On a class of holonomic $\mathcal{D}$-modules on symmetric matrices attached to the general linear group

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Abstract

We describe a class of regular holonomic $\mathcal{D}$-modules on complex symmetric matrices attached to the action of the general linear group.

1 Introduction

The theory of $\mathcal{D}$-modules is a generalization of the classical theory of linear algebraic differential equations of one variable (for instance the hypergeometric differential equation or the Legendre differential equation seen as a Gauss-Manin connection). It develops geometric aspects (monodromy, singularities) as well as algebraic geometry. Moreover this theory is central for other domains of mathematics: singularity theory, intersection cohomology and perversity, Lie groups, rigid analytic geometry. Perhaps the first systematic use of $\mathcal{D}$-modules appeared in [22]. Since then there have appeared several articles by Kashiwara and others. We should also mention the contribution of B. Malgrange. Last but not least we mention the work of A. Beilinson and J. Bernstein regarding the algebraic aspect of the theory. Among the $\mathcal{D}$-modules we single out a class of objects of utmost importance: holonomic $\mathcal{D}$-modules with regular singularities. One of the main problem in the $\mathcal{D}$-modules theory consists in the classification of these objects. Let us point out that several authors have taken an interest in it, notably L. Boutet de Monvel [1], P. Deligne [4], R. MacPherson and K. Vilonen [13] see also [3], [6], [14], [20], [23] etc. One knows by the Riemann-Hilbert correspondence (see. [8]) that there is a general equivalence between the category consisting of regular holonomic $\mathcal{D}_V$-modules with characteristic variety Σ and the category consisting of perverse sheaves on V (where V denotes a complex manifold) with microsupport Σ. This gives a classification of regular holonomic $\mathcal{D}$-modules theoretically, but in practice the classification of perverse sheaves is not always much simpler. A more accessible problem is as follows: given a complex manifold V on which a Lie group acts linearly with finitely many orbits $(V_j)_{j\in J}$; the problem is to classify regular holonomic $\mathcal{D}_V$-modules whose characteristic variety is contained in the union of conormal bundles ($\Sigma := \bigcup_{j\in J} \overline{T^*_{V_j}V}$) to these orbits. Closely equivalent: those that admit a good filtration stable by infinitesimal generators of the group. These modules form a full category denoted $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D})$. Here are some examples of such modules:

Example 1 *The $\mathcal{D}_V$-module $\mathcal{O}_V$ is generated by an homogeneous section of degree 0 namely $e_0 = 1_V$ satisfying the following relations:*

$$\sum_{1\le i\le j\le n} x_{ij}\partial_{ij}e_0 = 0 \text{ where } x_{ij} = x_{ji},\ \partial_{ij} := \frac{\partial}{\partial x_{ij}},\ X = (x_{ij}) \in V, \quad (1)$$

$$\det\left(\frac{1}{2}\widetilde{\partial}_{ij}\right)e_0 = 0, \text{ where } \widetilde{\partial}_{ii} := 2\partial_{ii} \text{ and } \widetilde{\partial}_{ij} := \partial_{ij} \text{ for } i \neq j. \quad (2)$$

Example 2 *The $\mathcal{D}_V$-module $\mathcal{B}_{\{0\}|V}$ (the Dirac mass supported by $\{0\}$) is generated by an homogeneous section $e_{-\frac{(n+1)}{2}n}$ of degree $-\frac{n(n+1)}{2}$ satisfying the equations:*

$$\sum_{1\le i\le j\le n} x_{ij}\partial_{ij}e_{-\frac{n(n+1)}{2}} = -\frac{n(n+1)}{2}e_{-\frac{n(n+1)}{2}} \quad \text{and } \det(X)\,e_{-\frac{n(n+1)}{2}} = 0. \quad (3)$$

Example 3 *The $\mathcal{D}_V$-module $\mathcal{O}_V\left(\frac{1}{\det(X)}\right)/\mathcal{O}_V$ is generated by an homogeneous section $e_{-n} = \frac{1}{\det(X)} \bmod \mathcal{O}_V$ of degree $-n$ such that*

$$\sum_{1\le i\le j\le n} x_{ij}\partial_{ij}e_{-n} = -ne_{-n} \quad \text{and} \quad \det(X)\,e_{-n} = e_0 = 1_V. \quad (4)$$

In this paper we consider the linear action of the general linear group $GL_n(\mathbb{C})$ on the vector space $V := S^2(\mathbb{C}^n)$ of complex symmetric matrices. There are $(n+1)$ orbits V_k $(0 \le k \le n)$: the set of rank k symmetric matrices in V. Note that here there is a natural algebra associated to this situation: the (graded) algebra $\mathcal{A}$ of (polynomial coefficients) differential operators acting on $\mathbb{C}[\det(X)]$ polynomials of the symmetric determinant, which is the quotient of the algebra $\overline{\mathcal{A}}$ of $SL_n(\mathbb{C})$-invariant operators on V by the annihilator of $\mathbb{C}[\det(X)]$ (see. section 3).
The main result of this paper is the theorem 18 saying that there is an equivalence of categories between the category $\mathrm{Mod}^{\mathrm{rh}}_{\Sigma}(\mathcal{D})$ consisting of regular holonomic $\mathcal{D}$-modules as above and the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ consisting of graded $\mathcal{A}$-modules of finite type for the Euler vector field on V. The algebra $\mathcal{A}$ is described simply by generators and relations (see. Proposition 8) thanks to Capelli identities given by R. Howe and T. Umeda (see. [7, p. 587, 11.2]). This also leads to the description of the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ as an “elementary” category consisting of “colored diagrams” of finite dimensional vector spaces and linear maps between them satisfying certain relations (Quiver category) on

which one can see what are the simple or indecomposable objects (see. section 6). Throught the paper we assume that the reader has some familiarity with the language of $\mathcal{D}$-modules. He may consult the very nice book [10] that provides a good account of an introduction to the general theory of $\mathcal{D}$-modules (see also [2]). Finally we should note that in previous papers the author has obtained similar results for $\mathcal{D}$-modules on $\mathbb{C}^n$ associated to the action of the orthogonal group (see. [16]), and on $M_n(\mathbb{C})$ the space of complex square matrices related to the action of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ (see. [17], [18], [19]).

2 Coherent $\mathcal{D}_V$-modules and homogeneous sections

Let $\mathcal{M}$ be a $\mathcal{D}_V$-module and $\theta := \sum_{1\leq i\leq j\leq n} x_{ij}\partial_{ij}$ the Euler vector field on the space of complex symmetric matrices $V := S^2(\mathbb{C}^n)$.

Definition 4 *We say that a section s in $\mathcal{M}$ is homogeneous if* $\dim_{\mathbb{C}} \mathbb{C}[\theta] s < \infty$. *This section is said to be homogeneous of degree $\lambda \in \mathbb{C}$, if there exists $j \in \mathbb{N}$ such that $(\theta - \lambda)^j s = 0$.*

We recall the following Theorem useful in the sequel:

Theorem 5 *([16, Theorem 1.3.]) If $\mathcal{M}$ is a coherent $\mathcal{D}_V$-module with a good filtration $(F_k\mathcal{M})_{k\in\mathbb{Z}}$ stable by θ. Then*
i) $\mathcal{M}$ is generated over $\mathcal{D}_V$ by finitely many global sections $(s_j)_{j=1,\cdots,q} \in \Gamma(V,\mathcal{M})$ such that $\dim_{\mathbb{C}} \mathbb{C}[\theta] s_j < \infty$,
ii) For any $k \in \mathbb{N}$, $\lambda \in \mathbb{C}$, the vector space $\Gamma(V, F_k\mathcal{M}) \bigcap \left[\bigcup_{p\in\mathbb{N}} \ker(\theta-\lambda)^p\right]$ of homogeneous global sections of $F_k\mathcal{M}$ of degree λ is finite dimensional.

We have the following definition:

Definition 6 *The action of the group G (preserving the good filtration) on a $\mathcal{D}_V$-module $\mathcal{M}$ is given by an isomorphism $u : p_1^+(\mathcal{M}) \xrightarrow{\sim} p_2^+(\mathcal{M})$ where $p_1 : G \times V \longrightarrow V$ is the projection on V and $p_2 : G \times V \longrightarrow V$, $(g, X) \longmapsto g \cdot X$ defines the action of G on V (satisfying the associativity conditions).*

Denote by $\widetilde{G}_{\mathbb{C}} := SL_n(\mathbb{C}) \times \mathbb{C}$ the universal covering of G and $\widetilde{G} := SL_n(\mathbb{C})$. Let $\mathcal{M}$ be an object in $\mathrm{Mod}_{\Lambda}^{\mathrm{rh}}(\mathcal{D}_V)$. We have the following proposition:

Proposition 7 *([16, Proposition 1.6.])The infinitesimal action of G on $\mathcal{M}$ lifts to an action of $\widetilde{G}_{\mathbb{C}}$ (resp. $\widetilde{G}$) on $\mathcal{M}$, compatible with the one of G on V and $\mathcal{D}_V$.*

3 $\widetilde{G}$-Invariant differential operators on $S^2(\mathbb{C}^n)$

This section consists in the description of the $\mathbb{C}$-algebra $\overline{\mathcal{A}} := \Gamma(V, \mathcal{D}_V)^{\widetilde{G}}$of (polynomial coefficients) differential operators on symmetric matrices V which are invariant under the action of the special linear group $\widetilde{G} := SL_n(\mathbb{C})$. It is shown in [7, p. 587, (11.2.5)] that the canonical generators of $\Gamma(V, \mathcal{D}_V)^{GL_n(\mathbb{C})}$ the algebra of $GL_n(\mathbb{C})$-invariant differential operators are the Capelli operators

$$P_k := \sum_{|I|=|J|=k} \det(X_{IJ}) \det\left(\widetilde{D}_{IJ}\right), \ 1 \le k \le n \tag{5}$$

where both I, J are subsets of $\mathbb{N}$ of cardinality k and X_{IJ} is the submatrix of X obtained by eliminating the rows not in I and the columns not in J with $\widetilde{D}_{IJ} := \frac{1}{2}\left(\widetilde{\partial}_{ij}\right)$, $\widetilde{\partial}_{ii} = 2\partial_{ii}$ and $\widetilde{\partial}_{ij} := \partial_{ij}$ for $i \ne j$. These operators commute:

$$\Gamma(V, \mathcal{D}_V)^{GL_n(\mathbb{C})} = \mathbb{C}[P_1, \cdots, P_n] \quad \text{(see.[7, p. 588])}. \tag{6}$$

Then we obtain generators of $\overline{\mathcal{A}}$ by adding symmetric determinants $\delta := \det(X)$ and $\widetilde{\Delta} := \det\left(\widetilde{D}\right)$. Note that here the Euler vector field $\theta = P_1 = \sum_{1 \le i \le j \le n} x_{ij}\partial_{ij}$. Denote by $\mathcal{J} \subset \overline{\mathcal{A}}$ the two sided ideal annihilating $\widetilde{G}$-invariant polynomials $\mathbb{C}[\delta]$.

Proposition 8 *$\overline{\mathcal{A}}$ is generated by $\delta, \theta, P_2, \cdots, P_{n-1}, \widetilde{\Delta}$ subject to the following relations modulo $\mathcal{J}$*

$$\begin{aligned}
[\theta, \delta] &= n\delta, \quad \left[\theta, \widetilde{\Delta}\right] = -n\widetilde{\Delta}, && (7)\\
[P_k, P_l] &= 0 \ \text{for} \ k, l = 1, \cdots, n, && (8)\\
\delta\widetilde{\Delta} &= \prod_{j=0}^{n-1}\left(\frac{\theta}{n} + \frac{j}{2}\right), && (9)\\
\widetilde{\Delta}\delta &= \prod_{j=0}^{n-1}\left(\frac{\theta}{n} + \frac{j+2}{2}\right), && (10)\\
\left[\widetilde{\Delta}, \delta\right] &= \left(\theta + \frac{n(n+1)}{4}\right)\prod_{j=2}^{n-1}\left(\frac{\theta}{n} + \frac{j}{2}\right), && (11)\\
P_k &= C_k^n \prod_{j=0}^{k-1}\left(\frac{\theta}{n} + \frac{j}{2}\right), && (12)\\
[P_k, \delta] &= (kC_k^n)\,\delta \prod_{j=1}^{k-1}\left(\frac{\theta}{n} + \frac{j}{2}\right), && (13)\\
\left[P_k, \widetilde{\Delta}\right] &= -kC_k^n \left\{\prod_{j=1}^{k-1}\left(\frac{\theta}{n} + \frac{j}{2}\right)\right\}\widetilde{\Delta} && (14)
\end{aligned}$$

Before starting the proof of Proposition 8 we should introduce the Bernstein-Sato polynomial associated with the symmetric determinants (see. [5, p. 74 Theorem]):

$$\det\left(\widetilde{D}_{IJ}\right)\delta^{\alpha}=\alpha\left(\alpha+\frac{1}{2}\right)\cdots\left(\alpha+\frac{k-1}{2}\right)\delta^{\alpha-1}; \tag{15}$$

In particular for $k=n$ ($P_n=\delta\widetilde{\Delta}$) we have the Cayley's determinantal Theorem:

$$\widetilde{\Delta}\delta^{\alpha}=\alpha\left(\alpha+\frac{1}{2}\right)\cdots\left(\alpha+\frac{n-1}{2}\right)\delta^{\alpha-1}. \tag{16}$$

This gives rise to the following formula:

$$P_k\delta^{\alpha}=C_n^k\alpha\left(\alpha+\frac{1}{2}\right)\cdots\left(\alpha+\frac{k-1}{2}\right)\delta^{\alpha}. \tag{17}$$

To compute invariants of differential operators on the space V, we very first remark that they should correspond (using symbols of differential operators) to invariants of $V\bigoplus V^*$.

Proof. of Proposition 8. Let us consider the non commutative $\mathbb{C}$-algebra $\overline{\mathcal{B}}:=\mathbb{C}\left\langle\delta,\theta,P_2,\cdots,P_{n-1},\widetilde{\Delta}\right\rangle$. To see that $\overline{\mathcal{A}}=\overline{\mathcal{B}}$ we first show $\mathrm{gr}\overline{\mathcal{A}}=\mathrm{gr}\overline{\mathcal{B}}$. Let V^* be the dual of V and (X,ξ) be matrices in $V\times V^*$. Put

$$X^t:=\begin{pmatrix} t & 0 & \cdots & 0\\ 0 & 1 & & \vdots\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & 1\end{pmatrix}\in V \quad\text{and}\quad \xi^{[t_j]}:=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & t_1 & 0 & & 0\\ \vdots & & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & t_{n-1} & 0\\ 0 & 0 & \cdots & 0 & t_n\end{pmatrix}\in V^*.$$

The invariant polynomials $p_k\left(X^t\!,\xi^{[t_j]}\right)$ are exactly the elementary symmetric polynomials $s_k(t_j):=\sum_{i_1<\cdots<i_k}t_{i_1}\cdots t_{i_k}$ for $1\le k\le n$. Let $f=f(X,\xi)$ be a polynomial in (X,ξ), there exists a polynomial q such that

$$f\left(X^t,\xi^{[t_j]}\right)=q\left(t,t_1,t_2,\cdots,t_n\right) \tag{18}$$

is a polynomial in variables $t,t_1,\cdots,t_n$. As the variables t_j remain in the same orbit if they are permuted, we can see that if f is invariant then $f\left(X^t,\xi^{[t_j]}\right)$ is a polynomial of t and the $s_k(t_j)$, $1\le k\le n$ i.e.

$$f\left(X^t,\xi^{[t_j]}\right)=\widetilde{q}\left(t,s_1(t_j),\cdots,s_n(t_j)\right). \tag{19}$$

Then the difference

$$f(X,\xi) - \widetilde{q}(\det(X), p_1, p_2, \cdots, p_n)$$

is a polynomial in (X,ξ) vanishing on the set ($(n+1)$-affine space) of $\left(X^t, \xi^{[t_j]}\right)$. Denote $\mathcal{R} := \bigsqcup_{t,t_j} \widetilde{G} \cdot \left(X^t, \xi^{[t_j]}\right)$ the union of orbits of points $\left(X^t, \xi^{[t_j]}\right)$ in one of affine spaces of $V \times V^*$. Assume f is invariant, then $f - p$ is invariant and vanishes on $\mathcal{R}$. Since the union $\mathcal{R}$ is open in $V \times V^*$, then $f - p$ vanishes everywhere.

$$\text{Thus} \qquad \mathrm{gr}\overline{\mathcal{A}} = \mathrm{gr}\overline{\mathcal{B}}. \tag{20}$$

Now, if Q is an invariant operator of degree $m \geq 0$ ($Q \in \overline{\mathcal{A}}$), its symbol $\sigma(Q)$ is also invariant and according to (20) there is a polynomial g such that

$$\sigma(Q)(X,\xi) = g(\det(X), p_1, \cdots, p_{n-1}, \det(\xi)). \tag{21}$$

Then Q can be written as the sum of an operator $T \in \overline{\mathcal{B}}$ (a polynomial in the P_k's, δ,$\widetilde{\Delta}$) and $R \in \overline{\mathcal{A}}$ an invariant operator of degree at most $m-1$:

$$Q = T\left(\delta, \theta, P_2, \cdots, P_{n-1}, \widetilde{\Delta}\right) + R, \text{ with } R \in \overline{\mathcal{A}},\ \deg(R) \leq m-1. \tag{22}$$

By recurrence on the degree of the operator we see that Q is a polynomial in the P_k's, δ, $\widetilde{\Delta}$ that is $Q \in \overline{\mathcal{B}}$. Therefore

$$\overline{\mathcal{A}} = \overline{\mathcal{B}}. \tag{23}$$

The remaining part is devoted to the proof of the relations (7),$\cdots$,(14). The formulas (7) are obvious since the symmetric determinant is an homogeneous polynomial of degree n. (8) holds since the P_k's commute (see. (6)). One obtains (9) and (10) (resp.(12)) using the Cayley formula (16) (resp. (17)). Then (13), (14) follows from (12). ■

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Denote by $\overline{\mathcal{J}} \subset \overline{\mathcal{A}}$ the ideal generated by the relations (7), (9), (10). Put $\mathcal{A} := \overline{\mathcal{A}}/\overline{\mathcal{J}}$ the quotient algebra of $\overline{\mathcal{A}}$ by $\overline{\mathcal{J}}$. We have the following corollary:

Corollary 9 *The quotient algebra $\mathcal{A}$ is generated by $\delta, \theta, \widetilde{\Delta}$ subject to the relations (7), (9), (10):*

$$\begin{aligned}
[\theta, \delta] &= n\delta, \quad \left[\theta, \widetilde{\Delta}\right] = -n\widetilde{\Delta}, \\
\delta\widetilde{\Delta} &= \prod_{j=0}^{n-1}\left(\frac{\theta}{n} + \frac{j}{2}\right), \\
\widetilde{\Delta}\delta &= \prod_{j=0}^{n-1}\left(\frac{\theta}{n} + \frac{j+2}{2}\right).
\end{aligned}$$

Proof. Let Q be in $\overline{\mathcal{A}}$, we decompose it into homogeneous components ($Q = \sum_{j\in\mathbb{Z}} Q_j$) Q_j of degree jn (i.e. $[\theta, Q_j] = jnQ_j$) so that if $j = 0$ then $Q_0 = \varphi(\theta)$ is a polynomial in θ. Indeed, Q_0 acts on $\mathbb{C}[\delta]$ then $Q_0 \in \mathbb{C}\left[\delta, \frac{\partial}{\partial(\delta)}\right]$ with $\delta\frac{\partial}{\partial(\delta)} = \frac{1}{n}\theta$. If $j > 0$ then $\widetilde{\Delta}^j Q_j = \psi(\theta)$ is a polynomial in θ because $\widetilde{\Delta}^j Q_j$ is homogeneous of degree 0. Likewise if $j < 0$ then $\delta^{-j}Q_j = \phi(\theta)$ is a polynomial in θ. Thus for any Q_j homogeneous of degree jn, its class modulo $\overline{\mathcal{J}}$ is of the form

$$Q_j \bmod \overline{\mathcal{J}} = \begin{cases} \delta^j \phi_j(\theta) & \text{if } j \geq 0 \\ \widetilde{\Delta}^{-j}\psi_j(\theta) & \text{if } j \leq 0 \end{cases} \tag{24}$$

where $\phi_j(\theta)$, $\psi_j(\theta)$ are (polynomials) homogeneous of degree 0. ■

4 Holonomic $\mathcal{D}_V$-modules and $\widetilde{G}$-invariant sections

In this section we prove that any regular holonomic $\mathcal{D}_V$-module whose characteristic variety is contained in Λ is generated by finitely many global sections which are invariant under the action of $\widetilde{G}$. This result is at the heart of the proof of the main theorem 18 established in the nex section.

Theorem 10 *Let $\mathcal{M}$ be an object in the category $Mod^{rh}_{\Lambda}(\mathcal{D}_V)$. Then $\mathcal{M}$ is generated by finitely many $\widetilde{G}$-invariant global sections.*

For the proof, first we use the fact that such $\mathcal{D}_V$-modules are essentially inverse images by the (symmetric) determinant map of $\mathcal{D}$-modules over $\mathbb{C}$. Next we introduce some quotient $\mathcal{D}_V$-modules with support on the closure of $\widetilde{G}$-orbits.

4.1 Inverse image of $\mathcal{D}_{\mathbb{C}}$-modules

Denote by $\overline{V}_k := \{X \in V,\ \mathrm{rank}(X) \leq k\}$ the set of symmetric matrices of rank at most k. Let $\mathcal{M}$ be a $\mathcal{D}_V$-module in $\mathrm{Mod}^{rh}_{\Lambda}(\mathcal{D}_V)$. Here we take the restriction of $\mathcal{M}$ to a section of the projection defined by the determinant map $\delta : V \longrightarrow \mathbb{C}$, $X \longmapsto \det(X)$. Then we can consider $\mathcal{M}$ as an inverse image by δ of a $\mathcal{D}_{\mathbb{C}}$-module $\mathcal{N}$ outside the hypersurface $\overline{V}_{n-1} := \{X \in V \,/\, \delta(X) = 0\}$. Recall that the determinant map δ is submersive out of $\overline{V}_{n-2}$. Let $i : \mathbb{C} \longrightarrow V$, $t \longmapsto X^t$ with

$$X^t := \begin{pmatrix} t & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

be a section of the map δ. Denote by $Z := i(\mathbb{C})$ its image.

Lemma 11 *Z is non characteristic for $\mathcal{M}$ i.e.* $\overline{T_Z^*V}\bigcap Ch(\mathcal{M})\subset T_V^*V$.

Proof. The line Z intersects the orbits V_{n-1}. So we prove that $\overline{T_Z^*V}\bigcap \overline{T_{V_{n-1}}^*V}$ is contained in the zero section T_V^*V. It suffices to check at the point X^0 $(t=0)$ which is the only point of the line Z above which the characteristic variety $\mathrm{Ch}(\mathcal{M})$ has a non zero covector $\xi_0\neq 0$. Note that this covector ξ_0 is parallel to $d\delta$(the conormal bundle to determinantal variety) and on the line Z we have $d\delta=dt\neq 0$, that is, $\xi_0\notin\overline{T_Z^*V}$. ■

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We should note that $\mathcal{M}$ is canonically isomorphic to $\delta^+i^+(\mathcal{M})$ in the neighborhood of the line Z since Z is non characteristic for $\mathcal{M}$ (see. Lemma 11). The sheaf $\mathcal{H}\mathrm{om}_{\mathcal{D}_V}\left(\mathcal{M},\,\delta^+i^+\mathcal{M}\right)$is constructible (see.[10]) and also locally constant on the fibers $\delta^{-1}(t)$, $t\in\mathbb{C}$. As the group $\widetilde{G}$ acts on the $\mathcal{D}_V$-modules $\mathcal{M}$ and $\delta^+i^+\mathcal{M}$, it acts also on the sheaf $\mathcal{H}\mathrm{om}_{\mathcal{D}_V}\left(\mathcal{M},\,\delta^+i^+\mathcal{M}\right)$ and because of the action of $\widetilde{G}$ the strata are the $(n+1)$ orbits of $\widetilde{G}$ that is $V_0,\,V_1,\,V_2,\,\cdots,\,V_n$ (see. [12]). The sheaf $\mathcal{H}\mathrm{om}_{\mathcal{D}_V}\left(\mathcal{M},\,\delta^+i^+\mathcal{M}\right)$ has a canonical section u defined in the neighborhood of Z (corresponding with the isomorphism $\mathcal{M}_{|Z}\stackrel{\sim}{\longrightarrow}\delta^+i^+(\mathcal{M})_{|Z}$ which induces the identity on Z). We have the following proposition:

Proposition 12 *The canonical isomorphism $u:\mathcal{M}_{|Z}\stackrel{\sim}{\longrightarrow}\delta^+i^+(\mathcal{M})_{|Z}$, defined in the neighborhood of Z such that $i^+.u=Id_{|Z}$, extends to $V\backslash\overline{V}_{n-2}$.*

Proof. The section $u:\mathcal{M}_{|Z}\stackrel{\sim}{\longrightarrow}\delta^+i^+(\mathcal{M})_{|Z}$is defined out of $\overline{V}_{n-2}$ (the singular part of the hypersurface $\overline{V}_{n-1}:=\delta^{-1}(0)$). We take a closer look at the orbits $V_n=V\backslash\delta^{-1}(0)$ and $V_{n-1}=\delta^{-1}(0)\backslash\overline{V}_{n-2}$. These orbits are simply connected (see. [15]) and their fundamental group $\pi_1(V_n)$ (resp. $\pi_1(V_{n-1})$) acts on the sheaf $\mathcal{H}\mathrm{om}_{\mathcal{D}_V}\left(\mathcal{M},\,\delta^+i^+\mathcal{M}\right)$. This sheaf is trivial on $V_n=\bigcup_{t\neq 0}\delta^{-1}(t)$ and on $V_{n-1}=\delta^{-1}(0)\backslash\overline{V}_{n-2}$. Then $u:\mathcal{M}_{|Z}\stackrel{\sim}{\longrightarrow}\delta^+i^+(\mathcal{M})_{|Z}$extends globally to the union $V_n\cup V_{n-1}=V\backslash\overline{V}_{n-2}$. ■

Note that the isomorphism of Proposition 12 holds out of $\overline{V}_{n-1}$:

$$\mathcal{M}_{|_{V\backslash\overline{V}_{n-1}}}\simeq\delta^+i^+(\mathcal{M})_{|_{V\backslash\overline{V}_{n-1}}}. \tag{25}$$

To extends this isomorphism of $\mathcal{D}_V$-modules we need to study their associated $\mathcal{D}_V$-modules of meromorphic sections with poles in the hypersurface $\overline{V}_{n-1}$ as follows.

4.1.1 Meromorphic $\mathcal{D}_V$-modules

Denote by $\overline{\mathcal{M}}:=\Gamma_{\left[V|\overline{V}_{n-1}\right]}(\mathcal{M})=\varinjlim_m\mathcal{H}\mathrm{om}_{\mathcal{O}_V}(\mathcal{I}^m,\mathcal{M})$ (where $\mathcal{I}$ is the defining ideal of $\overline{V}_{n-1}$) the algebraic module of meromorphic sections of $\mathcal{M}$ with pole in the hypersurface $\overline{V}_{n-1}$. We have a canonical homomorphism $\mathcal{M}\longrightarrow\overline{\mathcal{M}}$.

Proposition 13 *Let $\mathcal{N}$ be an holonomic $\mathcal{D}_{\mathbb{C}}$-module with regular singularity at $t = 0$. Assume that the operator of multiplication by t is invertible on $\mathcal{N}$ then*

i) the operator of multiplication by δ is invertible on the inverse image $\delta^+(\mathcal{N})$, in particular

ii) the canonical homomorphism

$$\delta^+(\mathcal{N}) \xrightarrow{\sim} \overline{\delta^+(\mathcal{N})} \tag{26}$$

is an isomorphism that is the meromorphic sections defined in $V \backslash \overline{V}_{n-1}$ extend to the whole V.

Proof. (i) follows from [11, Remark 1.1. p. 165] and [11, Lemma 1.2, p.166]. Next, recall [10] that we have an exact sequence $0 \to \Gamma_{[\overline{V}_{n-1}]}(\mathcal{M}) \to \mathcal{M} \to \overline{\mathcal{M}}$ where $\Gamma_{[\overline{V}_{n-1}]}(\mathcal{M}) = \varinjlim_m \mathcal{H}om_{\mathcal{O}_V}(\mathcal{O}_V/\mathcal{I}^m, \mathcal{M})$ is the subsheaf of $\mathcal{M}$ of sections annihilated by some power of $\mathcal{I}$. Since δ gives a bijection on $\delta^+(\mathcal{N})$, [11, Remark 1.1., p.165] asserts that $\mathcal{H}^k_{[\overline{V}_{n-1}]}\left(\delta^+(\mathcal{N})\right) = 0$ for any k. Then from the previous exact sequence, we get $\delta^+(\mathcal{N}) \simeq \overline{\delta^+(\mathcal{N})}$. ■

4.1.2 Extension of the isomorphism of $\mathcal{D}_V$-modules

From now on, we denote by $\mathcal{N} := i^+\mathcal{M}$ the restriction of the $\mathcal{D}_V$-module $\mathcal{M}$ to the transversal line Z. According to an argument of Kashiwara, since $\mathcal{M}$ and $\delta^+\mathcal{N}$ are regular holonomic and isomorphic out of $\overline{V}_{n-1}$ (see. (25)), then their corresponding meromorphic modules are also isomorphic that is

$$\overline{\mathcal{M}} \simeq \overline{\delta^+\mathcal{N}}\ . \tag{27}$$

Consider the left exact functor

$$\mathcal{M} \longrightarrow \overline{\mathcal{M}}\left(\simeq \overline{\delta^+\mathcal{N}}\right). \tag{28}$$

By using the basic fact that $\overline{\delta^+\mathcal{N}} \simeq \delta^+\mathcal{N}$ (see. relation (26) of Proposition 13) and the morphism (28), we deduce that there exists a morphism

$$v : \mathcal{M} \longrightarrow \delta^+\mathcal{N} \tag{29}$$

which is an isomorphism out of the hypersurface $\overline{V}_{n-1}$(see. relation (25)).

Lemma 14 *The image $v(\mathcal{M}) \subset \delta^+\mathcal{N}$ is a $\mathcal{D}_V$-module generated by its $\widetilde{G}$-invariant homogeneous global sections.*

4.2 Holonomic $\mathcal{D}_V$-modules with support on the $\widetilde{G}$-orbits

Here we introduce some subquotient modules of $\delta^+\left(\mathcal{O}_{\mathbb{C}}\left(\frac{1}{t}\right)\right)$ which will be used in the proof of Theorem 10 above.
Denote by $L := \delta^+\left(\mathcal{O}_{\mathbb{C}}\left(\frac{1}{t}\right)\right) = \mathcal{O}_V\left(\frac{1}{\delta}\right)$ the $\mathcal{D}_V$-module generated by its $\widetilde{G}$-invariant homogeneous sections $e_m := \delta^m$ (where $m \leq 0$) satisfying the following equations obtained from (16), (17)$(0 \leq k \leq n)$

$$\begin{aligned} \delta e_m &= e_{m+1}, && (30)\\ \widetilde{\Delta} e_m &= m\left(m+\frac{1}{2}\right)\cdots\left(m+\frac{n-1}{2}\right)e_{m-1}, && (31)\\ P_k e_m &= C_k^n m\left(m+\frac{1}{2}\right)\cdots\left(m+\frac{k-1}{2}\right)e_m, && (32)\end{aligned}$$

Let $L_m \subset L$ be submodules generated by e_{-m} $(m = 0, 1, \cdots, n)$ in $\mathcal{O}_V\left(\frac{1}{\delta}\right)$:

$$L_0 := \mathcal{O}_V \subset L_1 := \mathcal{D}_V\delta^{-1} \subset \cdots \subset L_n := \mathcal{D}_V\delta^{-n}. \tag{33}$$

We consider the quotient modules of $\mathcal{O}_V\left(\frac{1}{\delta}\right)$ by the L_{n-m-1} for $m = 0, \cdots, n$,

$$Q_m := \mathcal{O}_V\left(\frac{1}{\delta}\right)/L_{n-m-1} = \mathcal{O}_V\left(\frac{1}{\delta}\right)/\mathcal{D}_V\delta^{-(n-m-1)}. \tag{34}$$

Put $\widetilde{e}_k := e_{-(n-k)} \bmod P_{n-m-1}$for $k = 0, \cdots, m$. Then the Q_m are generated by the family $(\widetilde{e}_k)_{0\leq k\leq m}$ of invariant homogeneous sections of degree $-n(n-k)$ satisfying the following equations obtained thanks to the relations (30), (32), (31):

$$\begin{aligned} \theta\widetilde{e}_k &= -n(n-k)\widetilde{e}_k,\ 0 \leq k \leq m, && (35)\\ \delta\widetilde{e}_m &= 0, && (36)\\ \det(X_{IJ})\widetilde{e}_m &= 0,\ \ |I| = |J| = m+1, && (37)\\ \det\left(\widetilde{D}_{IJ}\right)\widetilde{e}_m &= 0,\ |I| = |J| = m. && (38)\end{aligned}$$

Proposition 15
(i) The $Q_m := \mathcal{O}\left(\frac{1}{\delta}\right)/P_{n-m-1}$ are $\mathcal{D}_V$-modules with support on $\overline{V}_m$.
(ii) Any section $s \in \Gamma\left(V\backslash\overline{V}_{m-1}, Q_m\right)$ of the $\mathcal{D}_V$-module Q_m in the complementary of $\overline{V}_{m-1}$ extends to the whole V $(m = 1, \cdots, n-1)$.

Proof. (ii) we should note that variety $\overline{V}_m$ is smooth out of $\overline{V}_{m-1}$ and normal along V_{m-1} for $m = 1, \cdots, n-1$. Next, the $\mathcal{D}_V$-module Q_m is the union of modules $\mathcal{O}_V\widetilde{e}_k$ $(0 \leq k \leq m)$ such that the associated graded modules $\mathrm{gr}(Q_m)$ is the sum of modules $\mathcal{O}_{T^*_{V_m}V}\widetilde{e}_k$ $(0 \leq k \leq m)$. In this case the property of extension here is true for functions because $\overline{V}_m$ is normal along $\overline{V}_{m-1}$. ■

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We are now ready to prove the Theorem10.

4.3 Proof of Theorem 10

Let us recall that the $Q_m := \mathcal{O}_V\left(\frac{1}{\delta}\right)/\mathcal{D}_V\delta^{-n+m+1}$ are subquotient modules generated by $(\widetilde{e}_k)_{0\le k\le m}$ and supported by $\overline{V}_m$, $m=0,\cdots,n$ (see. Proposition 15, (i)) .
We denote by $\widetilde{\mathcal{M}}\subset\mathcal{M}$ the submodule generated over $\mathcal{D}_V$ by $\widetilde{G}$-invariant homogeneous global sections (i.e. $\widetilde{\mathcal{M}}:=\mathcal{D}_V\left\{u\in\Gamma(V,\mathcal{M})^{\widetilde{G}},\dim_{\mathbb{C}}\mathbb{C}[\theta]\,u<\infty\right\}$).
We will see successively that the quotient module $\mathcal{M}/\widetilde{\mathcal{M}}$ is supported by $\overline{V}_m$ $(0\le m\le n-1)$, and the monodromy is trivial since $\overline{V_m}\backslash\overline{V}_{m-1}$ is simply connected.
To begin with, $\mathcal{M}/\widetilde{\mathcal{M}}$ is supported by $\overline{V}_{n-1}$: indeed $\mathcal{M}$ is isomorphic in $V\backslash\overline{V}_{n-1}$ to a $\mathcal{D}_V$-module $\delta^+(\mathcal{N})$ (see. relation (25) and Proposition 12). We may assume that the operator of multiplication by t is invertible on $\mathcal{N}$ such that there exists a morphism $v:\mathcal{M}\longrightarrow\delta^+(\mathcal{N})$ which is an isomorphism out of $\overline{V}_{n-1}$ (see. (29) and (25)). The image $v(\mathcal{M})$ is a submodule of $\delta^+(\mathcal{N})$ so it is generated by its $\widetilde{G}$-invariant homogeneous global sections (see. Lemma 14). If s is a $\widetilde{G}$-invariant homogeneous global section of a quotient of $\mathcal{M}$ then s lifts to an invariant homogeneous global section $\widetilde{s}$ of $\mathcal{M}$ ($\widetilde{s}\in\Gamma(V,\mathcal{M})^{\widetilde{G}}$, $\dim_{\mathbb{C}}\mathbb{C}[\theta]\,\widetilde{s}<\infty$). This means that $\mathcal{M}/\widetilde{\mathcal{M}}$ is supported by $\overline{V}_{n-1}$. Next, if $\mathcal{M}$ is supported by $\overline{V}_{n-1}$, it is isomorphic out of $\overline{V}_{n-2}$ to a direct sum of copies of Q_{n-1}, then there is a morphism $\mathcal{M}\longrightarrow Q_{n-1}^N$ whose sections extend (see. Proposition 15, (ii)) such that $\mathcal{M}/\widetilde{\mathcal{M}}$ is supported by $\overline{V}_{n-2}$ because the submodules of Q_{n-1} are also generated by their invariant homogeneous sections. In the same way by induction on m, if $\mathcal{M}$ is with support on $\overline{V}_m$ $(0\le m\le n-2)$ then there is a morphism $\mathcal{M}\longrightarrow Q_m^N$ which is an isomorphism out of $\overline{V}_{m-1}$, such that $\mathcal{M}/\widetilde{\mathcal{M}}$ is with support on $\overline{V}_{m-1}$ because the submodules of Q_m are also generated by their invariant homogeneous sections. Finally, if $\mathcal{M}$ is supported by V_0 (the Dirac module with support at the origin) then the result is obvious.

5 Main result

This section deals with the main theorem 18. Let $\mathcal{W}$ be the Weyl algebra on V and $\overline{\mathcal{A}}:=\Gamma(V,\mathcal{D}_V)^{\widetilde{G}}\subset\mathcal{W}$ the subalgebra of $\widetilde{G}$-invariant differential operators. We denoted $\mathcal{A}$ its quotient by $\overline{\mathcal{J}}$ i.e. $\mathcal{A}:=\overline{\mathcal{A}}/\overline{\mathcal{J}}$ where $\overline{\mathcal{J}}$ is the ideal generated by the relations (7), (9), (10) of Proposition 8. Then, we know from Corollary 9 that $\mathcal{A}$ is generated by the operators δ, θ, $\widetilde{\Delta}$ such that

$$\begin{aligned}
[\theta,\delta] &= n\delta,\quad \left[\theta,\widetilde{\Delta}\right]=-n\widetilde{\Delta},\\
\delta\widetilde{\Delta} &= \prod_{j=0}^{n-1}\left(\frac{\theta}{n}+\frac{j}{2}\right),\\
\widetilde{\Delta}\delta &= \prod_{j=0}^{n-1}\left(\frac{\theta}{n}+\frac{j+2}{2}\right).
\end{aligned}$$

Denote by $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the category consisting of graded $\mathcal{A}$-modules T of finite type such that $\dim_{\mathbb{C}} \mathbb{C}[\theta]\, u < \infty$ for $^{\forall} u \in T$. In other words, $T = \bigoplus_{\lambda \in \mathbb{C}} T_\lambda$ is a direct sum of $\mathbb{C}$-vector spaces ($T_\lambda := \bigcup_{p \in \mathbb{N}} \ker(\theta - \lambda)^p$ is finite dimensional) equipped with three endomorphisms δ, $\widetilde{\Delta}$, θ of degree n, $-n$, 0, respectively and satisfying the above relations, with $(\theta - \lambda)$ being a nilpotent operator on each T_λ.
Recall $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$ stands for the category whose objects are regular holonomic $\mathcal{D}_V$-modules whose characteristic variety is contained in Λ.
If $\mathcal{M}$ is an object in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$, denote by $\Psi(\mathcal{M})$ the submodule of $\Gamma(V, \mathcal{M})$ consisting of $\widetilde{G}$-invariant homogeneous global sections u of $\mathcal{M}$ such that $\dim_{\mathbb{C}} \mathbb{C}[\theta]\, u < \infty$. Recall that (see. Theorem 5)

$$\Psi(\mathcal{M})_\lambda := [\Psi(\mathcal{M})] \bigcap \left[\bigcup_{p \in \mathbb{N}} \ker(\theta - \lambda)^p \right]$$

is the $\mathbb{C}$-vector space of homogeneous global sections of degree λ of $\Psi(\mathcal{M})$ and $\Psi(\mathcal{M}) = \bigoplus_{\lambda \in \mathbb{C}} \Psi(\mathcal{M})_\lambda$. Then $\Psi(\mathcal{M})$ is an object in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$. Indeed, let $(s_1, \cdots, s_k)$ be a finite family of invariant homogeneous global sections generating the $\mathcal{D}_V$-module $\mathcal{M}$ (see. Theorem 10), we can see that the family $(s_1, \cdots, s_k)$ generates also $\Psi(\mathcal{M})$ as a $\mathcal{A}$-module: In fact, if $s = \sum_{j=1}^{k} p_j(X, D)\, s_j$ is an invariant section of $\mathcal{M}$ ($p_j \in \Gamma(V, \mathcal{D}_V)$), denote by $\widetilde{p_j}$ the average of p_j over $SU_n(\mathbb{C}) \times SU_n(\mathbb{C})$ (compact maximal subgroup of $\widetilde{G}$), then $\widetilde{p_j} \in \overline{\mathcal{A}}$. Let $f_j = f_j\left(\delta, \widetilde{\Delta}, \theta\right)$ be the class of $\widetilde{p_j}$ modulo $\overline{\mathcal{J}}$ that is $f_j \in \mathcal{A}$, then we also have $s = \sum_{j=1}^{k} \widetilde{p_j} s_j = \sum_{j=1}^{k} f_j s_j$.

Conversely, if T is an object in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$, one associates to it the $\mathcal{D}_V$-module

$$\Phi(T) := \mathcal{M}_0 \bigotimes_{\mathcal{A}} T \tag{39}$$

where $\mathcal{M}_0 := \mathcal{W}/\overline{\mathcal{J}}$ is a $(\mathcal{W}, \mathcal{A})$-module. Then $\Phi(T)$ is an object in the category $\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V)$.
Thus, we have defined two functors

$$\Psi : \mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V) \longrightarrow \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}),\ \Phi : \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}) \longrightarrow \mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V). \tag{40}$$

Now let us prove the two following lemmas:

Lemma 16 *The canonical morphism*

$$T \longrightarrow \Psi(\Phi(T)),\ t \longmapsto 1 \otimes t \tag{41}$$

is an isomorphism, and defines an isomorphism of functors $\mathrm{Id}_{\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})} \longrightarrow \Psi \circ \Phi$.

Proof. As above $\mathcal{M}_0 := \mathcal{W}/\overline{\mathcal{J}}$. Denote by ε (the class of $1_{\mathcal{W}}$ modulo $\overline{\mathcal{J}}$) the canonical generator of $\mathcal{M}_0$. Let $h \in \mathcal{W}$, denote by $\widetilde{h} \in \overline{\mathcal{A}}$ its average on $SU_n(\mathbb{C}) \times SU_n(\mathbb{C})$ and by φ the class modulo $\overline{\mathcal{J}}$ that is $\varphi \in \mathcal{A}$. Since ε is $\widetilde{G}$-invariant, we get $\widetilde{h\varepsilon} = \widetilde{h}\varepsilon = \varepsilon\varphi$. Moreover, we have $\widetilde{h}\varphi = 0$ if and only if $\widetilde{h} \in \overline{\mathcal{J}}$, in other words $\varphi = 0$. Therefore the average operator (over $SU_n(\mathbb{C}) \times SU_n(\mathbb{C})$) $\mathcal{W} \longrightarrow \overline{\mathcal{A}}$, $h \longmapsto \widetilde{h}$ induces a surjective morphism of $\mathcal{A}$-modules $v : \mathcal{M}_0 \longrightarrow \mathcal{A}$. More generally, for any $\mathcal{A}$-module T in the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ the morphism $v \otimes 1_T$ is a surjective map

$$v_T : \mathcal{M}_0 \bigotimes_{\mathcal{A}} T \longrightarrow \mathcal{A} \bigotimes_{\mathcal{A}} T = T \tag{42}$$

which is the left inverse of the morphism

$$u_T : T \longrightarrow \mathcal{M}_0 \bigotimes_{\mathcal{A}} T,\ t \longmapsto \varepsilon \otimes t \tag{43}$$

that is $(v \otimes 1_T) \circ (\varepsilon \otimes 1_T) = v(\varepsilon) = 1_T$. This means that the morphism u_T is injective. Next, the image of u_T is exactly the set of invariant sections of $\mathcal{M}_0 \bigotimes_{\mathcal{A}} T = \Phi(T)$ that is $\Psi(\Phi(T))$: indeed if $s = \sum_{i=1}^{p} h_i \otimes t_i$ is an invariant section in $\mathcal{M}_0 \bigotimes_{\mathcal{A}} T$, we may replace each h_i by its average $\widetilde{h_i} \in \mathcal{A}$, then we get

$$s = \sum_{i=1}^{p} \widetilde{h_i} \otimes t_i = \varepsilon \otimes \sum_{i=1}^{p} \widetilde{h_i} t_i \ \in \ \varepsilon \otimes T \tag{44}$$

that is $\sum_{i=1}^{p} \widetilde{h_i} t_i \in T$. Therefore the morphism u_T is an isomorphism from T to $\Psi(\Phi(T))$ and defines an isomorphism of functors. ■

Lemma 17 *The canonical morphism*

$$w : \Phi(\Psi(\mathcal{M})) \longrightarrow \mathcal{M}. \tag{45}$$

is an isomorphism and defines an isomorphism of functors $\Phi \circ \Psi \longrightarrow \mathrm{Id}_{\mathrm{Mod}_{\Lambda}^{\mathrm{rh}}(\mathcal{D}_V)}$.

Proof. As in the Theorem 10 the $\mathcal{D}_V$-module $\mathcal{M}$ is generated by a finite family of invariant homogeneous global sections $(s_i)_{i=1,\cdots,k} \in \Psi(\mathcal{M})$ so that the morphism w is surjective. Now denote by $\mathcal{Q}$ the kernel of the morphism $w : \Phi(\Psi(\mathcal{M})) \longrightarrow \mathcal{M}$. The $\mathcal{D}_V$-module $\mathcal{Q}$ is also generated by its invariant homogeneous global sections that is by $\Psi(\mathcal{Q})$. Then we get

$$\Psi(\mathcal{Q}) \subset \Psi[\Phi(\Psi(\mathcal{M}))] = \Psi(\mathcal{M}) \tag{46}$$

where we used $\Psi \circ \Phi = Id_{\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})}$ (see. the previous Lemma 16). Since the morphism $\Psi(\mathcal{M}) \longrightarrow \mathcal{M}$ is injective ($\Psi(\mathcal{M}) \subset \Gamma(V, \mathcal{M})$), we obtain$\Psi(\mathcal{Q}) = 0$.

Therefore $\mathcal{Q}=0$ (because $\Psi(\mathcal{Q})$ generates $\mathcal{Q}$) and the morphism w is injective. ■

Finally, we close this section by stating the following main result which is established by means of the previous lemmas:

Theorem 18 *The functors* Φ *and* Ψ *induce equivalence of categories*

$$\mathrm{Mod}^{\mathrm{rh}}_{\Lambda}(\mathcal{D}_V) \xrightarrow{\sim} \mathrm{Mod}^{\mathrm{gr}}(\mathcal{A}). \tag{47}$$

6 Colored diagrams associated with $\mathcal{A}$-modules

In the category $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$, objects are classified by finite colored diagrams of $\mathbb{C}$-linear maps. Actually a graded $\mathcal{A}$-module T in $\mathrm{Mod}^{\mathrm{gr}}(\mathcal{A})$ defines an infinite diagram consisting of finite dimensional vector spaces T_λ (with $(\theta-\lambda)$ being a nilpotent operator on each T_λ, $\lambda\in\mathbb{C}$) and linear maps between them deduced from the linear action of $\delta,\widetilde{\Delta},\theta$:

$$\cdots \rightleftarrows T_\lambda \underset{\widetilde{\Delta}}{\overset{\delta}{\rightleftarrows}} T_{\lambda+n} \rightleftarrows \cdots \tag{48}$$

satisfying the relations $(\theta-\lambda)\,T_\lambda \subset T_\lambda$,

$$\delta\widetilde{\Delta} = \frac{\theta}{n}\left(\frac{\theta}{n}+\frac{1}{2}\right)\cdots\left(\frac{\theta}{n}+\frac{n-1}{2}\right) \text{ on } T_\lambda, \tag{49}$$

$$\widetilde{\Delta}\delta = \left(\frac{\theta}{n}+1\right)\left(\frac{\theta}{n}+\frac{3}{2}\right)\cdots\left(\frac{\theta}{n}+\frac{n+1}{2}\right) \text{ on } T_\lambda. \tag{50}$$

Such a diagram is completely determined by a finite subset of objects and arrows.

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Ideed we have:

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A) If $\lambda\notin n\mathbb{Z}$, $\lambda\notin n\mathbb{Z}+\frac{n}{2}$ then the linear maps δ and $\widetilde{\Delta}$ are bijective:

$$T_{-\lambda} \quad\simeq\quad T_{\lambda+n}. \tag{51}$$

Therefore T is completely determined by one element T_λ equipped with the nilpotent action of $(\theta-\lambda)$.

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The remain part of the study is done in two parts according as n is even or odd.

B) If n is odd, we distinguishe two cases:

.

B.1 For $\lambda \in n\mathbb{Z}$. Then T is determined by degrees in $-\frac{n(n+1)}{2} \leq \lambda \leq 0$ that is a diagram of $\frac{n+3}{2}$ elements

$$T_{-\frac{n(n+1)}{2}} \underset{\widetilde{\Delta}}{\overset{\delta}{\rightleftarrows}} \cdots \rightleftarrows T_{-n} \underset{\widetilde{\Delta}}{\overset{\delta}{\rightleftarrows}} T_0. \tag{52}$$

In the other degrees δ or $\widetilde{\Delta}$ are bijective. Indeed, we have $T_0 \simeq \delta^k T_0 \simeq T_{nk}$ and $T_{-\frac{n(n+1)}{2}} \simeq \widetilde{\Delta}^k T_{-\frac{n(n+1)}{2}} \simeq T_{-\frac{n(n+1)}{2}-nk}$ ($k \in \mathbb{N}$) thanks to the relations (49), (50). The operator $\delta\widetilde{\Delta}$ (resp. $\widetilde{\Delta}\delta$) on T_λ has only one eigenvalue $\frac{\lambda}{n}(\frac{\lambda}{n}+\frac{1}{2})(\frac{\lambda}{n}+1)\cdots\left(\frac{\lambda}{n}+\frac{n-1}{2}\right)$ (resp. $(\frac{\lambda}{n}+1)(\frac{\lambda}{n}+\frac{3}{2})\cdots\left(\frac{\lambda}{n}+\frac{n+1}{2}\right)$) so that the equation (49) (resp.(50)) has a unique solution θ of eigenvalue λ if λ is not a critical value. Here $\lambda = 0,\ -\frac{n}{2},\ -n,\ \cdots,-\frac{n^2}{2},-\frac{n(n+1)}{2}$ thus it is always the case.

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B2. For $\lambda \in n\mathbb{Z}-\frac{n}{2}$. Then T is determined by degrees in $-\frac{n^2}{2} \leq \lambda \leq \frac{-n}{2}$ that is a diagram of n elements

.

$$T_{-\frac{n^2}{2}} \underset{\widetilde{\Delta}}{\overset{\delta}{\rightleftarrows}} \cdots \rightleftarrows T_{-\frac{3n}{2}} \underset{\widetilde{\Delta}}{\overset{\delta}{\rightleftarrows}} T_{-\frac{n}{2}}. \tag{53}$$

In the other degrees $\delta, \widetilde{\Delta}$ are bijective.

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C) If n is even , we distinguishe two cases:

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C.1 For $\lambda \in n\mathbb{Z}$. Then T is determined by degrees in $-\frac{n^2}{2} \leq \lambda \leq 0$ that is a diagram of $\left(\frac{n}{2}+1\right)$

$$T_{-\frac{n^2}{2}} \underset{\widetilde{\Delta}}{\overset{\delta}{\rightleftarrows}} \cdots \rightleftarrows T_{-n} \underset{\widetilde{\Delta}}{\overset{\delta}{\rightleftarrows}} T_0. \tag{54}$$

In the other degrees δ, $\widetilde{\Delta}$ are bijective.

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C.2. For $\lambda \in n\mathbb{Z}-\frac{n}{2}$. Then T is determined by degrees in $-\frac{n(n+1)}{2} \leq \lambda \leq \frac{-n}{2}$ thats is a diagram of $(n+1)$ elements

$$T_{-\frac{n(n+1)}{2}} \underset{\widetilde{\Delta}}{\overset{\delta}{\rightleftarrows}} \cdots \rightleftarrows T_{-\frac{3n}{2}} \underset{\widetilde{\Delta}}{\overset{\delta}{\rightleftarrows}} T_{-\frac{n}{2}}. \tag{55}$$

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In the other degrees δ, $\widetilde{\Delta}$ are bijective..

6.1 Some examples of irreducible diagrams

The following examples are provided to illustrate the theoretical results:

Example 19 *The irreducible $\mathcal{D}_V$-module $\mathcal{O}_V$ is generated by $e_0 := 1_V$ an homogeneous section of degree 0 such that $\theta e_0 = 0$ and $\widetilde{\Delta} e_0 = 0$. Then its associated graded $\mathcal{A}$-module has a basis (e_q) where $q = nk$ ($k \in \mathbb{N}$) such that $\widetilde{\Delta} e_0 = 0$ and satisfying the system:*

$$S_0 = \begin{cases} \theta e_q = q e_q \ (q = nk,\ k \in \mathbb{N}) \\ \delta e_q = e_{q+n}, \\ \widetilde{\Delta} e_q = \prod_{j=0}^{n-1} \left(\frac{q}{n} + \frac{j+2}{2}\right) e_{q-n} \end{cases} . \tag{56}$$

Since $\widetilde{\Delta} e_0 = 0$ (i.e $\widetilde{\Delta} T_0 = 0$), the arrows on the left of T_0 in the diagram vanish that is

$$0 \leftarrow T_0 \ \underset{\widetilde{\Delta}}{\overset{\delta}{\rightleftarrows}} \ T_n \ \rightleftarrows \cdots \tag{57}$$

Example 20 *The irreducible $\mathcal{D}_V$-module $\mathcal{B}_{\{0\}|V}$ is generated by $e_{-\frac{n(n+1)}{2}}$ an homogeneous section of degree $-\frac{n(n+1)}{2}$ such that $\theta e_{-\frac{n(n+1)}{2}} = -\frac{n(n+1)}{2} e_{-\frac{n(n+1)}{2}}$ and $\delta e_{-\frac{n(n+1)}{2}} = 0$. Its associated $\mathcal{A}$-module is obtained by Fourier transform of that of $\mathcal{O}_V$. So its basis is (e_q) where $q = -\frac{n(n+1)}{2} - nk$ ($k \in \mathbb{N}$) such that $\delta e_{-\frac{n(n+1)}{2}} = 0$ satisfying the system:*

$$S_1 = \begin{cases} \theta e_q = q e_q \ (q = -\frac{n(n+1)}{2} - nk,\ k \in \mathbb{N}) \\ \widetilde{\Delta} e_q = e_{q-n}, \\ \delta e_q = \prod_{j=0}^{n-1} \left(\frac{q}{n} + \frac{j}{2}\right) e_{q+n} \end{cases} \tag{58}$$

Since $\delta e_{-\frac{n(n+1)}{2}} = 0$ (i.e $T_{-\frac{n(n+1)}{2}} = 0$), the arrows at the right of $T_{-\frac{n(n+1)}{2}}$ in the diagram vanish that is

$$\cdots \rightleftarrows \ T_{-\frac{n(n+3)}{2}} \ \underset{\widetilde{\Delta}}{\overset{\delta}{\rightleftarrows}} \ T_{-\frac{n(n+1)}{2}} \rightarrow 0 \tag{59}$$

Example 21 *The irreducible $\mathcal{D}_V$-modules*

$$L^k := L_{n-k}/L_{n-k-1} := \mathcal{D}_V \delta^{-(n-k)} / \mathcal{D}_V \delta^{-(n-k-1)} \tag{60}$$

are supported by the closure of the obits $\overline{X}_k$ for $k = 0, \cdots, n$. Their corresponding diagrams are determined by only one element $T_{-n(n-k)}$ with the nilpotent action of $(\theta + n(n-k))$.

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References

[1] L. Boutet de Monvel, $\mathcal{D}$-modules holonômes réguliers en une variable, Mathématiques et Physique, Séminaire de L'ENS, Progr.Math., **37** Birkhäuser Boston, MA, (1972-1982), 313-321

[2] L. Boutet de Monvel, Revue sur la théorie des D-modules et modèles d'opérateurs pseudodifférentiels, Math. Phys. stud. 12, Kluwer Acad. Publ. (1991) 1-31

[3] T. Braden, M. Grinberg, Perverse sheaves on rank stratifications, Duke Math. J. **96**, no. 2 (1999), 317-362

[4] P. Deligne, Letter to R. Macpherson (1981)

[5] L. Garding, Extension of a formula by Cayley to symmetric determinants. Proc. Edinb. Math. Soc. Ser. **II**, **8**, (1947), 73-75

[6] A. Galligo, M. Granger, P. Maisonobe, $\mathcal{D}$-modules et faisceaux pervers dont le support singulier est un croisement normal, **I**, Ann. Inst. Fourier, **35 (1)** (1985), 1-48, **II** Astérisque, **130** (1985), 240-259

[7] R. Howe, T. Umeda, The Capelli identity, the double commutant theorem, and multiplicity free actions, Math. Ann., **290**, (1991) 565-619

[8] M. Kashiwara, The Riemann-Hilbert problem for holonomic systems, Publ. Res. Inst. Math. Sci. **20** (1984), 319-365

[9] M. Kashiwara, Algebraic study of systems of partial differential equations, Memo. Soc. Math. France*,* **63, (123** fascicule 4), (1995)

[10] M. Kashiwara, $\mathcal{D}$-modules and Microlocal calculus, Iwanami Series in Modern Mathematics, Translations of Mathematical Monographs, AMS, vol. **217**(2003)

[11] M. Kashiwara, T. Kawai, On the characteristic variety of a holonomic system with regular singularities**,** Adv. Math. **34** (1979), 163-184

[12] M. Kashiwara, T. Kawai, On holonomic systems of Microdifferential Equations **III**: Systems with regular singularities**,** Publ. Res. Inst. Math. Sci. **17** (1981), 813-979

[13] R. Macpherson, K. Vilonen, Elementary construction of perverse sheaves. Invent. Math. **84** (1986) 403-435

[14] R. Macpherson, K. Vilonen, Perverse sheaves with regular singularities along the curve $y^n = x^m$, Comment. Math. Helv. **63**, (1988), 89-102

[15] J. Milnor, Singular points of complex hypersurfaces, Annals of Math. Stud., vol. **61**, Princeton University Press

[16] P. Nang, $\mathcal{D}$-modules associated to the group of similitudes, Publ. Res. Inst. Math. Sci. **35** (2) (1999), 223-247

[17] P. Nang, On a class of holonomic $\mathcal{D}$-modules on $M_n(\mathbb{C})$ related to the action of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$, Adv. Math. (2008)

[18] P. Nang, $\mathcal{D}$-modules associated to determinantal singularities, Proc. Japan Acad. Ser. A Math. Sci. 80 (2004), no. 5, 74–78

[19] P. Nang, Algebraic description of $\mathcal{D}$-modules associated to 3×3-matrices, Bull. Sci. Math. 130 (2006), no. 1, 15–32

[20] L. Narvaez Macarro, Cycles évanescents et faisceaux pervers **I**: cas des courbes planes irréductibles, Compositio Math., **65**, (3) (1988) 321-347, **II**: cas des courbes planes réductibles, London Math. Soc. Lecture Notes Ser. **201** (1994), 285-32

[21] H. Rubenthaler, Invariant differential operators and infinite dimensional Howe-Type correspondence, Preprint ArXiv:0802.0440v1[math.RT] 4 Feb (2008)

[22] M. Sato, T. Kawai, M. Kashiwara, Microfunctions and pseudo differential equations. Hyperfunctions and pseudo diffrential equations (Proc. Conf., Katata 1971; dedicated to memory of André Martineau 1971) pp. 265-529. Lecture Notes in Math., Vol. **287**, Springer Berlin (1973)

[23] M.G.M. Van Doorn, Classification of $\mathcal{D}$-modules with regular singularities along normal crossings, Compositio Math., **60**, (1) (1986) 19-32

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