ON MATRIX DIFFERENTIAL EQUATIONS IN THE HOPF ALGEBRA OF RENORMALIZATION

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Abstract. We establish Sakakibara's differential equations [Sa04] in a matrix setting for the counter term (respectively renormalized character) in Connes–Kreimer's Birkhoff decomposition in any connected graded Hopf algebra, thus including Feynman rules in perturbative renormalization as a key example.

Contents

1. Introduction

Quantum field theory (QFT) unifies the fundamental principles of special relativity and quantum theory and provides the appropriate physical framework to describe phenomena at the smallest length scales respectively highest energies. Its mathematical structure is far from being as simple as that of its basic constituents. Moreover, up to now, perturbation theory is the most successful quantitative and qualitative approach to QFT. Although general agreement between theoretically predicted results in the perturbative regime of QFT and those experimentally measured has reached a satisfactory status, a truly non-perturbative understanding of the physics of quantum phenomena is mandatory, both for future advancements in terms of fundamental as well as calculational problems.

Perturbative QFT consist of two fundamental ingredients, the gauge principle and the concept of renormalization. The latter consists of an arbitrary regularization prescription, which parameterizes ultraviolet divergencies appearing in Feynman amplitudes and thereby renders them formally finite, together with a specific subtraction rule of those ill-defined expressions dictated by physical principles. Whereas both the gauge principle and the concept of renormalization experienced a rich development in theoretical physics, the former especially came to the fore of mathematical research with rich interactions between mathematicians and physicists. However, the latter suffered from the lack of an equally strong development of its mathematical aspects.

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Kreimer's recent findings [Krei98, Krei99a] mark a turning point in this context. He discovered a mathematical structure underlying renormalization in perturbative quantum field theory in terms of connected graded commutative Hopf algebras. Feynman rules are interpreted as Hopf algebra characters which associate to each Feynman graph its corresponding amplitude.

The concept of regularization in general introduces non-physical parameters. This process changes the nature of Feynman rules drastically, i.e., from linear multiplicative maps into the underlying base field, to algebra morphisms with image in a commutative unital algebra, e.g., Laurent series in dimensional regularization. Hence we identify regularized Feynman rules with a particular subclass of such maps from the Hopf algebra of Feynman graphs into a commutative unital algebra dictated by the regularization scheme.

Connes and Kreimer extended the results on the Hopf algebraic approach to perturbative renormalization by establishing the Hopf algebra of Feynman graphs including the concept of the renormalization group [CK99, CK00, CK01, CK02]. Moreover, Connes and Kreimer formulated in this picture the intricate process of perturbative renormalization in terms of an algebraic Birkhoff decomposition of regularized Feynman rules, using the minimal subtraction scheme in dimensional regularization.

In [EG05, EGGV06] it was shown how to organize the combinatorics of renormalization in terms of (pro-)nil- and unipotent triangular matrix representations with entries in a commutative Rota–Baxter algebra. A simple factorization of such matrices was derived using explicit non-recursive equations containing the renormalization scheme operator. This simple matrix decomposition offers a transparent picture of the process of renormalization in terms of the factorization of Feynman rules matrices.

In this work we would like to further develop the matrix calculus approach to perturbative renormalization in the abstract context of connected graded Hopf algebras. Any left coideal gives rise to a representation of the group of characters of the Hopf algebra by lower triangular unipotent matrices, the size of which being given by the dimension of the coideal. We investigate the matrix representation of two fundamental concepts which can be defined in this purely algebraic framework: the renormalization group and the beta-function. We retrieve then M. Sakakibara's differential equations involving the beta-function, giving to his approach the firm ground of triangular matrix calculus.

Before starting we should point the reader to the following papers [CaKe82, Col06, Del04, EK05, FG05, Krei03, Man01, tHVel73] and books [Col84, FGV01, IzZu80, Krei00, Muta87, Vas04] which are useful as introductory references, both with respect to perturbative QFT and renormalization theory, as well as its recently discovered Hopf-algebraic structures. Also, some readers may find it stimulating to leaf through the books by Brown [Br93] and Schweber [Schwe94] as well as the more recent one by Kaiser [Kai05] for some scientific-historical perspectives on Feynman graphs in QFT and renormalization theory. Schwinger's collection of reprints [Schwi58] contains many of the original articles marking the beginning of modern perturbative QFT and renormalization theory. Comprehensive treatments of Hopf algebras can be found in [Abe80, Sw69], see also the paper by Bergman [Berg85]. Other useful references are [ChPr95, FGV01, Kas95, Maj95, StSh93]. Hopf algebras in the context of combinatorics appeared in the work of Rota [Rota78], and Joni and Rota [JoRo79], see also [FG05, NiSw82, Schm95, SpDo97].

Let us briefly outline the organization of this paper. In section 2, after reminding Connes–Kreimer's Birkhoff decomposition of characters in the most general context of connected filtered Hopf algebras, we define the matrix representation associated with a left coideal, along the lines of [EG05], and write down the matrix counterpart of the Birkhoff decomposition. In section 3, we first define the renormalization group and the beta-function in the context of connected graded Hopf algebras, along the lines of [CK00] and [Man01], and then we describe the matrix counterparts of these notions. The key point is that the grading biderivation Y of the Hopf algebra can be represented by a diagonal matrix. The semidirect product of the group of characters with the associated one-parameter group of automorphisms can then be represented by (non-unipotent) lower-triangular matrices.

The two last subsections are devoted to a careful rewriting of some important results of M. Sakakibara ([Sa04]) in the matrix representation, yielding matrix differential equations for the beta-function.

2. The general set up

In the sequel k denotes the ground field with $char(k) = 0$ over which all algebraic structures are defined. Here the term algebra always means unital associative k-algebra, denoted by the triple $(\mathcal{A}, m_\mathcal{A}, \eta_\mathcal{A})$, where $\mathcal{A}$ is a k-vector space with a product $m_\mathcal{A} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a unit map $\eta_\mathcal{A} : k \to \mathcal{A}$. Similarly for coalgebras over k, denoted by the triple $(\mathcal{C}, \Delta_\mathcal{C}, \epsilon_\mathcal{C})$, where the coproduct map $\Delta_\mathcal{C} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is coassociative and $\epsilon_\mathcal{C} : \mathcal{C} \to k$ denotes the counit map. A subspace $\mathcal{J} \subset \mathcal{C}$ is called a left coideal if $\Delta_\mathcal{C}(\mathcal{J}) \subset \mathcal{C} \otimes \mathcal{J}$. A Hopf algebra, denoted by $(\mathcal{H}, m_\mathcal{H}, \eta_\mathcal{H}, \Delta_\mathcal{H}, \epsilon_\mathcal{H}, S)$, is a bialgebra together with the antipode $S : \mathcal{H} \to \mathcal{H}$, that is, it consists of an algebra and coalgebra structure in a compatible way and S is a k-linear map on $\mathcal{H}$ satisfying the Hopf algebra axioms [Abe80, Sw69]. In the following we omit subscripts for notational transparency if there is no danger of confusion, and denote algebras, coalgebras and Hopf algebras simply by $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{H}$, respectively.

2.1. **Connected filtered Hopf algebra.** Let $\mathcal{H}$ be a connected filtered bialgebra:

$$k = \mathcal{H}^{(0)} \subset \mathcal{H}^{(1)} \subset \cdots \subset \cdots \mathcal{H}^{(n)} \subset \cdots, \qquad \bigcup_{n \geq 0} \mathcal{H}^{(n)} = \mathcal{H},$$

and let $\mathcal{A}$ be any commutative algebra. The space $\mathcal{L}(\mathcal{H}, \mathcal{A})$ of linear maps from $\mathcal{H}$ to $\mathcal{A}$ together with the convolution product $f \star g := m_\mathcal{A} \circ (f \otimes g) \circ \Delta$, $f, g \in \mathcal{L}$:

$$\mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{f \otimes g} \mathcal{A} \otimes \mathcal{A} \xrightarrow{m_\mathcal{A}} \mathcal{A},$$

is an algebra with unit $e := \eta_\mathcal{A} \circ \epsilon$. For any $x \in \mathcal{H}^{(n)}$ we have, using a variant of Sweedler's notation [Sw69]:

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_{(x)} x' \otimes x'', \tag{1}$$

where the filtration degrees of x' and x'' are strictly smaller than n. Recall that by definition we call an element $x \in \mathcal{H}$ primitive if:

$$\bar{\Delta}(x) := \Delta(x) - x \otimes 1 - 1 \otimes x = 0.$$

The convolution product on $\mathcal{L}(\mathcal{H}, \mathcal{A})$ writes then with Sweedler's notation:

$$(f \star g)(x) = f(x)g(1) + f(1)g(x) + \sum_{(x)} f(x')g(x'') \in \mathcal{A}. \tag{2}$$

The filtration of $\mathcal{H}$ implies a decreasing filtration on $\mathcal{L}(\mathcal{H}, \mathcal{A})$ in terms of $\mathcal{L}^n := \{f \in \mathcal{L} \big| \ f_{|_{\mathcal{H}^{(n-1)}}} = 0\}$ and $\mathcal{L}(\mathcal{H}, \mathcal{A})$ is complete with respect to the induced topology [Man01]. The subset $\mathfrak{g}_0 := \mathcal{L}^1 \subset \mathcal{L}(\mathcal{H}, \mathcal{A})$ of linear maps α that send the bialgebra unit to zero, $\alpha(1) = 0$, forms a Lie algebra in $\mathcal{L}(\mathcal{H}, \mathcal{A})$. The exponential:

$$\exp^\star(\alpha) = \sum_k \frac{1}{k!} \alpha^{\star k}$$

makes sense and is a bijection from $\mathfrak{g}_0$ onto the group $G_0 = e + \mathfrak{g}_0$ of linear maps γ that send the bialgebra unit to the algebra unit, $\alpha(1) = 1_\mathcal{A}$ [Man01].

An infinitesimal character with values in $\mathcal{A}$ is a linear map $\xi \in \mathcal{L}(\mathcal{H}, \mathcal{A})$ such that for $x, y \in \mathcal{H}$:

$$\xi(xy) = \xi(x)e(y) + e(x)\xi(y). \tag{3}$$

We denote by $\mathfrak{g}_\mathcal{A} \subset \mathfrak{g}_0$ the linear space of infinitesimal characters. We call an $\mathcal{A}$-valued map ρ in $\mathcal{L}(\mathcal{H}, \mathcal{A})$ a character if for $x, y \in \mathcal{H}$:

$$\rho(xy) = \rho(x)\rho(y), \tag{4}$$

The set of such unital algebra morphisms is denoted by $G_\mathcal{A} \subset G_0$.

Let us now assume that $\mathcal{A}$ is a commutative algebra. It is easily verified then, see for instance [Man01], that the set $G_\mathcal{A}$ of characters from $\mathcal{H}$ to $\mathcal{A}$ forms a group for the convolution product, in fact it is the pro-unipotent *group*[1] *of $\mathcal{A}$-valued morphisms on the bialgebra $\mathcal{H}$.* And $\mathfrak{g}_{\mathcal{A}}$ in $\mathfrak{g}_0$ is the corresponding pro-nilpotent Lie algebra. The exponential map $\exp^\star$ restricts to a bijection between $\mathfrak{g}_{\mathcal{A}}$ and $G_{\mathcal{A}}$. The neutral element $e := \eta_{\mathcal{A}} \circ \epsilon$ in $G_{\mathcal{A}}$ is given by $e(1) = 1_{\mathcal{A}}$ and $e(x) = 0$ for $x \in \operatorname{Ker}\epsilon$. The inverse of $\varphi \in G_{\mathcal{A}}$ is given by composition with the antipode S:

$$\varphi^{\star-1} = \varphi \circ S. \tag{5}$$

Recall that the antipode $S : \mathcal{H} \to \mathcal{H}$ is the inverse of the identity for the convolution product on $\mathcal{L}(\mathcal{H}, \mathcal{H})$:

$$S \star Id = m \circ (S \otimes Id) \circ \Delta = \eta \circ \epsilon = Id \star S. \tag{6}$$

It always exists in a connected filtered bialgebra, hence any connected filtered bialgebra is a *connected filtered Hopf algebra*. The antipode is defined by:

$$S = \sum_{n \geq 0} (\eta \circ \epsilon - Id)^{\star n}. \tag{7}$$

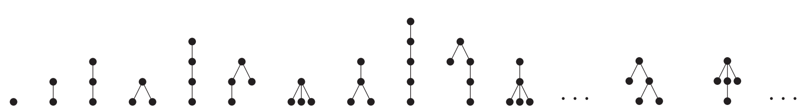
Recall that $\Delta^{(0)} := Id$ and for $n > 0$ $\Delta^{(n)} := (\Delta^{(n-1)} \otimes Id) \circ \Delta$. Equations (6) imply the following recursive formulas for the antipode starting with $S(1) = 1$ and for $x \in \operatorname{Ker}\epsilon$:

$$S(x) = -x - \sum_{(x)} S(x')x'', \tag{8}$$

$$S(x) = -x - \sum_{(x)} x'S(x''). \tag{9}$$

Example 1. Toy-model of decorated non-planar rooted trees: As a guiding example we will use the Hopf algebra of non-planar rooted trees established by Kreimer [Krei99a]. It provides the role model for the Hopf algebraic formulation of perturbative renormalization [CK99]. In fact, linear combinations of –decorated– non-planar rooted trees naturally encode the hierarchical structure of divergencies of a Feynman graph. Each vertex of a rooted tree represents a primitive divergence indicated by a decoration with a primitive one-particle (1PI) irreducible Feynman graph. Edges connecting such vertices encode thereby the nesting of subdivergencies, i.e., proper 1PI subgraphs sitting inside another 1PI graph. The root vertex is the overall divergence which contains those subdivergencies. Linear combinations of such decorated rooted trees may represent Feynman graphs with overlapping divergence structures [Krei99b].

By definition a rooted tree t is made out of vertices and nonintersecting oriented edges, such that all but one vertex have exactly one incoming line. We denote the set of vertices and edges of a rooted tree by $V(t)$, $E(t)$ respectively. The root is the only vertex with no incoming line. We draw the root on top of the tree. Let T denote the set of isomorphic classes of rooted trees. The empty tree is denoted by $1_{\mathcal{T}}$:



Let $\mathcal{T}$ be the k-vector space generated by T, which is graded by the number of vertices, $\deg(t) := |t| := |V(t)|$, $t \in \mathcal{T}$, with the convention that $\deg(1_{\mathcal{T}}) = 0$. Let $\mathcal{H}_{\mathcal{T}}$ be the graded commutative polynomial algebra of finite type over k generated by $\mathcal{T}$, $\mathcal{H}_{\mathcal{T}} := k[\mathcal{T}] = \bigoplus_{n \geq 0} k\mathcal{H}^{(n)}$. Monomials of trees are called forests. We extend $\deg(t_1 \dots t_n) := \sum_{i=1}^n \deg(t_i)$.

We will define a coalgebra structure on $\mathcal{T}$. The coproduct is defined in terms of *cuts* $c \subset E(t)$ on a tree $t \in \mathcal{T}$. A *primitive cut* is the removal of a single edge, $|c| = 1$, from the tree t. The tree t decomposes into two parts, denoted by the pruned part $P_c(t)$ and the rooted part $R_c(t)$, where the latter contains the original root vertex. An *admissible cut* of a rooted tree t is a set of primitive cuts, $|c| \geq 1$, such that along the unique path from the root to any vertex of t one encounters at most one cut.

[1]It is more precisely a group scheme, i.e. a functor $\mathcal{A} \mapsto G_{\mathcal{A}}$ from k-algebras to groups.

Let C_t be the set of all admissible cuts of the rooted tree $t \in \mathcal{T}$. We exclude the empty cut $c^{(0)}$: $P_{c^{(0)}}(t) = \emptyset$, $R_{c^{(0)}}(t) = t$, and the full cut $c^{(1)}$: $P_{c^{(1)}}(t) = t$, $R_{c^{(1)}}(t) = \emptyset$. Also let $C_t^{(01)}$ be $C_t \cup \{c^{(0)}(t), c^{(1)}(t)\}$. The *coproduct* is defined by:

$$\Delta(t) := t \otimes 1_{\mathcal{T}} + 1_{\mathcal{T}} \otimes t + \sum_{c \in C_t} P_c(t) \otimes R_c(t) = \sum_{c \in C_t^{(01)}} P_c(t) \otimes R_c(t). \tag{10}$$

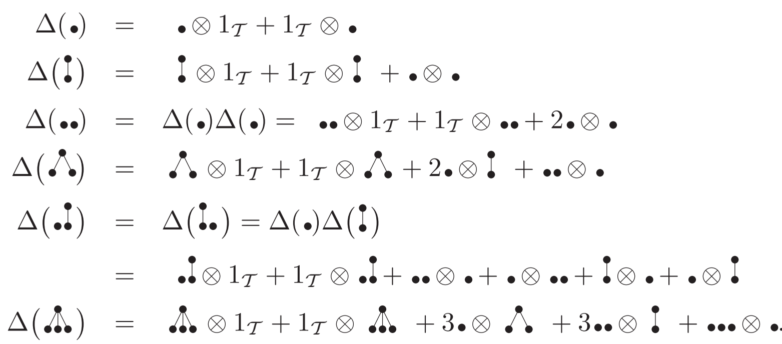
We shall call the rooted tree $R_c(t)$ the cotree (or cograph) corresponding to the admissible cut c on the tree t. One sees easily, that $\deg(t) = \deg(P_c(t)) + \deg(R_c(t))$, for all admissible cuts $c \in C_t$, and therefore:

$$\bar{\Delta}(t) = \sum_{c \in C_t} P_c(t) \otimes R_c(t) \in \sum_{p+q=\deg(t),\ p,\,q>0} \mathcal{H}^{(p)} \otimes \mathcal{H}^{(q)}.$$

Furthermore, this map is extended by definition to an algebra morphism on $\mathcal{H}_{\mathcal{T}}$:

$$\Delta\Big(\prod_{i=1}^{n} t_i\Big) := \prod_{i=1}^{n} \Delta(t_i).$$

The best way to get use to this particular coproduct is to present some examples:



One observes immediately that the vector space $\mathcal{T}$ defines a left coideal, that is :

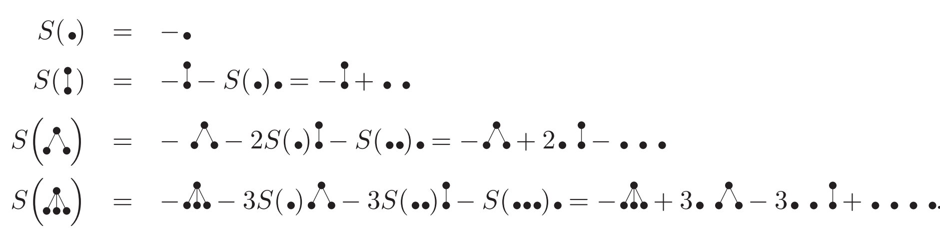
$$\mathcal{T} \xrightarrow{\Delta} \mathcal{H}_{\mathcal{T}} \otimes \mathcal{T}.$$

The experienced reader will easily recognize that this coproduct efficiently stores the so-called wood $\mathfrak{W}(\Gamma)$ for the Feynman graph Γ with its hierarchy of subdivergencies represented by the tree $t \sim \Gamma$. A wood simply contains all spinneys[2] of the graph [CaKe82, EG06]. The right-hand side of above coproduct consists of the cograph following form the contraction of the corresponding spinney denoted on the left-hand side.

Connes and Kreimer showed that $\mathcal{H}_{\mathcal{T}}$ with coproduct (10), and counit ϵ defined by $\epsilon(1) := 1_k$ and zero else is a connected $\mathbb{Z}_{\geq 0}$-graded commutative, non-cocommutative bialgebra of finite type and hence a Hopf algebra, with antipode S defined recursively by $S(1_{\mathcal{T}}) = 1_{\mathcal{T}}$ and:

$$S(t) := -t - \sum_{c \in C_t} S(P_c(t)) R_c(t).$$

Again, a couple of examples might be helpful here:



(11)

(12)

[2]A spinney is a –possibly non-connected– subgraph with one-particle irreducible components.

We chose a simple decoration of tree vertices by tree-factorials [Krei99a, KrDe99] defined as follows. Let $t \in \mathcal{T}$, each primitive cut, that is, each edge $c \in E(t)$, defines two rooted trees, i.e., the pruned tree $t_{v_c} := P_c(t)$ and $R_c(t)$. The root of the former is the vertex $v_c \in V(t)$ which had c as its incoming edge. The root of the cotree $R_c(t)$ is the original root. Let $w(t_{v_c}) := |V(t_{v_c})|$, then $\deg(t) = \deg(w(t_{v_c})) + \deg(R_c(t))$. The tree-factorial of t is defined by:

$$t^! := \prod_{\substack{c \in C_t^{(1)} \\ |c|=1}} w(t_{v_c}) = \prod_{v \in V(t)} w(t_v).$$

The second equality is clear as to each vertex we can associate the unique incoming edge $c \in E(t)$. As examples we mention:

$\bullet^! = 1$, $^! = 2$, $^! = 6$,  $^! = 3$,  $^! = 8$, and  $^! = 4$.

In the following we decorate each vertex $v \in V(t)$ of a rooted tree $t \in \mathcal{T}$ by its tree-factorial, $t^!_v$. For notational transparency we omit the decorations on the trees.

Recall the function:

$$\int_0^\infty y^{-az}(y+c)^{-1-bz}dy = B\big((a+b)z, 1-az\big)c^{-(a+b)z} \text{ , where } B(u,v) := \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},$$

where $c > 0$ and $\Gamma(a)$ is the usual Euler Gamma-function [Krei00]. We define now the family of functions:

$$B_n := B_n(z) := B(nz, 1-nz), \qquad n > 0.$$

Following [Krei99a, KrDe99] we define the following regularized toy-model character $\varphi = \varphi(a,\mu,z)$ from the Hopf algebra of integer decorated rooted trees $\mathcal{H}_\mathcal{T}$ to $\mathcal{A} := \mathbb{C}[z^{-1},z]][[\log(a/\mu)]]$, $a/\mu > 0$: $\varphi(1_\mathcal{T})(a,\mu,z) := 1_\mathcal{A}$,

$$\varphi(t)(a,\mu,z) := \left(\frac{a}{\mu}\right)^{-z|t|} \prod_{v \in V(t)} B_{w(t_v)}, \text{ for } \ \mathrm{t} \in \mathrm{Ker}\,\epsilon. \tag{13}$$

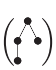
Here, a is assumed to be a dimensional *external* parameter, and μ is the so-called 't Hooft mass. The latter is an arbitrarily chosen parameter of the same dimension as a, such that the ratio a/μ is a positive number. The parameter μ introduces an external scale specific to dimensional regularization [Col84]. Let us give a few examples by applying $\varphi(a,\mu,z)$ to some trees. Defining $\alpha := \frac{a}{\mu}$ we find:

$$\varphi(\bullet)(a,\mu,z) = \alpha^{-z}B_1 = \alpha^{-z}B(z,1-z) = \alpha^{-z}\frac{\pi}{\sin \pi z} = \alpha^{-z}\big(\frac{1}{z} + \frac{\pi^2}{6}z + O(z^2)\big)$$

$\varphi($$)(a,\mu,z) = \alpha^{-2z}B_2B_1 = \alpha^{-2z}B(2z,1-2z)B(z,1-z),$

$\varphi\Big($$\Big)(a,\mu,z) = \alpha^{-3z}B_3B_1^2 = \alpha^{-3z}B(3z,1-3z)B(z,1-z)B(z,1-z),$

$\varphi\Big($$\Big)(a,\mu,z) = \alpha^{-4z}B_4B_1^3 = \alpha^{-4z}B(4z,1-4z)B(z,1-z)B(z,1-z)B(z,1-z),$

$\varphi\Big($$\Big)(a,\mu,z) = \alpha^{-4z}B_4B_2B_1B_1 = \alpha^{-4z}B(4z,1-4z)B(2z,1-2z)B^2(z,1-z).$

These types of models exemplify a rich structure capturing some aspects of real QFT calculations and we refer the reader to [BK99, Krei99a, KrDe99, Krei00] for more details on such toy-models.

For latter use we parameterize 't Hooft's mass:

$$\mu \to \mu(s) := e^s\mu, \ s \in \mathbb{R}, \tag{14}$$

such that $\alpha := a/\mu \to \alpha(s)$, and for a fixed a we define $\varphi(a,\mu,z) \to \varphi(\alpha(s),z) =: \varphi(s,z)$.

2.2. **Connes–Kreimer's Birkhoff decomposition of Hopf algebra characters.** Connes and Kreimer extended the work by Kreimer [Krei98, Krei99a] and established the connected graded commutative non-cocommutative Hopf algebra of Feynman graphs corresponding to a perturbative quantum field theory (pQFT).

Moreover, in the context of minimal subtraction as renormalization scheme in dimensional regularization Connes and Kreimer [CK99, CK00] discovered a unique Birkhoff type decomposition of Hopf algebra characters with values in the $\mathbb{C}$-algebra $\mathcal{A}$ of meromorphic functions capturing the process of renormalization in pQFT. Namely for any $\varphi \in G_{\mathcal{A}}$ we have:

$$\varphi = \varphi_-^{\star -1} \star \varphi_+, \tag{15}$$

where both φ_- and φ_+ belong to $G_{\mathcal{A}}$, and $\varphi_+(x) \in \mathcal{A}_+$ for any $x \in \mathcal{H}$, whereas $\varphi_-(x) \in \mathcal{A}_-$ for any $x \in \operatorname{Ker}\epsilon$. Here, $\mathcal{A}_-$ is the algebra of polynomials in $(z - z_0)^{-1}$ without constant term (the 'pole parts'), and $\mathcal{A}_+$ is the algebra of meromorphic functions which are holomorphic at z_0, corresponding to the splitting of the $\mathbb{C}$-algebra $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ of meromorphic functions. We denote by $\pi : \mathcal{A} \to \mathcal{A}_-$ the projection onto $\mathcal{A}_-$.

The components φ_- and φ_+ are given by recursive formulas. Suppose that $\varphi_-(x)$ and $\varphi_+(x)$ are known for $x \in \mathcal{H}^{(n-1)}$. Define for $x \in \mathcal{H}^{(n)}$ *Bogoliubov's preparation map*:

$$\bar{R} : x \longmapsto \varphi_- \star (\varphi - e)(x) = \varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x''). \tag{16}$$

The components in the Birkhoff decomposition are then given by:

$$\varphi_-(x) = -\pi\left(\bar{R}(x)\right), \tag{17}$$

$$\varphi_+(x) = (1_{\mathcal{A}} - \pi)\left(\bar{R}(x)\right). \tag{18}$$

The factor $\varphi_+(x)$ is the *renormalized character* whereas $\varphi_-(x)$ is the sum of *counter terms* one must add to $\bar{R}(x)$ to get $\varphi_+(x)$. In the example of minimal subtraction scheme the *renormalized value* of the character φ at $z_0 \in \mathbb{C}$ is the well-defined complex number $\varphi_+(z_0)$, whereas $\varphi(z_0)$ may not exist. The fact that φ_- and φ_+ are still characters relies on the Rota–Baxter property for the projection π, see e.g. [EGK04, EGK05, EG06]. Moreover, Connes–Kreimer's results do not depend on the type of regularization or subtraction scheme.

In terms of the toy-model character $\varphi(s,z) = \varphi(a, e^s\mu, z)$, see equation (13), with parameterized 't Hooft mass (14), mapping the Hopf algebra $\mathcal{H}_{\mathcal{T}}$ of non-planar integer decorated rooted trees to $\mathcal{A} := \mathbb{C}[z^{-1}, z]][[\log(\alpha(s))]]$, $\alpha(s) := a/(e^s\mu) > 0$, which decomposes into:

$$\mathcal{A} = z^{-1}\mathbb{C}[z^{-1}][[\log(\alpha(s))]] \oplus \mathbb{C}[z]][[\log(\alpha(s))]],$$

we find for the primitive tree $\bullet \in \mathcal{H}^{(1)}$ the counter term:

$$\begin{aligned} \varphi_-(\bullet)(s,z) &= -\pi\big(\bar{R}(\bullet)\big) = -\pi\big(\varphi(\bullet)\big) \\ &= -\pi(\alpha(s)^{-z}B_1) \in z^{-1}\mathbb{C}[z^{-1}] \end{aligned}$$

and for $\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix} \in \mathcal{H}^{(3)}$ we find:

$$\begin{aligned} \bar{R}\big(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}\big) &= \varphi\big(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}\big) + 2\varphi_-(\bullet)\varphi\big(\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}\big) + \varphi_-(\bullet\bullet)\varphi_-(\bullet) \\ &= \alpha^{-3z}B_3B_1^2 - 2\pi(\alpha^{-z}B_1)\alpha^{-2z}B_2B_1 + \pi(\alpha^{-2z}B_1^2)\alpha^{-z}B_1, \end{aligned}$$

such that the counter term is given by:

$$\varphi_-\big(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}\big)(s,z) = -\pi\Big(\varphi\big(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}\big) + 2\varphi_-(\bullet)\varphi\big(\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}\big) + \varphi_-(\bullet\bullet)\varphi_-(\bullet)\Big).$$

A detailed calculation shows that $\varphi_-\big(\begin{smallmatrix}&\bullet&\\ \bullet&&\bullet\end{smallmatrix}\big)(s,z) \in z^{-1}\mathbb{C}[z^{-1}]$. Hence any dependence on μ has disappeared. Feynman rule characters in dimensional regularization depend on the unit of mass μ. However, for the corresponding counter terms in their Birkhoff decomposition it is true in general that $\varphi_-(t)(s,z) \in z^{-1}\mathbb{C}[z^{-1}]$, $t \in \mathcal{H}_{\mathcal{T}}$. In terms of our toy-model character (13) with parameterized unit mass $\mu = \mu(s)$ (14) this may be summarized by saying that $\partial_s\varphi_-(t)(s,z) = 0$, $t \in \mathcal{H}_{\mathcal{T}}$. We will come back to this in section 3.

The reader should compare the above counter term with the expression for $S\big(\substack{\bullet\\ \diagup\diagdown\\ \bullet\quad\bullet}\big)$ in Equation (11). Eventually, the corresponding renormalized expression $\varphi_+\big(\substack{\bullet\\ \diagup\diagdown\\ \bullet\quad\bullet}\big) := (1_{\mathcal{A}} - \pi)\big(\bar{R}\big(\substack{\bullet\\ \diagup\diagdown\\ \bullet\quad\bullet}\big)\big)$ is then given by

$$\varphi_- \star \varphi\big(\substack{\bullet\\ \diagup\diagdown\\ \bullet\quad\bullet}\big) = \varphi_-\big(\substack{\bullet\\ \diagup\diagdown\\ \bullet\quad\bullet}\big) + \varphi\big(\substack{\bullet\\ \diagup\diagdown\\ \bullet\quad\bullet}\big) + 2\varphi_-(\bullet)\varphi\big(\substack{\bullet\\ |\\ \bullet}\big) + \varphi_-(\bullet\bullet)\varphi_-(\bullet).$$

The reader should verify that $\varphi_-(\bullet\bullet) = \varphi_-(\bullet)\varphi_-(\bullet)$, that is, $\varphi_- \in G_{\mathcal{A}}$, hence $\varphi_+ = \varphi_- \star \varphi \in G_{\mathcal{A}}$.

2.3. **The matrix representation.** In this section we recall the matrix representation of $\mathcal{L}(\mathcal{H}, \mathcal{A})$ associated with a left coideal. The next step consists in understanding the beta-function of Connes–Kreimer [CK01] in this matrix setting. We follow Sakakibara's approach [Sa04], giving his clever computations the concrete support of triangular matrices.

Let us start by retrieving some material from [EG05, EG06, EGM06]. Recall that the subalgebra $\mathcal{M}_n^\ell(\mathcal{A}) \subset \mathcal{M}_n(\mathcal{A})$ of lower triangular matrices in the algebra of $n \times n$ matrices with entries in the algebra $\mathcal{A}$, and with n finite or infinite has a decreasing filtration and is complete in the induced topology. Indeed, $\mathcal{M}_n^\ell(\mathcal{A})^m$ is the ideal of strictly lower triangular matrices with zero on the main diagonal and on the first $m-1$ subdiagonals, $m > 1$. We then have the decreasing filtration

$$\mathcal{M}_n^\ell(\mathcal{A}) \supset \mathcal{M}_n^\ell(\mathcal{A})^1 \supset \cdots \supset \mathcal{M}_n^\ell(\mathcal{A})^{m-1} \supset \mathcal{M}_n^\ell(\mathcal{A})^m \supset \cdots, m < n,$$

with

$$\mathcal{M}_n^\ell(\mathcal{A})^u\, \mathcal{M}_n^\ell(\mathcal{A})^v \subset \mathcal{M}_n^\ell(\mathcal{A})^{u+v}.$$

For $\mathcal{A}$ being commutative we denote by $\mathfrak{M}_n(\mathcal{A})$ the group of lower triangular matrices with unit diagonal which is $\mathfrak{M}_n(\mathcal{A}) = \mathbf{1} + \mathcal{M}_n^\ell(\mathcal{A})^1$. Here the $n \times n$, $n \leq \infty$, unit matrix is given by

$$\mathbf{1} := (\delta_{ij} 1_{\mathcal{A}})_{1 \leq i,j \leq n}. \tag{19}$$

Let $\mathcal{H}$ be a connected filtered Hopf algebra over k, let $\mathcal{A}$ be any commutative unital k-algebra, and let $\big(\mathcal{L}(\mathcal{H}, \mathcal{A}), \star\big)$ be the algebra of k-linear maps from $\mathcal{H}$ to $\mathcal{A}$ endowed with the convolution product. Let J be any left coideal of $\mathcal{H}$ (i.e. a vector subspace of $\mathcal{H}$ such that $\Delta(J) \subset \mathcal{H} \otimes J$).

We fix a basis $X = (x_i)_{i \in I}$ of the left coideal J (a *left subcoset* in the terminology of [EG05]). We suppose further that this basis is denumerable (hence indexed by $I = \mathbb{N}$ or $I = \{1, \cdots, m\}$) and *filtration ordered*, i.e. such that if $i \leq j$ and $x_j \in \mathcal{H}^{(n)}$, then $x_i \in \mathcal{H}^{(n)}$.

Definition 1. *The coproduct matrix in the basis X is the $|I| \times |I|$ matrix M with entries in $\mathcal{H}$ defined by :*

$$\Delta(x_i) = \sum_{j \in I} M_{ij} \otimes x_j.$$

Lemma 1. *The coproduct matrix is lower-triangular with diagonal terms equal to* 1*.*

Proof. Suppose $x_i \in \mathcal{H}^{(n)}$ and $x_i \notin \mathcal{H}^{(n-1)}$. Then it is well-known that (see e.g. [FG05], [Man01]):

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i + \text{ terms of filtration degree } \leq n-1.$$

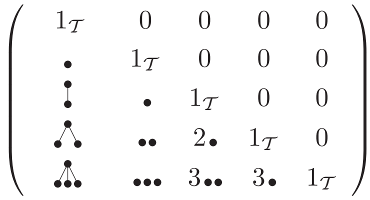
Then clearly $M_{ii} = 1$, and moreover if $M_{ij} \neq 0$ and $i \neq j$, then $x_j \in \mathcal{H}^{(n-1)}$. If $i < j$ this implies $x_i \in H^{(n-1)}$, which contradicts the hypothesis. Hence $M_{ij} = 0$ if $i < j$. $\square$

Recalling the example of rooted trees and choosing the following subset $\mathcal{T}' \subset \mathcal{T}$ of rooted trees with the displayed linear order:

$$\mathcal{T}' := \Big\{ t_1 := 1_{\mathcal{T}},\ t_2 := \bullet,\ t_3 := \substack{\bullet\\ |\\ \bullet},\ t_4 := \substack{\bullet\\ \diagup\diagdown\\ \bullet\quad\bullet},\ t_5 := \substack{\bullet\\ \diagup\,|\,\diagdown\\ \bullet\,\bullet\,\bullet} \Big\}. \tag{20}$$

The 5×5 coproduct matrix of Definition 1 is then given by:

(21) $M =$



Observe that for each tree $t_i \in \mathcal{T}'$ all cotrees in $\bar{\Delta}(t_i)$ are of degree strictly lower than $\deg(t_i)$ and are contained in $\mathcal{T}'$, hence $\mathcal{T}'$ forms a left coideal in $\mathcal{H}_{\mathcal{T}}$.

Now define $\Psi_J : \mathcal{L}(\mathcal{H}, \mathcal{A}) \to \mathrm{End}_{\mathcal{A}}(\mathcal{A} \otimes J)$ by :

$$\Psi_J[f](x_j) = \sum_i f(M_{ij}) \otimes x_i. \tag{22}$$

In other words, the matrix of $\Psi_J[f]$ is given by $\big(f(M_{ij})\big)_{i,j \in I}$.

Proposition 2. *The map Ψ_J defined above is an algebra homomorphism. Its transpose does not depend on the choice of the basis.*

Proof. The second statement is straightforward: let us denote by ${}^{\top}\Psi_J$ the transpose of Ψ_J, i.e. the map defined by:

$$^{\top}\Psi_J[f](x_i) = \sum_j f(M_{ij}) \otimes x_j.$$

For any $x \in J$ we have then, using Sweedler's notation:

$$^{\top}\Psi_J[f](x) = \sum_{(x)} f(x_{(1)}) \otimes x_{(2)}.$$

Hence ${}^{\top}\Psi_J[f]$ has an intrinsic expression as the composition of the three maps below:

$$\mathcal{A} \otimes J \xrightarrow{Id_{\mathcal{A}} \otimes \Delta} \mathcal{A} \otimes \mathcal{H} \otimes J \xrightarrow{Id_{\mathcal{A}} \otimes f \otimes Id_J} \mathcal{A} \otimes \mathcal{A} \otimes J \xrightarrow{m_{\mathcal{A}} \otimes Id_J} \mathcal{A} \otimes J.$$

We have to show for any $f, g \in \mathcal{L}(\mathcal{H}, \mathcal{A})$:

$$\Psi_J[f \star g] = \Psi_J[f]\Psi_J[g]. \tag{23}$$

It is shown in [EG05] that ${}^{\top}\Psi_J$ is an anti-homomorphism, which proves the claim. We give here an alternative proof: Using coassociativity $(Id \otimes \Delta) \circ \Delta(x_i) = (\Delta \otimes Id) \circ \Delta(x_i)$, we immediately get :

$$\Delta(M_{ij}) = \sum_{k=0}^{|I|} M_{ik} \otimes M_{kj}.$$

Hence,

$$\begin{aligned} \Psi_J[f \star g](x_j) &= \sum_i (f \star g)(M_{ij}) \otimes x_i \\ &= \sum_i \sum_k f(M_{ik}) g(M_{kj}) \otimes x_i \\ &= \Psi_J[f] \circ \Psi_J[g](x_j), \end{aligned}$$

which proves (23). □

The Lie algebra of infinitesimal characters is mapped into a Lie subalgebra $\widehat{\mathfrak{g}}_{\mathcal{A}}$ in $\mathcal{M}^{\ell}_{|I|}(\mathcal{A})^1$. This is immediately seen by applying definition (22), since elements in $\mathfrak{g}_{\mathcal{A}}$ map the unit to zero due to relation (3). Whereas characters in the group $G_{\mathcal{A}}$ are mapped to the subgroup $\widehat{G}_{\mathcal{A}} \subset \mathfrak{M}_{|I|}(\mathcal{A})$.

The toy-model character $\varphi = \varphi(s,z)$ in equation (13) with parameterized 't Hooft mass, applied to the coproduct matrix (21) gives

$$\widehat{\varphi}(s,z) = \begin{pmatrix} 1_{\mathcal{A}} & 0 & 0 & 0 & 0 \\ \alpha(s)^{-z}B_1 & 1_{\mathcal{A}} & 0 & 0 & 0 \\ \alpha(s)^{-2z}B_2B_1 & \alpha(s)^{-z}B_1 & 1_{\mathcal{A}} & 0 & 0 \\ \alpha(s)^{-3z}B_3B_1^1 & \alpha(s)^{-2z}B_1^2 & 2\alpha(s)^{-z}B_1 & 1_{\mathcal{A}} & 0 \\ \alpha(s)^{-4z}B_4B_1^3 & \alpha(s)^{-3z}B_1^3 & 3\alpha(s)^{-2z}B_1^2 & 3\alpha(s)^{-z}B_1 & 1_{\mathcal{A}} \end{pmatrix}$$

Recall that $\alpha = \alpha(s) = a/\mu(s)$, where $\mu(s) = e^s\mu$.

The following remarks should be useful latter. The coproduct matrix M with entries in $\mathcal{H}$ can be seen as the image of the identity map under $\Psi_J : \mathcal{L}(\mathcal{H},\mathcal{H}) \to \mathrm{End}_{\mathcal{H}}(\mathcal{H}\otimes J)$, i.e. :

$$\Psi_J[Id](x_j) = \sum_i Id(M_{ij}) \otimes x_i. \tag{24}$$

The equations (6) imply for the matrix representation of the antipode $S \in \mathcal{L}(\mathcal{H},\mathcal{H})$:

$$\begin{aligned} \Psi_J[S \star Id](x_j) &= \Psi_J[S] \circ \Psi_J[Id](x_j) \\ &= \sum_i \sum_k S(M_{ik}) Id(M_{kj}) \otimes x_i \\ &= \Psi_J[\eta \circ \epsilon](x_j) \\ &= \sum_i \eta \circ \epsilon(M_{ij}) \otimes x_i \\ &= \sum_i \mathbf{1}\delta_{ij} \otimes x_i. \end{aligned}$$

Here $\mathbf{1}$ denotes the $|I| \times |I|$ unit matrix, $(\delta_{ij}1_{\mathcal{T}})_{1\le i,j\le |I|}$. Hence, $\Psi_J[S] = M^{-1}$, and the inverse can be calculated readily in terms of the geometric series:

$$\Psi_J[S] = M^{-1} = \mathbf{1} + \sum_{k>0} (-1)^k (M - \mathbf{1})^k.$$

Using the bijection $\exp^\star$ between $\mathfrak{g}_{\mathcal{A}}$ and $G_{\mathcal{A}}$ we may write any $\varphi \in G_{\mathcal{A}}$ as an exponential of the element $\alpha = \log^\star(\varphi)$ in $\mathfrak{g}_{\mathcal{A}}$. In terms of matrices we find $\log(\Psi[\varphi]) = \varphi(\log(\Psi[Id]))$, using the fact that φ is a character. Such that:

$$\log(\Psi[Id]) = \log(M) = \sum_{k>0} (-1)^k \frac{(M - \mathbf{1})^k}{k}$$

defines the matrix of the so-called *normal coordinates*.

2.4. **The matrix form of Connes–Kreimer's Birkhoff decomposition.** Suppose that the commutative target space algebra $\mathcal{A}$ in $\mathcal{L}(\mathcal{H},\mathcal{A})$ splits into two subalgebras:

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+, \tag{25}$$

where the unit $1_{\mathcal{A}}$ belongs to $\mathcal{A}_+$. Let us denote by $\pi : \mathcal{A} \to \mathcal{A}_-$ the projection onto $\mathcal{A}_-$ parallel to $\mathcal{A}_+$. One readily verifies that π is an idempotent *Rota–Baxter operator* [EG06], that is, it satisfies the relation:

$$\pi\big(\pi(a)b + a\pi(b) - ab\big) = \pi(a)\pi(b). \tag{26}$$

Indeed, let $a, b \in \mathcal{A}$:

$$\pi(a)b + a\pi(b) - ab = \pi(a)\pi(b) - (Id_{\mathcal{A}} - \pi)(a)(Id_{\mathcal{A}} - \pi)(b)$$

such that applying π on both sides gives relation (26), since it eliminates the term $(Id_{\mathcal{A}} - \pi)(a)(Id_{\mathcal{A}} - \pi)(b)$ without changing the term $\pi(a)\pi(b)$, as $\pi(Id_{\mathcal{A}} - \pi)(a) = 0$ and $\mathcal{A}_- = \pi(\mathcal{A})$, $\mathcal{A}_+ = (Id_{\mathcal{A}} - \pi)(\mathcal{A})$ are subalgebras.

In fact, this is a special case of an additive decomposition theorem characterizing Rota–Baxter algebras which was proven by Atkinson for general, not necessarily associative algebras in [Atk63].

Theorem 3. (Atkinson, [Atk63]) *Let $\mathcal{A}$ be a k-algebra. A k-linear operator $R : \mathcal{A} \to \mathcal{A}$ satisfies the Rota–Baxter relation (26) if and only if the following two statements are true. Firstly, $\mathcal{A}_+ := R(\mathcal{A})$ and $\mathcal{A}_- := \tilde{R}(\mathcal{A})$ are subalgebras in $\mathcal{A}$. Secondly, for $x, y, z \in \mathcal{A}$, $R(x)R(y) = R(z)$ implies $\tilde{R}(x)\tilde{R}(y) = -\tilde{R}(z)$. Here we denoted the map $\tilde{R} := (Id_{\mathcal{A}} - R)$.*

The case of an idempotent Rota–Baxter map implies $\mathcal{A}_- \cap \mathcal{A}_+ = \{0\}$. In the context of perturbative renormalization in QFT where the regularization prescription implies the target space algebra $\mathcal{A}$, the corresponding splitting of $\mathcal{A}$ into a direct sum of two subalgebras is called a *renormalization scheme*. For example, the *minimal renormalization scheme* in *dimensional regularization* corresponds to the splitting of the $\mathbb{C}$-algebra $\mathcal{A}$ of meromorphic functions in which $\mathcal{A}_-$ is the algebra of polynomials in $(z - z_0)^{-1}$ without constant term (the "pole parts"), and $\mathcal{A}_+$ is the algebra of meromorphic functions which are holomorphic at z_0.

The following theorem describes a multiplicative decomposition for associative unital Rota–Baxter algebras and was observed by Atkinson in [Atk63], see also [EGM06].

Theorem 4. (Atkinson, [Atk63]) *Let $\mathcal{A}$ be an associative unital Rota–Baxter algebra with Rota–Baxter map R. Suppose $\mathcal{A}$ to be have a decreasing filtration and to be complete in the induced topology. Assume X and Y in $\mathcal{A}$ to be solutions of the equations:*

$$X = 1_{\mathcal{A}} - R(X\ a) \text{ and } Y = 1_{\mathcal{A}} - \tilde{R}(a\ Y), \tag{27}$$

for $a \in \mathcal{A}^{(1)}$. Then we have the following factorization

$$X(1_{\mathcal{A}} + a)Y = 1_{\mathcal{A}}, \quad \textit{such that}\ \ 1_{\mathcal{A}} + a = X^{-1}Y^{-1}. \tag{28}$$

For an idempotent Rota–Baxter map this factorization is unique.

Proof. The Rota–Baxter relation (26) yields for any $\alpha, \beta \in \mathcal{A}$:

$$R\big(\alpha\tilde{R}(\beta)\big) + \tilde{R}\big(R(\alpha)\beta\big) = R(\alpha)\tilde{R}(\beta). \tag{29}$$

We then simply calculate the product XY and use equation (29), with $\alpha = Xa$ and $\beta = aY$:

$$\begin{aligned} XY &= \big(1_{\mathcal{A}} - R(X\ a)\big)\ \big(1_{\mathcal{A}} - \tilde{R}(a\ Y)\big) \\ &= 1_{\mathcal{A}} - R(X\ a) - \tilde{R}(a\ Y) + R(X\ a)\tilde{R}(a\ Y) \\ &= 1_{\mathcal{A}} - R\Big(Xa\ \big(1_{\mathcal{A}} - \tilde{R}(a\ Y)\big)\Big) - \tilde{R}\Big(\big(1_{\mathcal{A}} - R(X\ a)\big)\ a\ Y\Big) \\ &= 1_{\mathcal{A}} - R(XaY) - \tilde{R}(XaY) \\ &= 1_{\mathcal{A}} - XaY. \end{aligned}$$

Hence, we obtain the factorization in (28). The uniqueness for an idempotent Rota–Baxter map is easy to show [EGK05, EGM06]. □

Let us come back to the matrix representation of $\mathcal{L}(\mathcal{H}, \mathcal{A})$ with a splitting $\mathcal{A}$ (25) via Ψ_J (22). We define a Rota–Baxter map R on $\Psi_J[\mathcal{L}] \subset \mathcal{M}^{\ell}_{|I|}(\mathcal{A})$ by extending the Rota–Baxter map π on $\mathcal{A}$ entrywise, i.e., for the matrix $\tau = (\tau_{ij}) \in \mathcal{M}^{\ell}_{|I|}(\mathcal{A})$, define:

$$\mathrm{R}(\tau) = \big(\pi(\tau_{ij})\big). \tag{30}$$

Theorem 5. [EG05] *Then the triple* $\Big(\mathcal{M}^{\ell}_{|I|}(\mathcal{A}), \mathrm{R}, \{\mathcal{M}^{\ell}_{|I|}(\mathcal{A})^{l}\}_{l<|I|}\Big)$ *forms a non-commutative complete filtered Rota–Baxter algebra with idempotent Rota–Baxter map* R.

Atkinson's multiplicative decomposition immediately implies a factorization of $\widehat{G}_{\mathcal{A}} \subset \mathfrak{M}^{\ell}_{|I|}(\mathcal{A})$ into the subgroups:

$$\widehat{G}^{-}_{\mathcal{A}} \subset \mathbf{1} + \mathrm{R}\big(\mathcal{M}^{\ell}_{|I|}(\mathcal{A})^1\big) \subset \mathfrak{M}^{\ell}_{|I|}(\mathcal{A})$$

and

$$\widehat{G}^{+}_{\mathcal{A}} \subset \mathbf{1} + \tilde{\mathrm{R}}\big(\mathcal{M}^{\ell}_{|I|}(\mathcal{A})^1\big) \subset \mathfrak{M}^{\ell}_{|I|}(\mathcal{A}),$$

that is, for each $\widehat{\varphi} := \Psi[\varphi] \in \widehat{G}_{\mathcal{A}}$, $\varphi \in G_{\mathcal{A}}$ there exist unique $\widehat{\varphi}_- \in \widehat{G}^-_{\mathcal{A}}$ and $\widehat{\varphi}_+ \in \widehat{G}^+_{\mathcal{A}}$, such that:

$$\widehat{\varphi} = \widehat{\varphi}_-^{-1}\ \widehat{\varphi}_+. \tag{31}$$

We immediately see that $\widehat{\varphi}_-$ and $\widehat{\varphi}_+^{-1}$ are unique solutions of the equations (27) in Theorem 4:

$$\widehat{\varphi}_- = \mathbf{1} - \mathrm{R}\Big(\widehat{\varphi}_-\ (\widehat{\varphi} - \mathbf{1})\Big), \tag{32}$$

$$\widehat{\varphi}_+^{-1} = \mathbf{1} - \tilde{\mathrm{R}}\Big((\widehat{\varphi} - \mathbf{1})\ \widehat{\varphi}_+^{-1}\Big). \tag{33}$$

Moreover, after some simple algebra using the factorization $\widehat{\varphi} = \widehat{\varphi}_-^{-1}\widehat{\varphi}_+$:

$$\widehat{\varphi}_+(\widehat{\varphi}^{-1} - \mathbf{1})\ = \widehat{\varphi}_- - \widehat{\varphi}_+ = -\widehat{\varphi}_-(\widehat{\varphi} - \mathbf{1})$$

we immediately get the recursion for $\widehat{\varphi}_+$ [EG05]:

$$\widehat{\varphi}_+ = \mathbf{1} - \tilde{\mathrm{R}}\Big(\widehat{\varphi}_+\ (\widehat{\varphi}^{-1} - \mathbf{1})\Big). \tag{34}$$

and hence we see that

$$\widehat{\varphi}_+ = \mathbf{1} + \tilde{\mathrm{R}}\Big(\widehat{\varphi}_-\ (\widehat{\varphi} - \mathbf{1})\Big). \tag{35}$$

The matrix entries of $\widehat{\varphi}_-$ and $\widehat{\varphi}_+^{-1}$ can be calculated without recursions using $\sigma := \widehat{\varphi}$ from the equations:

$$\begin{aligned}
(\widehat{\varphi}_-)_{ij} &= -\pi(\sigma_{ij}) - \sum_{k=2}^{j-i} \sum_{i>l_1>l_2>\cdots>l_{k-1}>j} (-1)^{k+1}\pi\Big(\pi(\cdots\pi(\sigma_{il_1})\sigma_{l_1l_2})\cdots\sigma_{l_{k-1}j}\Big)\\
(\widehat{\varphi}_+^{-1})_{ij} &= -\tilde{\pi}((\sigma^{-1})_{ij}) - \sum_{k=2}^{j-i} \sum_{i>l_1>l_2>\cdots>l_{k-1}>j} (-1)^{k+1}\tilde{\pi}\Big(\tilde{\pi}(\cdots\tilde{\pi}((\sigma^{-1})_{il_1})(\sigma^{-1})_{l_1l_2})\cdots(\sigma^{-1})_{l_{k-1}j}\Big),
\end{aligned}$$

where $\tilde{\pi} := 1_{\mathcal{A}} - \pi$. The matrix entries of $\widehat{\varphi}_+$ follow from the first formula, i.e., the one for the entries in $\widehat{\varphi}_-$, by replacing π by $-\tilde{\pi}$. We may therefore define the matrix:

$$\widehat{\bar{R}}[\varphi] := \widehat{\varphi}_-\ (\widehat{\varphi} - \mathbf{1}) \tag{36}$$

such that:

$$\widehat{\varphi}_- = \mathbf{1} - \mathrm{R}\Big(\widehat{\bar{R}}[\varphi]\Big)\ \text{ and }\ \widehat{\varphi}_+ = \mathbf{1} + \tilde{\mathrm{R}}\Big(\widehat{\bar{R}}[\varphi]\Big). \tag{37}$$

In fact, equations (35) and (33) may be called *Bogoliubov's matrix formulae* for the counter term and renormalized Feynman rules matrix, $\widehat{\varphi}_-$, $\widehat{\varphi}_+$, respectively. Equation (36) is the matrix form of Bogoliubov's $\bar{R}$- or preparation map (16), e.g. see [Col84] :

$$\widehat{\bar{R}}[\varphi] := \Psi_J[\bar{R}] = \Psi_J[\varphi_- \star (\varphi - e)]. \tag{38}$$

Here, φ_- is the counter term character (17) of the algebraic *Birkhoff decomposition* of Connes and Kreimer [CK99, CK00], which we mentioned in the foregoing section. To phrase it differently, the above matrix factorization follows via the representation Ψ_J from Connes–Kreimer's Birkhoff factorization (15) on the group $G_{\mathcal{A}}$ of $\mathcal{A}$-valued Hopf algebra characters.

Remark 6. We may apply the result from [EGK04, EGK05, EG05, EGM06], see also [Man01], to the above matrix representation of $\mathfrak{g}_{\mathcal{A}}$ respectively $G_{\mathcal{A}}$. In these references a unique non-linear map χ was established on $\mathfrak{g}_{\mathcal{A}}$ which allows to write the characters φ_- and φ_+ as exponentials. In the matrix picture we hence find for $\widehat{Z} \in \widehat{\mathfrak{g}}_{\mathcal{A}}$ and $\widehat{\varphi} = \exp(\widehat{Z}) \in \widehat{G}_{\mathcal{A}}$:

$$\widehat{\varphi} = \exp\big(\mathrm{R}(\chi(\widehat{Z}))\big)\exp\big(\tilde{\mathrm{R}}(\chi(\widehat{Z}))\big). \tag{39}$$

The matrices $\widehat{\varphi}_- := \exp\big(-\mathrm{R}(\chi(\widehat{Z}))\big)$ and $\widehat{\varphi}_+^{-1} := \exp\big(-\tilde{\mathrm{R}}(\chi(\widehat{Z}))\big)$ are in $\widehat{G}^-_{\mathcal{A}}$ and $\widehat{G}^+_{\mathcal{A}}$, respectively, and solve Bogoliubov's matrix formulae in (37).

3. The matrix representation of the beta-function

It is the goal of this section to establish a matrix representation of the beta-function as it appears in the work of Connes and Kreimer [CK01]. Once we have achieved this we reformulate in a transparent manner Sakakiabara's findings [Sa04] hereby providing a firm ground for his calculations. In the next paragraph we review the main points of the beta-function calculus in the Hopf algebra context following mainly the paper [Man01].

3.1. **The beta-function in the Hopf algebra of renormalization.** From now on, $k = \mathbb{C}$ stands for the complex numbers, and $\mathcal{A}$ will denote the algebra of meromorphic functions in one complex variable z endowed with the minimal subtraction scheme at $z_0 = 0$. Hence $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$, where $\mathcal{A}_+$ is the subalgebra of meromorphic functions which are holomorphic at 0, and $\mathcal{A}_-$ stands for the polynomials in z^{-1} without constant term. We moreover suppose that the Hopf algebra $\mathcal{H}$ is graded, with filtration coming from the graduation, i.e. :

$$\mathcal{H}^{(n)} = \bigoplus_{0 \le k \le n} \mathcal{H}_k.$$

The grading induces a biderivation Y defined on homogeneous elements by :

$$\begin{aligned} Y : \mathcal{H}_n &\longrightarrow \mathcal{H}_n \\ x &\longmapsto nx. \end{aligned}$$

Exponentiating Y we get a one-parameter group θ_t of automorphisms of the Hopf algebra $\mathcal{H}$, defined on $\mathcal{H}_n$ by :

$$\theta_t(x) = e^{nt}x. \tag{40}$$

The map $\varphi \mapsto \varphi \circ Y$ is a derivation of $\big(\mathcal{L}(\mathcal{H},\mathcal{A}),\star\big)$, and $\varphi \mapsto \varphi \circ \theta_t$ is an automorphism of $\big(\mathcal{L}(\mathcal{H},\mathcal{A}),\star\big)$ for any complex t. We will rather consider the one-parameter group $\varphi \mapsto \varphi \circ \theta_{tz}$ of automorphisms of the algebra $\big(\mathcal{L}(\mathcal{H},\mathcal{A}),\star\big)$ i.e. :

$$\varphi^t(x)(z) := e^{tz|x|}\varphi(x)(z). \tag{41}$$

Differentiating at $t = 0$ we get:

$$\frac{d}{dt}\Big|_{t=0}\varphi^t = z(\varphi \circ Y). \tag{42}$$

We denote by $G^{\Phi}_{\mathcal{A}}$ the set of those characters $\varphi \in G$ such that the negative part of the Birkhoff decomposition of φ^t does not depend on t, namely:

$$G^{\Phi}_{\mathcal{A}} = \Big\{\varphi \in G_{\mathcal{A}} \Big| \ \frac{d}{dt}(\varphi^t)_- = 0\Big\}.$$

In particular the dimensional-regularized Feynman rules verify this property: in physical terms, the counter terms do not depend on the choice of the arbitrary mass-parameter μ ('tHooft's mass) one must introduce in dimensional regularization in order to get dimensionless expressions (see [CK00]). We also denote by $G^{\Phi}_{\mathcal{A}_-}$ the elements φ of $G^{\Phi}_{\mathcal{A}}$ such that $\varphi = \varphi_-^{\star -1}$. Recall from [Man01] that there is a bijection $\widetilde{R} : G_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}$ defined by:

$$\varphi \circ Y = \varphi \star \widetilde{R}(\varphi). \tag{43}$$

Since composition on the right with Y is a derivation for the convolution product, the map $\widetilde{R}$ verifies a cocycle property:

$$\widetilde{R}(\varphi \star \psi) = \widetilde{R}(\psi) + \psi^{\star -1} \star \widetilde{R}(\varphi) \star \psi. \tag{44}$$

We summarize some key results of [CK01] in the following proposition:

Proposition 7.

(1) *For any $\varphi \in G_{\mathcal{A}}$ there is a one-parameter family h_t in $G_{\mathcal{A}}$ such that $\varphi^t = \varphi \star h_t$, and we have:*

$$\dot{h}_t := \frac{d}{dt} h_t = h_t \star z\widetilde{R}(h_t) + z\widetilde{R}(\varphi) \star h_t. \tag{45}$$

(2) *$z\widetilde{R}$ restricts to a bijection from $G^{\Phi}_{\mathcal{A}}$ onto $\mathfrak{g}_{\mathcal{A}} \cap \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$. Moreover it is a bijection from $G^{\Phi}_{\mathcal{A}\ -}$ onto those elements of $\mathfrak{g}_{\mathcal{A}}$ with values in the constants, i.e. :*

$$\mathfrak{g}^c_{\mathcal{A}} = \mathfrak{g}_{\mathcal{A}} \cap \mathcal{L}(\mathcal{H}, \mathbb{C}).$$

(3) *For $\varphi \in G^{\Phi}_{\mathcal{A}}$, the constant term of h_t, defined by:*

$$F_t(x) = \lim_{z \to 0} h_t(x)(z) \tag{46}$$

is a one-parameter subgroup of $G_{\mathcal{A}} \cap \mathcal{L}(\mathcal{H}, \mathbb{C})$, the scalar-valued characters of $\mathcal{H}$.

Proof. For any $\varphi \in G_{\mathcal{A}}$ one can write:

$$\varphi^t = \varphi \star h_t \tag{47}$$

with $h_t \in G_{\mathcal{A}}$. From (47), (42) and (43) we immediately get:

$$\varphi \star \dot{h}_t = \varphi \star h_t \star z\widetilde{R}(\varphi \star h_t).$$

Equation (45) then follows from the cocycle property (44). This proves the first assertion.

Now take any character $\varphi \in G^{\Phi}_{\mathcal{A}}$ with Birkhoff decomposition $\varphi = \varphi_-^{\star -1} \star \varphi_+$ and write the Birkhoff decomposition of φ^t :

$$\begin{aligned}
\varphi^t &= (\varphi^t)_-^{\star -1} \star (\varphi^t)_+ \\
&= (\varphi_-)^{\star -1} \star (\varphi^t)_+ \\
&= (\varphi \star \varphi_+^{\star -1}) \star (\varphi^t)_+ \\
&= \varphi \star h_t,
\end{aligned}$$

with h_t taking values in $\mathcal{A}_+$. Then $z\widetilde{R}(\varphi)$ also takes values in $\mathcal{A}_+$, as a consequence of equation (45) at $t = 0$. Conversely, suppose that $z\widetilde{R}(\varphi)$ takes values in $\mathcal{A}_+$. We show that h_t also takes values in $\mathcal{A}_+$ for any t, which immediately implies that φ belongs to $G^{\Phi}_{\mathcal{A}}$.

For any $\gamma \in \mathfrak{g}_{\mathcal{A}}$, let us introduce the linear transformation U_γ of $\mathfrak{g}_{\mathcal{A}}$ defined by :

$$U_\gamma(\delta) := \gamma \star \delta + z\delta \circ Y.$$

If γ belongs to $\mathfrak{g}_{\mathcal{A}} \cap \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ then U_γ restricts to a linear transformation of $\mathfrak{g}_{\mathcal{A}} \cap \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$.

Lemma 8. *For any $\varphi \in G_{\mathcal{A}}$, $n \in \mathbb{N}$ we have :*

$$z^n \varphi \circ Y^n = \varphi \star U^n_{z\widetilde{R}(\varphi)}(e).$$

Proof. Case $n = 0$ is obvious, $n = 1$ is just the definition of $\widetilde{R}$. We check thus by induction, using again the fact that composition on the right with Y is a derivation for the convolution product :

$$\begin{aligned}
z^{n+1} \varphi \circ Y^{n+1} &= z(z^n \varphi \circ Y^n) \circ Y \\
&= z\big(\varphi \star U^n_{z\widetilde{R}(\varphi)}(e)\big) \circ Y \\
&= z(\varphi \circ Y) \star U^n_{z\widetilde{R}(\varphi)}(e) + z\varphi \star \big(U^n_{z\widetilde{R}(\varphi)}(e) \circ Y\big) \\
&= \varphi \star \big(z\widetilde{R}(\varphi) \star U^n_{z\widetilde{R}(\varphi)}(e) + zU^n_{z\widetilde{R}(\varphi)}(e) \circ Y\big) \\
&= \varphi \star U^{n+1}_{z\widetilde{R}(\varphi)}(e).
\end{aligned}$$

□

Let us go back to the proof of Proposition 7. According to Lemma 8 we have for any t, at least formally:

$$\varphi^t = \varphi \star \exp(tU_{z\widetilde{R}(\varphi)})(e). \tag{48}$$

We still have to fix the convergence of the exponential just above in the case when $z\widetilde{R}(\varphi)$ belongs to $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$. Let us consider the following decreasing bifiltration of $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$:

$$\mathcal{L}_+^{p,q} = (z^q \mathcal{L}(\mathcal{H}, \mathcal{A}_+)) \cap \mathcal{L}^p,$$

where $\mathcal{L}^p$ is the set of those $\alpha \in \mathcal{L}(\mathcal{H}, \mathcal{A})$ such that $\alpha(x) = 0$ for any $x \in \mathcal{H}$ of degree $\leq p-1$. In particular $\mathcal{L}^1 = \mathfrak{g}_0$. Considering the associated filtration :

$$\mathcal{L}_+^n = \sum_{p+q=n} \mathcal{L}_+^{p,q},$$

we see that for any $\gamma \in \mathfrak{g}_0 \cap \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ the transformation U_γ increases the filtration by 1, i.e :

$$U_\gamma(\mathcal{L}_+^n) \subset \mathcal{L}_+^{n+1}.$$

The algebra $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ is not complete with respect to the topology induced by this filtration, but the completion is $\mathcal{L}(\mathcal{H}, \widehat{\mathcal{A}_+})$, where $\widehat{\mathcal{A}_+} = \mathbb{C}[[z]]$ stands for the formal series. Hence the right-hand side of (48) is convergent in $\mathcal{L}(\mathcal{H}, \widehat{\mathcal{A}_+})$ with respect to this topology. Hence for any $\gamma \in \mathcal{L}(\mathcal{H}, \mathcal{A}_+)$ and for φ such that $z\widetilde{R}(\varphi) = \gamma$ we have $\varphi^t = \varphi \star h_t$ with $h_t \in \mathcal{L}(\mathcal{H}, \widehat{\mathcal{A}_+})$ for any t. On the other hand we already know that h_t takes values in meromorphic functions for each t. So h_t belongs to $\mathcal{L}(\mathcal{H}, \mathcal{A}_+)$, which proves the first part of the second assertion. Equation (45) at $t = 0$ reads:

$$z\widetilde{R}(\varphi) = \dot{h}(0) = \frac{d}{dt}\Big|_{t=0} (\varphi^t)_+. \tag{49}$$

For $\varphi \in G^\Phi_{\mathcal{A}_-}$ we have, thanks to the property $\varphi(\operatorname{Ker} \epsilon) \subset \mathcal{A}_-$:

$$\begin{aligned}
h_t(x) = (\varphi^t)_+(x) &= (I-\pi)\Big(\varphi^t(x) + \sum_{(x)} \varphi^{\star -1}(x')\varphi^t(x'')\Big) \\
&= t(I-\pi)\big(z|x|\varphi(x) + z\sum_{(x)} \varphi^{\star -1}(x')\varphi(x'')|x''|\big) + O(t^2) \\
&= t\operatorname{Res}(\varphi \circ Y) + O(t^2),
\end{aligned}$$

hence:

$$\dot{h}(0) = \operatorname{Res}\,(\varphi \circ Y). \tag{50}$$

From equations (42), (43) and (50) we get:

$$z\widetilde{R}(\varphi) = \operatorname{Res}\,(\varphi \circ Y) \tag{51}$$

for any $\varphi \in G^\Phi_{\mathcal{A}_-}$, hence $z\widetilde{R}(\varphi) \in \mathfrak{g}^c$. Conversely let β in $\mathfrak{g}^c$. Consider $\psi = \widetilde{R}^{-1}(z^{-1}\beta)$. This element of $G_\mathcal{A}$ verifies by definition, thanks to equation (43) :

$$z\psi \circ Y = \psi \star \beta.$$

Hence for any $x \in \operatorname{Ker} \epsilon$ we have :

$$z\psi(x) = \frac{1}{|x|}\Big(\beta(x) + \sum_{(x)} \psi(x')\beta(x'')\Big).$$

As $\beta(x)$ is a constant (as a function of the complex variable z) it is easily seen by induction on $|x|$ that the right-hand side evaluated at z has a limit when z tends to zero. Thus $\psi(x) \in \mathcal{A}_-$, and then :

$$\psi = \widetilde{R}^{-1}\Big(\frac{1}{z}\beta\Big) \in G^\Phi_{\mathcal{A}_-},$$

which proves assertion (2).

Let us prove assertion (3): Equation $\varphi^t = \varphi \star h_t$ together with $(\varphi^t)^s = \varphi^{t+s}$ yields:

$$h_{s+t} = h_s \star (h_t)^s. \tag{52}$$

Taking values at $z = 0$ immediately yields the one-parameter group property:

$$F_{s+t} = F_s \star F_t \tag{53}$$

thanks to the fact that the evaluation at $z = 0$ is an algebra morphism. □

We can now give a definition of the beta-function :

Definition 2. *For any $\varphi \in G^\Phi_{\mathcal{A}}$, the beta-function of φ is the generator of the one-parameter group F_t defined by equation (46) in Proposition 7. It is the element of the dual $\mathcal{H}^\star$ defined by:*

$$\beta(\varphi) = \frac{d}{dt}\Big|_{t=0} F_t(x)$$

for any $x \in \mathcal{H}$.

Proposition 9. *For any $\varphi \in G^\Phi_{\mathcal{A}}$ the beta-function of φ coincides with the one of the negative part $\varphi_-^{\star-1}$ in the Birkhoff decomposition. It is given by any of the three expressions:*

$$\begin{aligned} \beta(\varphi) &= \operatorname{Res}\ \widetilde{R}(\varphi) \\ &= \operatorname{Res}\ (\varphi_-^{\star-1} \circ Y) \\ &= -\operatorname{Res}\ (\varphi_- \circ Y). \end{aligned}$$

Proof. The third equality will be derived from the second by taking residues on both sides of the equation:

$$0 = \widetilde{R}(e) = \widetilde{R}(\varphi_-) + \varphi_-^{\star-1} \star \widetilde{R}(\varphi_-^{\star-1}) \star \varphi_-,$$

which is a special instance of the cocycle formula (44). Suppose first $\varphi \in G^\Phi_{\mathcal{A}_-}$, hence $\varphi_-^{\star-1} = \varphi$. Then $z\widetilde{R}(\varphi)$ is a constant according to assertion 2 of Proposition 7. The proposition then follows from equation (50) evaluated at $z = 0$, and equation (51). Suppose now $\varphi \in G^\Phi_{\mathcal{A}}$, and consider its Birkhoff decomposition. As both components belong to $G^\Phi_{\mathcal{A}}$ we apply Proposition 7 to them. In particular we have:

$$\begin{aligned} \varphi^t &= \varphi \star h_t, \\ (\varphi_-^{\star-1})^t &= \varphi_-^{\star-1} \star v_t, \\ (\varphi_+)^t &= \varphi_+ \star w_t, \end{aligned}$$

and equality $\varphi^t = (\varphi_-^{\star-1})^t \star (\varphi_+)^t$ yields:

$$h_t = (\varphi_+)^{\star-1} \star v_t \star \varphi_+ \star w_t. \tag{54}$$

We denote by F_t, V_t, W_t the one-parameter groups obtained from h_t, v_t, w_t, respectively, by letting the complex variable z go to zero. It is clear that $\varphi^+\big|_{z=0} = e$, and similarly that W_t is the constant one-parameter group reduced to the co-unit ε. Hence equation (54) at $z = 0$ reduces to:

$$F_t = V_t, \tag{55}$$

hence the first assertion. the cocycle equation (44) applied to the Birkhoff decomposition reads:

$$\widetilde{R}(\varphi) = \widetilde{R}(\varphi_+) + (\varphi_+)^{\star-1} \star \widetilde{R}(\varphi_-^{\star-1}) \star \varphi_+.$$

Taking residues of both sides yields:

$$\operatorname{Res}\ \widetilde{R}(\varphi) = \operatorname{Res}\ \widetilde{R}(\varphi_-^{\star-1}),$$

which ends the proof. □

Definition 3. *The one-parameter group $F_t = V_t$ above is the renormalization group of φ* [CK01].

Remark 10. It is possible to reconstruct φ_- from $\beta(\varphi)$ using a scattering-type formula ([CK01, Co02, CM04b, Man01]). Hence φ_- (i.e. the divergence structure of φ) is uniquely determined by its residue.

3.2. **The matrix representation of the grading derivation.** As in reference [CK01], denote by Z_0 the derivation of the algebra $\mathcal{L}(\mathcal{H},\mathcal{A})$ (which is also a derivation of the the Lie algebra $\mathfrak{g}_\mathcal{A}$) given by $\alpha \mapsto \alpha \circ Y$. Let $\widetilde{\mathcal{L}}$ be the semi-direct product of $\mathcal{L}(\mathcal{H},\mathcal{A})$ with Z_0, and let $\widetilde{\mathfrak{g}_\mathcal{A}}$ be the semi-direct product $\mathfrak{g}_\mathcal{A} \rtimes \mathbb{C}.Z_0$. Similarly let $\widetilde{G}_\mathcal{A} = G_\mathcal{A} \rtimes \mathbb{C}$ be the semi-direct product of the group $G_\mathcal{A}$ by the one-parameter group $\theta_t = \exp tZ_0$ of automorphisms. Let J be any *graded* left coideal of $\mathcal{H}$. We suppose further that the filtration-ordered basis $(x_i)_{i\in I}$ of J is graded. i.e. made of homogeneous elements. The degree of x_i will be denoted by $|x_i|$. We want to extend the matrix representation Ψ_J from $\mathcal{L}(\mathcal{H},\mathcal{A})$ to $\widetilde{\mathcal{L}}$. This can be done by representing Z_0 by a diagonal matrix.

Proposition 11. *The correspondence $\Psi_J : \widetilde{\mathcal{L}} \to \mathrm{End}_\mathcal{A}(\mathcal{A}\otimes J)$ defined as in Paragraph 2.3 on $\mathcal{L}(\mathcal{H},\mathcal{A})$, and such that:*

$$\Psi_J[Z_0](x_i) = |x_i|.x_i \tag{56}$$

is an algebra morphism.

Proof. We only have to show the equality[3]:

$$\big[\Psi_J[f],\, \Psi_J[-Z_0]\big] = \Psi_J([f,-Z_0]) = \Psi_J(f\circ Y). \tag{57}$$

This follows by a direct computation:

$$\begin{aligned}
\big[\Psi_J[f],\, \Psi_J[-Z_0]\big](x_j) &= -\Psi_J[f]\Psi_J[Z_0](x_j) + \Psi_J[Z_0]\Psi_J[f](x_j)\\
&= \sum_i (|x_i| - |x_j|) f(M_{ij}) \otimes x_i,
\end{aligned}$$

whereas:

$$\begin{aligned}
\Psi_J[f\circ Y](x_j) &= \sum_i (f\circ Y)(M_{ij}) \otimes x_i\\
&= \sum_i |M_{ij}| f(M_{ij}) \otimes x_i.
\end{aligned}$$

By definition of the coproduct matrix, and thanks to the fact that $\mathcal{H}$ is graded, the coefficients M_{ij} are homogeneous of degree $|x_i| - |x_j|$, which finishes the proof. □

By taking exponentials of the above diagonal matrices, we of course get a matrix representation of the one-parameter group θ_t (40), namely:

$$\Psi_J[\exp tZ_0](x_i) = e^{t|x_i|} x_i. \tag{58}$$

3.3. **Matrix differential equations.** We fix a graded coideal J of $\mathcal{H}$, and we therefore introduce the following notation:

$$\widehat{f} := \Psi_J[f]. \tag{59}$$

Keeping the notations of Paragraph 3.1 we have then, as a consequence of Propositions 2 and 11:

$$\widehat{\varphi^t}(z) = e^{tz\widehat{Z_0}} \widehat{\varphi}(z) e^{-tz\widehat{Z_0}}, \tag{60}$$

as an equality of size $|I|$ square matrices with coefficients in meromorphic functions of the complex variable z. With respect to the example left coideal generated by $\mathcal{T}' \subset \mathcal{T}$ in (20) we find:

$$\widehat{Z_0} = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&2&0&0\\ 0&0&0&3&0\\ 0&0&0&0&4 \end{pmatrix} \quad\text{and}\quad e^{tz\widehat{Z_0}} = \begin{pmatrix} 1_\mathcal{A}&0&0&0&0\\ 0&e^{zt}&0&0&0\\ 0&0&e^{2zt}&0&0\\ 0&0&0&e^{3zt}&0\\ 0&0&0&0&e^{4zt} \end{pmatrix} \tag{61}$$

[3]There is a minus sign in front of Z_0 due to the fact that we have put it on the right inside of the bracket, reflecting the fact that the action of the one-parameter group has been written on the right.

We now change the general notations slightly and consider $\widehat{\varphi}$ as a function of the variable $t \in \mathbb{C}$ with values in $\mathcal{A} \otimes J$ (i.e. as a matrix-valued function of both variables (t, z)). More precisely we put:

$$\widehat{\varphi}(t,z) = e^{tz\widehat{Z_0}}\widehat{\varphi}(0,z)e^{-tz\widehat{Z_0}},$$

where $\widehat{\varphi}(0,z)$ stands for the old $\widehat{\varphi}(z)$. Now $\widehat{\varphi}_-(t)$ (resp. $\widehat{\varphi}_+(t)$) will stand for the negative (resp. positive) component of the Birkhoff decomposition of $\widehat{\varphi}(t)$ (31), for any $t \in \mathbb{C}$. Using the toy-model character $\varphi = \varphi(s,z)$ in equation (13) with parameterized 't Hooft mass, we find explicitly:

$$\begin{aligned}\widehat{\varphi^t}(z) = \widehat{\varphi}(t,z) \quad &= \begin{pmatrix} 1_{\mathcal{A}} & 0 & 0 & 0 & 0 \\ (e^{-t}\alpha(0))^{-z}B_1 & 1_{\mathcal{A}} & 0 & 0 & 0 \\ (e^{-t}\alpha(0))^{-2z}B_2B_1 & (e^{-t}\alpha(0))^{-z}B_1 & 1_{\mathcal{A}} & 0 & 0 \\ (e^{-t}\alpha(0))^{-3z}B_3B_1^2 & (e^{-t}\alpha(0))^{-2z}B_1^2 & 2(e^{-t}\alpha(0))^{-z}B_1 & 1_{\mathcal{A}} & 0 \\ (e^{-t}\alpha(0))^{-4z}B_4B_1^3 & (e^{-t}\alpha(0))^{-3z}B_1^3 & 3(e^{-t}\alpha(0))^{-2z}B_1^2 & 3(e^{-t}\alpha(0))^{-z}B_1 & 1_{\mathcal{A}} \end{pmatrix} \\ &= e^{tz\widehat{Z_0}}\begin{pmatrix} 1_{\mathcal{A}} & 0 & 0 & 0 & 0 \\ \alpha(0)^{-z}B_1 & 1_{\mathcal{A}} & 0 & 0 & 0 \\ \alpha(0)^{-2z}B_2B_1 & \alpha(0)^{-z}B_1 & 1_{\mathcal{A}} & 0 & 0 \\ \alpha(0)^{-3z}B_3B_1^2 & \alpha(0)^{-2z}B_1^2 & 2\alpha(0)^{-z}B_1 & 1_{\mathcal{A}} & 0 \\ \alpha(0)^{-4z}B_4B_1^3 & \alpha(0)^{-3z}B_1^3 & 3\alpha(0)^{-2z}B_1^2 & 3\alpha(0)^{-z}B_1 & 1_{\mathcal{A}} \end{pmatrix}e^{-tz\widehat{Z_0}} \end{aligned} \tag{62}$$

Hence, we observe the simple transformation on $\widehat{\varphi}(z) = \widehat{\varphi}(t,z)$, where t parameterizes 't Hooft's unit mass (14), i.e., $\alpha(t) := \frac{a}{e^t\mu} = e^{-t}\alpha(0) > 0$:

$$\widehat{\varphi}^0(z) := \widehat{\varphi}(0,z) \xrightarrow{Ad[e^{tz\widehat{Z_0}}]} \widehat{\varphi}^t(z) = \widehat{\varphi}(t,z).$$

We introduce the auxiliary matrix:

$$A := \widehat{\varphi}e^{tz\widehat{Z_0}}, \tag{63}$$

as well as its Birkhoff decomposition:

$$A = A_-^{-1}A_+, \quad \text{with} \quad A_- = \widehat{\varphi}_- \text{ and } A_+ = \widehat{\varphi}_+e^{tz\widehat{Z_0}}. \tag{64}$$

From the obvious equality:

$$A(t) = e^{tz\widehat{Z_0}}Ae^{-tz\widehat{Z_0}} = e^{tz\widehat{Z_0}}A(0)$$

we get by differentiating with respect to t:

$$\frac{d}{dt}A = \dot{A} = z\widehat{Z_0}A. \tag{65}$$

Using the Birkhoff decomposition of A then yields:

$$z\widehat{Z_0}\widehat{\varphi}_-^{-1}A_+ = \dot{\widehat{\varphi}_-^{-1}}A_+ + \widehat{\varphi}_-^{-1}\dot{A}_+. \tag{66}$$

Multiplying both sides with $\widehat{\varphi}_-$ on the left and with A_+^{-1} on the right we get:

$$\widehat{\varphi}_-(z\widehat{Z_0})\widehat{\varphi}_-^{-1} = \widehat{\varphi}_-\dot{\widehat{\varphi}_-^{-1}} + \dot{A}_+A_+^{-1}. \tag{67}$$

Suppose now that $\varphi(0)$ belongs to $G_{\mathcal{A}}^{\Phi}$, which implies for the corresponding matrix $\widehat{\varphi}(0) = \widehat{\varphi}_-^{-1}\widehat{\varphi}_+$:

$$\frac{d}{dt}\widehat{\varphi}_- = \dot{\widehat{\varphi}}_- = 0. \tag{68}$$

Then equation (67) reduces to:

$$\widehat{\varphi}_-(z\widehat{Z_0})\widehat{\varphi}_-^{-1} = \dot{A}_+A_+^{-1}. \tag{69}$$

3.4. **The renormalization group and the beta-function in the matrix setting.** Keeping the notations of the previous paragraph, it is clear that if $\varphi(0)$ belongs to $G^{\Phi}_{\mathcal{A}}$, then $\varphi(t) \in G^{\Phi}_{\mathcal{A}}$ for any t, and moreover the renormalization group and the beta-function of $\varphi(t)$ do not depend on t. We can then talk about the renormalization group and the beta-function of φ without mentioning a particular value of t.

Theorem 12. *The matrix representation of the beta-function reads:*

$$\widehat{\beta}(\varphi) = \widehat{\varphi}_-(z\widehat{Z_0})\widehat{\varphi}_-^{-1} - z\widehat{Z_0}. \tag{70}$$

Proof. This is a direct computation of (scalar-valued) matrices, see Eq. (55) and Definition 3:

$$\widehat{F}_t = \lim_{z\to 0}\Big(\widehat{\varphi}_-(z)e^{zt\widehat{Z_0}}\widehat{\varphi}_-^{-1}(z)e^{-zt\widehat{Z_0}}\Big). \tag{71}$$

The limit exists by Proposition 7. The term inside the bracket on the right-hand side is holomorphic at zero as a function of z, and so is its derivative with respect to t. The operation $\frac{\partial}{\partial t}$ commutes then with evaluating at $z=0$, and we get by definition of the beta-function:

$$\begin{aligned}\widehat{\beta}(\varphi) &= \lim_{z\to 0}\frac{\partial}{\partial t}\Big(\widehat{\varphi}_-(z)e^{zt\widehat{Z_0}}\widehat{\varphi}_-^{-1}(z)e^{-zt\widehat{Z_0}}\Big)\big|_{t=0}\\ &= \lim_{z\to 0}\Big(\widehat{\varphi}_-(z)z\widehat{Z_0}\widehat{\varphi}_-^{-1}(z) - z\widehat{Z_0}\Big).\end{aligned}$$

Now, subtracting $z\widehat{Z_0}$ on both sides of Equation (69) gives an expression on the left-hand side which admits a limit when $z\to\infty$, and a term on the right-hand side which admits a limit when $z\to 0$. Hence the term:

$$\widehat{\varphi}_-(z)z\widehat{Z_0}\widehat{\varphi}_-^{-1}(z) - z\widehat{Z_0} = \widehat{\varphi}_-(z)\big[\widehat{\varphi}_-^{-1}(z), -z\widehat{Z_0}\big]$$

is a matrix with constant coefficients, which proves the theorem. □

Coming back to Remark 6 we find immediately for the matrix beta-function the simple equation in the Lie algebra $\widehat{\mathfrak{g}}_{\mathcal{A}}$, in accordance with results in [Man01]:

$$\begin{aligned}\widehat{\beta}(\varphi) &= \exp\big(\mathrm{R}(\chi(\widehat{Z}))\big)(z\widehat{Z_0})\exp\big(-\mathrm{R}(\chi(\widehat{Z}))\big) - z\widehat{Z_0}\\ &= z\big[\mathrm{R}(\chi(\widehat{Z})),\widehat{Z_0}\big] + \frac{z}{2!}\big[\mathrm{R}(\chi(\widehat{Z})),[\mathrm{R}(\chi(\widehat{Z})),\widehat{Z_0}]\big] + \frac{z}{3!}\big[\mathrm{R}(\chi(\widehat{Z})),[\mathrm{R}(\chi(\widehat{Z})),[\mathrm{R}(\chi(\widehat{Z})),\widehat{Z_0}]]\big] + \cdots\\ &= z\sum_{n>0}\frac{1}{n!}\mathrm{ad}\big[\mathrm{R}(\chi(\widehat{Z}))\big]^{(n)}(\widehat{Z_0})\end{aligned} \tag{72}$$

The following corollary is a direct consequence of Equation (69).

Corollary 13.

$$\dot{A}_+A_+^{-1} = \widehat{\beta}(\varphi) + z\widehat{Z_0}. \tag{73}$$

and for the renormalization matrix we get

Corollary 14.

$$\widehat{\varphi}_+(t,z) = e^{t(\widehat{\beta}(\varphi)+z\widehat{Z_0})}\widehat{\varphi}_+(z,0)e^{-tz\widehat{Z_0}}. \tag{74}$$

Proof. From Corollary 13 we get:

$$A_+(t) = e^{t(\widehat{\beta}(\varphi)+z\widehat{Z_0})}A_+(0),$$

which immediately proves the claim. Alternatively one readily observes that:

$$\begin{aligned}\widehat{\varphi}_+(t,z) = \widehat{\varphi}_-\widehat{\varphi^t} &= \widehat{\varphi}_-e^{tz\widehat{Z_0}}\widehat{\varphi}(0,z)e^{-tz\widehat{Z_0}}\\ &= \widehat{\varphi}_-e^{tz\widehat{Z_0}}\widehat{\varphi}_-^{-1}\widehat{\varphi}_+(0,z)e^{-tz\widehat{Z_0}}\\ &= e^{t\widehat{\varphi}_-(z\widehat{Z_0})\widehat{\varphi}_-^{-1}}\widehat{\varphi}_+(z,0)e^{-tz\widehat{Z_0}}\\ &= e^{t(\widehat{\beta}(\varphi)+z\widehat{Z_0})}\widehat{\varphi}_+(z,0)e^{-tz\widehat{Z_0}},\end{aligned}$$

where we used the well-known fact $\exp(A)\exp(B)\exp(-A) = \exp(\exp(A)B\exp(-A))$. □

Corollary 15. (Connes–Kreimer's scattering-type formula [CK99])

$$\widehat{\varphi}_- = \lim_{t\to+\infty} e^{-t(\frac{\widehat{\beta}(\varphi)}{z}+\widehat{Z_0})}e^{t\widehat{Z_0}}. \tag{75}$$

Proof. We again adapt Sakakibara's computation ([Sa04] § 2) to our matrix setting:

$$\begin{aligned} e^{-t(\frac{\widehat{\beta}(\varphi)}{z}+\widehat{Z_0})}e^{t\widehat{Z_0}} &= e^{-t(\widehat{\varphi}_-\widehat{Z_0}\widehat{\varphi}_-^{-1})}e^{t\widehat{Z_0}} \\ &= \widehat{\varphi}_- e^{-t\widehat{Z_0}}\widehat{\varphi}_-^{-1}e^{t\widehat{Z_0}} \\ &= \Psi_J\big[\varphi_- \star \theta_{-t}\big(\varphi_-^{\star-1}\big)\big]. \end{aligned}$$

Now we have for any homogeneous $x \in \mathcal{H}$ of degree ≥ 1:

$$\begin{aligned} \lim_{t\to+\infty}\big(\varphi_- \star \theta_{-t}(\varphi_-^{\star-1})\big)(x) &= \lim_{t\to+\infty}\Big(\varphi_-(x) + e^{-t|x|}\varphi_-^{-1}(x) + \sum_{(x)}\varphi_-(x')e^{-t|x''|}\varphi_-^{-1}(x'')\Big) \\ &= \varphi_-(x). \end{aligned}$$

Replacing x with the matrix coefficients M_{ij} proves then the corollary. □

This result becomes evident on the level of matrices when going back to equation (62). Assume for a moment $z \in \mathbb{R}$ positive. One observes by replacing t by $-t$ that in the first equality on the right-hand side each entry has the form $(\widehat{\varphi^{-t}}_{ij})_{i\geq j} = (\exp(-tz|M_{ij}|)\widehat{\varphi}_{ij})_{i\geq j}$ with M_{ij} being the coproduct matrix in (21). Hence, with $|M_{ii}| = 0$ we see immediately that $(\widehat{\varphi^{-t}}_{ij}) \xrightarrow{t\to\infty} \mathbf{1}$.

Remark 16. Considering the Proposition 9 in the matrix setting we have an alternative matrix representation of the beta-function:

$$\widehat{\beta}(\varphi) = [\mathrm{Res}\ \ \widehat{\varphi}_-,\, \widehat{Z}_0]. \tag{76}$$

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