph to the CC-polygon $\vec{v}$ as follows. First, connect two points v_{ab}, v_{cd} of $\vec{v}$ of degree zero with an interval. For each point of $\vec{v}$ of degree one, draw an interval (external edge = leaf), perpendicular to the x-axis, starting from the point and ending on the interval $(v_{ab}v_{cd})$. Choosing the interval starting from the point v_{da} as the root edge, one obtains a planar rooted tree as in Figure 6. One can see that the resulting planar rooted

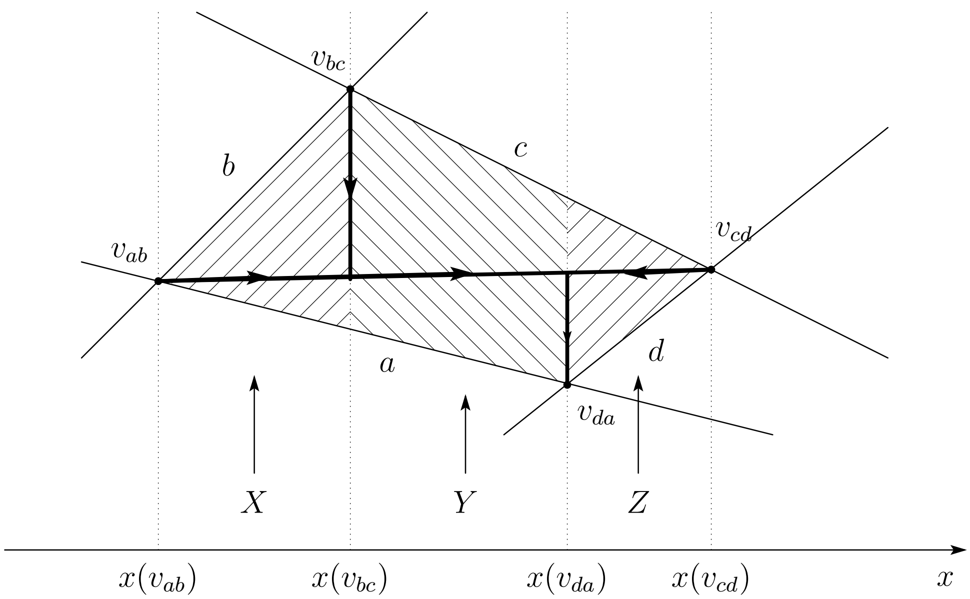


FIGURE 6. CC-polygon $\vec{v} = (\overset{\circ}{v}_{ab}, v_{bc}, \overset{\circ}{v}_{cd}, v_{da})$.

tree corresponds to the one in the first term of the right hand side of eq.(5.5). We shall show that the second term of the right hand side of eq.(5.5) in fact vanishes and the first term derives the area $Area(\vec{v})$. Let us calculate the first term. As in Figure 6, we divide the CC-polygon $\vec{v}$ into three by the lines through v_{bc} and v_{da} both of which are perpendicular to the x-axis. The areas between $x(v_{ab})$ and $x(v_{bc})$, $x(v_{bc})$ and $x(v_{da})$, $x(v_{da})$ and $x(v_{cd})$ are denoted X, Y, Z, respectively. First, one gets $m(\mathbf{e}_{ab}, \mathbf{e}_{bc}) = \pm e^{-X}\delta_{v_{bc}}$. We know $h_{ac}\delta_{v_{bc}} = \pm e^{f_{ac} - f_{ac}(x(v_{bc}))} \cdot \vartheta_{v_{bc}}$. Then, $P_{ad}m(-h_{ac}\delta_{v_{bc}}, \mathbf{e}_{cd})$ is $\mathbf{e}_{ad}$ times the value of the product of $-h_{ac}\delta_{v_{bc}}$ and $\mathbf{e}_{cd}$ at the point $x(v_{da}) \in \mathbb{R}$:

$$\begin{aligned} P_{ad}m(-h_{ac}\delta_{v_{bc}}, \mathbf{e}_{cd}) &= \pm \left(e^{f_{ac}(x(v_{da})) - f_{ac}(x(v_{bc}))} \cdot e^{f_{cd}(x(v_{da})) - f_{cd}(x(v_{cd}))} \right) \cdot \mathbf{e}_{ad} \\ &= \pm \left(e^{-Y} \cdot e^{-Z} \right) \cdot \mathbf{e}_{ad}, \end{aligned}$$

where note that $f_{ac}(x(v_{bc})) - f_{ac}(x(v_{da})) = Y$ and $f_{cd}(x(v_{cd})) - f_{cd}(x(v_{da})) = Z$. Combining all these together, we obtain the first term in the right hand side of eq.(5.5): $\pm e^{-X-(Y+Z)}\mathbf{e}_{ad}$.

In a similar way, one can see that the second term vanishes. The product $m(\mathbf{e}_{bc}, \mathbf{e}_{cd})$ is proportional to $\delta_{v_{bc}}$, and its image by h_{bd} is proportional to $e^{f_{bd} - f_{bd}(x(v_{bc}))} \cdot \vartheta_{v_{bc}}^{-1}$, whose value at the point $x(v_{da}) \in \mathbb{R}$ is equal to zero. Therefore, $P_{ad}m(\mathbf{e}_{ab}, -h_{bd}\delta_{v_{bc}})$ vanishes due to the projection P_{ad}.

This example shows how the transversal A_∞ products are derived in general; the HPT machinery defines a higher product m_k in terms of the sum of values associated to planar rooted k-trees over all the k-trees, but only the one compatible with the k-tree associated to the corresponding CC-polygon survives and produces the area of the CC-polygon. This phenomenon can be found in the original Morse homotopy theory [2, 5], and also its extension [21] where the area of the polygon is taken into account as above.

The following is the first example of non-transversal products (3.5).

Example 2. $\vec{v} = (v_{ab}, \mathring{v}_{bc}, \mathring{v}_{cd}, v_{da})$, $x(v_{bc}) < x(v_{ab}) < x(v_{da}) < x(v_{cd})$. Let us calculate the non-transversal A_∞-product $m_4(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd}, \mathbf{e}_{da})$. This is again described as the sum of the values associated to trivalent planar rooted 4-trees in the framework of HPT. In a similar way as in the transversal case above, the 4-tree giving nonzero value is again the one corresponding to the CC-polygon only. Here, the 4-tree corresponding to the CC-polygon $\vec{v}$ is obtained as follows. Connect the two degree zero points v_{bc} and v_{cd} with an interval. For each degree one point (in this case v_{ab} and v_{da}), draw an interval, perpendicular to the x-axis, starting from the point and ending on the interval $(v_{bc}v_{cd})$. Then, we need a root edge; we add an edge, perpendicular to the x-axis, starting from a point on the interval $(v_{bc}v_{cd})$ between $x(v_{ab})$ and $x(v_{da})$ and ending on the interval $(v_{ab}v_{da})$ (Figure 7). One can check that only the multilinear map corresponding to this

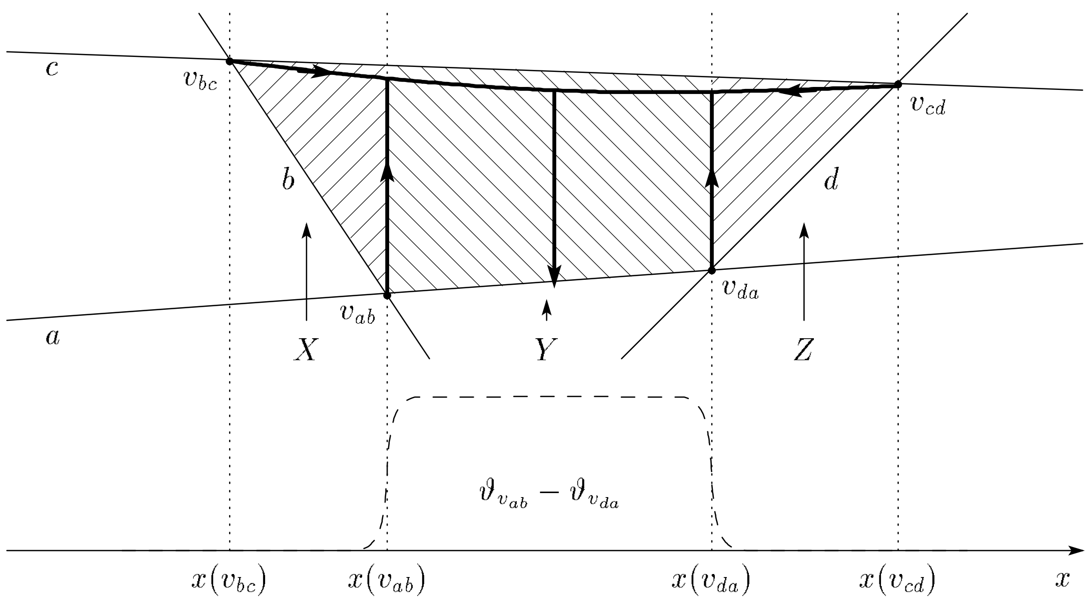


FIGURE 7. CC-polygon $\vec{v} = (v_{ab}, \mathring{v}_{bc}, \mathring{v}_{cd}, v_{da})$.

4-tree is nonzero and it turns out to be $\pm e^{-(X+Y+Z)}(\vartheta_{v_{ab}} - \vartheta_{v_{da}})$.

Example 3. $\vec{v} = (\mathring{v}_{ab}, v_{bc}, \mathring{v}_{cd}, v_{da})$, $x(v_{ab}) < x(v_{bc}) < x(v_{da}) < x(v_{cd})$. Consider the non-transversal A_∞-product $m_4(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd}, \mathbf{e}_{da})$. In this case, there exist two choices of the 4-trees corresponding to the CC-polygon $\vec{v}$. As in the previous example, we need to add an appropriate root edge. One can see that there exist two choices (i) and (ii) of the root edge as in Figure 8. Actually, for the multilinear maps associated to 4-trees by HPT, only those corresponding to these two 4-trees give nonzero contribution. Recall that $P_{aa} = \mathrm{Id}_{\Omega'_{aa}}$ and one has

$$\begin{aligned} m_4(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd}, \mathbf{e}_{da}) = &\pm m(\mathbf{e}_{ab}, -h_{ba}m(\mathbf{e}_{bc}, -h_{ca}m(\mathbf{e}_{cd}, \mathbf{e}_{da}))) \\ &\pm m(-h_{ac}m(\mathbf{e}_{ab}, \mathbf{e}_{bc}), -h_{ca}m(\mathbf{e}_{cd}, \mathbf{e}_{da}))), \end{aligned}$$

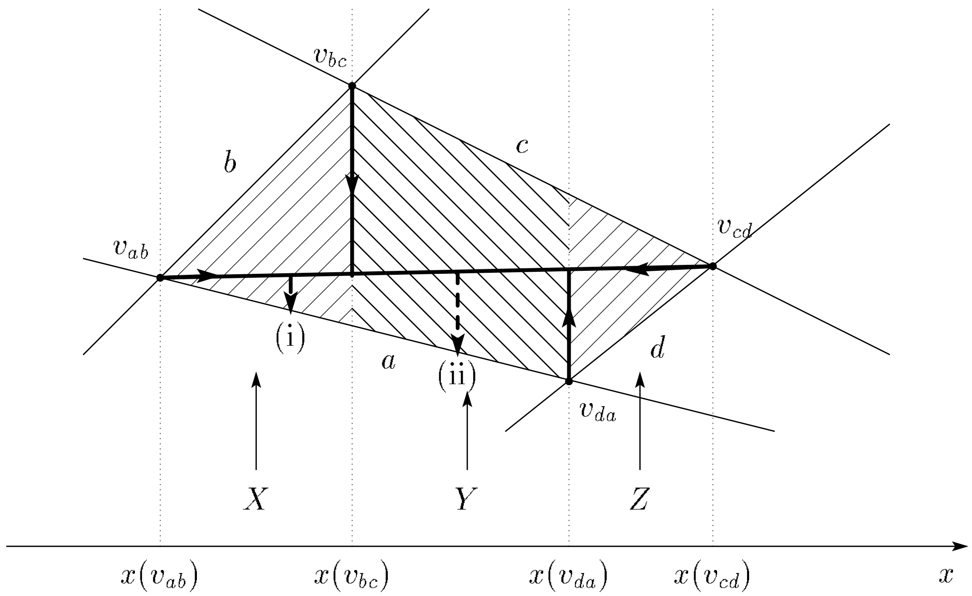


FIGURE 8. CC-polygon $\vec{v} = (\mathring{v}_{ab}, v_{bc}, \mathring{v}_{cd}, v_{da})$ with two choices of the associated 4-trees for the non-transversal A_∞-product $m_4(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd}, \mathbf{e}_{da})$.

where the first (resp. second) term of the right hand side of the equation above corresponds to the 4-tree with the root edge (i) (resp. (ii)). These two terms turn out to be

$$m(\mathbf{e}_{ab}, -h_{ba}m(\mathbf{e}_{bc}, -h_{ca}m(\mathbf{e}_{cd}, \mathbf{e}_{da}))) = \pm e^{-(X+(Y+Z))}(\vartheta_{v_{ab}} - \vartheta_{v_{bc}}),$$
$$m(-h_{ac}m(\mathbf{e}_{ab}, \mathbf{e}_{bc}), -h_{ca}m(\mathbf{e}_{cd}, \mathbf{e}_{da}))) = \pm e^{-(X+Y+Z)}(\vartheta_{v_{bc}} - \vartheta_{v_{da}}),$$

where the signs $\pm$ actually agree with each other and the result is

$$m_4(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd}, \mathbf{e}_{da}) = \pm e^{-(X+Y+Z)}(\vartheta_{v_{ab}} - \vartheta_{v_{da}}).$$

Example 4. $\vec{v} = (\mathring{v}_{ab}, (v_{bc})^{\otimes d}, \mathring{v}_{cd}, v_{da})$, $x(v_{ab}) < x(v_{bc}) < x(v_{da}) < x(v_{cd})$. Consider the non-transversal A_∞-product

$$m_{1+d+1}(\mathbf{e}_{ab}, (\mathbf{e}_{bc})^{\otimes d}, \mathbf{e}_{cd}) := m_{1+d+1}(\mathbf{e}_{ab}, \overbrace{\delta_{v_{bc}}, \dots, \delta_{v_{bc}}, \mathbf{e}_{bc}, \delta_{v_{bc}}, \dots, \delta_{v_{bc}}}^{d}, \mathbf{e}_{cd}).$$

In fact, the result is independent of the order of $\delta_{v_{bc}}$'s and $\mathbf{e}_{bc}$, though $\delta_{v_{bc}}$'s in the left (resp. right) hand side of $\mathbf{e}_{bc}$ are elements in V^1_{bb} (resp. V^1_{cc}). The corresponding CC-polygon is $\vec{v}$. Then, the situation is the same as Example 1 for a transversal A_∞-product except that we have d elements associated to the point v_{bc}. One obtains

$$m_{1+d+1}(\mathbf{e}_{ab}, (\mathbf{e}_{bc})^{\otimes d}, \mathbf{e}_{cd}) = \pm P_{ad}m(-h_{ac}(w), \mathbf{e}_{cd}),$$

where $h_{ac}(w) \in V^0_{ac}$ is given by

$$\pm h_{ac}m(\dots - h_{ac}m(-h_{ac}m(-h_{ab}m(\dots - h_{ab}m(\mathbf{e}_{ab}, \delta_{v_{bc}}), \dots, \delta_{v_{bc}}), \mathbf{e}_{bc}), \delta_{v_{bc}}), \dots, \delta_{v_{bc}}).$$

Since the final result is independent of the order of $\delta_{v_{bc}}$ and $\mathbf{e}_{bc}$, let us try to calculate this w in the case all $\delta_{v_{bc}}$ is in the left hand side of the element $\mathbf{e}_{bc}$. Then,

$$\begin{aligned} w &= \pm -h_{ac}m(-h_{ab}m(\dots - h_{ab}m(\mathbf{e}_{ab}, \delta_{v_{bc}}), \dots, \delta_{v_{bc}}), \mathbf{e}_{bc}) \\ &= \pm e^{-X} \cdot h_{ac}m(\overbrace{(h_{ab}\delta_{v_{bc}} \cdot \dots \cdot (h_{ab}\delta_{v_{bc}} \cdot (h_{ab}\delta_{v_{bc}}))}^{d-1} \dots), \mathbf{e}_{bc}) \\ &= \pm e^{-X}\frac{1}{d!}(\vartheta_{v_{bc}})^d, \end{aligned}$$

where, in the second line we omit denoting the product $m : \Omega'^0_{ab} \otimes \Omega'^1_{bb} \to \Omega'^1_{ab}$. In the third equality, recall that $\delta_{v_{bc}} = d\vartheta_{v_{bc}}$ and $h_{ab}d_{ab}(\vartheta_{v_{bc}})^k = (\vartheta_{v_{bc}})^k$, and then we used the formula $\delta_{v_{bc}} \cdot (\vartheta_{v_{bc}})^{k-1} = (1/k)\, d(\vartheta_{v_{bc}})^k$ for $k = 1, 2, \ldots$. The remaining calculation is the same as Example 1 and we obtain $m_{1+d+1}(\mathbf{e}_{ab}, (\mathbf{e}_{bc})^{\otimes d}, \mathbf{e}_{cd}) = \pm(1/d!)\, e^{-(X+Y+Z)}\mathbf{e}_{ad}$.

Example 5. $\vec{v} = ((\overset{\circ}{v}_{ab})^{\otimes(d_-,d_+)}, v_{bc}, \overset{\circ}{v}_{cd}, v_{da})$, $x(v_{ab}) < x(v_{bc}) < x(v_{da}) < x(v_{cd})$. Consider the non-transversal A_∞-product

$$m_{d+1+1}((\mathbf{e}_{ab})^{\otimes(d_-,d_+)}, \mathbf{e}_{bc}, \mathbf{e}_{cd}) := m_{d+1+1}(\overbrace{\delta_{v_{ab}}, \ldots, \delta_{v_{ab}}}^{d_-}, \mathbf{e}_{ab}, \overbrace{\delta_{v_{ab}}, \ldots, \delta_{v_{ab}}}^{d_+}, \mathbf{e}_{bc}, \mathbf{e}_{cd}),$$

where $d := d_- + d_+$. By HPT, there exist $d!/((d_-)!(d_+)!)$ number of $(d+1+1)$-trees which give nonzero contribution to the corresponding $(d+1+1)$-linear maps. The number $d!/((d_-)!(d_+)!)$ comes from the number of the orders that one acts $-h_{ab}m(\delta_{v_{ab}}, *)$ to $\mathbf{e}_{ab}$ d_- times and $-h_{ab}m(*, \delta_{v_{ab}})$ to $\mathbf{e}_{ab}$ d_+ times. In fact, the result does not depend on the order and one can describe

$$m_{d+1+1}((\mathbf{e}_{ab})^{\otimes(d_-,d_+)}, (\mathbf{e}_{bc}), \mathbf{e}_{cd}) = \frac{d!}{(d_-)!(d_+)!} P_{ad}m(-h_{ac}m(w, \mathbf{e}_{bc}), \mathbf{e}_{cd}),$$

where $w \in V^0_{ab}$ is given by

$$\begin{aligned} &\pm (-h_{ab}m(\delta_{v_{ab}}, *))^{d_-} \circ (-h_{ab}m(*, \delta_{v_{ab}}))^{d_+}\, \mathbf{e}_{ab} \\ &= \pm\, \overbrace{(h_{ab}\delta_{v_{ab}} \cdot \cdots (h_{ab}\delta_{v_{ab}} \cdot (h_{ab}\delta_{v_{ab}}))}^{d} \cdots) \\ &= \pm \frac{1}{d!}\left(\vartheta_{v_{ab}} - \frac{1}{2}\right)^d. \end{aligned}$$

Here, in a similar way as in the previous example (Example 4), in the second line we omit denoting the product $m : \Omega'^1_{aa} \otimes \Omega'^0_{ab} \to \Omega'^1_{ab}$ or $m : \Omega'^0_{ab} \otimes \Omega'^1_{bb} \to \Omega'^1_{ab}$. The difference of the situation here from that in the previous example is that here $h_{ab}\delta_{v_{ab}} = \vartheta_{v_{ab}} - (1/2)$ and we have the formula $h_{ab}\delta_{v_{ab}}(\vartheta_{v_{ab}} - (1/2))^k = (1/(k+1))(\vartheta_{v_{ab}} - (1/2))^{k+1}$, which is used in the third equality of the equation above. The remaining calculations are similar to those in Example 1; we finally obtain $P_{ad}m(-h_{ac}m(w, \mathbf{e}_{bc}), \mathbf{e}_{cd}) = \pm(1/(2^d d!))e^{-(X+Y+Z)}\mathbf{e}_{ad}$ and then

$$m_{d+1+1}((\mathbf{e}_{ab})^{\otimes(d_-,d_+)}, \mathbf{e}_{bc}, \mathbf{e}_{cd}) = \frac{1}{2^d (d_-)!(d_+)!}\mathbf{e}_{ad}.$$

Example 6. $\vec{v} = ((\overset{\circ}{v}_{ab})^{\otimes(d_-,d_+)}, v_{bc}, \overset{\circ}{v}_{cd}, (v_{da})^{\otimes d'})$, $x(v_{bc}) < x(v_{ab}) < x(v_{cd}) < x(v_{da})$. Consider the non-transversal A_∞-product

$$m_{d+1+1+d'}((\mathbf{e}_{ab})^{\otimes(d_-,d_+)}, \mathbf{e}_{bc}, \mathbf{e}_{cd}, (\mathbf{e}_{da})^{\otimes d'}).$$

This can be calculated by combining the arguments in Examples 3, 4, and 5. As in Example 3, the A_∞-product is given as the sum

$$\begin{aligned} &m_{d+1+1+d'}((\mathbf{e}_{ab})^{\otimes(d_-,d_+)}, \mathbf{e}_{bc}, \mathbf{e}_{cd}, (\mathbf{e}_{da})^{\otimes d'}) \\ &= \pm m(w, -h_{ab}m(\mathbf{e}_{bc}, w')) \pm m(-h_{ac}m(w, \mathbf{e}_{bc}), w'), \end{aligned} \tag{5.6}$$

where w is just the w in the previous example (Example 5), and w' is given in a similar way as in Example 4:

$$w = \pm\frac{1}{(d_-)!(d_+)!}\left(\vartheta_{v_{ab}} - \frac{1}{2}\right)^d, \qquad w' = \pm\frac{1}{(d')!}(\vartheta^{-1}_{v_{da}})^{d'}.$$

Here recall that $\vartheta^{-1}_{v_{da}} := 1-\vartheta_{v_{da}}$. Then, the two terms in the second line of eq.(5.6) turn out to be

$$m(w,-h_{ab}m(\mathbf{e}_{bc},w'))=\pm\frac{1}{(d_-)!(d_+)!(d')!}e^{-(X+Y+Z)}\ (\vartheta^{+1}_{v_{ab}})^d\cdot(\vartheta^{-1}_{v_{bc}}),$$
$$m(-h_{ac}m(w,\mathbf{e}_{bc}),w')=\pm\frac{1}{(d_-)!(d_+)!(d')!}e^{-(X+Y+Z)}(\vartheta_{v_{bc}})\cdot(\vartheta^{-1}_{v_{da}})^{d'}.$$

The signs in fact agree with each other so that $(\vartheta_{v_{ab}})^d\cdot(\vartheta^{-1}_{v_{bc}})+(\vartheta_{v_{bc}})\cdot(\vartheta^{-1}_{v_{da}})^{d'}=\big((\vartheta_{v_{ab}})^d-(\vartheta_{v_{bc}})\big)+\Big((\vartheta_{v_{bc}})-\Big(1-(\vartheta^{-1}_{v_{da}})^{d'}\Big)\Big)=(\vartheta_{v_{ab}})^d\cdot(\vartheta^{-1}_{v_{da}})^{d'}$, and we finally obtain

$$m_{d+1+1+d'}((\mathbf{e}_{ab})^{\otimes(d_-,d_+)},\mathbf{e}_{bc},\mathbf{e}_{cd},(\mathbf{e}_{da})^{\otimes d'})=\pm\frac{1}{(d_-)!(d_+)!(d')!}(\vartheta^{+1}_{v_{ab}})^d\cdot(\vartheta^{-1}_{v_{da}})^{d'}.$$

One can also check the case that v_{bc} has multiplicity d'', $(v_{bc})^{\otimes d''}$, where, after using the following identity $\sum_{k=0}^d C_{k,d-k}(\vartheta_{v_{bc}})^k(\vartheta^{-1}_{v_{bc}})^{d-k}=(\vartheta_{v_{bc}}+\vartheta^{-1}_{v_{bc}})^d=1$, we finally obtain just $(1/(d'')!)$ times the result above.

• **The sign** The sign is determined precisely as follows. In order to simplify the sign in HPT, one may first consider the suspension $s(\mathcal{C}'_{DR}(\mathfrak{F}_N))$ of $\mathcal{C}'_{DR}(\mathfrak{F}_N)$. Then, apply HPT to $s(\mathcal{C}'_{DR}(\mathfrak{F}_N))$ and obtain the A_∞-products of $s(\mathcal{C}(\mathcal{F}_N))$. Finally, as the desuspension of $s(\mathcal{C}(\mathcal{F}_N))$ one obtains the A_∞-products of $\mathcal{C}(\mathfrak{F}_N)$.

To see how the sign is determined, it is enough to demonstrate the calculations in the examples of transversal A_∞-products below. As we saw in eq.(5.5), a transversal A_∞-product m_n, $n\ge 2$, is described in terms of trivalent planar tree graphs, where the number of m and $-h_{ab}$ for some $a\neq b\in\mathfrak{F}_N$ are $(n-1)$ and $(n-2)$, respectively. Since the sign problem for m_2 is obvious, let us consider the case $n\ge 3$. We obtain the sign from the following three parts.

- For any product $m(w,w')$ in a tree graph, the degree $(|w|,|w'|)$ in $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ is $(0,1)$ or $(1,0)$. The suspension $s:\mathcal{C}'_{DR}(\mathfrak{F}_N)\to s(\mathcal{C}'_{DR}(\mathfrak{F}_N))$ leads to sign $(-1)^I$ for I the number of the products $m(w,w')$ with degree of type $(0,1)$ (see eq.(2.2)).
- Associated to each internal edge, we have $-h_{ab}(\delta_v)$ for some $a\neq b\in\mathfrak{F}_N$ and $v\in S_{all}$. The arrow of the internal edge is oriented from the left to the right or from the right to the left. Then, we have sign $(-1)^J$ where J is the number of the internal edges oriented from the left to the right. (Compare this argument with eq.(5.1) and eq.(5.2) with $n=1$.)
- In the process of the desuspension $s(\mathcal{C}(\mathfrak{F}_N))\to\mathcal{C}(\mathfrak{F}_N)$, an A_∞-product $m_n(w_1,\dots,w_n)$ gets sign $(-1)^K$ with $K=n-i$ if w_i, for some $1\le i\le n$, is the only degree zero element, and $K=(n-i)+(n-j)$ if w_i and w_j, for some $1\le i<j\le n$, are the only degree zero elements (see eq.(2.2)). Note that by degree counting (Lemma 3.5) there are not more than two degree zero elements in $\{w_1,\dots,w_n\}$.

Thus, $(-1)^{I+J+K}$ is the sign we finally obtain.

Let us consider the examples of the transversal A_∞-products $m_n(w_1,\dots,w_n)$ with two degree zero elements w_i and w_j for some $1\le i<j\le n$. The corresponding tree graph is described as in Figure 9 (a) and (b) when the corresponding CC-polygon $\vec{v}$ has $\sigma(v)=-1$ and $\sigma(v)=-1$, respectively. Here, note that $n=k+k'+l+l'+2$. ($i=k'+1$ and $j=k'+1+k+l+1$ for case (a), and $i=l+1$ and $j=l+1+l'+k'+1$ for case (b).) For the case (a), one obtains

$$I=k+l'+1,\qquad J=k+k',\qquad K=l'+(k+l+1+l')$$

and hence the sign is $(-1)^{I+J+K}=(-1)^{k+k'+l+l'+2}=(-1)^n$. For the case (b), one obtains

$$I=k+l'+1,\qquad J=k+k',\qquad K=k+(l'+k'+1+k)$$

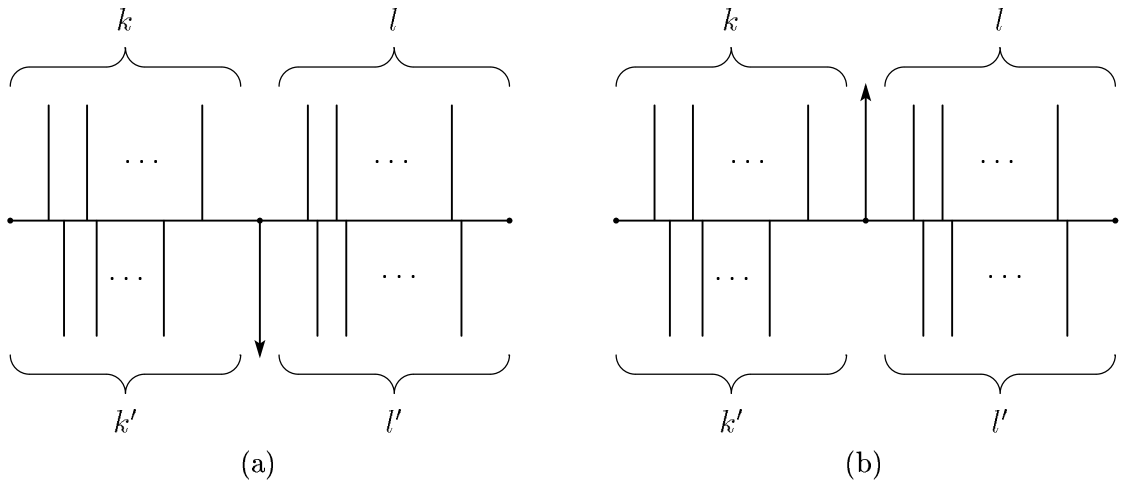


FIGURE 9. The tree graphs corresponding to CC-polygons $\vec{v}$ with (a): $\sigma(\vec{v}) = -1$ and (b): $\sigma(\vec{v}) = +1$.

and hence the sign is $(-1)^{I+J+K} = 1$. One can see that in both cases the results agree with the definition of the transversal A_∞-products in eq.(3.4). The calculation for the transversal A_∞-product $m_n(w_1, \ldots, w_n)$ with only one degree zero element in $\{w_1, \ldots, w_n\}$ is similar.

6. Concluding remarks

We can apply the arguments in the present paper to two-tori directly. Then, we can discuss the homological mirror symmetry for two-tori including non-transversal A_∞-products. In this case, note that we can include the identity morphism in the two-torus analog of the graded vector space V_{aa}, and further the DG category has a canonical nondegenerate inner product (canonical pairing of the Serre functor) defining cyclicity (see [24, 15] in noncommutative tori setting). Then, the dual of the identity morphism will be a natural representative of the cohomology of V_{aa}^1. Thus, if we start from finitely many objects, we can obtain an example of finite dimensional minimal A_∞-algebras by applying HPT again to the two-torus analog of the graded vector space V_{aa} of the A_∞-category $\mathcal{C}(\mathfrak{F}_N)$. From this viewpoint, the A_∞-category $\mathcal{C}(\mathfrak{F}_N)$ we constructed in this paper is an intermediate step. In particular, in $\mathbb{R}^2$ case, the A_∞-structure is not equipped with cyclicity in the sense as in [15]; though cyclicity for an A_∞-structure is defined by a *non-degenerate* inner product, the inner product defined naturally in this case becomes degenerate on V_{aa}.

The generalization of the story of the present paper ($\mathbb{R}$ case) to $\mathbb{R}^n$ case is also an important issue, where, though the generalization of the DG-category $\mathcal{C}_{DR}(\mathfrak{F}_N)$ is straightforward (see [16, 17]), we need to define a higher dimensional analog of the DG-category $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ so that HPT can be applied to it. The construction of the higher dimensional analogue of $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ is not straightforward, [3] but it seems not still impossible. This higher dimensional generalization also enables us to consider nontrivial noncommutative deformation of the A_∞-categories (see [16, 17]).

The reader might notice that elements in V_{aa}^1 played a special role in the present paper. In fact, the elements in V_{aa}^1 is related to open string background, *i.e.*, the solutions of the Maurer-Cartan equation of the A_∞-structure (see [6, 18]). In $\mathbb{R}^2$ case, the Maurer-Cartan equation will be trivial, which implies that all elements in V_{aa}^1 can be the solution of the Maurer-Cartan equation. Then, nontrivial deformation of lines to curves in $\mathbb{R}^2$ can also be taken into account in this framework.

[3] For instance, even if we consider affine Lagrangians only, the orbits of the gradient flow are not affine. Since the action of the homotopy operator h_{ab} is defined by the orbits, those non-affine orbits cause various subtleties as pointed out by K. Fukaya to the author. Another approach to construct an A_∞-structure of a Fukaya category on a torus fibration is discussed in [4] where we can avoid this kind of subtleties.

Finally, instead of the application to tori, we hope to apply the arguments of this paper to more general manifolds since the A_∞-categories in this $\mathbb{R}^2$ case and their higher dimensional generalization, if it could be done, might be thought of a local construction of the A_∞-categories which should be defined on the whole manifolds.

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