TOWARD ZETA FUNCTIONS AND COMPLEX DIMENSIONS OF MULTIFRACTALS

MICHEL L. LAPIDUS AND JOHN A. ROCK

ABSTRACT. Multifractals are inhomogeneous measures (or functions) which are typically described by a full spectrum of real dimensions, as opposed to a single real dimension. Results from the study of fractal strings in the analysis of their geometry, spectra and dynamics via certain zeta functions and their poles (the complex dimensions) are used in this text as a springboard to define similar tools fit for the study of multifractals such as the binomial measure. The goal of this work is to shine light on new ideas and perspectives rather than to summarize a coherent theory. Progress has been made which connects these new perspectives to and expands upon classical results, leading to a healthy variety of natural and interesting questions for further investigation and elaboration.

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1. Introduction

The concept of *multifractals* arises from the study of objects in physics, geology, chemistry, economy and crystal growth (among others) which are believed to be more fully described by a spectrum of real dimensions rather than with a single real value such as the Hausdorff or the Minkowski dimension. Chapter 9 of [29] provides an excellent introduction to some natural occurrences of these multifractal objects, and Chapter 17 of [2] and Appendix B of [27] provide mathematically heuristic descriptions of their construction and properties. Essentially, each dimension in the spectrum corresponds in a specific manner to a regularity value that describes the multi-scale behavior of the object in question. In our case, we focus on a binomial measure on the unit interval whose support is the ternary Cantor set. Ultimately we would like to take into account the oscillations intrinsic to such objects by allowing a spectrum of *complex dimensions*, perhaps one for each regularity value.

Section 2 features the construction of our primary example of a multifractal (a binomial measure with support on the Cantor set) and recalls some heuristic results on measures of this type from [2] and [27]. Also, the definition of *regularity* is recalled as it appears in [19]. Regularity is key to the description of the multi-scale behavior of multifractal measures and functions.

Section 3 provides a brief introduction to and a summary of some results from the analysis of fractals in the study of the geometry, spectra and dynamics of fractal strings via certain zeta functions and their poles (the complex dimensions) in the classical sense of [13], [9], [15], and [16]. Further results and analysis from the theory of complex dimensions of fractal strings can be found in [3, 11, 12, 15, 16].

Section 4 summarizes some of the recent results from [14] and [28]. The definition of the *partition zeta function* and some illustrative theorems on the connection to current results from other approaches to multifractal analysis (see, e.g., [4, 7, 17, 24]) are given and discussed, in particular the further solidification of the heuristic results described in [2] and [27] and revisited in Section 2.

Section 5 reviews the slightly less recent results (from [10] and [28]) on the use of multifractal zeta functions to illuminate some topological properties of fractal strings. These results generalize and expand the theory of complex dimensions of ordinary fractal strings from [15, 16].

Section 6 closes with a collection of natural questions that arise from the work and deserve further attention. Also, suggestions for further research are given and briefly discussed.

Overall, the goal of this work is to shine some light on new ideas, not to summarize a coherent theory. The proofs and full development of theorems have been left out for brevity but can be found in the appropriate references.

2. A Multifractal Measure on the Cantor Set

This section begins with the construction of a simple example of a multifractal measure. A binomial measure μ can be constructed by adding a mass distribution to the construction of the Cantor set which consists of a countable intersection of a nonincreasing sequence of closed intervals whose lengths tend to zero. Specifically, in addition to removing open middle thirds, weight is added at each stage. On the remaining closed intervals of each stage of the construction, place $1/3$ of the weight on the left interval and $2/3$ on the right, ad infinitum. See Figure 1. The measure found in the limit, denoted μ, is a multifractal measure.

Figure 1. *Constructing a binomial measure μ on the Cantor set.*

An integral notion of this text which stems from other approaches to multifractal analysis is *regularity* (or *coarse Hölder exponent*). This notion and some illustrative theorems on its application to a mathematical formalism for multifractals can be found in [19]. In our case, as well as that of [10, 14, 18, 19, 28], regularity allows for the breakdown of a given multifractal measure by observing its behavior at different scales.

Definition 2.1. *Let* $\mathbf{X}([0,1])$ *denote the space of closed subintervals of* $[0, 1]$. *The* *regularity* $A(U)$ *of a Borel measure* μ *with* $U \in \mathbf{X}([0,1])$ *and range* $[0, \infty]$ *is*

$$A(U) = \frac{\log \mu(U)}{\log |U|},$$

where $|\cdot| = \lambda(\cdot)$ *is the Lebesgue measure on* $\mathbb{R}$.

Equivalently, $A(U)$ is the exponent α that satisfies

$$|U|^\alpha = \mu(U).$$

Note that regularity can be considered for any interval, whether open, closed or neither.

Collecting intervals according to their regularity is key to the developments of the multifractal and partition zeta functions (defined in sections 5 and 4 respectively), which mirror that of the geometric zeta functions (in section 3) in certain respects. Thus, the following notation is helpful: Let $U \in \mathcal{R}(\alpha)$ if and only if $A(U) = \alpha$. Regularity values α in the extended real numbers $[-\infty, \infty]$ will be considered. In the extreme cases,

$$\alpha = \infty = A(U) \Leftrightarrow \mu(U) = 0 \text{ and } |U| > 0,$$

and

$$\alpha = -\infty = A(U) \Leftrightarrow \mu(U) = \infty \text{ and } |U| > 0.$$

Fixing the regularity α allows for the definition of a generalization of geometric zeta functions called *multifractal zeta functions.* These functions, originally from [10] and discussed in Section 5, yield additional topological information for fractal strings and suggest a way of conducting multifractal analysis for Borel measures on the unit interval. The properties of these multifractal zeta functions have not yet been shown to relate significantly to current results in multifractal analysis. However, as discussed in Section 4, the *partition zeta functions* first described in [28] and later in [14] do relate to the results described in [2] and [27] mentioned

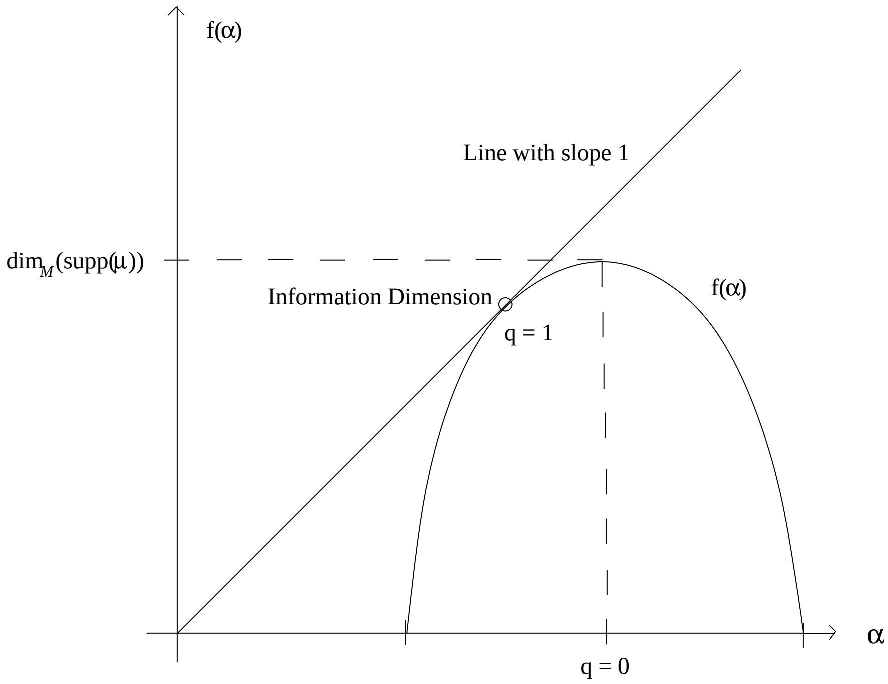

Figure 2. *The multifractal spectrum curve of* $f(\alpha)$ *for the measure* μ, *as found in* [2], *page 259.*

below. A similar approach to this type of analysis appears in the work-in-progress [18], where very nice connections to classical results also exist and are explained.

Current preliminary results on the measure μ can be found in Chapter 17 of [2]. To briefly summarize, a Legendre transform pair using the *multifractal spectrum* $f(\alpha)$ of regularity α and parameter q results in the curve in Figure 2. Roughly speaking, each regularity value α corresponds to a collection of sets which have the same regularity value at ever-decreasing scales and combine to yield a dimension (of sorts) for that regularity value. In particular, when the parameter $q = 1$, the regularity value yields the information dimension of the measure μ, and when $q = 0$, the regularity value yields the Minkowski dimension of the support of μ. This multifractal spectrum is rediscovered as the abscissa of convergence function $\sigma(\alpha)$ for the partition zeta functions of the measure μ, as described in Section 4 of this work.

Multifractal analysis has not yet been defined in a common, strict sense. Indeed, different authors provide a variety of different approaches arising from both mathematics and applications. For a physics perspective, see, e.g., [22, 26, 29] and part of [17]. For a mathematical perspective, see Chapter 17 of [2] and Appendix B of [27] for an introduction to the subject matter, and see [29] from an applications standpoint. For even more on the mathematics side, see [4, 5, 6, 7, 17, 21, 23, 24, 25].

Before elaborating on the connections between the results described in this work and the results described in [2] and [27], the next section discusses the techniques used in the study of fractal strings that motivate the definitions of the multifractal and partition zeta functions.

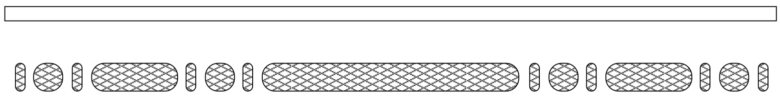

Figure 3. *An approximation of the Cantor String* Ω_1.

3. Fractal Strings

A *fractal string*, in the classical sense of [13], [9], [15] and [16], is a bounded, open subset of the real line denoted by $\Omega = \cup_{j=1}^{\infty}(a_j, b_j)$. Such objects consist of an at most countable collection of disjoint open intervals. When there is a countably infinite collection of disjoint intervals in Ω contained in $[0,1]$ and the boundary $\partial\Omega$ is equal to the complement Ω^c in the usual topology of $[0,1]$, the resulting set is often a fractal subset of the real line. An example of such an object which is directly pertinent to this work is the Cantor String. Its complement in $[0,1]$ is the classic Cantor set. See Figure 3 for a finite approximation of the Cantor String. Throughout this text[1], we assume that a fractal string Ω has total length 1, is a subset of $[0,1]$, and has boundary $\partial\Omega$ equal to Ω^c, the complement of Ω in $[0,1]$.

Important geometric, spectral and dynamical information is contained in the collection of the lengths of the intervals which constitute Ω. This collection is denoted $\mathcal{L}$. Thus, $\mathcal{L} = \{\ell_j\}_{j=1}^{\infty}$, where the ℓ_j are the lengths of the disjoint open intervals (a_j, b_j). Often it is best to consider $\mathcal{L}$ as a set of distinct lengths $\{l_n\}_{n=1}^{\infty}$ with multiplicities $\{m_n\}_{n=1}^{\infty}$. The results of this section depend only on the sequence of lengths $\mathcal{L}$ and not on the topological configuration of the disjoint open intervals of the fractal string Ω which generate them. As such, and by abuse of notation, the lengths $\mathcal{L}$ may be referred to as the fractal string in place of the open set Ω. A discussion regarding the differences between fractal strings with varying topological configuration but identical lengths can be found in Section 5. The independence of the results presented in this section from the topological configuration of the relevant fractal strings is a key motivation for the definitions that follow in Section 4. The lengths themselves retain much information regarding the fractal strings, and a similar comment can be applied to the lengths associated with the construction in Section 4 of a multifractal measure such as μ from Section 2.

In this work, the key notion of dimension is the *Minkowski dimension*, whereas in many other works on fractals and multifractals, the Hausdorff dimension is prominent. The following develops the definition of the Minkowski dimension.

The one-sided volume of the tubular neighborhood of radius ε of $\partial\Omega$ is

$$V(\varepsilon) = \lambda(\{x \in \Omega \mid dist(x, \partial\Omega) < \varepsilon\}), \tag{1}$$

where $\lambda(\cdot) = |\cdot|$ denotes the Lebesgue measure. The *Minkowski dimension* of $\partial\Omega$, or simply of $\mathcal{L}$, is

$$\dim_M(\partial\Omega) = D = D_{\mathcal{L}} := \inf\{d \geq 0 \mid \limsup_{\varepsilon \to 0^+} V(\varepsilon)\varepsilon^{d-1} < \infty\}. \tag{2}$$

Note that one may refer directly to the Minkowski dimension of the sequence of lengths $\mathcal{L}$.

The equation below describes a relationship between the Minkowski dimension D of (the boundary of) a fractal string with lengths $\mathcal{L}$ and the sum of each of its

[1] Theorem 5.2 is an exception.

lengths with exponent $\sigma \in \mathbb{R}$. This was first observed in [9] using a key result of Besicovitch and Taylor [1], and a direct proof can be found in [16], pp. 17–18.

$$D = D_{\mathcal{L}} = \inf\left\{\sigma \in \mathbb{R} \;\middle|\; \sum_{j=1}^{\infty} \ell_j^{\sigma} < \infty\right\}. \tag{3}$$

$D_{\mathcal{L}}$ is the abscissa of convergence of the Dirichlet series $\sum_{j=1}^{\infty} \ell_j^s$, where $s \in \mathbb{C}$. This Dirichlet series is the *geometric zeta function* of $\mathcal{L}$ and it is the function that has been generalized in [10, 14, 28] using notions from multifractal analysis.

Definition 3.1. *The* <u>*geometric zeta function*</u> *of a fractal string* Ω *with lengths* $\mathcal{L}$ *is*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{n=1}^{\infty} m_n l_n^s,$$

where $\mathrm{Re}(s) > D_{\mathcal{L}}$.

When possible, such a function can be meromorphically continued to a region W in the complex plane. Such an extension may reveal a family of poles which are called the *complex dimensions* of the string $\mathcal{L}$, denoted $\mathcal{D}_{\mathcal{L}}$, which has the following definition.

Definition 3.2. *The set of* <u>*complex dimensions*</u> *of a fractal string* Ω *with lengths* $\mathcal{L}$ *is*

$$\mathcal{D}_{\mathcal{L}}(W) = \{\omega \in W \;\mid\; \zeta_{\mathcal{L}} \text{ has a pole at } \omega\}.$$

Many interesting results stem from the investigation of the geometric zeta functions and complex dimensions of fractal strings. For instance, the following theorem characterizes the Minkowski measurability of a fractal string and can be found in [15, 16]. First, we define some of the terms used in the theorem.

If $\lim_{\varepsilon\to 0^+} V(\varepsilon)\varepsilon^{d-1}$ exists and is positive and finite for some d, then $d = D$ and we say that $\mathcal{L}$ is *Minkowski measurable*. The *Minkowski content* of $\mathcal{L}$ is then defined by $\mathcal{M}(D,\mathcal{L}) := \lim_{\varepsilon\to 0^+} V(\varepsilon)\varepsilon^{D-1}$. The Minkowski dimension[2] can be expressed in terms of the upper box dimension

$$\limsup_{\varepsilon\to 0^+} \frac{N_\varepsilon(F)}{-\log \varepsilon}, \tag{4}$$

where $N_\varepsilon(F)$ is the smallest number of cubes with side length ε that cover a nonempty bounded subset F of $\mathbb{R}^r$. In [8], it is shown that if $F = \partial\Omega$ is the boundary of a bounded open set Ω, then $r - 1 \leq \dim_H(F) \leq \dim_M(F) \leq r$ where r is the Euclidean dimension of the ambient space, $\dim_H(F)$ is the Hausdorff dimension of F and $\dim_M(F) = D$ is the Minkowski dimension of F. In this work, we have $r = 1$ and thus $0 \leq \dim_H(F) \leq \dim_M(F) \leq 1$. The specific conditions under which the theorem below holds are too complicated to concisely describe in this work, however the full theorem and context can be found in Chapter 8 of [16].

Theorem 3.3. *If a fractal string* Ω *with lengths* $\mathcal{L}$ *satisfies certain mild conditions, then the following statements are equivalent:*

(1) D *is the only complex dimension of* Ω *with real part* $D_{\mathcal{L}}$*, and it is simple.*
(2) $\partial\Omega$ *is Minkowski measurable.*

[2] Defined as previously in Eq. (2), except 1 is replaced with r.

The complex dimensions of the Cantor String are easily found. The distinct lengths are $l_n = 3^{-n}$ with multiplicities $m_n = 2^{n-1}$ for every $n \in \mathbb{N}$. Hence, for $\mathrm{Re}(s) > \log_3 2$,

$$\zeta_{\mathcal{L}}(s) = \zeta_{CS}(s) = \sum_{n=1}^{\infty} 2^{n-1} 3^{-ns} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

The last equation holds for all $s \in \mathbb{C}$ after analytic continuation, hence

$$\mathcal{D}_{\mathcal{L}} = \mathcal{D}_{CS} = \left\{ \log_3 2 + \frac{2im\pi}{\log 3} \mid m \in \mathbb{Z} \right\}.$$

Remark 3.4. Theorem 3.3 applies to self-similar strings (fractal strings whose boundary is a self-similar set). The Cantor String is self-similar, thus Theorem 3.3 indicates that the Cantor String is *not* Minkowski measurable.

The texts [15, 16] (specifically Chapter 8 of [16]) also contain the following key result, which uses the complex dimensions of a fractal string in a formula for the volume of the inner ε-neighborhoods of the fractal string.

Theorem 3.5 (Tube Formula)**.** *The volume of the one-sided tubular neighborhood of radius ε of the boundary of a fractal string Ω with lengths $\mathcal{L}$ is given (under mild hypotheses) by the following explicit formula with error term:*

$$V(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}}(W) \cup \{0\}} \operatorname{res}\left(\frac{\zeta_{\mathcal{L}}(s)(2\varepsilon)^{1-s}}{s(1-s)}; \omega \right) + \mathcal{R}(\varepsilon),$$

where the error term can be estimated by $\mathcal{R}(\varepsilon) = \mathcal{O}(\varepsilon^{1-\sigma})$ as $\varepsilon \to 0^+$ with σ being the upper bound of the graph of a certain bounded, real-valued continuous function. When the horizontal and vertical axes are exchanged, the graph of this function is the left part of the boundary of the region W.

Remark 3.6. If $\mathcal{L}$ is a self-similar string, the hypotheses of Theorem 3.5 are always satisfied and its conclusion holds with $\mathcal{R}(\varepsilon) \equiv 0$. This is the case for the Cantor String, for example.

In order to illustrate Theorem 3.5 in a very simple situation, we give the concrete form of the tube formula for the Cantor String (see [16, Eq.(1.14), p.15]):

$$V_{CS}(\varepsilon) = \frac{1}{2\log 3} \sum_{n=-\infty}^{\infty} \frac{(2\varepsilon)^{1-D-in\mathbf{p}}}{(D+in\mathbf{p})(1-D-in\mathbf{p})} - 2\varepsilon,$$

for all $0 < \varepsilon \leq 1/2$, where $D = \log_3 2$ is the Minkowski dimension of the Cantor String (and the Cantor set) as above, and $\mathbf{p} = 2\pi/\log 3$ is its oscillatory period.

Important geometric and spectral information regarding the structure of a fractal string is contained in its sequence of lengths $\mathcal{L}$. The real parts of the complex dimensions are related to the amplitudes of the oscillations in the volume $V(\varepsilon)$ of the tubular neighborhood of the string, while the imaginary parts coincide with the frequencies. Other examples elaborating this philosophy are provided in [15, 16] as well as [11, 12], where some higher–dimensional counterparts to Theorem 3.5 in the case of the Koch snowflake curve and self-similar systems (and tilings) can be found.

The strength of these results and their independence from the topological configuration of the fractal string Ω, and hence their complete dependence on the lengths $\mathcal{L}$ alone, is a key motivation for defining the partition zeta functions in the next section in terms of sequences of lengths with certain properties. The next section translates the construction of the geometric zeta function of a fractal string into the construction of a family of partition zeta functions for a multifractal measure such as the measure μ from Section 2.

4. Partition Zeta Functions

The *partition zeta functions* described in this section were first defined in [28] and will be discussed further in [14]. These functions are defined for any Borel measure on the unit interval along with a family of partitions and are parameterized by the regularity values attained by such measures. The construction is very similar to that of the geometric zeta function in the sense that the functions are series whose terms come from a sequence of properly defined lengths. For geometric zeta functions the terms are derived from the lengths of the disjoint intervals of a given fractal string. However, for partition zeta functions the terms are derived from a sequence of lengths from a family of partitions which exhibit the same regularity.

The families of partitions we consider satisfy certain requirements and occur quite naturally in the construction of multinomial measures such as the binomial measure. Consider an ordered family of partitions $\mathfrak{P} = \{\mathcal{P}_n\}_{n=1}^{\infty}$ of $[0,1]$, each of which splits the unit interval into finitely many subintervals. The order is given by the relation $\mathcal{P}_n \succ \mathcal{P}_{n+1}$ taken to mean that each of the intervals P_{n+1}^k which comprise the partition $\mathcal{P}_{n+1}$ is a subinterval of some interval in $\mathcal{P}_n$[3].

Definition 4.1. *For a measure μ on the interval $[0,1]$ with an ordered family of partitions $\mathfrak{P}$, the* <u>*partition zeta function with regularity α*</u> *is*

$$\zeta_{\mathfrak{P}}^{\mu}(\alpha, s) = \sum_{n=1}^{\infty} \sum_{A(P_n^k)=\alpha} |P_n^k|^s,$$

where the inner sum is taken over the intervals P_n^k in the partition $\mathcal{P}_n$ which have regularity $A(P_n^k) = \alpha \in [-\infty, \infty]$, and $\mathrm{Re}(s)$ *is large enough.*

Recall the construction of the measure μ from Section 2. The breakdown of mass and length readily defines a *natural family of partitions* $\mathfrak{P}$ for this measure μ as simply the closed intervals and their complements in the construction of the Cantor set. The mass breakdown allows for the separation of the individual intervals in $\mathfrak{P}$ into collections according to their regularity, found with respect to the measure μ. At each stage, the intervals with the same regularity α have multiplicities given by binomial coefficients. In turn, their lengths and multiplicities constitute the terms in the definition of the partition zeta function with that regularity.

To determine the intervals which have the same regularity, note that the collection of all intervals from every partition in the family $\mathfrak{P}$ is a countable set. Thus, the regularity values attained on these intervals are a function of the ordered pair of integers (k_1, k_2) (which satisfy the properties mentioned below) as follows:

$$\alpha := \alpha(k_1, k_2) = \frac{\log(2^{nk_1}/3^{nk_2})}{\log(1/3^{nk_2})} = 1 - \frac{k_1}{k_2}\log_3 2,$$

[3] To avoid trivial situations, we further assume that the mesh of the sequence of partitions $\mathcal{P}_n$ tends to zero.

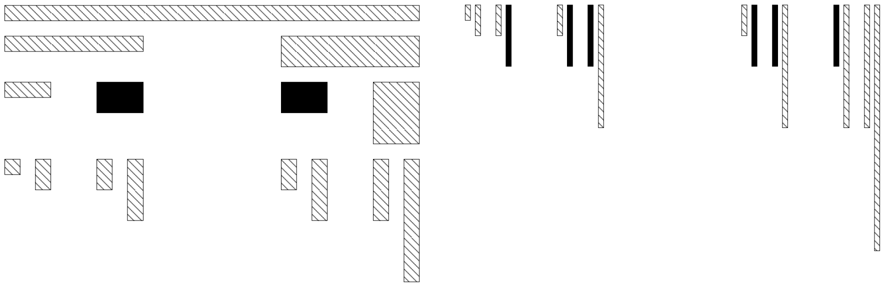

Figure 4. *Construction of the multifractal binomial measure μ, with emphasis on the intervals with regularity $\alpha(1,2)$.*

for all $n \in \mathbb{N}$ and where $k_1 \in \mathbb{N} \cup \{0\}$, $k_2 \in \mathbb{N}$, $k_1 \leq k_2$ and k_1 and k_2 are necessarily relatively prime (denoted $(k_1, k_2) = 1$), except when $k_1 = 0$ or 1 and $k_2 = 1$. The integers k_j relate to the measure and regularity of intervals in that, roughly, k_1 is the number of times an interval falls to the right and gets 2/3 of the mass after k_2 stages during the construction.

The regularity value $\alpha(k_1, k_2)$ with fixed (k_1, k_2) as above only occurs in the partitions $\mathcal{P}_{nk_2}$ for each $n \in \mathbb{N}$, with multiplicity $\binom{nk_2}{nk_1}$. In summation,

$$\zeta^{\mu}_{\mathfrak{P}}(\alpha(k_1, k_2), s) = \zeta^{\mu}_{\mathfrak{P}}(\alpha(k_2 - k_1, k_2), s) = \sum_{n=1}^{\infty} \binom{nk_2}{nk_1} 3^{-k_2 ns}.$$

See Figure 4 for the first several intervals with regularity $\alpha(1, 2)$.

There is a notion of multifractal spectrum which stems immediately from this set up and is reminiscent of similar results on multifractals found in Chapter 17 of [2], especially the graphs of the spectra in Figures 17.2 on page 259 (reproduced in Section 2 of this work) and 17.3 on page 261. In our context, the spectrum $\sigma(\alpha)$ is defined as the function which yields the abscissa of convergence of the partition zeta function with regularity α, $\zeta^{\mu}_{\mathfrak{P}}(\alpha, s)$, for all α. See Figure 5 for an approximation of the graph for this function and compare to Figure 2 which contains the graph of the multifractal spectrum $f(\alpha)$ from [2].

This graph exhibits some interesting properties, such as its maximum coincides with the Minkowski dimension of the support of μ. Further, this structure holds in greater generality. If, in the construction of the Cantor set, the initial length is replaced by some h^{-1} and the smaller weight by a w^{-1}, we have the following theorem. The proof is omitted, but can be found in [14, 28].

Theorem 4.2. *As a function of the regularity values, the abscissa of convergence function σ associated with the partition zeta function of the measure $\mu(h, w)$ with $h \geq 2$ and $w > 2$ has the form*

$$\begin{aligned}(5) \qquad \sigma(\alpha) \;&=\; \frac{(\alpha - \log_h w)}{\log_h (w-1)} \cdot \log_h \left(\frac{-(\alpha - \log_h w)}{\log_h (w-1)} \right) \\ &\quad - \left(1 + \frac{(\alpha - \log_h w)}{\log_h (w-1)}\right) \cdot \log_h \left(1 + \frac{(\alpha - \log_h w)}{\log_h (w-1)}\right).\end{aligned}$$

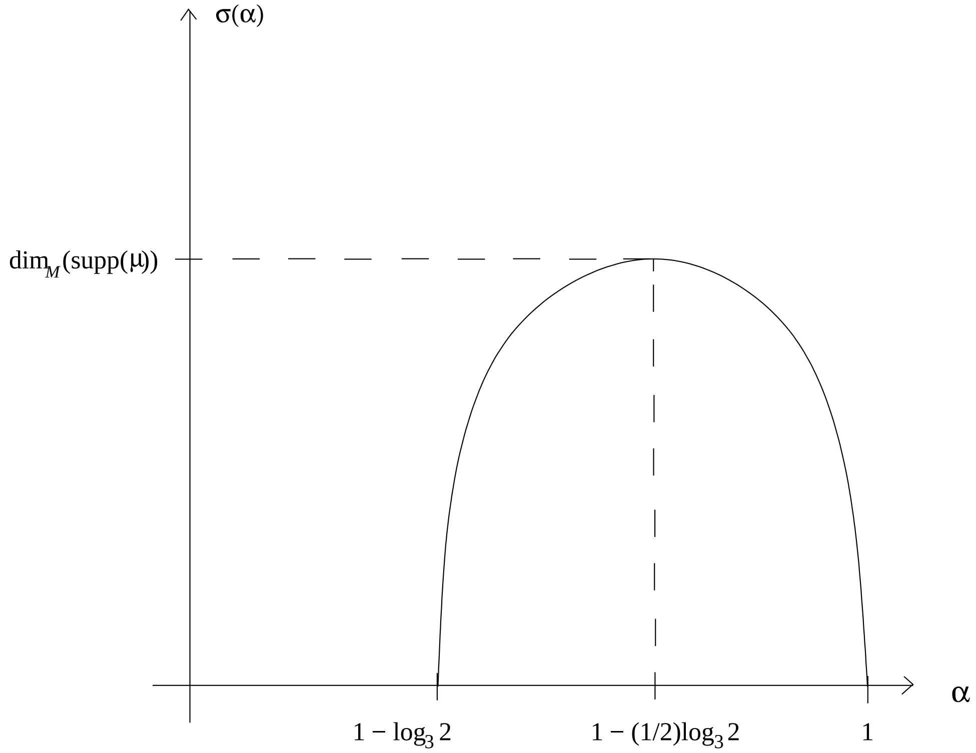

Figure 5. σ as a function of α for the measure μ.

As the abscissa of convergence function, σ is defined on a dense subset of the interval $[\log_h w - \log_h (w-1), \log_h w]$, and it attains its maximum at

$$\alpha = \alpha(1,2) = \log_h w - (1/2)\log_h (w-1). \tag{6}$$

This maximum value coincides with the Minkowski dimension of the support of the measure $\mu(h,w)$. That is,

$$\begin{aligned} \dim_M(supp(\mu(h,w)) &= \max\{\sigma(\alpha) \mid \alpha = \alpha(k_1,k_2), (k_1,k_2) = 1\} \\ &= \log_h 2. \end{aligned} \tag{7}$$

Theorem 4.2 almost contains the following multifractal spectrum $f(\alpha)$ discussed on page 934 of Appendix B in [27] as a specific case:

$$\begin{aligned} \sigma(\alpha) = f(\alpha) = \quad & -\frac{\alpha_{max} - \alpha}{\alpha_{max} - \alpha_{min}} \log_2\left(\frac{\alpha_{max} - \alpha}{\alpha_{max} - \alpha_{min}}\right) \\ & -\frac{\alpha - \alpha_{min}}{\alpha_{max} - \alpha_{min}} \log_2\left(\frac{\alpha - \alpha_{min}}{\alpha_{max} - \alpha_{min}}\right), \end{aligned}$$

where α_{max} and α_{min} denote the maximum and minimum regularity values attained by the measure in question. To fit the measure from page 934 of Appendix B in

[27] to our setting, take $h = 2$ and $w = 3$. The pertinent regularity values become

$$\alpha := \alpha(k_1, k_2) = \frac{\log(2^{nk_1}/3^{nk_2})}{\log(1/2^{nk_2})} = \log_2 3 - \frac{k_1}{k_2},$$

where α_{max} and α_{min} are attained when $k_1 = 0$ with $k_2 = 1$ and $k_1 = k_2 = 1$, respectively. We say that f is "almost" a specific case because the equation $\sigma(\alpha) = f(\alpha)$ only holds on a dense and discrete subset of $[\alpha_{min}, \alpha_{max}]$, as opposed to the full interval on which f is defined.

In addition, the graph of the spectrum $f(\alpha)$ depicted in Section 2 of this work and Chapter 17 of [2] also stems from an equation such as (5) from Theorem 4.2 (see page 261 of [2]).

Among the common features of these graphs are the coincidence of the maximum height of the curve with the Minkowksi dimension of the support of the underlying measure and the symmetry about the vertical line that passes through this maximum height. The symmetry of $\sigma(\alpha)$ is evident from the equality of the binomial coefficients $\binom{nk_2}{nk_1}$ and $\binom{nk_2}{n(k_2-k_1)}$ in the respective partition zeta functions.

The distinctions between the abscissa of convergence function σ of this text and the multifractal spectrum f of [2] and [27] lie in their developments. The function σ follows directly from the partition zeta functions defined by the weighted partitions which define the measure μ, whereas f follows from the same heuristic development but takes its values from an appropriate Legendre transform.

Further generalizations to measures with multiplicative structure similar to that of μ have been made, but for brevity we shall merely mention their existence. Also, in [18], another type of zeta function which describes multifractal measures in a manner very similar to the partition zeta functions has been defined and investigated. The results contained in that paper provide even more connections between the analysis of fractal strings via zeta functions and this new approach to multifractal analysis.

It is important to note that the partition zeta functions do not yield the geometric zeta function as some kind of special case. Indeed, there is no underlying fractal string or closed set that results from a construction like that of the Cantor set for the intervals of $\mathfrak{P}$ and a fixed regularity. So, although they do not recover the geometric zeta function for fractal strings, the partition zeta functions provide some interesting information for multifractal measures, further solidifying the existing results described, for example, in [2] and [27].

The next section describes a precursor to the partition zeta function and the similar zeta function in [18]. Despite its name, the multifractal zeta function does not connect to multifractal analysis as thoroughly as these other zeta functions, but it is a generalization of the geometric zeta functions of fractal strings and provides topological information which can not be obtained from their complex dimensions.

5. Multifractal (or Topological) Zeta Functions

The multifractal zeta function, which made its first appearance in [10], was initially developed to investigate the properties of multifractal measures. Its definition also relies on the notion of regularity, but the lengths come from a much larger and more complicated family than the family given by $\mathfrak{P}$ for the partition zeta functions. This larger collection creates many computational and theoretical difficulties, yet two regularity values ($\pm\infty$) yield new and existing results for fractal strings. The results

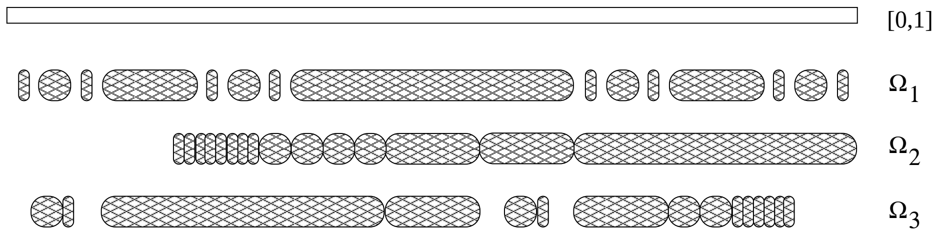

Figure 6. *Three strings with the same lengths $\mathcal{L}$, the lengths of the Cantor String Ω_1, but different topological configuration.*

presented in this section are originally from [10] (viewed from a slightly different perspective), and can also be found in [28] (viewed from the current perspective of this work).

In order to bring fractal strings into a framework that uses regularity, an appropriate measure must be defined. For a fractal string Ω in the unit interval, the measure μ_Ω is the measure which has a unit point-mass at every endpoint of the fractal string. Thus, any interval which does not contain an endpoint of Ω does not have mass, hence its regularity is ∞. On the other hand, any interval which contains a neighborhood of a limit point of the endpoints of Ω has infinite mass, hence its regularity is $-\infty$. These measures combine with the multifractal zeta functions to recover and extend the results obtained for fractal strings through the geometric zeta functions.

Using intervals whose lengths appear in a sequence $\mathcal{N}$ (which decreases to zero) and collecting them according to their regularity α allow for the definition of multifractal zeta function given below, where $k_n(\alpha)$ is the number of new disjoint intervals $K_p^n(\alpha)$ which arise at stage n. The intervals $K_p^n(\alpha)$ do not necessarily have length in $\mathcal{N}$, rather they are the disjoint intervals of the set which is the union of all closed intervals of length $\eta_n \in \mathcal{N}$ and the same regularity α.

Definition 5.1. *The multifractal zeta function of a measure μ, sequence $\mathcal{N}$, associated regularity value $\alpha \in [-\infty, \infty]$ and is given by*

$$\zeta_{\mathcal{N}}^{\mu}(\alpha, s) = \sum_{n=1}^{\infty} \sum_{p=1}^{k_n(\alpha)} |K_p^n(\alpha)|^s$$

for $\mathrm{Re}(s)$ *large enough.*

In this setting, we have the following theorem. The full proof can be found in [10, 28]. The basic idea of the proof is that an interval with regularity $\alpha = \infty$ has no mass, thus this interval must be a subset of the complement of the support of the measure, the fractal string $\Omega_\mu = (supp(\mu))^c$. The decreasing sequence $\mathcal{N}$ ensures that every disjoint open interval in this fractal string is recovered, in turn enabling us to recover the geometric zeta function. Unlike some of the other results on fractal strings mentioned in this work, the following theorem does not require the fractal string to have total length 1, to be a subset of $[0, 1]$, nor to have boundary equal to the complement in the smallest compact interval which contains the fractal string.

Theorem 5.2. *The multifractal zeta function of a positive Borel measure μ, any sequence $\mathcal{N}$ such that $\eta_n \searrow 0$ and regularity $\alpha = \infty$ is the geometric zeta function* *of* $\Omega_\mu = (supp(\mu))^c$ *(where the complement is taken in the smallest compact interval containing* $supp(\mu)$*), with lengths* $\mathcal{L}_\mu$*. That is,* $\zeta_{\mathcal{N}}^{\mu}(\infty, s) = \zeta_{\mathcal{L}_\mu}(s)$*.*

When a fractal string which has a boundary (complement in the unit interval) that is a perfect set, such as the Cantor String, we get the following theorem with omitted proof. The full development and proof can be found in [10, 28]. The regularity value $-\infty$ allows us to distinguish between fractal strings with identical lengths $\mathcal{L}$ and, hence, the same Minkowski dimension, but with different topological arrangements. See Figure 6 for approximations of three fractal strings which have the same $\mathcal{L}$ (the lengths of the Cantor String Ω_1), but have obviously distinct topological properties. In light of the following theorem, one may refer to the multifractal zeta function of a measure μ_Ω with regularity $\alpha = -\infty$ as the *topological* zeta function of the fractal string Ω.

Theorem 5.3. *Let* Ω *be a fractal string with sequence of lengths* $\mathcal{L}$ *and perfect boundary. Suppose that* $\mathcal{N}$ *is a sequence which decreases to zero such that* $l_n > \eta_n \geq l_{n+1}$ *and* $l_n > 2\eta_n$*, for all* $n \in \mathbb{N}$*. Then*

$$\zeta_{\mathcal{N}}^{\mu_\Omega}(\infty, s) = \zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} m_n l_n^s, \text{ and} \tag{8}$$

$$\zeta_{\mathcal{N}}^{\mu_\Omega}(-\infty, s) = h(s) + \sum_{n=2}^{\infty} m_n (l_n - 2\eta_n)^s, \tag{9}$$

where $h(s)$ *is the entire function given by* $h(s) = \sum_{p=1}^{k_1(-\infty)} |K_p^1(-\infty)|^s$*.*

For $q = 1, 2, 3$ and fractal strings Ω_q, Theorem 5.3 yields

$$\zeta_{\mathcal{N}}^{\mu_q}(\infty, s) = \zeta_{CS}(s) = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

However, the multifractal zeta functions corresponding to the regularity value $-\infty$ have the following forms, where the function h_3 is entire:

$$\begin{aligned}
\zeta_{\mathcal{N}}^{\mu_1}(-\infty, s) &= 2\left(\frac{1}{3} + \frac{1}{9}\right)^s + \sum_{n=2}^{\infty} 2^{n-1} \left(\frac{1}{3^n} - \frac{2}{3^{n+1}}\right)^s \\
&= 2\left(\frac{4}{9}\right)^s + \frac{2}{27^s}\left(\frac{1}{1 - 2 \cdot 3^{-s}}\right), \\
\zeta_{\mathcal{N}}^{\mu_2}(-\infty, s) &= \eta_1^s = \frac{1}{9^s}, \\
\zeta_{\mathcal{N}}^{\mu_3}(-\infty, s) &= h_3(s) + \sum_{n=2}^{\infty} m_n \left(l_{2n-1} + l_{2n} - 2\eta_{2n-1}\right)^s \\
&= h_3(s) + \sum_{n=2}^{\infty} 2^{n-1} \left(\frac{1}{3^{2n-1}} + \frac{1}{3^{2n}} - \frac{2}{3^{2n}}\right)^s \\
&= h_3(s) + \left(\frac{2^{s+1}}{81^s}\right)\left(\frac{1}{1 - 2 \cdot 9^{-s}}\right).
\end{aligned}$$

More definitively, it follows from the above discussion that the poles (complex dimensions) of these multifractal zeta functions differ completely:

$$\begin{aligned}
\mathcal{D}_{\mathcal{N}}^{\mu_1}(-\infty) &= \mathcal{D}_{CS} = \left\{ \log_3 2 + \frac{2i\pi m}{\log 3} \ \mid \ m \in \mathbb{Z} \right\}, \\
\mathcal{D}_{\mathcal{N}}^{\mu_2}(-\infty) &= \emptyset, \qquad \text{and} \\
\mathcal{D}_{\mathcal{N}}^{\mu_3}(-\infty) &= \left\{ \log_9 2 + \frac{2i\pi m}{\log 9} \ \mid \ m \in \mathbb{Z} \right\}.
\end{aligned}$$

Thus, multifractal zeta functions for at least two regularity values provide useful information about the properties of fractal strings when certain measures are considered, specifically the measures which have unit mass at every endpoint of the disjoint open intervals which define the fractal string. These multifractal zeta functions were the starting point for the development of the more refined and relevant (with respect to multifractal analysis) partition zeta functions.

6. Conclusion

There are many questions that arise in this new investigation of the application of zeta functions to fractal and multifractal analysis. For instance, when and where do the partition zeta functions have a meromorphic extension? And what are their poles? Furthermore, what are the ramifications? In light of the results with the complex dimensions of fractal strings in [15, 16], can we capture other oscillations intrinsic to multifractals in terms of these zeta functions? Can the theory be extended to higher-dimensional multifractals, as was done in the case of ordinary self-similar fractals in [11, 12]? And what about multifractals which are not the result of a multiplicative process?

Although the multifractal zeta functions do not (conveniently) provide significant information regarding a multifractal analysis of measures, they allow for new results on the *topological* properties of fractal strings to be obtained. In particular, such results can not be obtained through use of the geometric zeta functions of ordinary fractal strings alone. With this in mind, can we randomize the theory in order to deal with more realistic examples from the point of view of applications, as was done in [3] for ordinary fractal strings?

In general, what other information can be uncovered in the world of fractals by means of such zeta functions?

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