FROM GLOBAL TO LOCAL

AN INDEX BOUND FOR UMBILIC POINTS ON SMOOTH CONVEX SURFACES

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ABSTRACT. We prove that the $\mathbb{Z}/2$-valued index of an isolated umbilic point on a $C^{3+\alpha}$-smooth convex surface in Euclidean 3-space is less than 2. This follows from a localization of the authors' proof of the global Carathéodory conjecture.

The link between the two is a semi-local technique that we term *totally real blow-up*. Topologically, given a real surface in a complex surface, the totally real blow-up is the connect sum of the real surface with an embedded real projective plane. We show that this increases the sum of the complex indices of the real surface by 1, and hence cancels isolated hyperbolic complex points.

This leads to a reduction of the local result to the global result (the non-existence of embedded Lagrangian surfaces with a single complex point), which proves that the umbilic index for smooth surfaces is less than 2.

Comparison of our smooth result with that of Hans Hamburger in the real analytic case (stating that the index of an isolated umbilic point on a real analytic convex surface is less than or equal to 1) suggests the existence of "exotic" umbilic points of index 3/2.

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A series of expository video clips explaining the methods and results of this paper can be found at the following link: `http://www.youtube.com/watch?v=ybop3dETUjc`.

In this paper the authors' proof of the global Carathéodory conjecture [4] is extended to prove the following local index bound:

Main Theorem:
The index of any isolated umbilic point on a $C^{3+\alpha}$-smooth convex surface S in Euclidean 3-space is less than 2.

Here the index (an element of $\mathbb{Z}/2$) is the winding number of the principal foliation about the isolated umbilic point, and $C^{3+\alpha}$ is the usual Hölder space with $\alpha \in (0,1)$. This bound does not preclude the existence of a smooth umbilic of index 3/2, which is ruled out by the work of Hamburger in the real analytic case [5], and points to a difference between the smooth and real analytic categories for surfaces in $\mathbb{R}^3$. The proof depends upon the reformulation of questions regarding umbilic points on convex surfaces in $\mathbb{R}^3$ to questions regarding complex points on Lagrangian sections in TS^2 with its canonical neutral Kähler structure. Thus, the result is equivalent to:

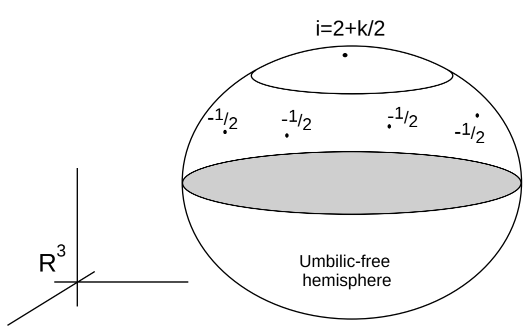
Main Theorem (Reformulation):
The index of any isolated complex point on a $C^{2+\alpha}$-smooth Lagrangian section Σ of TS^2 is less than 4.

The section Σ is the set of oriented normal lines to the surface S, considered as a surface (with a loss of one derivative) in the 4-manifold TS^2 of all oriented lines in Euclidean 3-space.

The Main Theorem is proven as follows. Suppose for the sake of contradiction that there exists a $C^{2+\alpha}$-smooth Lagrangian section of TS^2 containing an isolated complex point of index $4+k$ for $k \geq 0$. Join a neighbourhood containing the complex point to a totally real Lagrangian hemisphere by the addition of a Lagrangian annulus (for an explicit example of such a hemisphere see [4]).

Generically, this annulus contains isolated elliptic and hyperbolic complex points, where the complex index is $+1$ and -1, respectively. By Lai's index formula (see [2]) the sum of the complex indices on Σ is $\chi(T\Sigma)+\chi(N\Sigma) = 4$, and so the sum of the hyperbolic and elliptic indices is $-k$.

A straight-forward application of the h-principle proves that elliptic and hyperbolic complex points can be cancelled pair-wise in the Lagrangian category, leaving exactly k hyperbolic points in the annulus. Denote the resulting Lagrangian section by Σ'. A corresponding surface in $\mathbb{R}^3$ that is orthogonal to the oriented lines of Σ' is shown schematically below with its umbilics and their indices.



A key step in the transition from the global Carathéodory conjecture to the local index bound is to connect sum the Lagrangian section Σ' with k copies of $\mathbb{R}P^2$, thereby removing hyperbolic complex points. This construction can be applied to remove hyperbolic complex points on any real surface in a complex surface. The analogy with blowing-up in algebraic geometry motivates us to call it a *totally real blow-up*, where, in contrast to the standard notion of blowing up, the ambient complex surface is left unchanged by our operation.

Removal of the hyperbolic points yields $\Sigma_1 = \Sigma' \# k\mathbb{R}P^2 \subset TS^2$, a compact embedded surface containing a single complex point and a totally real Lagrangian hemisphere. The proof now follows the global argument used in [4]. On the one hand, the existence of holomorphic discs with boundary lying on the totally real Lagrangian hemisphere implies that the co-kernel of the $\bar{\partial}$-operator is non-zero. On the other hand, the infinite dimensional Sard-Smale theorem for surfaces with a single complex point implies that the co-kernel is zero. This contradiction implies that a Lagrangian section containing a complex point of index $4+k$ cannot exist.

In the next section we summarize the background geometry required to prove the Main Theorem. In section 2 we show how to remove hyperbolic complex points by connect sum with embedded real projective planes. Section 3 contains the proof of the Main Theorem and the final section briefly discusses the result.

1. Background

In this section we summarize the geometry of neutral Kähler TS^2. Further details can be found in [3] and references therein.

The space of oriented lines in $\mathbb{R}^3$ can be identified with TS^2 by noting that an oriented line can be identified with a pair of orthogonal vectors $(\vec{U}, \vec{V})$, the first of which is a unit vector (the direction of the oriented line):

$$\{(\vec{U}, \vec{V}) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |\vec{U}| = 1 \quad \vec{U} \cdot \vec{V} = 0 \} = TS^2.$$

By lifting the standard complex coordinate ξ (obtained by stereographic projection from the south pole on S^2) we get complex coordinates (ξ, η) on $T^{10}S^2 \cong TS^2$. In particular, identify $(\xi, \eta) \in \mathbb{C}^2$ with the vector

$$\eta \frac{\partial}{\partial \xi} + \bar{\eta} \frac{\partial}{\partial \bar{\xi}} \in T_\xi S^2.$$

In other words, the coordinate ξ represents the direction vector $\vec{U}$ of the oriented line, while η determines the perpendicular distance vector $\vec{V}$. The canonical projection $\pi : TS^2 \to S^2$, $\pi(\xi, \eta) = \xi$, maps an oriented line to its direction.

The neutral Kähler structure on TS^2 consists of a complex structure $\mathbb{J}$ for which these coordinates are holomorphic:

$$\mathbb{J}\left(\frac{\partial}{\partial \xi}\right) = i \frac{\partial}{\partial \xi} \qquad \mathbb{J}\left(\frac{\partial}{\partial \eta}\right) = i \frac{\partial}{\partial \eta},$$

together with a compatible symplectic 2-form Ω and a metric $\mathbb{G}$ of signature $(2,2)$, which have the following local expressions in (ξ, η)-coordinates:

$$\Omega = 4(1+\xi\bar{\xi})^{-2} \mathbb{R}\mathrm{e} \left(d\eta \wedge d\bar{\xi} - \frac{2\bar{\xi}\eta}{1+\xi\bar{\xi}} d\xi \wedge d\bar{\xi} \right),$$

$$\mathbb{G} = 4(1+\xi\bar{\xi})^{-2} \mathbb{I}\mathrm{m} \left(d\bar{\eta} d\xi + \frac{2\bar{\xi}\eta}{1+\xi\bar{\xi}} d\xi d\bar{\xi} \right).$$

Consider now a real surface Σ in TS^2. A point γ is said to be *complex* if the complex structure of TS^2 preserves the tangent space of Σ at γ: $\mathbb{J} : T_\gamma\Sigma \to T_\gamma\Sigma$. In coordinates, a point γ on Σ given by $\nu \mapsto (\xi(\nu,\bar{\nu}), \eta(\nu,\bar{\nu}))$ is complex iff

$$\partial\eta\bar{\partial}\xi - \bar{\partial}\eta\partial\xi|_\gamma = 0.$$

On the other hand a surface Σ is *Lagrangian* at a point γ if $\Omega|_\Sigma = 0$ at γ. A Lagrangian surface is a surface which is Lagrangian at all of its points.

Proposition 1.1. *Let S be a convex surface in $\mathbb{R}^3$ and $\Sigma \subset TS^2$ be the surface formed by the oriented normals to S. Then Σ is a Lagrangian section of $\pi : TS^2 \to S^2$.*

Conversely, if Σ is a Lagrangian section, then there exists a 1-parameter family of convex surfaces in $\mathbb{R}^3$ which are orthogonal to the oriented lines of Σ.

Moreover, $p \in S$ is umbilic iff the oriented normal to S through p is a complex point on the surface Σ formed by the oriented normals of S.

Of particular importance are the surfaces in TS^2 that arise as a graph of a section of the projection $\pi : TS^2 \to S^2$:

Proposition 1.2. *A section $\xi \mapsto (\xi, \eta = F(\xi,\bar{\xi}))$ is Lagrangian iff there exists a real function (called the* support function *of the corresponding surface in $\mathbb{R}^3$) $r : S^2 \to \mathbb{R}$ such that*

$$\frac{\partial r}{\partial \xi} = \frac{2\bar{F}}{(1+\xi\bar{\xi})^2},$$

where the support function is defined up to $r \to r+C$ (which yields parallel surfaces).

A point is complex iff at the point $\bar{\partial}F = 0$.

A point on a Lagrangian section is complex iff the corresponding point on an orthogonal surface in $\mathbb{R}^3$ is umbilic.

An isolated umbilic point p on a convex surface S is a singularity of the principal foliation of the surface and, as such, has an index $i(p) \in \mathbb{Z}/2$ (a half-integer because the foliation may not be orientable).

On the other hand, an isolated complex point γ on a real surface Σ in a complex surface also has an index $I(\gamma) \in \mathbb{Z}$, see [2]. In our case, these are related by

Proposition 1.3. *Let S be a convex surface in $\mathbb{R}^3$ containing an isolated umbilic point p and let $\Sigma \subset TS^2$ be the surface determined by the oriented normal lines of S with corresponding isolated complex point γ. Then*

$$I(\gamma) = 2i(p).$$

Following the standard convention for complex points we adopt the terminology:

Definition 1.4. An isolated complex point on a real surface is *elliptic* if the index is equal to 1, and is *hyperbolic* if the index is equal to -1.

2. Totally Real Blow-up

Consider the connected sum of a real surface Σ in TS^2 with a copy of $\mathbb{R}P^2$. That is, remove discs from Σ and $\mathbb{R}P^2$ and identify the boundary circles. A copy of the real projective plane with a disc removed is called a *cross-cap*. It can also be viewed as an annulus with the inner boundary curve antipodally identified.

We term this operation *totally real blow-up*: "blow-up" because, at the topological level, it is the real analogue of complex blowing-up (connect sum of a complex surface with $\overline{\mathbb{C}P^2}$), and "totally real" because it removes certain types of complex points. Exactly which type can be removed is established in the next result:

Proposition 2.1. *Let $\Sigma \subset \mathbb{M}$ be an embedded real surface in a complex surface with a single hyperbolic complex point $\gamma \in \Sigma$. Then the surface $\tilde{\Sigma} = \Sigma \# \mathbb{R}P^2$ given by removing a neighbourhood of γ and joining a cross-cap can be smoothed so that $\tilde{\Sigma}$ is totally real and embedded.*

Proof. Choose local holomorphic coordinates (ξ, η) on $\mathbb{M}$ so that the surface in the neighbourhood of the complex point at $\xi = 0$ is given by

$$\eta = \alpha\bar{\xi}^2 + \beta\xi\bar{\xi} + o(|\xi|^3),$$

for complex numbers α, β. As the complex point is hyperbolic we have that $|\alpha| > 2|\beta|$ (see [2]) and so by a compactly supported deformation this can be reduced to the form

$$\eta = \alpha\bar{\xi}^2,$$

without the creation of any further complex points.

Fix constants ϵ and R_0 such that $1 - \epsilon < R_0 < 1$, and consider the surface $\tilde{\Sigma}$ defined by

$$\xi = (1 - \nu\bar{\nu})\nu \qquad \text{for} \qquad 1 - \epsilon \leq |\nu| \leq 1,$$

$$\eta = \begin{cases} \alpha(1 - \nu\bar{\nu})^2\bar{\nu}^2 & \text{for} \quad 1 - \epsilon \leq |\nu| \leq R_0, \\ \alpha(a + b(1 - \nu\bar{\nu}) + c(1 - \nu\bar{\nu})^2)\bar{\nu}^2 & \text{for} \quad R_0 \leq |\nu| \leq 1. \end{cases}$$

This surface is C^1-smooth for the following choice of constants a and b:

$$a = (c - 1)(1 - R_0^2)^2 \qquad b = 2(1 - c)(1 - R_0^2),$$

and c to be determined. Moreover, it is the connected sum $\tilde{\Sigma} = \Sigma \# \mathbb{R}P^2$ and is easily seen to be embedded.

We now show that for certain values of the constant c, the surface $\tilde{\Sigma}$ is totally real. Recall, a point γ on a real surface is complex iff

$$\partial\eta\bar{\partial}\xi - \bar{\partial}\eta\partial\xi|_\gamma = 0.$$

Computing this for the surface above, we find that for $1 - \epsilon \leq |\nu| \leq R_0$ there are no complex points, while for $R_0 \leq |\nu| \leq 1$ we have

$$\begin{aligned}\partial\eta\bar{\partial}\xi - \bar{\partial}\eta\partial\xi = & -2[1 + (c-1)R_0^4 - (c(3 + 2R_0^2)R_0^2 + (5 + 2R_0^2)(1 - R_0^2))\nu\bar{\nu} \\ & + (5(1 - R_0^2) + c(2 + 5R_0^2))\nu^2\bar{\nu}^2 + 3c\nu^3\bar{\nu}^3]\bar{\nu}.\end{aligned}$$

This is zero when $\nu = 0$, which is outside of the range for the parameter, so we exclude this value.

Consider then the other factor with $x = \nu\bar{\nu}$ and $y = R_0^2$:

$$g(x, y) := 1 + (c-1)y^2 - (c(3+2y)y + (5+2y)(1-y))x + (5(1-y) + c(2+5y))x^2 + 3cx^3$$

The function g has the following properties:

(1) $g(1, 1) = 0$,
(2) $\partial_x g|_{(1,1)} = 0$ and $\partial_y g|_{(1,1)} = 0$,
(3) $\partial_x^2 g|_{(1,1)} = 4c$, $\partial_y^2 g|_{(1,1)} = 2(c - 1)$ and $\partial_x\partial_y g|_{(1,1)} = -3(c - 1)$.

Thus $\partial_x^2 g \partial_y^2 g - (\partial_x\partial_y g)^2|_{(1,1)} = (9-c)(c-1)$ and so $(1,1)$ is local minimum for g if $1 < c < 9$. Thus for this range of c, there exists $\epsilon > 0$ s.t. $g(x,y) > 0$ for $1-\epsilon < y^2 < 1$ and $R_0 \leq x^2 \leq 1$. For c outside of this range, circles of complex points occur which contribute zero to the total complex index and may be removed by a generic perturbation.

We conclude that $\tilde{\Sigma}$ is totally real. □

Note 2.2. To get a picture of the totally real blow-up of a hyperbolic point in terms of oriented lines in $\mathbb{R}^3$, consider the real surface in TS^2 given by the section $\xi \mapsto (\xi, \eta = (1+\xi\bar{\xi})^2\bar{\xi}^2)$. It is easily checked that this surface is Lagrangian and has a single hyperbolic complex point at $\xi = 0$:

$$\mathbb{I}\mathrm{m}\frac{\partial}{\partial\xi}\left(\frac{\eta}{(1+\xi\bar{\xi})^2}\right) = 0 \qquad \frac{\partial\eta}{\partial\bar{\xi}}(0) = 0 \qquad i(0) = \lim_{R\to 0}\tfrac{1}{8\pi i}\int_0^{2\pi}\frac{\partial}{\partial\theta}\ln\left(\frac{\bar{\partial}\eta}{\partial\bar{\eta}}\right)d\theta = -\tfrac{1}{2},$$

where $\xi = Re^{i\theta}$. Thus it must correspond to a 1-parameter family of convex surfaces in $\mathbb{R}^3$ which contain an isolated umbilic point of index $-1/2$. Let us construct these surfaces explicitly.

The support function is found by integrating the defining equation in Proposition 1.2 and the result is

$$r = \tfrac{2}{3}(\xi^3 + \bar{\xi}^3) + C.$$

To reconstruct these surfaces recall the fundamental correspondence relation [4] in Euclidean coordinates (x^1, x^2, x^3):

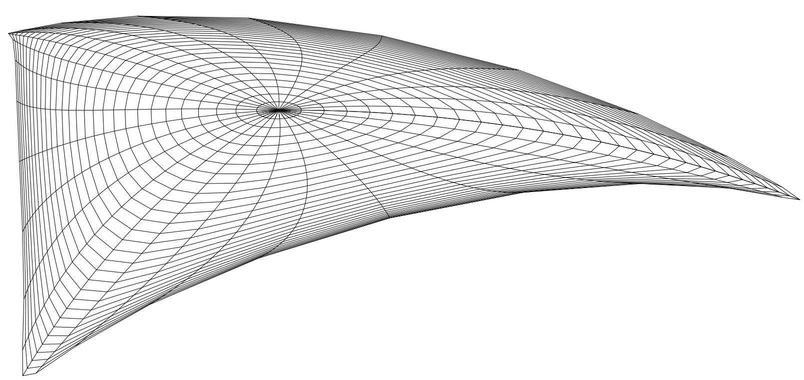
$$x^1 + ix^2 = \frac{2(\eta - \bar{\eta}\xi^2) + 2\xi(1+\xi\bar{\xi})r}{(1+\xi\bar{\xi})^2}, \qquad x^3 = \frac{-2(\eta\bar{\xi} + \bar{\eta}\xi) + (1-\xi^2\bar{\xi}^2)r}{(1+\xi\bar{\xi})^2},$$

Therefore the 1-parameter family of convex surfaces, parameterized by the inverse of their Gauss maps, is

$$x^1 + ix^2 = \frac{2(3\bar{\xi}^2 - \xi^4 + 5\xi\bar{\xi}^3 - 3\xi^5\bar{\xi})}{3(1+\xi\bar{\xi})} + \frac{2c\xi}{1+\xi\bar{\xi}}$$

$$x^3 = -\frac{4(\xi+\bar{\xi})(\xi - \xi\bar{\xi} + \xi)(1+2\xi\bar{\xi})}{3(1+\xi\bar{\xi})} + \frac{c(1-\xi\bar{\xi})}{1+\xi\bar{\xi}}$$

Below is a plot of one of these surfaces with curves indicating the images of $|\xi|$=constant and of arg(ξ)=constant.



Umbilic of index -1/2

Let us now construct the cross-cap. Consider the map $h : [R_0, 1] \times S^1 \to TS^2$, which takes $\nu = Re^{i\theta}$ to

$$\xi = (1-\nu\bar{\nu})\nu \qquad\qquad \eta = \bar{\nu}^2.$$

This an embedded cross-cap since the image of $\{1\} \times S^1$ is a circle with antipodal points identified.

These two surfaces can be joined along a circle by fixing a constant Gauss radius R_0 such that $3^{-\frac{1}{2}} < R_0 < 1$, and defining the surface $\tilde{\Sigma} = \Sigma \# \mathbb{R}P^2$ by

$$\xi = (1-\nu\bar{\nu})\nu \qquad \text{for} \qquad 3^{-\frac{1}{2}} < |\nu| \leq 1,$$

$$\eta = \begin{cases} (1+(1-\nu\bar{\nu})^2\nu\bar{\nu})^2(1-\nu\bar{\nu})^2\bar{\nu}^2 & \text{for} \quad 3^{-\frac{1}{2}} < |\nu| \leq R_0, \\ (a+b(1-\nu\bar{\nu})+c(1-\nu\bar{\nu})^2)\bar{\nu}^2 & \text{for} \quad R_0 \leq |\nu| \leq 1. \end{cases}$$

This surface is C^2-smooth for the following choice of constants

$$a = -(1-R_0^2)^4(5+2R_0^2-46R_0^4+54R_0^6-21R_0^8),$$

$$b = -2(1-R_0^2)^3(6-51R_0^4+61R_0^6-24R_0^8),$$

$$c = -6+18R_0^2+42R_0^4-180R_0^6+225R_0^8-126R_0^{10}+28R_0^{12}.$$

Moreover, it is easily seen to be totally real and may be smoothed to $C^{2+\alpha}$.

In $\mathbb{R}^3$, this surface can be visualized as a 2-parameter family of oriented lines, or a 1-parameter family of ruled surfaces which doubly covers a cylinder. On the following page we illustrate a sequence of ruled surfaces corresponding to circles of constant Gauss radius $|\xi|$ on the cross-cap.

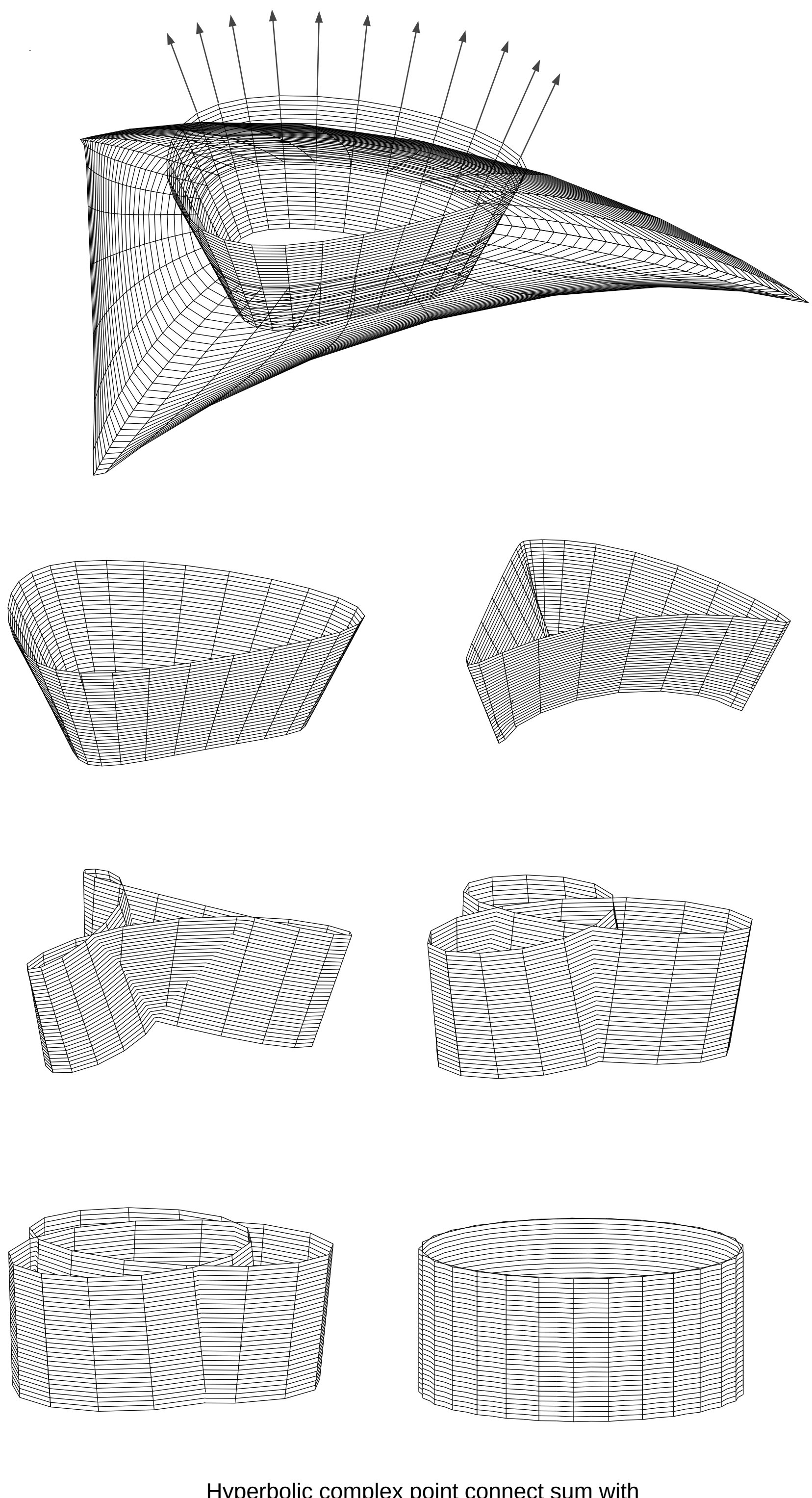
Note 2.3. Removing the hyperbolic complex point means that the sum of the indices of the complex points γ_j on Σ has increased by 1. This can be seen in $\mathbb{R}^3$ as follows.

Recall Lai's formula for this sum:

$$\sum_j I(\gamma_j) = \chi(T\Sigma) + \chi(N\Sigma).$$

Adding a cross-cap reduces $\chi(T\Sigma)$ by 1 and so a totally real blow-up must increase $\chi(N\Sigma)$ by 2.

To see that this is indeed the case, note that $\chi(N\Sigma)$ can be identified with the number of intersection points of Σ with a small perturbation of Σ. Perturbing the cross-cap by a translation in $\mathbb{R}^3$ orthogonal to the central cylinder (addition of a certain quadratic holomorphic section in TS^2), the ruling of the cylinder and its perturbation coincide at exactly 2 lines. The rest of the cross-cap does not intersect its perturbation and so, replacing the disc by the cross-cap increases $\chi(N\Sigma)$ by 2, as claimed.



Hyperbolic complex point connect sum with
an RP^2 of oriented lines in R^3

3. Proof of the Main Theorem

3.1. **Reformulation.** Suppose, for the sake of contradiction, that there exists an open $C^{3+\alpha}$ convex surface containing an isolated umbilic point of index $i = 2+k/2$, for $k \geq 0$. Close up the surface to form a closed convex surface S by joining it to an umbilic-free hemisphere. Here hemisphere refers to the range of the Gauss map of the surface $\pi_S : S \to S^2$ and umbilic-free hemispheres are simple to construct (for an explicit example see [4]).

This closed surface S will have umbilic points in the annulus, which are generically isolated and whose indices sum to $-k/2$. In fact, these generic umbilic points will be those classified by Darboux in 1896 [1], what we term *elliptic* umbilic points of index 1/2 and *hyperbolic* umbilic points of index $-1/2$ (in-line with the complex terminology).

The reformulation outlined in Section 1 means that the set of oriented normal lines to this surface form a global Lagrangian section Σ in TS^2 with 1 complex point of index $I = 4+k$, and isolated elliptic and hyperbolic complex points whose indices sum to $-k$.

3.2. **Cancellation of Complex Points.** By a small deformation of Σ we cancel the elliptic complex points so that k hyperbolic complex points remain. Such cancellation can be carried out in general for complex points on real surfaces in a complex surface (see for example Corollary 9.5.2 in [2]). However, in our case we want to ensure that the deformation remains Lagrangian. This follows from:

Lemma 3.1. *Let Σ be a smooth simply connected Lagrangian section with non-empty totally real boundary. Suppose that the sum of the complex indices on Σ is zero. Then there exists a smooth Lagrangian section Σ' with the following properties:*

(1) $\Sigma = \Sigma'$ *on* $\Sigma - K$ *for* K *a compact set in the interior of* Σ*,*
(2) Σ' *is totally real.*

Proof. Let $U = \pi(\Sigma)$ and pull back the jet-bundles over Σ to ones over U.

In particular, the support function $r : U \to \mathbb{R}$ gives a section in the 2-jet bundle of $\mathbb{R} \oplus Hom(\mathbb{R},\mathbb{R}) \oplus Hom(\mathbb{R}^2,\mathbb{R}^2)$ over U. This section is locally given by the map

$$\xi \mapsto (r, r_1, r_2, r_{11}, r_{12}, r_{21}, r_{22}),$$

where we set $\xi = x^1 + ix^2$ and a subscript denotes differentiation. Sections that arise in such a manner are called *holonomic* - see for example [8]. A non-holonomic section is of the form

$$\xi \mapsto (r, X_1, X_2, Y_1, Y_2, Y_3, Y_4),$$

for functions $\{X_j\}, \{Y_j\}$ on U. The boundary is totally real and so $|\partial\partial r|^2 \neq 0$ on ∂U. Note that

$$|\partial\partial r|^2 = (r_{11} - r_{22})^2 + (r_{12} + r_{21})^2.$$

Since the index sum is zero, the winding number of $\partial\partial r$ around the boundary is zero. Thus there exists a non-holonomic section of this bundle such that $(Y_1-Y_4)^2 + (Y_2+Y_3)^2 \neq 0$ on U which agrees with the original section in a neighbourhood of the boundary.

Consider the differential relation $(Y_1 - Y_4)^2 + (Y_2 + Y_3)^2 \neq 0$. The convex hull of the complement of this relation is the full fibre (i.e. the relation is ample) and so by the relative h-principle, there exists a holonomic section Σ' which is totally real and agrees with the original section on the boundary.

The surface Σ' has the properties (1) to (2) above.

□

Using this Lemma we can cancel pairs of hyperbolic and elliptic complex points by a perturbation. We thus arrive at a Lagrangian section Σ' with an isolated complex point of index $4+k$ and k hyperbolic complex points, which also contains a totally real Lagrangian hemisphere.

3.3. **Removal of Hyperbolic Points.** Now remove the complex points of index -1 from the surface by totally real blow-up, as described in Proposition 2.1. That is, we modify Σ' by removing a small disc containing the complex point of index -1, and attaching $\mathbb{R}P^2$ with a disc removed (a cross-cap).

Removing each of the k complex points yields a closed embedded surface $\Sigma_1 = \Sigma'\#k\mathbb{R}P^2$ containing a single isolated complex point (of index $I = 4+k$), which is Lagrangian outside of the k copies of $\mathbb{R}P^2$.

In summary, the sum of the indices of the complex points γ_j on Σ_1 is

$$\begin{aligned}\sum_j I(\gamma_j) &= \chi(T\Sigma_1) + \chi(N\Sigma_1)\\ &= \chi(T(\Sigma'\#k\mathbb{R}P^2)) + \chi(N(\Sigma'\#k\mathbb{R}P^2))\\ &= \chi(T\Sigma') - k + \chi(N\Sigma') + 2k\\ &= 4+k,\end{aligned}$$

which is the index of the single complex point on Σ_1.

3.4. **Global Argument.** A minor modification of the global arguments of [4] can now be applied to Σ_1. We sketch the arguments.

Definition 3.2. Let

$$\mathcal{S}_0 = \{\Sigma \mid \Sigma \subset TS^2 \text{ embedded } C^{2+\alpha} \text{ surface containing the point (0,0) }\}$$

and the space of sections

$$\Gamma_0(\mathbb{J}(T\Sigma)) = \left\{ \, v \in \Gamma(\mathbb{J}(T\Sigma)) \ \middle| \ v \in C^{1+\alpha}, \ \ v(\gamma_0) = 0, \, \right\}.$$

Let $\Sigma_1 = \Sigma'\#k\mathbb{R}P^2$ be the closed surface in TS^2 with a single complex point of index $4+k$ obtained as above, and by a translation and rotation suppose that the complex point lies at $(0,0) \in TS^2$.

The neighbourhood of Σ_1 in $\mathcal{S}_0$ can be modeled by the sections Γ_0:

Proposition 3.3. *There exists $\epsilon > 0$ and $\Phi : B_\epsilon(0) \subset \Gamma_0(\mathbb{J}(T\Sigma_1)) \to \mathcal{U} \subset \mathcal{S}_0$ so that $\mathcal{U} = \Phi(B_\epsilon(0))$ is a Banach manifold.*

Proof. The proof follows along the lines of the arguments in section 2.2 of [4]. □

In fact, we are interested in Lagrangian variations of Σ_1.

Definition 3.4. Let

$$\Gamma_0^{\text{lag}}(\mathbb{J}(T\Sigma)) = \left\{ \, v \in \Gamma(\mathbb{J}(T\Sigma)) \ \middle| \ v \in C^{1+\alpha}, \ \ v(\gamma_0) = 0, \ \ d(\mathbb{J}(v) \lrcorner \Omega) = 0 \, \right\}.$$

Proposition 3.5. $\Gamma_0^{lag}(\mathbb{J}(T\Sigma_1)) \subset \Gamma_0(\mathbb{J}(T\Sigma_1))$ *is a Banach subspace.*

Corollary 3.6. *There exists* $\epsilon > 0$ *and* $\Phi : B_\epsilon^{lag}(0) \subset \Gamma_0^{lag}(\mathbb{J}(T\Sigma_1)) \to \mathcal{U}^{lag} \subset \mathcal{U}$ *so that* $\mathcal{U}^{lag} = \Phi(B_\epsilon^{lag}(0))$ *is a Banach submanifold.*

Now the argument proceeds as in [4]. Namely we conclude that for an open dense set of $\mathcal{U}^{\rm lag}$, the co-kernel of the Cauchy-Riemann operator is zero. However, by mean curvature flow we can construct holomorphic discs whose boundary lie on any totally real Lagrangian hemisphere, contradicting the surjectivity of the $\bar{\partial}$-operator.

Thus no $C^{2+\alpha}$ Lagrangian surface with an isolated complex point of index $I = 4 + k$ for $k \geq 0$ exists. Equivalently, there does not exist a $C^{3+\alpha}$ convex surface containing an isolated umbilic point of index $i = 2 + k/2$ for $k \geq 0$.

4. Discussion

Since the early 1920's, attempts to prove the global Conjecture have sought to establish a local index bound, usually with the inclusion of the assumption of real analyticity on the surface (originally due to Hamburger [5] and more recently by Ivanov [6]). Conflating this bound with a much later conjecture of Loewner (related to [7]), the umbilic index bound most often sought is $i \leq 1$.

The Main Theorem represents a reversal of these historical attempts. That is, the results in this paper establish a local index bound as a consequence of global restrictions on the $\bar{\partial}$-operator. Remarkably, this opens up a gap between what we claim is a sharp result in the smooth category (umbilic index less than 2) with Hamburger's result in the real analytic category (umbilic index less than or equal to 1).

Thus, we are led to the possibility of isolated umbilic points of index 3/2 on smooth, non-real analytic surfaces. The existence and implications of such *exotic* umbilic points will be considered in a future paper.

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