NONARCHIMEDEAN CANTOR SET AND STRING

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Dedicated to Professor Vladimir Arnold, on the occasion of his jubilee

ABSTRACT. We construct a nonarchimedean (or *p-adic*) analogue of the classical ternary Cantor set $\mathcal{C}$. In particular, we show that this nonarchimedean Cantor set $\mathcal{C}_3$ is self-similar. Furthermore, we characterize $\mathcal{C}_3$ as the subset of 3-adic integers whose elements contain only $0's$ and $2's$ in their 3-adic expansions and prove that $\mathcal{C}_3$ is naturally homeomorphic to $\mathcal{C}$. Finally, from the point of view of the theory of fractal strings and their complex fractal dimensions [7, 8], the corresponding nonarchimedean Cantor string resembles the standard archimedean (or *real*) Cantor string perfectly.

1. Introduction

Our goal in this article is to provide a good *nonarchimedean* (or *p-adic*) analogue of the classic Cantor ternary set $\mathcal{C}$ and to show that it satisfies a counterpart of some of the key properties of $\mathcal{C}$ in this nonarchimedean context. We also show that the corresponding p-adic fractal string, called the nonarchimedean Cantor string and denoted by $\mathcal{CS}_3$, is an exact analogue of the ordinary archimedean Cantor string, a central example in the theory of real (or archimedean) fractal strings and their complex dimensions [7, 8]. Furthermore, we compute the geometric zeta function of $\mathcal{CS}_3$ and the associated complex fractal dimensions.

In a forthcoming paper [9], we will develop a general framework for formulating a theory of self-similar p-adic (or nonarchimedean) strings and their complex fractal dimensions. Besides answering a natural mathematical question, these results may be useful in various aspects of mathematical physics, including p-adic quantum mechanics and string theory, where extensions from the archimedean to the nonarchimedean setting have been extensively explored [12].

1.1. p-adic numbers.

Let $p \in \mathbb{N}$ be a fixed prime number. For any nonzero $x \in \mathbb{Q}$, we can always write $x = p^v \cdot a/b$, with $a, b \in \mathbb{Z}$ and for some unique $v \in \mathbb{Z}$ so that p does not divide ab. The *p-adic norm* is a function $|\cdot|_p : \mathbb{Q} \longrightarrow [0, \infty)$ given by

$$|x|_p = p^{-v} \quad \text{and} \quad |0|_p = 0.$$

One can verify that $|\cdot|_p$ is indeed a norm on $\mathbb{Q}$. Furthermore, it satisfies a *strong triangle inequality*: for any $x, y \in \mathbb{Q}$, we have $|x + y|_p \leq \max\{|x|_p, |y|_p\}$; the induced metric is therefore called an ultrametric. This inequality is called the *nonarchimedean property* because for each $x \in \mathbb{Q}$, $|nx|_p$ will never exceed $|x|_p$ for any $n \in \mathbb{N}$. The metric completion of $\mathbb{Q}$ with respect to the p-adic norm is

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the field of p-adic numbers $\mathbb{Q}_p$. More concretely, there is a unique representation of every $z \in \mathbb{Q}_p$: $z = a_v p^v + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots$, for some $v \in \mathbb{Z}$ and $a_i \in \{0, 1, \ldots, p-1\}$ for all $i \geq v$. An important subspace of $\mathbb{Q}_p$ is the unit ball, $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \ : \ |x|_p \leq 1\}$, which can also be represented as follows:

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \cdots \mid a_i \in \{0, 1, \ldots, p-1\}, \ \forall i \geq 0\}.$$

Using this *p-adic expansion*, we can see that

$$\mathbb{Z}_p = \bigcup_{c=0}^{p-1} (c + p\mathbb{Z}_p), \tag{1}$$

where $c + p\mathbb{Z}_p := \{y \in \mathbb{Q}_p : |y - c|_p \leq 1/p\}$. Moreover, by the nonarchimedean property of the p-adic norm, $\mathbb{Z}_p$ is closed under addition and hence is a ring. It is called the ring of *p-adic integers* and $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. Note that $\mathbb{Z}_p$ is compact and thus complete. (For general references on p-adic analysis, see, e.g., [4, 11].) It is also known that there are topological models of $\mathbb{Z}_p$ in the Euclidean space $\mathbb{R}^d$ as fractal spaces such as the Cantor set and the Sierpińsky gasket [11, §I.2.5]. In fact, $\mathbb{Z}_p$ is homeomorphic to the ternary Cantor set. It is thus natural to wonder what exactly is the *nonarchimedean* (or *p-adic*) analogue of the ternary Cantor set. We will answer this question in §2.

1.2. **Ternary Cantor set.** The classical ternary Cantor set, denoted by $\mathcal{C}$, is the set that remains after iteratively removing the open middle third subinterval(s) from the closed unit interval $C_0 = [0, 1]$. The construction is illustrated in Figure 1. Hence, the ternary (or archimedean) Cantor set $\mathcal{C}$ is equal to $\bigcap_{n=0}^{\infty} C_n$.

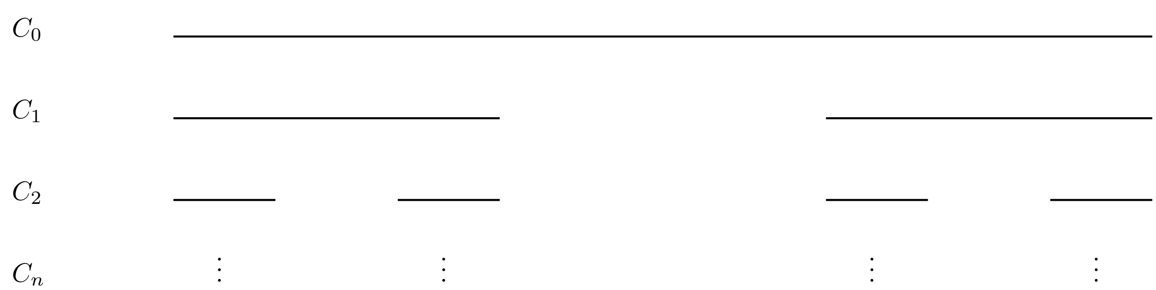

FIGURE 1. Construction of the archimedean Cantor set $\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$.

For comparison with our results in the nonarchimedean case, we state without proof the following well-known results (see, e.g., [1, Ch. 9] and [2, p. 50]):

Theorem 1.1. *The ternary Cantor set $\mathcal{C}$ is self-similar. More specifically, it is the unique nonempty, compact invariant set in $\mathbb{R}$ generated by the family $\{\Phi_1, \Phi_2\}$ of similarity contraction mappings of $[0, 1]$ into itself, where $\Phi_1(x) = x/3$ and $\Phi_2(x) = x/3 + 2/3$. That is,*

$$\mathcal{C} = \Phi_1(\mathcal{C}) \cup \Phi_2(\mathcal{C}).$$

Theorem 1.2. *The Cantor set is characterized by the ternary expansion of its elements as*

$$\mathcal{C} = \left\{ c \in [0, 1] \ : \ c = a_0 + \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots, \ a_i \in \{0, 2\}, \forall i \geq 0 \right\}.$$

We note that, as usual, we choose the nonrepeating ternary expansion here. Such a precaution will not be needed in §2 for the elements of $\mathbb{Q}_3$ because the 3-adic expansion is unique.

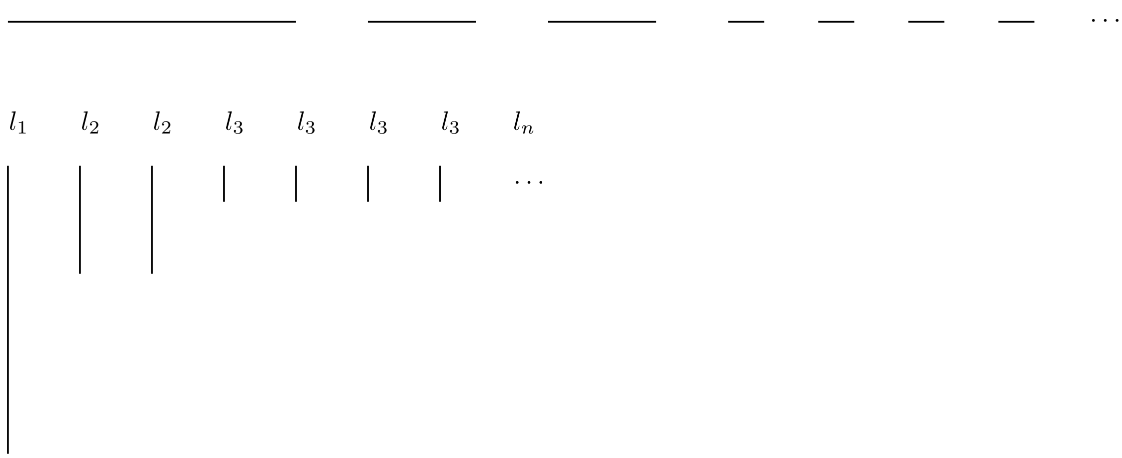

FIGURE 2. Cantor string (above); Cantor string viewed as a fractal harp (below).

1.3. **Cantor fractal string.** The *archimedean* (or *real*) *Cantor string* $\mathcal{CS}$ is defined as the complement of the ternary Cantor set in the closed unit interval $[0,1]$. By construction, the topological boundary of $\mathcal{CS}$ is the ternary Cantor set $\mathcal{C}$. The Cantor string is one of the simplest and most important examples in the research monographs [7, 8] by Lapidus and van Frankenhuijsen. Indeed, it is used throughout those books to illustrate and motivate the general theory; see also, e.g., [5] and [6]. From the point of view of the theory of fractal strings and their complex dimensions [7,8], it suffices to consider the sequence $\{l_n\}_{n\in\mathbb{N}}$ of lengths associated to $\mathcal{CS}$. More specifically, these are the lengths of the intervals of which the bounded open set $\mathcal{CS}\subset\mathbb{R}$ is composed. Accordingly, the Cantor string consists of $1=m_1$ interval of length $l_1=1/3$, $2=m_2$ intervals of length $l_2=1/9$, $4=m_3$ intervals of length $l_3=1/27$, and so on; see Figure 2.

Important information about the geometry of $\mathcal{CS}$, e.g., the Minkowski dimension and the Minkowski measurability ([5–8]), is contained in its *geometric zeta function*

$$\zeta_{\mathcal{CS}}(s) := \sum_{n=1}^{\infty} m_n\cdot l_n^s = \sum_{n=1}^{\infty}\frac{2^{n-1}}{3^{ns}} = \frac{3^{-s}}{1-2\cdot 3^{-s}} \qquad \text{for } \Re(s)>D, \tag{2}$$

where $D=\log 2/\log 3$ is the *Minkowski dimension* of the ternary Cantor set. In addition, $\zeta_{\mathcal{CS}}$ can be extended to a meromorphic function on the entire complex plane $\mathbb{C}$, as given by the last expression in (2). The corresponding set of poles of $\zeta_{\mathcal{CS}}$ is then given by

$$\mathcal{D}_{\mathcal{CS}} = \{D+\imath\nu\mathbf{p} \mid \nu\in\mathbb{Z}\}, \tag{3}$$

where $\mathbf{p}=2\pi/\log 3$ is the *oscillatory period* of $\mathcal{CS}$. Here and henceforth, we let $\imath:=\sqrt{-1}$. The set $\mathcal{D}_{\mathcal{CS}}$ is called the set of *complex dimensions* of the Cantor string; see Figure 5 in §3.

The general theme of the monographs [7, 8] is that *the complex dimensions describe oscillations in the geometry and the spectrum of a fractal string.* In particular, there are oscillations of order D in the geometry of $\mathcal{CS}$ and therefore its boundary, the Cantor set, is *not* Minkowski measurable; see [6], [8, §1.1.2].

In §3, we will obtain a nonarchimedean (or p-adic) analogue of the Cantor string and establish its main properties.

2. Nonarchimedean Cantor set

Let $\mathbb{Z}_p$ be the set of p-adic integers. The *p-adic ball* with center $a \in \mathbb{Q}_p$ and radius p^{-n}, $n \in \mathbb{Z}$, is the set

$$a + p^n\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^{-n}\}.$$

Two interesting *"nonarchimedean phenomena"* are that each point of the p-adic ball is a center and a p-adic ball is both open and closed. Moreover, every interval[1] in $\mathbb{Q}_p$ can be canonically decomposed into p equally long subintervals, as in (1).

Consider the ring of 3-adic integers $\mathbb{Z}_3$. In a procedure reminiscent of the construction of the classic Cantor set, we construct the *nonarchimedean Cantor set*. First, we subdivide $T_0 = \mathbb{Z}_3$ into 3 equally long subintervals. We then remove the "middle" third $T_1 = 1 + 3\mathbb{Z}_3$ and repeat this process with each of the remaining subintervals. Finally, we define the nonarchimedean Cantor set $\mathcal{C}_3$ to be $\bigcap_{n=0}^{\infty} T_n$; see Figure 3. The nonarchimedean analogue of Theorem 1.1 is given by Theorem 2.1:

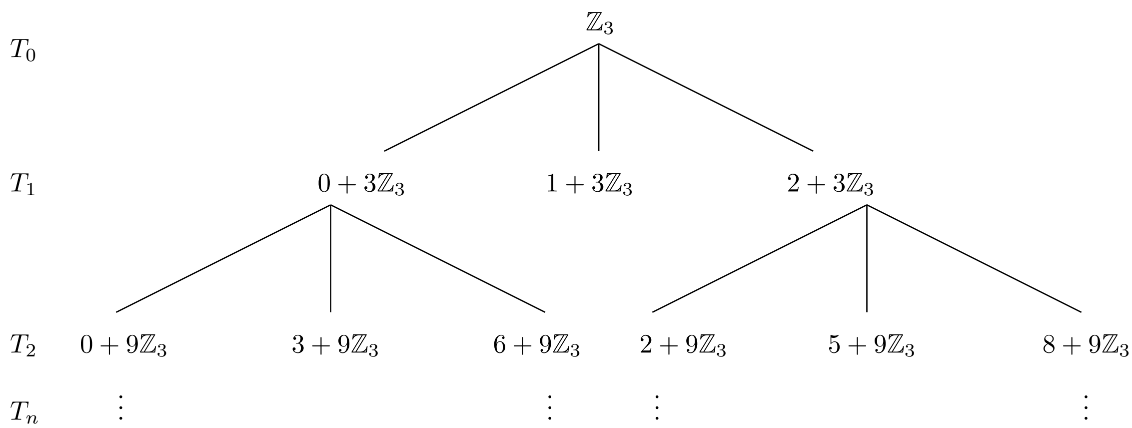

Figure 3. Construction of the nonarchimedean Cantor set $\mathcal{C}_3 = \bigcap_{n=0}^{\infty} T_n$.

Theorem 2.1. *The nonarchimedean Cantor set $\mathcal{C}_3$ is self-similar. More specifically, it is the unique nonempty, compact invariant set in $\mathbb{Q}_p$ generated by the family $\{\Psi_1, \Psi_2\}$ of similarity contraction mappings of $\mathbb{Z}_3$ into itself, where*

$$\Psi_1(x) = 3x \quad \textit{and} \quad \Psi_2(x) = 3x + 2. \tag{4}$$

That is,

$$\mathcal{C}_3 = \Psi_1(\mathcal{C}_3) \cup \Psi_2(\mathcal{C}_3).$$

Proof. From Figure 4, we can see that $\Psi_1(T_n) \cup \Psi_2(T_n) = T_{n+1}$ for all $n \geq 0$. Since each Ψ_i is injective $(i = 1, 2)$, we have $\Psi_i(\mathcal{C}_3) = \Psi_i(\bigcap_n T_n) = \bigcap_n \Psi_i(T_n)$. Therefore,

$$\Psi_1(\mathcal{C}_3) \cup \Psi_2(\mathcal{C}_3) = \bigcap_{n=0}^{\infty} (\Psi_1(T_n) \cup \Psi_2(T_n)) = \bigcap_{n=0}^{\infty} T_{n+1} = \mathcal{C}_3.$$

The Contraction Mapping Principle, applied to the complete metric space of all nonempty compact subsets of $\mathbb{Z}_3$,[2] equipped with the Hausdorff metric induced by the 3-adic norm, shows that there

[1] We shall sometimes call the ball $a + p^n\mathbb{Z}_p$ an "interval".

[2] Recall from §1.1 that $\mathbb{Z}_3$ itself is a complete metric space.

is a unique invariant set of the family of similarity contraction mappings $\{\Psi_1, \Psi_2\}$. We refer to Hutchinson's paper [2] for a detailed argument in the case of arbitrary complete metric spaces.

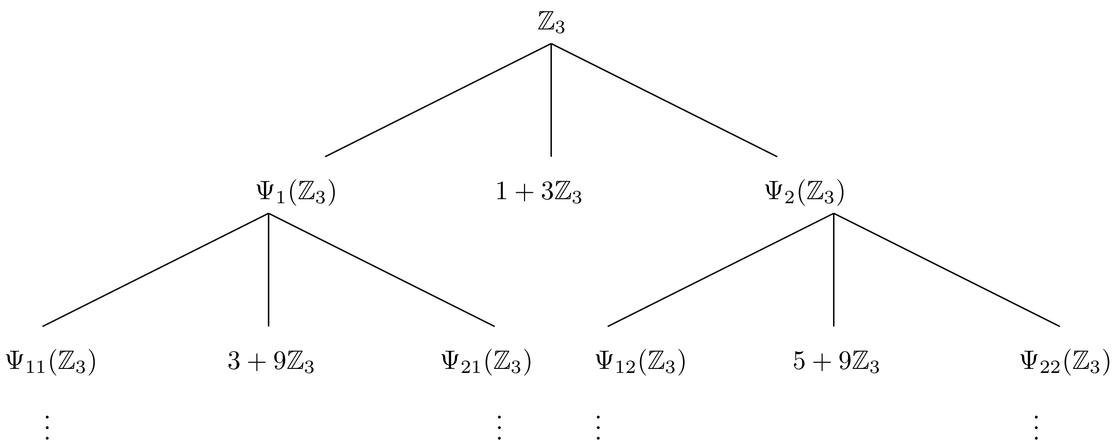

FIGURE 4. Construction of the nonarchimedean Cantor set via an Iterated Function Scheme (IFS).

Theorem 2.2. *Let $W_k = \{1,2\}^k$ be the set of all finite words, on 2 symbols, of a given length $k \geq 0$. Then*

$$\mathcal{C}_3 = \bigcap_{k=0}^{\infty} \bigcup_{w \in W_k} \Psi_w(\mathbb{Z}_3),$$

where $\Psi_w := \Psi_{w_k} \circ \cdots \circ \Psi_{w_1}$ for $w = (w_1, \ldots, w_k) \in W_k$ and the maps Ψ_{w_i} are as in Equation (4).

Proof. For each $k = 0, 1, 2, \ldots$, we have that

$$\bigcup_{w \in W_k} \Psi_w(\mathbb{Z}_3) = T_k.$$

Hence, in light of Theorem 2.1, the result follows at once from the definition of $\mathcal{C}_3$; see Figure 4.

The following result is the nonarchimedean analogue of Theorem 1.2:

Theorem 2.3. *The nonarchimedean Cantor set is characterized by the 3-adic expansion of its elements. That is,*

$$\mathcal{C}_3 = \left\{ \kappa \in \mathbb{Z}_3 \mid \kappa = a_0 + a_1 3 + a_2 3^2 + \cdots, a_i \in \{0,2\}, \ \forall i \geq 0 \right\}.$$

Proof. First of all, observe that the inverses of Ψ_1 and Ψ_2 are, respectively,

$$\Psi_1^{-1}(x) = \frac{x}{3} \quad \text{and} \quad \Psi_2^{-1}(x) = \frac{x-2}{3}.$$

Secondly, it is clear that for $a_i' \in \{0,1,2\}$ and $i = 0, 1, 2 \ldots$,

$$a_0' + a_1' 3 + a_2' 3^2 + \cdots \in 1 + 3\mathbb{Z}_3 \Leftrightarrow a_0' = 1. \tag{5}$$

Now, let $\kappa = a_0 + a_1 3 + a_2 3^2 + \cdots \in \mathbb{Z}_3$ and suppose that some coefficients in its 3-adic expansion are 1's. We will show that κ must then be in the image of $1 + 3\mathbb{Z}_3$ under some composition of the maps Ψ_1 and Ψ_2. Let $l \in \mathbb{N}$ be the first index such that $a_l = 1$. Hence, $a_j = 0$ or 2 for all $j < l$. If

$a_0 = 0$, then we apply Ψ_1^{-1} to κ, and if $a_0 = 2$, then we apply Ψ_2^{-1} to κ. In both cases, we have that

$$\Psi_i^{-1}(\kappa) = a_1 + a_2 3 + \cdots + a_l 3^{l-1} + a_{l+1} 3^l + \cdots . \tag{6}$$

Depending on whether $a_1 = 0$ or 2, we apply Ψ_1^{-1} or Ψ_2^{-1}, respectively, to the above 3-adic expansion (6). Proceeding in this manner, we will get

$$\Psi_w^{-1}(\kappa) = a_l + a_{l+1} 3 + \cdots ,$$

which is in $1 + 3\mathbb{Z}_3$ for some $k \in \mathbb{N}$ and $w \in W_k$. Thus $\kappa \in \Psi_w(1 + 3\mathbb{Z}_3)$. Since $(1 + 3\mathbb{Z}_3) \cap \mathcal{C}_3 = \emptyset$ and Ψ_w is injective, we deduce that $\kappa \notin \mathcal{C}_3$. Therefore, all of the digits of $\kappa \in \mathcal{C}_3$ must lie in $\{0, 2\}$.

Conversely, suppose that all of the coefficients in $\kappa = a_0 + a_1 3 + a_2 3^2 + \cdots \in \mathbb{Z}_3$ are $0's$ or $2's$. Then, by the above observation (5), $\kappa \notin 1 + 3\mathbb{Z}_3$. Moreover, $\kappa \notin \Phi_w(1 + 3\mathbb{Z}_3)$ for any $w \in W_k, k = 0, 1, 2, \ldots$, since none of the coefficients a_i is equal to 1. That is,

$$\kappa \notin \bigcup_{k=0}^{\infty} \bigcup_{w \in W_k} \Phi_w(1 + 3\mathbb{Z}_3) =: B.$$

But $\mathcal{C}_3 \cap B = \emptyset$ and $\mathcal{C}_3 \cup B = \mathbb{Z}_3$, as can be seen in Equation (8) and Theorem 3.3. Hence, $\kappa \in \mathcal{C}_3$, as desired.

Theorem 2.4. *The ternary Cantor set* $\mathcal{C}$ *and the nonarchimedean Cantor set* $\mathcal{C}_3$ *are homeomorphic.*

Proof. Let $\phi : \mathcal{C} \to \mathcal{C}_3$ be the map sending

$$\sum_{i=0}^{\infty} a_i 3^{-i} \mapsto \sum_{i=0}^{\infty} a_i 3^i, \tag{7}$$

where $a_i \in \{0, 2\}$ for all $i \geq 0$. We note that on the left-hand side of (7), we use the ternary expansion in $\mathbb{R}$, whereas on the right-hand side we use the 3-adic expansion in $\mathbb{Q}_3$. Then, clearly, ϕ is a continuous bijective map from $\mathcal{C}$ to $\mathcal{C}_3$. Since both $\mathcal{C}$ and $\mathcal{C}_3$ are compact spaces in their respective natural metric topologies, ϕ is a homeomorphism.

Remark 2.5. *In view of Theorem 2.4, like its archimedean counterpart* $\mathcal{C}$, *the nonarchimedean Cantor set* $\mathcal{C}_3$ *is totally disconnected, uncountably infinite and has no isolated points.*

3. Nonarchimedean Cantor string

The *nonarchimedean* (or *p-adic*) *Cantor string* is defined to be

$$\mathcal{CS}_3 := (1 + 3\mathbb{Z}_3) \cup (3 + 9\mathbb{Z}_3) \cup (5 + 9\mathbb{Z}_3) \cup \cdots = \mathbb{Z}_3 \backslash \mathcal{C}_3, \tag{8}$$

the complement of $\mathcal{C}_3$ in $\mathbb{Z}_3$; see the "middle" parts of Figure 3. Therefore, by analogy with the relationship between the archimedean Cantor set and Cantor string, the nonarchimedean Cantor set $\mathcal{C}_3$ can be thought of as some kind of "boundary" of the nonarchimedean Cantor string. Certainly, $\mathcal{C}_3$ is not the topological boundary of $\mathcal{CS}_3$ because the latter boundary is empty.

Since $\mathbb{Q}_p$ is a locally compact group, there is a unique translation invariant Haar measure μ_H, normalized so that $\mu_H(\mathbb{Z}_p) = 1$, and hence $\mu_H(a + 3^n\mathbb{Z}_3) = 3^{-n}$; see [4], [11]. As in the real case in §1.3, we may identify $\mathcal{CS}_3$ with the sequence of lengths $l_n = 3^{-n}$ with multiplicities $m_n = 2^{n-1}$ for $n \in \mathbb{N}$.

The following theorem provides the exact analogue of Equations (2) and (3):

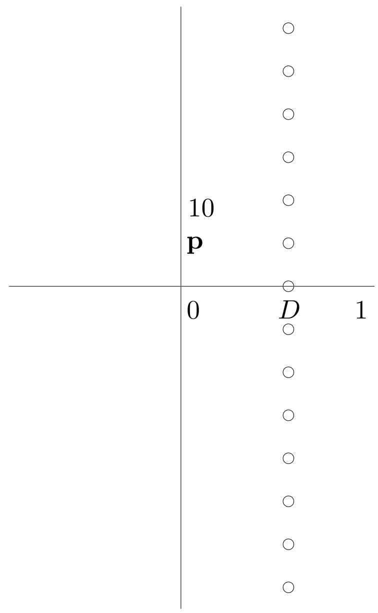

FIGURE 5. The set of complex dimensions, $\mathcal{D}_{\mathcal{CS}} = \mathcal{D}_{\mathcal{CS}_3}$, of the archimedean and nonarchimedean Cantor strings, $\mathcal{CS}$ and $\mathcal{CS}_3$.

Theorem 3.1. *The geometric zeta function of the nonarchimedean Cantor string is meromorphic in all of* $\mathbb{C}$ *and is given by*

$$\zeta_{\mathcal{CS}_3}(s) = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}. \tag{9}$$

Hence, the set of complex dimensions of $\mathcal{CS}_3$ *is given by*

$$\mathcal{D}_{\mathcal{CS}_3} = \{D + \imath\nu\mathbf{p} \mid \nu \in \mathbb{Z}\}, \tag{10}$$

where $D = \log 2/\log 3$ *is the dimension of* $\mathcal{CS}_3$ *and* $\mathbf{p} = 2\pi/\log 3$ *is its oscillatory period.*

Proof. By definition (see [9, 10]), the geometric zeta function of $\mathcal{CS}_3$ is given by

$$\begin{aligned} \zeta_{\mathcal{CS}_3}(s) &:= (\mu_H(1 + 3\mathbb{Z}_3))^s + (\mu_H(3 + 9\mathbb{Z}))^s + (\mu_H(5 + 9\mathbb{Z}_3))^s + \cdots \\ &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}} \qquad \text{for } \Re(s) > \log 2/\log 3. \end{aligned}$$

Furthermore, the meromorphic extension of $\zeta_{\mathcal{CS}_3}$ to all of $\mathbb{C}$ is given by the last expression in the above equation. The *complex dimensions* of $\mathcal{CS}_3$, defined as the poles of $\zeta_{\mathcal{CS}_3}$, are all the solutions ω of the equation $1 - 2 \cdot 3^{-\omega} = 0$. These are precisely of the form

$$\omega = \frac{\log 2}{\log 3} + \imath\nu\frac{2\pi}{\log 3}, \qquad \nu \in \mathbb{Z}.$$

Remark 3.2. *In [9, 10], we prove that* D *is the Minkowski dimension of* $\mathcal{CS}_3 \subset \mathbb{Z}_3$. *Clearly,* D *is also the abscissa of convergence of the Dirichlet series defining* $\zeta_{\mathcal{CS}_3}$.

The following result was used in the second part of the proof of Theorem 2.3:

Theorem 3.3. *With the same notation as in Theorem 2.2, we have that*

$$\mathcal{CS}_3 = \bigcup_{k=0}^{\infty} \bigcup_{w \in W_k} \Psi_w(1+3\mathbb{Z}_3).$$

Proof. For each $k = 0, 1, 2, \ldots$, we let $\widetilde{T_{k+1}} = \mathbb{Z}_3 \backslash T_{k+1}$, the complement of T_{k+1} in $\mathbb{Z}_3$. Then

$$\widetilde{T_{k+1}} = \bigcup_{w \in W_k} \Psi_w(1+3\mathbb{Z}_3).$$

Hence, in light of Theorem 2.1, we have that

$$\bigcup_{k=0}^{\infty} \bigcup_{w \in W_k} \Psi_w(1+3\mathbb{Z}_3) = \bigcup_{k=0}^{\infty} \widetilde{T_{k+1}} = \widetilde{\bigcap T_{k+1}} = \widetilde{\mathcal{C}_3} = \mathcal{CS}_3,$$

by the definitions of $\mathcal{C}_3$ and $\mathcal{CS}_3$. See Figure 6.

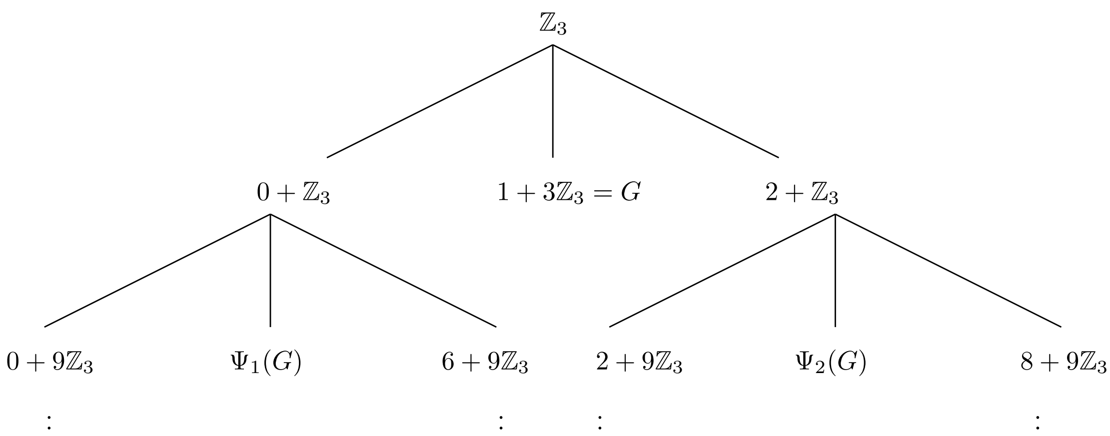

FIGURE 6. Construction of the nonarchimedean Cantor string via an IFS.

Remark 3.4. *The above theorem shows that $G = 1+3\mathbb{Z}_3$ is the* generator *of the nonarchimedean Cantor string. This is a particular case of a more general construction of self-similar p-adic fractal strings [9, 10]. Moreover, $\mathcal{CS}_3$ is* not *Minkowski measurable as a subset of $\mathbb{Q}_3$. In fact, in contrast to the archimedean case ([8], Theorems 8.23 and 8.36), self-similar p-adic fractal strings are* never *Minkowski measurable.*

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