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INSTANTONS BEYOND TOPOLOGICAL THEORY II

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Abstract. The present paper is the second part of our project in which we describe quantum field theories with instantons in a novel way by using the "infinite radius limit" (rather than the limit of free field theory) as the starting point. The theory dramatically simplifies in this limit, because the correlation functions of all, not only topological (or BPS), observables may be computed explicitly in terms of integrals over finite-dimensional moduli spaces of instanton configurations. In Part I we discussed in detail the one-dimensional (that is, quantum mechanical) models of this type. Here we analyze the supersymmetric two-dimensional sigma models and four-dimensional Yang–Mills theory, using the one-dimensional models as a prototype. We go beyond the topological (or BPS) sectors of these models and consider them as full-fledged quantum field theories. We study in detail the space of states and find that the Hamiltonian is not diagonalizable, but has Jordan blocks. This leads to the appearance of logarithms in the correlation functions. We find that our theories are in fact logarithmic conformal field theories (theories of this type are of interest in condensed matter physics). We define jet-evaluation observables and consider in detail their correlation functions. They are given by integrals over the moduli spaces of holomorphic maps, which generalize the Gromov–Witten invariants. These integrals generally diverge and require regularization, leading to an intricate logarithmic mixing of the operators of the sigma model. A similar structure arises in the four-dimensional Yang–Mills theory as well.

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1. Introduction

Many two- and four-dimensional quantum field theories have two kinds of coupling constants: the actual coupling g, which in particular counts the loops in the perturbative calculations, and the topological coupling, ϑ, the theta-angle, which is the chemical potential for the topological sectors in the path integral. These couplings can be combined into the complex coupling τ and its complex conjugate $\overline{\tau}$.

For example, in the four-dimensional gauge theory one combines the Yang–Mills coupling g and the theta-angle ϑ as follows:

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2}, \qquad \overline{\tau} = \frac{\vartheta}{2\pi} - \frac{4\pi i}{g^2}. \tag{1.1}$$

For the two-dimensional sigma model with the complex target space X, endowed with a Hermitian metric $g_{i\bar{j}}$ and a $(1,1)$ type two-form $B_{i\bar{j}}$ one defines

$$\tau_{i\bar{j}} = B_{i\bar{j}} + i g_{i\bar{j}}, \qquad \overline{\tau} = B_{i\bar{j}} - i g_{i\bar{j}} \tag{1.2}$$

If $dB = 0$, then the two-form B plays the role of the theta-angle.

1.1. **Weak coupling limit.** We wish to study the dependence of the theory on $\tau, \overline{\tau}$ as if they were two separate couplings, not necessarily complex conjugate to each other. The resulting theory should greatly simplify in the limit when

$$\overline{\tau} \to -i\infty \ , \qquad \tau \ \text{ is fixed}\,. \tag{1.3}$$

This is the weak coupling limit, in which the theta-angle has a large imaginary part. In two-dimensional sigma models, it is also known as the "infinite radius limit". This limit has been studied in the literature since the early days of the theory of instantons, see, e.g., [15].

The reason for this simplification is that the path integral of the theory in this limit, as described by a first-order Lagrangian, represents the "delta-form" supported on the instanton moduli space, which is a union of *finite-dimensional* components labeled by "instanton numbers". Therefore the correlation functions are expressed as linear combinations of integrals over these finite-dimensional components.

The idea is to use the theory in this weak coupling limit as the *starting point* for investigating more realistic models corresponding the finite values of coupling constants. In other words, we first describe the theory in this limit, and then develop a *perturbation theory* around this point in the space of couplings in order to reach the theories defined for other values of the coupling constants. In the framework of this perturbation theory we could, in principle, compute all correlation functions in terms of finite-dimensional integrals. A more detailed discussion of this point, along with a summary of our project, may be found in [27].

We view this as a viable alternative to the conventional approach in quantum field theory of using the perturbation theory around a Gaussian point describing a free field theory. The advantage of our approach is that we do not need to impose from the beginning a linear structure on the space of fields. On the contrary, the non-linearity is preserved and is reflected in the geometry of the moduli space of instantons. This is why we believe that our approach may be beneficial for understanding some of the hard dynamical questions, such as confinement, that have proved elusive in the conventional formalism.

The 4D *S-duality* and its 2D analogue, the *mirror symmetry*, give us another tool for connecting the theory at our special limit to the theories in the physical range of coupling constants. In a physical theory, in which $\overline{\tau}$ is complex conjugate to τ, the S-duality sends $\tau \mapsto -1/\tau$. It is reasonable to expect that S-duality still holds when we complexify the coupling constants $\tau, \overline{\tau}$. It should then act as follows:

$$\tau \mapsto -1/\tau, \qquad \overline{\tau} \mapsto -1/\overline{\tau}.$$

Now observe that applying this transformation to $\overline{\tau} = -i\infty$ and finite τ, we obtain $\tau' = -1/\tau$ and $\overline{\tau}' = 0$. These coupling constants are already within the range of physical values, in the sense that both the coupling constant g and the theta-angle ϑ are finite! Therefore we hope that our calculations in the theory with $\overline{\tau} = -i\infty$ could be linked by S-duality (or mirror symmetry) to exact non-perturbative results in a physical theory beyond the topological sector.

1.2. **Topological sector.** In supersymmetric models there is an important class of observables, called the *topological*, or *BPS observables*, whose correlation functions are independent of $\overline{\tau}$. They commute with the supersymmetry charge $\mathcal{Q}$ of the theory and comprise the *topological sector* of the theory. Their correlation functions are closely related to the Gromov–Witten and Donaldson invariants, in two-dimensional sigma models and four-dimensional Yang–Mills theory, respectively. The perturbation away from the point $\overline{\tau} = -i\infty$ is given by a $\mathcal{Q}$-exact operator, and therefore the correlation functions of the BPS observables (which are $\mathcal{Q}$-closed) remain unchanged when we move away from the special point. This is the secret of success of the computation of the correlation functions of the BPS observables achieved in recent years in the framework of topological field theory: the computation is actually done in the theory at $\overline{\tau} = -i\infty$, but because of the special properties of the BPS observables the answer remains the same for other values of the coupling constant. But for general observables the correlation functions do change in a rather complicated way when we move away from the special point.

We would like to go *beyond the topological sector* and consider more general correlation functions of non-BPS observables. Multiple reasons for this are described in detail in the Introductions to our earlier papers [26, 27], and we will not repeat them here. Clearly, if we wish to use the model in the limit $\overline{\tau} = -i\infty$ as a launchpad for studying the models at more general values of the coupling constants, we must first understand it as a full-fledged quantum field theory, beyond its topological sector.

In [26], to which we will henceforth refer as "Part I", we have launched a program of systematic study of the $\overline{\tau} \to -i\infty$ limit of the instanton models in one, two and four dimensions. In Part I we have described in detail the one-dimensional models. These are supersymmetric quantum mechanical models on Kähler manifolds X equipped with a Morse function f. Here we analyze the supersymmetric two-dimensional sigma models and four-dimensional Yang–Mills theory, using the one-dimensional models as a prototype.

1.3. **Sigma models.** Let us describe briefly our results concerning two-dimensional supersymmetric sigma models. The first step is to recast these models in the framework of the quantum mechanical models that we have studied in Part I. For a fixed Riemann surface Σ the space of bosonic fields in the supersymmetric sigma model with the target manifold X is $\mathrm{Maps}(\Sigma, X)$, the space of maps $\Sigma \to X$. If we choose Σ to be the cylinder $I \times \mathbb{S}^1$, then we may interpret $\mathrm{Maps}(\Sigma, X)$ as the space of maps from the interval I to the loop space $LX = \mathrm{Maps}(\mathbb{S}^1, X)$. Thus, we may think of the two-dimensional sigma model on the cylinder with the target X as the quantum mechanical model on the loop space LX. Hence it is natural to try to write the Lagrangian of the sigma model in such a way that it looks exactly like the Lagrangian of the quantum mechanical model on LX with a Morse function f.

It turns out that if X is a Kähler manifold, this is "almost" possible. However, there are two caveats. First of all, our function f is the so-called Floer function [21] which has non-isolated critical points corresponding to the constant loops in LX, so is in fact a Morse–Bott function. We can deal with this problem by deforming this function so that it only has isolated critical points, corresponding to the constant loops whose values are critical points of a Morse function on X (this amounts to considering the sigma model in the background of a gauge field, as we explain in Section 7). The second, and more serious, issue is that the Floer function f is not single-valued on LX, but only on the universal (abelian) cover $\widetilde{LX}$. In other words, it is an example of a Morse–Novikov function, or, more properly, Morse–Bott–Novikov function. Because of that, the instantons are identified with gradient trajectories of the pull-back of f to $\widetilde{LX}$.

Suppose that $I = \mathbb{R}$ with a coordinate t. In the limits $t \to \pm\infty$ a gradient trajectory tends to the critical points of f on $\widetilde{LX}$, which are the preimages of constant maps in LX. Therefore the gradient trajectory may be interpreted as the map of the cylinder compactified by two points at $\pm\infty$ to X, or equivalently, a map $\mathbb{CP}^1 \to X$. Moreover, the condition that it corresponds to a gradient trajectory of the Floer function f simply means that this map is *holomorphic*. Thus, we obtain that the instantons of the two-dimensional sigma model are holomorphic maps $\mathbb{CP}^1 \to X$, and more generally, $\Sigma \to X$, where Σ is an arbitrary compact Riemann surface.

In order to describe the structure of the space of states of the two-dimensional sigma model in the infinite radius limit, we first need to generalize our results obtained in Part I to the case of Morse–Novikov and Morse–Bott–Novikov functions. (Actually, examples of such functions arise already for finite-dimensional manifolds with non-trivial fundamental groups.) Essentially, this amounts to considering the universal (abelian) cover (which is $\widetilde{LX}$ in the case of sigma model). One also needs to impose an equivariance condition on the states of the model corresponding to the action of the (abelianized) fundamental group $H_2(X, \mathbb{Z})$ on $\widetilde{LX}$. Besides those changes, the structure of the space of states in the limit $\overline{\tau} \to \infty$ is similar to the one that we have observed in our analysis of quantum mechanical models in Part I.

There are spaces of "in" and "out" states with a canonical pairing between them. Each of them is isomorphic to the space of semi-infinite "delta-forms" on the loop space LX supported on the subset of boundary values of holomorphic maps from the disc (which is the interior of the circle for the "in" states and the exterior for the "out" states). These spaces of delta-forms may be described in terms of the familiar Fock representations of the chiral and anti-chiral $\beta\gamma, bc$-systems (the chiral-anti-chiral de Rham complex of X).

However, this ismorphism is not canonical because of the instanton effects which cause non-trivial self-extensions between the spaces of semi-infinite delta-forms. These extensions are similar to the extensions arising in quantum mechanics which we have studied in detail in Part I. One consequence of this is that the Hamiltonian is not diagonalizable, but has Jordan blocks. Another consequence is that our models are logarithmic conformal field theories.

1.4. **Logarithmic structure of the correlation functions.** The logarithmic features of our model are revealed upon the computation of the correlation functions. As we discussed above, in the infinite radius limit the path integral localizes on the moduli space of instantons, which are holomorphic maps $\Sigma \to X$. Because we are dealing here with a Morse–Bott–Novikov function, this moduli space now has infinitely many connected components labeled by $\beta \in H_2(X, \mathbb{Z})$ (the moduli space is non-empty only if the integral of the Kähler class ω of X over β is non-negative). All of them are finite-dimensional.

The simplest class of observables of this model consists of the evaluation observables corresponding to differential forms on X. Their correlation functions are given by integrals of their pull-backs to the moduli spaces of holomorphic maps, or, more precisely, their Kontsevich compactifications, under the evaluation maps.

Such correlation functions have been studied extensively in the literature for the BPS observables, corresponding to closed differential forms on X. They are expressed in terms of the Gromov-Witten invariants of X. However, in order to describe the structure of the sigma model as a full-fledged quantum field theory (and, for instance, observe that the Hamiltonian is non-diagonalizable), we must go beyond the topological sector of the model and study correlation function of more general, non-BPS, observables. These observables include evaluation observables corresponding to differential forms on X that are not closed.

In this paper we introduce an even larger class of non-BPS observables which we call the *jet-evaluation observables*. They keep track of not only the value of a map $\Sigma \to X$ at a point $p \in \Sigma$, but also its derivatives with respect to a local coordinate at p. These observables correspond to differential forms not on X, but on its jet space JX, the space of jets of holomorphic maps from a small disc to X. Their correlation functions are signficantly more complicated than those of the evaluation observables. They are again given by integrals over the (compactified) moduli spaces of holomorphic maps $\Sigma \to X$, but now these integrals may diverge at the boundary divisors and require regularization. We have encountered the necessity of the regularization of instanton integrals in Part I when studying one-dimensional quantum-mechanical models. An immediate consequence of this regularization is the *logarithmic mixing of operators*: the naive perturbative jet-evaluation operators are not well-defined as operators of the full quantum field theory. They are only well-defined up to the addition of their "logarithmic partners". This ambiguity is reflected in the ambiguity of the regularization of our integrals. Roughly speaking, to obtain a true operator of the non-perturbative sigma model, we need to take a jet-evaluation observable in its perturbative definition together with a consistent set of regularization rules for all of these integrals.

Together, operators and their logarithmic partners span finite-dimensional subspaces in the space of operators (or, equivalently, states), on which the Hamiltonian acts as a Jordan block. The logarithmic mixing is also responsible for the appearance of logarithmic terms in the operator product expansion (OPE) of these operators, as we will see in explicit examples below.

We note that the upper-triangular entries in the Jordan blocks are proportional to the instanton parameters of the model, so their appearance is caused by the instanton effects. When we move away from the point $\overline{\tau} = -i\infty$ (that is, to finite radius), anti-instantons contributions arise which cause the appearance of non-zero lower-triangular entries, and the Hamiltonian becomes diagonalizable. In other words, Jordan blocks appear because there are instantons, but no anti-instantons, in the infinite radius limit.

Thus, we find that the two-dimensional supersymmetric sigma model in the infinite radius limit is a *logarithmic conformal field theory* with central charge 0. Theories of this kind have been studied extensively recently (see, e.g., [34, 38, 35, 47, 16, 41] and references therein), in part because of their applications to condensed matter physics. The theory has a large chiral algebra, which is nothing but the space of global sections of the chiral de Rham complex of X introduced in [40]. (The chiral algebra is not affected by logarithms; only operators depending on both holomorphic and anti-holomorphic coordinates are so affected.)

To illustrate these phenomena, we consider explicitly two important examples: the first is the sigma model with an elliptic curve as the target. This is essentially a free field theory which may be described at both finite and infinite radii. We explain what the passage to the infinite radius means for these models. The second case of interest is when the target manifold is $\mathbb{P}^1$. In [25] this model was described as a deformation of the free field theory by *holomortex operators*. Here we revisit and rederive this description and present numerous examples of correlation functions, OPE and logarithmic mixing in this model.

In Part III of this paper we will consider similar structures in the $\mathcal{N} = (0, 2)$ supersymmetric, as well as purely bosonic, sigma models. One of the motivations for the study of these models is that when the target manifold is the flag variety of a simple Lie group G, this model has an affine Kac–Moody algebra symmetry of critical level (see [19, 23]), even though the theory is not conformally invariant. The correlation functions in this model, coupled to a gauge field, may be viewed as solutions of systems of linear differential equations on the moduli space of G-bundles on the worldsheet, which arise naturally in the geometric Langlands correspondence.

1.5. **Yang–Mills theory.** The next step in our program is the investigation of the four-dimensional gauge theory. Again, we start with the supersymmetric model, which is a twisted $\mathcal{N} = 2$ super-Yang–Mills theory with gauge group G on a four-dimensional manifold $\mathbf{M}^4$, in the limit $\overline{\tau} \to -i\infty$ with fixed

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2}.$$

Suppose that $\mathbf{M}^4 = \mathbb{R} \times M^3$, where M^3 is a compact three-dimensional manifold. Let t denote the coordinate along the $\mathbb{R}$ factor. Then the Yang-Mills theory may be interpreted as quantum mechanics on the space $\mathcal{A}/\mathcal{G}$ of gauge equivalence classes of G-connections on M^3, with the Morse–Novikov function being the Chern–Simons functional [3, 51]. However, there is again a new element, compared to the previously discussed theories, and that is the appearance of gauge symmetry. The quotient $\mathcal{A}/\mathcal{G}$ has complicated singularities because the gauge group $\mathcal{G}$ has non-trivial stabilizers in the space $\mathcal{A}$. For this reason we should consider the *gauged Morse theory* on the space $\mathcal{A}$ of connections itself.

This theory is defined as follows. Let X be a manifold equipped with an action of a group G and a G-invariant Morse function f. Then the gradient vector field $v^\mu \partial_{x^\mu}$, where $v^\mu = h^{\mu\nu}\partial_{x^\nu} f$ commutes with the action of G. Denote by $\mathcal{V}^\mu_a \partial_{x^\mu}$ the vector fields on X corresponding to basis elements J^a of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. We define a gauge theory generalization of the gradient trajectory: it is a pair $(x(t) : \mathbb{R} \to X, A_t(t)dt \in \Omega^1(\mathbb{R}, \mathfrak{g}^*))$, which is a solution of the equation

$$\frac{dx^\mu}{dt} = v^\mu(x(t)) + \mathcal{V}^\mu_a(x(t)) A^a_t(t). \tag{1.4}$$

The group of maps $g(t) : \mathbb{R} \to G$ acts on the space of solutions by the formula

$$g : (x(t), A_t(t)dt) \mapsto \left(g(t)\cdot x(t),\ g^{-1}(t)\partial_t g(t) + g^{-1}(t)A_t(t)g(t)\right),$$

and the moduli space of gradient trajectories is, naively, the quotient of the space of solutions of (1.4) by this action. However, because this action has non-trivial stabilizers and the ensuing singularities of the quotient, it is better to work equivariantly with the moduli space of solutions of the equations (1.4).

In Section 8 we develop a suitable formalism of equivariant integration on the moduli space of gradient trajectories of the gauged Morse theory. We then apply this formalism to the case when $X = \mathcal{A}$, the space of connections on a three-manifold M^3 and f is the Chern–Simons functional (note that this formalism may also used to define gauged sigma models in two dimensions, see Section 7.6). The corresponding equivariant integrals give us the correlation functions of evaluation observables of the Yang-Mills theory in our weak coupling limit $\overline{\tau} \to -i\infty$. In the case of the BPS observables these correlation functions are the *Donaldson invariants* [51]. They comprise the topological (or BPS) sector of the theory.

We obtain more general (off-shell) correlation functions by considering more general, i.e., non-BPS, observables. We will present some sample computations of these off-shell correlation functions which exhibit the same effects as in one- and two-dimensional models considered above. In particular, we will observe the appearance of the logarithm function in the correlation functions. Moreover, we find the same kind of logarithmic mixing that we have observed in two-dimensional sigma models. We also present an example of instanton corrections to the OPE of some natural observables in the Yang–Mills theory which involve logarithm. Thus, we conclude that the supersymmetric Yang–Mills theory in the weak coupling limit is a logarithmic CFT in four dimensions.

1.6. **Contents.** The paper is organized as follows. In Section 2 we give the definition of the two-dimensional sigma models and its infinite radius limit. We then interpret it as quantum mechanics on the the loop space of the target manifold X, equipped with the Floer function. The multi-valuedness of the Floer function requires certain modifications in the analysis of quantum mechanical models from Part I. Hence we discuss in Section 3 the general structure of quantum mechanics on non-simply connected manifolds. Having developed the general theory, we go back to our main example, the sigma models, in Section 4. We describe its space of states in terms of certain spaces of delta-forms supported on semi-infinite ascending manifolds of the Floer function inside the universal covering of the loop space of X. Algebraically, these are given in terms of Fock modules over the $\beta\gamma$-bc-systems. In particular, the chiral algebra of the sigma model may be identified with the chiral de Rham complex introduced in [40]. We isolate an important subspace in the space of states: the space of differential forms on the jet space of X. The corresponding operators are the jet-evaluation observables which we study in detail in the subsequent sections. We also consider two examples: when X is an elliptic curve and $\mathbb{P}^1$.

In Section 5 we take up the correlation functions. We first recall the definition of a simplest ones, which are the Gromov–Witten invariants. Then we explain the difference between the BPS (or topological) and non-BPS observables and explain why it is important to go beyond the topological sector of the model. We give examples of correlation functions of non-BPS observables and show how to extract from them some unusual features of our model, such as the non-diagonalizability of the Hamiltonian. The next section, Section 6, plays a special role. Here we look closely at the correlation functions and the OPE of the jet-evaluation observables introduced in Section 4. We give examples of logarithmic mixing of observables and explain the underlying reasons for this phenomenon using the geometry of the moduli spaces of stable maps and the description of the space of states in terms of consecutive extensions of spaces of delta-forms.

In Section 7 we consider the sigma models in the background of a $\mathbb{C}^\times$-gauge field. These models correspond to a deformation of the Morse–Bott–Novikov function to a Morse–Novikov function, with isolated critical points. This leads to simplifications in the description of the space of states of the model and allows us to make a closer contact with the results of Part I. In particular, we consider the case of the sigma model with the target $\mathbb{P}^1$ and relate the formulas for the Hamiltonian involving the so-called Cousin–Grothendieck operators obtained in Part I to the description of the sigma model of $\mathbb{P}^1$ as a deformation of the free theory by holomortex operators.

In Section 8 we consider the four-dimensional supersymmetric Yang–Mills theory. We explain how to modify the formalism developed in the previous sections in the presence of gauge invariance. We then compute explicitly non-BPS correlation functions of the $SU(2)$ model in the one instanton sector using the ADHM construction of the moduli space of anti-self-dual connections. We show that these correlation functions exhibit the same kind of logarithmic behavior that we have seen the one- and two-dimensional models.

Finally, in Section 9 we present our conclusions and outlook.

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2. Two-dimensional sigma models

In this section we give the definition of the supersymmetric two-dimensional sigma model in the infinite radius limit. We then interpret it as quantum mechanics on the loop space, equipped with a Morse–Bott–Novikov function.

2.1. **Definition of the supersymmetric sigma model.** Let X be a compact Kähler manifold. We will denote by $X^a, a = 1, \ldots, N = \dim X$, local holomorphic coordinates on X, and by $X^{\overline{a}} = \overline{X^a}$ their complex conjugates. We write the metric as $g_{a\bar{b}} dX^a dX^{\overline{b}}$ and the Kähler form as

$$\omega_K = \frac{i}{2} g_{a\bar{b}} dX^a \wedge dX^{\overline{b}}. \tag{2.1}$$

We consider the type A twisted $\mathcal{N} = (2,2)$ supersymmetric sigma model on a Riemann surface Σ with the target manifold X. Given a map $\Phi : \Sigma \to X$, we consider the pull-backs of X^a and $X^{\overline{a}}$ as functions on Σ, denoted by the same symbols. We also have fermionic fields ψ^a $\psi^{\overline{a}}, a = 1, \ldots, N$, which are sections of $\Phi^*(T^{1,0}X)$ and $\Phi^*(T^{0,1}X)$, respectively, and and π_a and $\pi_{\overline{a}}$, which are sections of $\Phi^*(\Omega^{1,0}X) \otimes \Omega^{1,0}\Sigma$ and $\Phi^*(\Omega^{0,1}X) \otimes \Omega^{0,1}\Sigma$, respectively. The Levi-Civita connection on TX corresponding to the metric $g_{a\bar{b}}$ induces a connection on $\Phi^*(TX)$. The corresponding covariant derivatives have the form

$$D_{\overline{z}}\psi^a = \partial_{\overline{z}}\psi^a + \partial_{\overline{z}}X^b \cdot \Gamma^a_{bc}\psi^c,$$
$$D_z\psi^{\overline{a}} = \partial_z\psi^{\overline{a}} + \partial_z X^{\overline{b}} \cdot \Gamma^{\overline{a}}_{\overline{b}\overline{c}}\psi^{\overline{c}},$$

where $\Gamma^a_{bc} = g^{a\overline{b}}\partial_b g_{c\overline{b}}$.

The standard action of the supersymmetric sigma model is

$$\int_\Sigma \left(\frac{1}{2}\lambda(g_{a\overline{b}}(\partial_{\overline{z}}X^a\partial_z X^{\overline{b}} + \partial_z X^a \partial_{\overline{z}} X^{\overline{b}}) + i\pi_a D_{\overline{z}}\psi^a + i\pi_{\overline{a}}D_z\psi^{\overline{a}} + \frac{1}{2}\lambda^{-1}R^{a\overline{b}}{}_{c\overline{d}}\pi_a\pi_{\overline{b}}\psi^c\psi^{\overline{d}}\right) d^2z, \tag{2.2}$$

We are now in the same position in which we were in quantum mechanics (see formula (2.6) in Part I), and our analysis will proceed along the same lines.

We start by describing the instantons and the anti-instantons of this model. Applying the "Bogomolny trick" as in the quantum mechanical model of (see Part I, Section 2.3), we may rewrite the bosonic part of the action as

$$\lambda\int_\Sigma d^2z\, |\partial_{\overline{z}}X|^2 + \lambda\int_\Sigma \Phi^*(\omega_K) = \lambda\int_\Sigma g_{a\overline{b}}\partial_{\overline{z}}X^a\partial_z X^{\overline{b}} d^2z + \lambda\int_\Sigma \frac{i}{2}g_{a\overline{b}}dX^a\wedge dX^{\overline{b}},$$

or as

$$\lambda\int_\Sigma d^2z\, |\partial_z X|^2 - \lambda\int_\Sigma \Phi^*(\omega_K) = \lambda\int_\Sigma g_{a\overline{b}}\partial_z X^a\partial_{\overline{z}} X^{\overline{b}} d^2z - \lambda\int_\Sigma \frac{i}{2}g_{a\overline{b}}dX^a\wedge dX^{\overline{b}},$$

(here ω_K is the Kähler form given by formula (2.1)). Thus, we see that in a given topological sector, where $\int_\Sigma \Phi^*\omega_K$ is fixed, the absolute minima of the action are given by the holomorphic maps (satisfying $\partial_{\overline{z}}X^a = 0$) or the anti-holomorphic maps (satisfying $\partial_z X^a = 0$). These are the instantons and the anti-instantons of the sigma models, just like the gradient trajectories and the anti-gradient trajectories were the instantons and the anti-instantons in quantum mechanics. Similarly, to their quantum mechanical counterparts, the contributions of the instantons and anti-instantons to the path integral are suppressed by the exponential factor[1] $e^{-\lambda\left|\int_\Sigma \Phi^*(\omega_K)\right|}$.

The next step is to add to the action (2.2) the term

$$\int_\Sigma \Phi^*(B) = \int_\Sigma B_{a\overline{b}}dX^a\wedge dX^{\overline{b}},$$

where

$$B = B_{a\overline{b}}dX^a\wedge dX^{\overline{b}}$$

is a closed two-form on X, called the *B-field*. Our goal is to enhance the effect of the instantons and further suppress the anti-instantons. To this end we choose the B-field to be of the form

$$B = -\lambda\omega_K + \tau,$$

where

$$\tau = \tau_{a\overline{b}}dX^a\wedge dX^{\overline{b}}$$

[1]For holomorphic maps $\int_\Sigma \Phi^*(\omega_K) \geq 0$ and for anti-holomorphic maps $\int_\Sigma \Phi^*(\omega_K) \leq 0$.

is another closed two-form.[2] Note that it is analogous to the term $(-\lambda - i\tau)\int_I df$ that we added to the Lagrangian of the quantum mechanical model (see Sections 2.3–2.4 of Part I). The resulting action reads

$$\int_\Sigma d^2z \left(\lambda|\partial_{\overline{z}}X|^2 + i\pi_a D_{\overline{z}}\psi^a + i\pi_{\overline{a}}D_z\psi^{\overline{a}} + \frac{1}{2}\lambda^{-1}R^{a\overline{b}}{}_{c\overline{d}}\pi_a\pi_{\overline{b}}\psi^c\psi^{\overline{d}}\right) + \int_\Sigma \tau_{a\overline{b}}dX^a \wedge dX^{\overline{b}}. \tag{2.3}$$

The holomorphic maps (i.e., the instantons of our model) are the absolute minima of this action, and they are no longer suppressed in the path integral, whereas the anti-holomorphic maps (anti-instantons) are now doubly suppressed by the exponential factor $e^{-2\lambda\left|\int_\Sigma \Phi^*(\omega_K)\right|}$.

2.2. **Infinite radius limit.** We now wish to take the limit $\lambda \to \infty$, in which the metric on X becomes very large (hence the name "infinite radius limit"). In this limit the instantons survive, but the anti-instantons disappear. In terms of the coupling constants

$$\tau_{a\overline{b}} = B_{a\overline{b}} + \frac{i}{2}\lambda g_{a\overline{b}}, \qquad \overline{\tau}_{a\overline{b}} = B_{a\overline{b}} - \frac{i}{2}\lambda g_{a\overline{b}}$$

it is the limit in which $\overline{\tau}_{a\overline{b}} \to -i\infty$, but the $\tau_{a\overline{b}}$'s are kept finite.

As in quantum mechanics, in order to implement this limit we first pass to the first order Langrangian (see [52, 5]):

$$\int_\Sigma \Big(-ip_a\partial_{\overline{z}}X^a - ip_{\overline{a}}\partial_z\overline{X^a} + \lambda^{-1}g^{a\overline{b}}p_a p_{\overline{b}} + i\pi_a D_{\overline{z}}\psi^a + i\pi_{\overline{a}}D_z\psi^{\overline{a}} + \frac{1}{2}\lambda^{-1}R^{a\overline{b}}{}_{c\overline{d}}\pi_a\pi_{\overline{b}}\psi^c\psi^{\overline{d}}\Big)\, d^2z + \int_\Sigma \tau_{a\overline{b}}dX^a \wedge d\overline{X^b} \tag{2.4}$$

(compare with the action (2.12) in Part I). For finite values of λ, by eliminating the momenta variables $p_a, p_{\overline{a}}$ using the equations of motion, we obtain precisely the action (2.3). Therefore the two actions are equivalent for finite values of λ. But now we can take the limit $\lambda \to \infty$ in the new action. The resulting action is

$$-i\int_\Sigma \left(p_a\partial_{\overline{z}}X^a + p_{\overline{a}}\partial_z X^{\overline{a}} - \pi_a D_{\overline{z}}\psi^a - \pi_{\overline{a}}D_z\psi^{\overline{a}}\right) d^2z + \int_\Sigma \tau_{a\overline{b}}dX^a \wedge dX^{\overline{b}} \tag{2.5}$$

(compare with formula (2.13) of Part I).

As in the quantum mechanical model, we may redefine the momentum variables as follows:

$$p_a \mapsto p'_a = p_a + \Gamma^b_{ac}\pi_b\psi^c, \qquad p_{\overline{a}} \mapsto p'_{\overline{a}} = p_{\overline{a}} + \Gamma^{\overline{b}}_{\overline{a}\overline{c}}\pi_{\overline{b}}\psi^{\overline{c}}. \tag{2.6}$$

Then the action (2.5) becomes

$$-i\int_\Sigma \left(p'_a\partial_{\overline{z}}X^a + p'_{\overline{a}}\partial_z X^{\overline{a}} - \pi_a\partial_{\overline{z}}\psi^a - \pi_{\overline{a}}\partial_z\psi^{\overline{a}}\right) d^2z + \int_\Sigma \tau_{a\overline{b}}dX^a \wedge dX^{\overline{b}} \tag{2.7}$$

[2] The action is CPT invariant if $\overline{B} = -B$, i.e., if B is purely imaginary. However, like in quantum mechanics, we break CPT invariance by considering a complex B-field with the real part equal to $-\lambda\omega_K$.

(compare with formula (2.16) in Part I). However, the transformation formulas for the new momenta variables under the changes of coordinates now become more complicated (see Section 4.1). They are similar to the quantum mechanical formulas (2.17) in Part I.

2.3. **Sigma model in the infinite radius limit as a CFT.** The theory with the action (2.7) appears to be a conformal field theory (CFT). Indeed, up to a topological term $\int \Phi^*\tau$, the action is a curved version of the beta-gamma-bc system. In particular, the stress-energy tensor is given by the formula:

$$T = i(p'_a \partial_z X^a + \pi_a \partial_z \psi^a) \tag{2.8}$$

which is invariant under the coordinate transformations. In the perturbative limit, where we consider the sigma model with the target space a single coordinate patch $U \approx \mathbb{C}^n$, the stress-tensor (2.8) forms a Virasoro algebra with the vanishing central charge. At the same time, the following OPEs are obtained from the action (2.7):

$$\begin{aligned} &p_a(z)X^b(w) \sim -\frac{i\delta_a^b}{z-w}, \qquad \pi_a(z)\psi^b(w) \sim -\frac{i\delta_a^b}{z-w},\\ &p_{\overline{a}}(\overline{z})X^{\overline{b}}(\overline{w}) \sim -\frac{i\delta_{\overline{a}}^{\overline{b}}}{\overline{z}-\overline{w}}, \qquad \pi_{\overline{a}}(\overline{z})\psi^{\overline{b}}(\overline{w}) \sim -\frac{i\delta_{\overline{a}}^{\overline{b}}}{\overline{z}-\overline{w}}. \end{aligned} \tag{2.9}$$

However, including non-perturbative effects makes the theory a logarithmic CFT, and induces interesting corrections to the operator product expansions derived from (2.9).

2.4. **Sigma model as quantum mechanics on the loop space.** We now interpret the supersymmetric sigma model in the infinite radius limit as a quantum mechanical model on the loop space LX. Let Σ be a cylinder $I \times \mathbb{S}^1$, where I is an interval. We will view a map $(I \times \mathbb{S}^1) \to X$ as a map $I \to LX$, where $LX = \mathrm{Maps}(\mathbb{S}^1, X)$ is the loop space of X.

We introduce coordinates t on I and σ on $\mathbb{S}^1$, so that σ is periodic with the period 2π. Then the holomorphic coordinate on Σ is $z = t + i\sigma$. Using local holomorphic and anti-holomorphic coordinates X^a and $X^{\overline{a}}$ on an open subset $U \subset X$, we may represent a map $\mathbb{S}^1 \to U$ by the Fourier series

$$X^a(\sigma) = \sum_{n\in\mathbb{Z}} X^a_n e^{-in\sigma}, \qquad X^{\overline{a}}(\sigma) = \sum_{n\in\mathbb{Z}} X^{\overline{a}}_n e^{in\sigma}. \tag{2.10}$$

Therefore we may use $X^a_n, X^{\overline{a}}_n$ as local holomorphic and anti-holomorphic coordinates on the loop space LX. Let $p'_{a,n}$ and $p'_{\overline{a},n}$ be the Fourier coefficients of the corresponding expansions of the momenta variables p'_a and $p'_{\overline{a}}$, dual to X^a_{-n} and $X^{\overline{a}}_{-n}$, respectively. Note that we have

$$\overline{X^a_n} = X^{\overline{a}}_n, \qquad \overline{p'_{a,n}} = p'_{\overline{a},n}.$$

Similarly, expanding in Fourier series, we obtain the fermionic variables

$$\psi^a_n, \psi^{\overline{a}}_n, \pi_{a,n}, \pi_{\overline{a},n}, \qquad n \in \mathbb{Z}.$$

In terms of these variables, the action (2.7) on the cylinder may be rewritten as follows (we ignore for a moment the last term $\int_\Sigma \Phi^*(\tau)$):

$$-i\int_I \left(\sum_{n\in\mathbb{Z}} p'_{a,n}\left(\frac{dX^a_n}{dt}+nX^a_n\right)+\sum_{n\in\mathbb{Z}} p'_{\overline{a},n}\left(\frac{dX^{\overline{a}}_n}{dt}+nX^{\overline{a}}_n\right)\right.$$
$$\left.-\sum_{n\in\mathbb{Z}}\pi_{a,n}\left(\frac{d\psi^a_n}{dt}+n\psi^a_n\right)-\sum_{n\in\mathbb{Z}}\pi_{\overline{a},n}\left(\frac{d\psi^{\overline{a}}_n}{dt}+n\psi^{\overline{a}}_n\right)\right)dt$$

(up to the inessential factor of π). We recognize in this formula the action (see formula (1.1) of Part I)

$$S=-i\int_I\left(p'_A\left(\frac{dX^A}{dt}-v^A\right)+p'_{\overline{A}}\left(\frac{dX^{\overline{A}}}{dt}-\overline{v^A}\right)+\right.$$
$$\left.-\pi_A\left(\frac{d\psi^A}{dt}-\frac{\partial v^A}{\partial X^B}\psi^B\right)-\pi_{\overline{A}}\left(\frac{d\psi^{\overline{A}}}{dt}-\frac{\partial\overline{v^A}}{\partial X^{\overline{B}}}\psi^{\overline{B}}\right)\right)dt-i\tau\int df, \tag{2.11}$$

of the quantum mechanical model associated to the loop space LX and the vector field

$$v=-\sum_{n\in\mathbb{Z}}nX^a_n\frac{\partial}{\partial X^a_n}-\sum_{n\in\mathbb{Z}}nX^{\overline{a}}_n\frac{\partial}{\partial X^{\overline{a}}_n} \tag{2.12}$$

In other words, we have written

$$p'_a\partial_{\overline{z}}X^a=\frac{1}{2}p'_a(\partial_tX^a+i\partial_\sigma X^a),\qquad p'_{\overline{a}}\partial_zX^{\overline{a}}=\frac{1}{2}p'_a(\partial_tX^{\overline{a}}-i\partial_\sigma X^{\overline{a}}).$$

Thus, the above vector field v (which is real) corresponds to the vector field $i\partial_\sigma$ when acting on holomorphic coordinates and to $-i\partial_\sigma$ when acting on anti-holomorphic coordinates. Note that ∂_σ is the natural vector field on LX corresponding to infinitesimal rotation of the loop. Therefore this vector field comes from the natural $U(1)$-action on LX corresponding to the loop rotation. Since X is a complex manifold, we may complexify this action to a $\mathbb{C}^\times$-action. The vector field v then comes from the action of the subgroup $\mathbb{R}^\times\subset\mathbb{C}^\times$. The gradient flow is therefore given by the equations

$$\partial_tX^a+i\partial_\sigma X^a=\partial_{\overline{z}}X^a=0, \tag{2.13}$$

$$\partial_tX^{\overline{a}}-i\partial_\sigma X^{\overline{a}}=\partial_zX^{\overline{a}}=0. \tag{2.14}$$

These are the Cauchy–Riemann equations for the map $\Phi:\Sigma=I\times\mathbb{S}^1\to X$, and so the gradient trajectories are the holomorphic maps with given boundary conditions.

Thus, we have interpreted the two-dimensional sigma model on the cylinder as a quantum mechanical model of the type that we have studied previously. However, in order to apply our results we need to represent the vector field v given by formula (2.12) as the *gradient vector field* of a function f on LX: $v=\nabla f$.[3] Here we consider a natural metric on LX induced by the metric g on X (and the measure $d\sigma$ on $\mathbb{S}^1$). A

[3]Most importantly, our construction involves multiplication of the wave-functions by $e^{\lambda f}$, so we do need to have f.

tangent vector at a point $\gamma : \mathbb{S}^1 \to X$ is a section of the vector bundle $\gamma^*(TX)$. Given two such sections η_1, η_2, we define their scalar product as

$$\int_{\mathbb{S}^1} \langle \eta_1, \eta_2 \rangle_g d\sigma.$$

In order to construct the function f, we take the contraction of the above vector field v with this metric on LX. This gives us a one-form β on LX, and if $v = \nabla f$, then $\beta = df$. If $H_1(LX, \mathbb{Q}) = 0$, then such f would exist. When we considered above quantum mechanical models on a Kähler manifold X we had assumed that the set of zeros of the vector field v on X was non-empty. It is known [22] that in this case $H_1(X, \mathbb{Q}) = 0$ (actually, even $H_1(X, \mathbb{Z}) = 0$), and so the function f exists.

However, for a Kähler manifold X, the group $H_1(LX, \mathbb{Q})$ is always non-trivial. The best we can do in this case is to construct a *multi-valued* Morse function, also known as *Morse–Novikov function* [45], f on LX whose gradient is the vector field v. This function becomes single valued when pulled back to a covering $\widetilde{LX}$ of LX. The critical points of this function are the preimages of constant loops in $\widetilde{LX}$, so this is strictly speaking not a Morse–Novikov function, but a *Morse–Bott–Novikov function*. Another important phenomenon is that at the critical points the Hessian of the function f has infinitely many positive and negative eigenvalues, so only the relative index of two critical points is well-defined. This indicates that the corresponding Morse–Novikov complex computes "semi-infinite" cohomology of LX. These properties require that we make some adjustments in our construction.

Before explaining how the theory changes when we take into account all of these phenomena, we recall how to construct this function in our case. This construction goes back to the work of Floer [21] (for the connection to two-dimensional sigma models, see, e.g., [48]). Let us assume for simplicity that X is simply-connected, and so LX is connected. Then any loop $\gamma : \mathbb{S}^1 \to X$ can be contracted to a point, and hence γ may be extended to a map $\widetilde{\gamma} : D \to X$, where D is a two-dimensional disc with the boundary $\mathbb{S}^1$. Now set

$$f(\widetilde{\gamma}) = \int_D \widetilde{\gamma}^*(\omega_K). \tag{2.15}$$

How does f depend on the choice of $\widetilde{\gamma}$ extending a fixed loop γ? Let $\widetilde{\gamma}'$ be another such extension. Then gluing $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ together along the boundary we obtain a map $\Phi_{\widetilde{\gamma},\widetilde{\gamma}'} : \mathbb{S}^2 \to X$. It is clear from the definition that

$$f(\widetilde{\gamma}) = f(\widetilde{\gamma}') + \int_{\mathbb{S}^2} \Phi^*_{\widetilde{\gamma},\widetilde{\gamma}'}(\omega_K).$$

Thus, as a function of $\gamma \in LX$, the function f is defined up to the addition of an integral of the Kähler form ω_K over cycles in $H_2(X, \mathbb{Z})$ represented by two-dimensional spheres. Let $\widetilde{LX}$ be the space of equivalence classes of maps $\widetilde{\gamma} : D \to X$ modulo the following equivalence relation: we say that $\widetilde{\gamma} \sim \widetilde{\gamma}'$ if $\widetilde{\gamma}|_{\partial D} = \widetilde{\gamma}'|_{\partial D}$ and $\widetilde{\gamma}$ is homotopically equivalent to $\widetilde{\gamma}'$ in the space of all maps $D \to X$ which coincide with $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ on the boundary circle ∂D. We have the obvious map $\widetilde{LX} \to LX$, which realizes $\widetilde{LX}$ as a covering of LX. The group of deck transformations is naturally identified with

$H_2(X, \mathbb{Z})$. Under our assumption that $\pi_1(X)$ is trivial, we have $H_2(X, \mathbb{Z}) = \pi_2(X) = \pi_1(LX)$, and $\widetilde{LX}$ is the universal cover of LX.

The function f defined by formula (2.15) is a single valued function on $\widetilde{LX}$. It is easy to see that its differential *df* is the pull-back of the one-form β obtained by contracting the metric on LX with the vector field v. Indeed, consider the evaluation map $\mathrm{ev} : LX \times \mathbb{S}^1 \to X$ and the projection $\mathrm{pr} : LX \times \mathbb{S}^1 \to LX$. Then *df* is the pull-back to $\widetilde{LX}$ of the one-form on LX which is the push-forward of $\mathrm{ev}^*(\omega_K)$ with respect to pr. The value of this one-form on LX on a tangent vector $\eta \in T_\gamma(LX) = \Gamma(\mathbb{S}^1, \gamma^*(TX))$ is the integral

$$\int_{\mathbb{S}^1} \langle \gamma_* \eta, \omega_K \rangle .$$

Using formula (2.1), we find that in terms of the local holomorphic coordinates X_n^a on the loop space this one-form is equal to

$$-\sum_{n \in \mathbb{Z}} n g_{a\bar{b}} (X_n^a dX_n^{\bar{b}} + n X_n^{\bar{b}} dX_n^a),$$

which is precisely the contraction of the metric on LX and v. Thus, the vector field v is indeed the gradient of the (multi-valued) function f.

Now we can identify the action (2.7) with the quantum mechanical action (2.11) on the loop space LX equipped with the multi-valued function f given by formula (2.15). In the next section we will discuss how the multi-valuedness of this function changes the structure of the model.

3. Quantum mechanics with Morse–Novikov functions

In this section X denotes a compact finite-dimensional manifold, which is the target space of a quantum mechanical model. We will focus on the effects in this model caused by non-simply connectedness of X. For compact Kähler manifolds considered in Part I the (abelianized) fundamental group is always trivial. Therefore we will consider here the more general case of a smooth real Riemannian manifold X.

The quantum mechanics on a non-simply connected manifold has some interesting new features. They stem from the fact that the space of paths on X is disconnected. Thus the path integral, computing the amplitudes of propagation from one point on X to another involves a sum over the connected components, which may be labeled by the fundamental group $\pi_1(X)$. We will discuss how the instanton calculus developed in Part I should be adjusted in this more general setting, with the aim of applying the results to the case of the loop spaces that is relevant to the two-dimensional sigma models (where the role of X is played by LX, the loop space).

3.1. **Path integral analysis on non-simply connected manifolds.** We study supersymmetric quantum mechanics on a smooth real compact Riemannian manifold X, with a non-trivial fundamental group $\pi_1(X) \neq 0$. In designing the path integral describing the kernel $G(x_i, x_f)$ of the evolution operator of getting from a point $x_i \in X$ to a point $x_f \in X$ in this situation, a physicist faces a choice: either to fix a homotopy class of paths connecting the points x_i and x_f, or to sum over all homotopy classes with some weights. It is a well-known fact (see, e.g., [13]) that one should sum over all homotopy classes in order to get a unitary theory. However, there are different ways of summing over them, which correspond to different sectors of the theory.

In order to define these sectors, it is convenient to introduce a closed one-form $b \in Z^1(X)$ and change the action of the theory as follows:

$$S \longrightarrow S_b = S + 2\pi i \int x^* b \tag{3.1}$$

The modification (3.1) does not modify the equations of motion. In fact, it only knows about the homotopy class of a path. Moreover, up to boundary terms, it only depends on the class $[b] \in Z^1(X)/Z^1_{\mathbb{Z}}(X)$. The space of states becomes a direct sum, (or rather a direct integral

$$\int_{[b]\in Z^1(X)/Z^1_{\mathbb{Z}}(X)} \mathcal{H}_{[b]} \tag{3.2}$$

of orthogonal spaces, known as *ϑ-vacuum superselection sectors*). Each of these spaces is isomorphic to the standard space of L^2 differential forms on X, but the action of the Hamiltonian on $\mathcal{H}_b$ depends on b:

$$H_b = \{\mathcal{Q}_b, \mathcal{Q}_b^*\},$$

where $\mathcal{Q}_b = \mathcal{Q}_0 + 2\pi i b\wedge$ and $\mathcal{Q}_b^*$ is its adjoint. We will analyze this in more detail below.

Note that changing the one-form b by the *real* exact form $b \to b + d\alpha$, does not change the physics of the problem, as this change can be compensated by the unitary transformation

$$\Psi \longrightarrow e^{-2\pi i\alpha} \cdot \Psi \tag{3.3}$$

of the wave-functions. In particular, the spectrum of the Hamiltonian does not change. This is why in (3.2) we have only the finite-dimensional space

$$Z^1(X)/Z^1_{\mathbb{Z}}(X) = H^1(X,\mathbb{R})/H^1(X,\mathbb{Z}) \tag{3.4}$$

which labels different ϑ-sectors.

Now we shall generalize the standard discussion and allow *complex-valued* closed one-forms $b \in Z^1_{\mathbb{C}}(X)$. We will then discuss the corresponding modification of the space of states.

Let us recall the set-up of Part I, Section 2.3. We start with the Lagrangian of a quantum mechanical model on X given by formula (2.6) of Part I:

$$\begin{aligned} S = \int_I \Bigg(&\frac{1}{2}\lambda g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + \frac{1}{2}\lambda g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} + \\ &i\pi_\mu D_t\psi^\mu - i g^{\mu\nu}\frac{D^2 f}{Dx^\nu Dx^\alpha}\pi_\mu\psi^\alpha + \frac{1}{2}\lambda^{-1}R^{\mu\nu}_{\alpha\beta}\pi_\mu\pi_\nu\psi^\alpha\psi^\beta\Bigg)\, dt. \end{aligned} \tag{3.5}$$

Next, we modify it by adding the term $-i\vartheta\int_I df$, where f is a Morse function. We write $\vartheta = \tau - i\lambda$, where λ is the (real) parameter in front of the metric in the action (3.5). Then we wish to take the limit $\lambda \to +\infty$ with τ finite and fixed. We observe (see Part I, Section 3.2) that for finite λ the correlation functions of this model are equivalent to the correlation functions of the model described by the Lagrangian (3.5) with the additional term $-i\tau\int_I df$. However, there is a price to pay: we need to rescale the wave-functions of the "in" states by $e^{\lambda f}$, rescale the wave-functions of the "out" states by $e^{-\lambda f}$ and conjugate the observables by $e^{\lambda f}$. By making these transformations we relate the correlation functions of the theory with the term $-i\vartheta \int_I df$, where ϑ has a large imaginary part, to those of the theory with the term $-i\tau \int_I df$, where τ is finite.

Let us revisit our formulas from Part I in the case of simply-connected X. The supercharges and the Hamiltonian of the theory with the action (3.5), acting on the space of L^2 differential forms on X, read as follows (see formulas (2.4)–(2.4) of Part I)

$$\begin{aligned} \mathcal{Q} &= d_\lambda = e^{-\lambda f} d e^{\lambda f} = d + \lambda df \wedge \\ \mathcal{Q}^* &= (d_\lambda)^* = \frac{1}{\lambda} e^{\lambda f} d^* e^{-\lambda f} = \frac{1}{\lambda} d^* + \iota_v, \\ H &= \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^*\} = \frac{1}{2}\left(-\lambda^{-1}\Delta + \lambda \|df\|^2 + (\mathcal{L}_v + \mathcal{L}_v^*)\right), \end{aligned}$$

where $v = \nabla f$ (recall that for a vector field ξ we denote by $\mathcal{L}_\xi$ its Lie derivative acting on differential forms).

Next, consider the model with the action (3.5) plus the term $-\lambda \int_I df$. Then the supercharges and the Hamiltonian take the form (see formulas (3.4)–(3.6) of Part I):

$$Q_\lambda = \widetilde{\mathcal{Q}} = e^{\lambda f} \mathcal{Q} e^{-\lambda f} = d, \tag{3.6}$$

$$Q_\lambda^* = \widetilde{\mathcal{Q}}^* = e^{\lambda f} \mathcal{Q}^* e^{-\lambda f} = 2\iota_v + \frac{1}{\lambda} d^*, \tag{3.7}$$

$$H_\lambda = e^{\lambda f} H e^{-\lambda f} = \frac{1}{2}\{Q_\lambda, Q_\lambda^*\} = \mathcal{L}_v - \frac{1}{2\lambda}\Delta. \tag{3.8}$$

Now let us also add the term $-i\tau \int_I df$ to the previous action (so the net result is the action (3.5) plus the term $-i\vartheta \int_I df$). The corresponding supercharges and Hamiltonian read

$$\begin{aligned} Q_{\lambda,\tau} &= e^{i\tau f} Q_\lambda e^{-i\tau f} = d - i\tau df \wedge, \\ Q_{\lambda,\tau}^* &= e^{i\tau f} Q_\lambda^* e^{-i\tau f} = 2\iota_v + \frac{1}{\lambda}(d^* + i\tau i_v), \end{aligned}$$

$$H_{\lambda,\tau} = e^{i\tau f} H_\lambda e^{-i\tau f} = \mathcal{L}_v - i\tau \ \|v\|^2 + \frac{1}{2\lambda}\left(-\Delta + \tau\left(\mathcal{L}_v + \mathcal{L}_v^*\right) + i\tau^2 \ \|v\|^2\right). \tag{3.9}$$

In the case of simply-connected X, the Hamiltonians (3.9) and (3.8) are related by conjugation with $e^{i\tau f}$ (and likewise for the supercharges). Therefore in the case of simply-connected manifold X, this finite τ-term is not important (see the discussion at the end of of Section 2.4 of Part I). That is why in most of the discussion of Part I we had dropped this term. But on non-simply connected manifolds this term is important as it corresponds to the choice of the "ϑ-vacuum" sector. Therefore we need to include it in our formulas in this case.

3.2. **Maximal abelian cover.** In all of the above formulas the Hamiltonian and the supercharges depend only on the vector field v. If $H_1(X, \mathbb{Z}) = 0$ (in particular, for simply-connected X) there exists a function f on X such that $v = \nabla f$, and we have used this function in the above formulas. Let us suppose now that $H_1(X, \mathbb{Z}) \neq 0$. Then v may still be written as ∇f, but f may be multi-valued, i.e., defined on the

$H_1(X,\mathbb{Z})$-cover $\widetilde{X}$ of X. This is the maximal abelian cover of X.[4] The points of $\widetilde{X}$ may be described as pairs $(x,[I])$, where $x \in X$ and $[I]$ is an equivalence class of a path I connecting x with another, fixed, reference point $x_0 \in X$. The equivalence relation identifies two paths I and I' if the image of their difference in $H_1(X,\mathbb{Z})$ is equal to 0. The group $H_1(X,\mathbb{Z})$ naturally acts on $\widetilde{X}$.

Let b_v be the closed one-form obtained by contracting v and the metric g on X (thus, $b_v = df$ in the simply-connected case). Then the sought-after function f on $\widetilde{X}$ is constructed as follows: its value at the point $(x,[I]) \in \widetilde{X}$ is equal to $\int_I b_v$. Then we have $v = \nabla f$, in the sense that the vector field ∇f on $\widetilde{X}$ is $H_1(X,\mathbb{Z})$-invariant and corresponds to the vector field v on X. Note also that df is the pull-back of b_v to $\widetilde{X}$. In what follows, by a slight abuse of notation, we will sometimes write df for b_v.

Let us consider the case when f is a function on $\widetilde{X}$ that is not $H_1(X,\mathbb{Z})$-invariant, and hence gives rise to a multi-valued function on X. In this case conjugation by $e^{i\tau f}$ is problematic, since it maps differential forms on X (i.e., $H_1(X,\mathbb{Z})$-invariant forms on $\widetilde{X}$) to differential forms on $\widetilde{X}$, which are τ-equivariant. These are the forms $\omega \in \Omega^\bullet(\widetilde{X})$ such that

$$\mu^*\omega = \exp\left(i\tau \int_\mu df\right)\omega, \qquad \mu \in H_1(X,\mathbb{Z}). \tag{3.10}$$

We denote the space of such forms by $\Omega^\bullet_\tau(X)$.

Thus, we find that there are two possible pictures for the Hamiltonian formalism with non-zero τ (at finite λ for now): we may either consider the space $\Omega^\bullet(X)$ with the Hamiltonian $H_{\lambda,\tau}$ (which tends to $\mathcal{L}_v + i\tau\|v\|^2$ in the limit $\lambda \to \infty$), or the space $\Omega^\bullet_\tau(X)$ with the Hamiltonian H_λ (which tends to $\mathcal{L}_v$ in the limit $\lambda \to \infty$). Given an eigenstate Ψ of $H_{\lambda,\tau}$ in $\Omega^\bullet(X)$, we obtain an eigenstate $e^{-i\tau f}\Psi$ of H_λ in $\Omega^\bullet_\tau(X)$.

However, as in the simply-connected case (see Part I, Section 3), in the limit $\lambda \to \infty$ both spaces decouple into spaces of "in" and "out" states and undergo a violent transformation which results in certain spaces of delta-forms. In the first picture, those should be defined on X, and in the second – on $\widetilde{X}$, with the two pictures again related by the operator of multiplication by $e^{i\tau f}$.

3.3. **Ground states.** The first question to consider is the structure of the ground states. Here we encounter the following puzzle. In the case of simply-connected X there were two obvious ground states (for all finite values of λ): the zero form 1 and the top form $e^{2\lambda f}\mathrm{vol}_g$ (see Part I, Section 3.5). But now the analogue of the "in" ground state 1 in $\Omega^\bullet(X)$ is the eigenfunction $e^{i\tau f}$ of $H_{\lambda,\tau}$. However, this function is not periodic, and hence it does not belong to the space of states (the corresponding function in $\Omega^\bullet_\tau(X)$ would be 1, which is not τ-periodic, and hence also not allowed). So, naively, we could conclude that the ground state which is of degree 0 as a differential form is absent in the spectrum of the theory. A similar analysis shows that the top degree form would be absent as well.

[4]We will assume that $H_1(X,\mathbb{Z})$ has no torsion; otherwise, we have to take the quotient of $H_1(X,\mathbb{Z})$ by its torsion subgroup.

However, it may be easily shown by the path integral analysis that these non-periodic wave-functions are in fact the correct ground states of the $\lambda = \infty$ theory.

The point is that we should view the function $e^{-i\tau f}$ on X not as a function, but as a *generalized function* (or *distribution*) on X. As we explained in Part I, Section 4, including such generalized functions into the spectrum is inevitable for the theory in the limit $\lambda \to \infty$. Now we see that for non-simply laced manifolds X one has to allow distributions with mild singularities (such as the jump of f across some hypersurface) even for finite values of λ in order to have a consistent Hamiltonian formulation of the theory. This type of singularity is similar, and actually milder, than the delta-form type singularities we have encountered in the study of the excited states in the limit $\lambda \to \infty$ in case of simply-connected X, and which we will encounter below for non-simple connected X as well. It might lead, in the worst case, to the Jordan block form of the Hamiltonian (3.9) at $\lambda = \infty$.

Since it is not *a priori* obvious whether to allow such jumps of the eigenfunctions or not, it is instructive to analyze this phenomenon in detail in the simplest possible case, that of a circle $\mathbb{S}^1$. This will be done in the next section and will help us to confirm that sometimes such discontinuous periodic functions are indeed bona fide eigenfunctions of the Hamiltonian.

3.4. **Example:** $X = \mathbb{S}^1$**.** As an example, take $X = \mathbb{S}^1$, with the coordinate $x \sim x + 2\pi$, and the Morse–Novikov function

$$f(x) = \mu x + \cos(x). \tag{3.11}$$

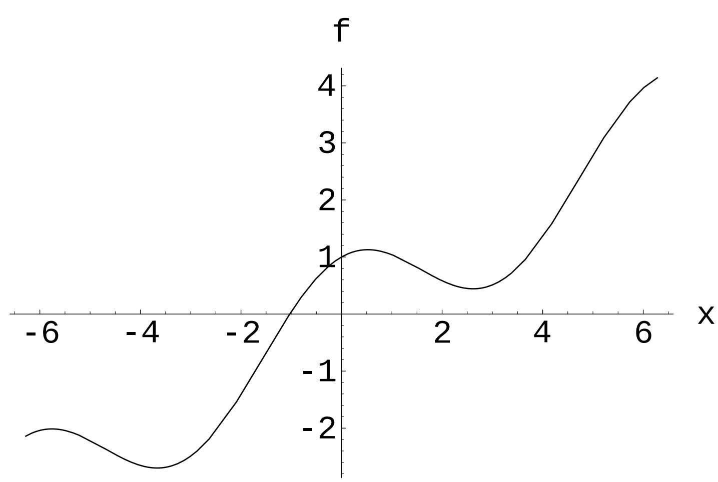

Figure 1. Morse–Novikov function on $\mathbb{S}^1$

3.4.1. *Act I.* We start with the model described by the action (3.5) with the additional term $(-i\tau - \lambda)\int_I df$ in the limit $\lambda \to \infty$. We rewrite the corresponding action in the first order formalism, following Part I, Section 2.3, like this:

$$S = -i \int_I p(\dot{x} - V(x))dt - i \int_I V(x)\dot{x}dt,$$

where our vector field is $v = V(x)\partial_x$, with

$$V(x) = \mu - \sin(x).$$

Suppose first that $0 < |\mu| < 1$. Then it has two critical points: x_+ and x_-, $\sin x_\pm = \mu$, with $\cos x_+ < 0$, $\cos x_- > 0$. Let us choose the branch $f_+(x)$ of $f(x)$ such that it is smooth at x_+ and has a jump at x_-:

$$f(x_- - 0) = f(x_- + 0) + 2\pi\mu.$$

Let us also define the branch $f_-(x)$, which is smooth at x_-, and jumps by $2\pi\mu$ across x_+.

The Hamiltonian at $\lambda = \infty$ is given by the first order operator:

$$H_{\infty,\tau} = V(x)\left(\partial_x - i\tau V(x)\right). \tag{3.12}$$

acting on the degree zero differential forms, i.e., functions on $\mathbb{S}^1$. It annihilates the ground state wave-function:

$$\Psi^{(0)\mathrm{in}}(x) = e^{-i\tau f_+(x)}.$$

Indeed, the $\delta(x - x_-)$ term in $H_{\infty,\tau}\Psi^{(0)\mathrm{in}}$, due to the jump of $f(x)$, is multiplied by $V(x_-) = 0$. The second ground state, a one-form, is given by the delta-form supported at the attractive critical point x_-:

$$\Psi^{(1)\mathrm{in}}(x) = e^{i\tau f_-(x_-)}\ \delta(x - x_-)dx.$$

Analogously, the "out" states are built using the branch $f_-(x)$:

$$\Psi^{(0)\mathrm{out}}(x) = e^{i\tau f_-(x)} \tag{3.13}$$

$$\Psi^{(1)\mathrm{out}}(x) = e^{i\tau f_+(x_+)}\delta(x - x_+)dx. \tag{3.14}$$

This is in complete analogy with the analysis on $\mathbb{CP}^1$ presented in Part I, Section 3.5.

Now let us briefly discuss the excited states. By analogy with the analysis of Part I, Section 3.7, we have two series of (generalized) "in" eigenstates of the operator $H_{\infty,\tau}$ on the zero degree forms corresponding to the two critical points (and similarly for one-forms). The first of them is found to be proportional to

$$\Psi^{(0)\mathrm{in}}_{n,-} = \frac{1}{(n-1)!}\partial_w^{n-1}\left(e^{-i\tau f_-(x)}\delta(w)\right), \qquad n > 0,$$

where w is a new coordinate on $\mathbb{S}^1$ near $x = x_-$, such that

$$w = e^{\kappa_- y}, \quad y = \int_0^x \frac{dx}{V(x)}, \tag{3.15}$$

and the constant κ_- is chosen so that the map $x \mapsto w$ is a local diffeomorphism near $x = x_-$:

$$\kappa_- = V'(x_-) \tag{3.16}$$

In our example $\kappa_- = \sqrt{1-\mu^2}$.

The corresponding eigenvalue of the Hamiltonian (3.12) is given by:

$$E_{n,-} = \sqrt{1-\mu^2}\ n\ , \quad n > 0 \tag{3.17}$$

(the zero energy state is given by the wave-function $\Psi^{(0)\mathrm{in}}$).

The second series of excited states on the the zero-forms are generalized eigenstates of the Hamiltonian (3.12) given by the formula

$$\Psi^{(0)\mathrm{in}}_{n,+} = z^n e^{-i\tau f_+(x)},$$

where z is another coordinate on $\mathbb{S}^1$, which is related by the local diffeomorphism to x near x_+, where it vanishes:

$$z = e^{\kappa_+ y}, \qquad \kappa_+ = V'(x_+). \tag{3.18}$$

In our example $\kappa_+ = -\kappa_-$ and

$$z = \frac{1}{w}.$$

The Hamiltonian has Jordan blocks mixing the states $\Psi^{(0)\mathrm{in}}_{n,+}$ and $\Psi^{(0)\mathrm{in}}_{n+1,-}$.

There is also a similar picture for excited "in" states among one-forms, and also for the "out" states.

3.4.2. *Intermezzo.* Consider a more general Morse–Novikov function

$$f(x) = \mu x + \varphi(x)$$

with sufficiently small μ so that the vector field $V(x) = f'(x)$ has multiple zeroes x_α on $\mathbb{S}^1$:

$$\mu + \varphi'(x_\alpha) = 0.$$

The spectrum of the theory depends crucially on the "weights"

$$\kappa_\alpha = V'(x_\alpha).$$

The spectrum can be either simple or degenerate. If the weights have "resonances", as in the above example, the Hamiltonian has Jordan blocks, familiar from our analysis in Part I. But for generic weights the spectrum is simple, and hence no Jordan blocks arise. Note that we may set $\mu = 0$.

3.4.3. *Act II.* Next, we increase $|\mu|$. Once $|\mu| > 1$, the vector field $V(x)\partial_x$ has no zeroes, and hence it can be brought to the normal form

$$V(x)\partial_x = \frac{2\pi}{T}\partial_y, \tag{3.19}$$

where T is the period of revolution:

$$T = \int_0^{2\pi} \frac{dx}{V(x)}, \tag{3.20}$$

and the periodic coordinate $y \sim y + 2\pi$ is given by the integral (3.15). The period T in our case is given by:

$$T = \frac{2\pi}{\sqrt{\mu^2 - 1}}. \tag{3.21}$$

Miraculously enough, we can now find perfectly smooth eigenfunctions of the Hamiltonian $H_{\infty,\tau}$:

$$\Psi^{(0)\mathrm{in}}_n = e^{-i\tau\left(f(x) - \mu\frac{y}{T}\right) + iny}, \qquad n \in \mathbb{Z}, \tag{3.22}$$

with the eigenvalues

$$E_n = i\sqrt{\mu^2 - 1}\,(n - \tau\mu)\,. \tag{3.23}$$

Thus, we find that the spectrum is discrete and has a non-zero imaginary part.

3.4.4. *Act III.* Let us now compare our results with the standard example of the free particle on the circle, which can be solved for finite λ. This is somewhat similar to testing the $\lambda = \infty$ theory against the harmonic oscillator (see Part I, Sects. 3.3–3.4).

So, we take again $X = \mathbb{S}^1$, with the metric λdx^2, $x \sim x + 2\pi$. We consider the function $f = \mu x$. In a sense, this function corresponds to the limit $\mu \to \infty$ of the function from Section 3.4.3, where we neglect the contribution of the cosine function $\cos(x)$. The Lagrangian, with the ϑ-term, is given by:

$$S_\vartheta = \int \frac{\lambda}{2}\left((\dot{x}^2 - \mu^2) + i\vartheta\mu\dot{x}\right) dt, \tag{3.24}$$

where, as before $\vartheta = \tau - i\lambda$. The quantization of the model (3.24) is done in a standard way, leading to the eigenstates, labeled by the integer $n \in \mathbb{Z}$,

$$\Psi_n(x) = \frac{1}{\sqrt{2\pi}} \exp inx,$$

of the Hamiltonian $H_{\lambda,\tau}$ with the energy

$$E_n = \frac{(n - \vartheta\mu)^2}{2\lambda} + \frac{1}{2}\lambda\mu^2\,, \qquad n \in \mathbb{Z}. \tag{3.25}$$

For real ϑ the spectrum is real, positive, discrete, and bounded from below. Now let us rewrite (3.25) in terms of $\tau = \vartheta + i\lambda$:

$$E_{n,\lambda} = i\mu(n + \tau\mu) + \frac{(n + \tau\mu)^2}{2\lambda}.$$

Clearly, the $\lambda \to \infty$ limit of this expression with τ kept finite gives (3.23) up to the replacement

$$\mu \to \sqrt{\mu^2 - 1}\,.$$

3.4.5. *Corrections in* $\frac{1}{\lambda}, \frac{1}{\mu}$. The Hamiltonian for $f(x) = \mu x + \cos(x)$ can be obtained from the one for $f(x) = \mu x$ by perturbing it with the term $\sin(x)\partial_x$ (after the conjugation by $e^{i\tau f(x)}$). The second order perturbation theory gives:

$$E_n = (n + \tau\mu)\,(i\mu + \epsilon_n)\left(1 + \frac{2}{(2i\mu + 4\epsilon_n)^2 - \lambda^{-2}}\right)\,, \tag{3.26}$$

where $\epsilon_n = \frac{n+\tau\mu}{2\lambda}$. In the limit $\lambda \to \infty$ we get

$$E_n \sim i\mu(n + \tau\mu)(1 - \frac{1}{2\mu^2}),$$

which is the expansion of $\sqrt{\mu^2 - 1}$. If we expand (3.26) in $\frac{1}{\lambda}$, the first correction is given by:

$$E_n \sim (n + \tau\mu)\left(i\mu(1 - \frac{1}{2\mu^2}) + \epsilon_n(1 + \frac{1}{2\mu^2})\right).$$

Note that there are two ways to describe the situation here. In one approach, we deal with L^2 functions on $X = \mathbb{S}^1$, and the Hamiltonian is ϑ-dependent:

$$H_{\lambda,\tau} = \frac{1}{2\lambda}\left(-i\partial_x + (\tau - i\lambda)\mu\right)^2 + \frac{1}{2}\lambda\mu^2.$$

In the second approach, we lift the wave-functions to the covering space $\widetilde{X} = \mathbb{R}$ and perform the gauge transformation

$$\Psi(x) = e^{i\vartheta f}\widetilde{\Psi}(x), \tag{3.27}$$

which maps the Hamiltonian to

$$H_\lambda = -\frac{1}{2\lambda}\partial_x^2 + \frac{1}{2}\lambda\mu^2,$$

making it τ-independent. The price to pay for the simpler form of the Hamiltonian is the non-trivial $\pi_1(X) = \mathbb{Z}$-equivariance condition on the resulting wave-functions $\widetilde{\Psi}$:

$$\widetilde{\Psi}(x + 2\pi) = e^{-2\pi i\vartheta\mu}\widetilde{\Psi}(x), \tag{3.28}$$

which leads to the same spectrum (3.25) if the appropriate analytic conditions are imposed.

3.4.6. *Act IV.* Let us now discuss a puzzle. It is *well-known* that in the presence of the periodic potential, the energy levels split to form continuous bands [13], [1]:

$$E(\vartheta) = E_0 + Ke^{-S_0}\cos\vartheta, \tag{3.29}$$

where one assumes a generic periodic potential $U(x)$:

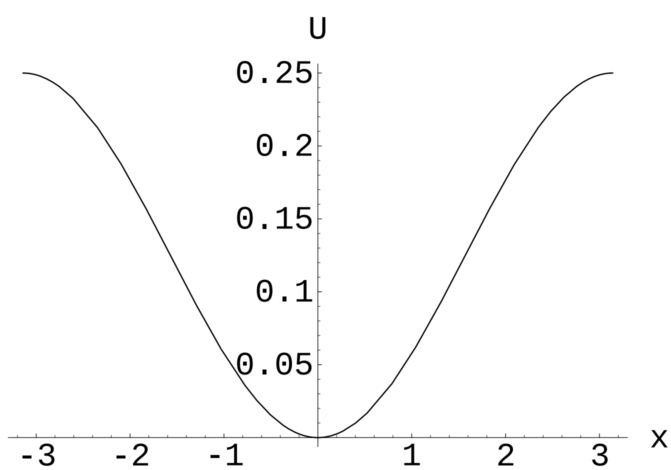

Figure 2. A typical periodic potential, i.e., a potential on $\mathbb{S}^1$.

and S_0 is the instanton action, i.e.

$$S_0 = \int \sqrt{U(x)}dx$$

Now let us write

$$U(x) = \lambda f'(x)^2$$

and take our limit: $\lambda \to \infty$, $\vartheta \to -i\infty$, τ finite. The spectrum (3.29) becomes complex:

$$E(\tau) \sim E_0 + K e^{-\tau} \tag{3.30}$$

This result should surprise us, since we are not in the situation of act III (see Section 3.4.4) where the function $f(x)$ had no critical points and the spectrum of Hamiltonian turned out to be imaginary. In the case at hands the function $f(x)$ clearly has the critical point(s) (which correspond(s) to the minima of $U(x)$). What is going on here?

The resolution of the paradox is quite instructive. The superpotential $f(x)$ in this example is not a Morse function! Indeed, if we want to insist on the periodicity of df, not just $\|df\|^2$ then we have to admit that near the minimum of $U(x)$ (in our example it is unique) the function $f(x)$ behaves as:

$$f(x) \sim x|x| \ ,$$

or, if we want to achieve the same qualitative behavior with C^∞-functions, $f(x) \sim x^3$, a A_2 singularity. Once we perturb it a little bit, two Morse critical points will occur, and the spectrum will change dramatically – the eigenvectors will be replaced by the adjoint eigenvectors as in the act I (Section 3.4.1).

3.5. **From path integral to the space of states.** Having analyzed in detail the case $X = \mathbb{S}^1$, we turn to a general non-simply connected target manifold X.

Let θ be a (complex) one-form on X. Consider the quantum mechanical model described by the action (3.5) plus the term $-i\int_I \theta$. Let us write

$$\theta = \tau - i\lambda df, \tag{3.31}$$

where τ is another one-form. Recall that f is *a priori* a multi-valued function. In this formula, df really stands for b_v, the contraction of the vector field v with the metric g. Formula (3.31) is a small generalization of the formulas above in which we had $\tau df, \tau \in \mathbb{C}$, for the one-form τ, so that $\theta = \vartheta df$. In fact, the one-form τ does not have to be proportional to df.

The Hilbert space of states of this model is the space of L^2 differential forms on X. The question before us is to describe what happens with this space in the limit when $\lambda \to +\infty$, and τ is fixed.

As discussed in Part I, Section 3.2, the values of the "in" states of the theory at points of X may be constructed as path integrals over maps from the half-line $I = (-\infty, 0]$ to X. The value of the wave-function corresponding to a state Ψ at $x \in X$ is given by the path integral over those trajectories for which the end point $0 \in I$ goes to x. Now, as explained in Part I, Section 3.2, the theory with the term $-i\int_I \theta$ is equivalent to the theory without this term, but in which the "in" states are multiplied by $e^{i\int_{x_-}^{x}\theta}$ and the "out" states – by $e^{-i\int_{x_-}^{x}\theta}$ (here x_- is the boundary condition at $-\infty \in I$ corresponding to the choice of the vacuum). If θ were exact, these states would be well-defined functions on X (as in our analysis in Part I). But we are now interested in the case when θ is not exact. Then this integral depends on the choice of the integration contour I going from x_- to x.

In fact, if we replace I by another path I' starting and ending at the same points, the action will change by the factor $e^{i\int_{[I]-[I']}\theta}$, where we integrate θ over the closed loop obtained by gluing together I and $-I'$. Thus, the corresponding path integral is naturally a function not on X, but on $\widetilde{X}$, the $H_1(X,\mathbb{Z})$-covering $\widetilde{X}$ of X (as before, we assume here that $H_1(X,\mathbb{Z})$ has no torsion). As explained above, in Section 3.2, the points of $\widetilde{X}$ are pairs $(x,[I])$, where $x\in X$ and $[I]$ is a homotopy class of a path I connecting x with another, fixed, point $x_0\in X$ (corresponding to the boundary condition at $-\infty$). The equivalence relation identifies two paths I and I' if the image of their difference in $H_1(X,\mathbb{Z})$ is equal to 0. Note that the group $H_1(X,\mathbb{Z})$ naturally acts on $\widetilde{X}$.

The "in" states of the model with the term $-i\int_I\theta$ are therefore the differential forms Ψ on $\widetilde{X}$ satisfying the equivariance condition

$$\gamma^*(\Psi)=e^{i\int_\gamma\theta}\Psi,\qquad \gamma\in H_1(X,\mathbb{Z}). \tag{3.32}$$

In the physical model θ is a real one-form. In this case the space of such states carries a natural structure of a Hilbert space with the hermitean inner product given by the formula

$$\langle\Phi|\Psi\rangle=\int_X(\star\overline{\Phi})\wedge\Psi.$$

The equivariance conditions on Ψ and Φ give rise to factors that are inverse to each other in this case, and so their product is a differential form on $\widetilde{X}$ that is a pull-back of a form on X which we can integrate over X (this is the appropriate version of the L^2 condition in the non-simply connected case). In other words, the operation of complex conjugation and Hodge star identifies the spaces of "in" and "out" states in this case.

The existence of a hermitean inner product is expected in a model described by a CPT invariant action, as explained in Part I, Section 3.2. Our action is CPT invariant precisely when θ is real. Note that there is a special case of this construction when the lattice of periods $\int_\gamma\theta$ in $\mathbb{C}$ is equal to $2\pi\mathbb{Z}$. Then the equivariance condition (3.32) means that Ψ is a differential form on X, so the space of states is not affected. Therefore we obtain a family of hermitean theories parametrized by $H^1(X,\mathbb{R})/2\pi H^1(X,\mathbb{Z})$.

For our purposes, however, we wish to consider a complex one-form $\theta=-i\lambda df+\tau$, where λ is a real parameter which coincides with the factor in front of the metric in the action (3.5). Let us assume that τ is a real one-form. We wish to take λ to $+\infty$, while keeping τ fixed. Adding this θ-term breaks the CPT invariance of the action, so that we have, as before, separate spaces of "in" and "out" states. What are these spaces?

Let f be the function on $\widetilde{X}$ defined by the formula $f(x,[I])=\int_I\beta$. Let us assume that f has only isolated non-degenerate critical points on $\widetilde{X}$. In this case we will say that f is a *Morse–Novikov function on* X (see [45]). Note that while f is not well-defined on X, the gradient vector field ∇f and the corresponding $\mathbb{C}^\times$-action ϕ descend to X, by our assumption. Therefore the critical points of f on $\widetilde{X}$ are the preimages of the fixed points $x_\alpha,\alpha\in A$, of ϕ on X. Let S_α be the set of preimages of x_α in $\widetilde{X}$. For each $\alpha\in A$, S_α is a torsor over $H_1(X,\mathbb{Z})$; in other words, $H_1(X,\mathbb{Z})$ acts on S_α simply transitively, even though S_α cannot be canonically identified with $H_1(X,\mathbb{Z})$.

As in Part I, we obtain the vectors in the "in" space of states by multiplying the differential forms Ψ on $\widetilde{X}$ satisfying equivariance condition (3.32) by $e^{\lambda f}$. These newly rescaled states $\widetilde{\Psi}$ then satisfy the equivariance condition

$$\gamma^*(\widetilde{\Psi}) = e^{i\int_\gamma \tau}\widetilde{\Psi}, \qquad \gamma \in H_1(X,\mathbb{Z}). \tag{3.33}$$

Likewise, the vectors in the "out" space are obtained by multiplying the differential forms Φ on $\widetilde{X}$ satisfying the equivariance condition

$$\gamma^*(\Phi) = e^{-i\int_\gamma \theta}\Phi, \qquad \gamma \in H_1(X,\mathbb{Z}),$$

by $e^{-\lambda f}$. These new states $\widetilde{\Phi}$ satisfy the equivariance condition

$$\gamma^*(\widetilde{\Phi}) = e^{-i\int_\gamma \tau}\widetilde{\Phi}, \qquad \gamma \in H_1(X,\mathbb{Z}). \tag{3.34}$$

Because the equivariance conditions (3.33) and (3.34) are opposite to each other, the wedge product $\Phi \wedge \Psi$ is the pull-back of a differential form on X, which we integrate over X. Hence we obtain a well-defined pairing between the spaces of "in" and "out" states.

3.6. **Spaces of delta-forms.** The limit $\lambda \to \infty$ is now described in the same way as in the case when f is a Morse function on X. Quasi-classical analysis, as in Part I, Section 3.8, shows that before multiplication by $e^{\lambda f}$, the wave-functions are concentrated near the critical points of the Morse–Novikov function on $\widetilde{X}$, where they are approximately given by Gaussian type distributions. However, after we multiply them by $e^{\lambda f}$, the terms in the Gaussian distribution corresponding to the positive eigenvalues of the Hessian of f at the critical point disappear, whereas the ones corresponding to the negative eigenvalues get doubled. As the result, we obtain delta-forms supported at the strata of the decomposition of $\widetilde{X}$ into the ascending manifolds with respect to f (this analysis applies to the "in" states; for the "out" states we obtain delta-forms supported at the descending manifolds).

Let $\widetilde{X}_{\alpha,\mu}, \alpha \in A, \mu \in S_\alpha$, (resp., $\widetilde{X}^{\alpha,\mu}$) be the strata of the decomposition of $\widetilde{X}$ into the union of ascending (resp., descending) manifolds of the function f. The strata $\widetilde{X}_{\alpha,\mu}, \mu \in S_\alpha$, are the inverse images in $\widetilde{X}$ of the strata $X_\alpha \subset X$ of the decomposition

$$X = \bigsqcup_{\alpha \in A} X_\alpha \tag{3.35}$$

coming from the $\mathbb{C}^\times$-action ϕ on X, and similarly for the strata $\widetilde{X}^{\alpha,\mu}$. Let $\widetilde{\mathcal{H}}^{\rm in}_{\alpha,\mu}$ be the space of "delta-forms" supported on $\widetilde{X}_{\alpha,\mu} \subset \widetilde{X}$, defined as in Part I, Section 3.8. Set

$$\widetilde{\mathcal{H}}^{\rm in}_\alpha = \prod_{\mu \in S_\alpha} \widetilde{\mathcal{H}}^{\rm in}_{\alpha,\mu}.$$

The group $H_1(X,\mathbb{Z})$ naturally acts on $\widetilde{\mathcal{H}}^{\rm in}_{\alpha,\mu}$ by shifting the index μ by $\gamma \in H_1(X,\mathbb{Z})$ (recall that S_α is an $H_1(X,\mathbb{Z})$-torsor).

The space of "in" states of our model in the limit $\lambda \to \infty$ is then the subspace of those vectors $(\widetilde{\Psi}^{\rm in}_{\alpha,\mu})$ in $\bigoplus_{\alpha\in A} \widetilde{\mathcal{H}}^{\rm in}_\alpha$ that satisfy the τ-equivariance condition

$$\widetilde{\Psi}^{\rm in}_{\alpha,\mu+\gamma} = e^{i\int_\gamma \tau}\widetilde{\Psi}^{\rm in}_{\alpha,\mu}, \qquad \gamma \in H_1(X,\mathbb{Z}). \tag{3.36}$$

Therefore for a fixed α all vectors $\widetilde{\Psi}^{\mathrm{in}}_{\alpha,\mu}, S_\alpha$, are determined once we know one of them (for a particular μ). Thus, we find that the space of states is non-canonically isomorphic to $\bigoplus_{\alpha\in A} \mathcal{H}^{\mathrm{in}}_\alpha$, where $\mathcal{H}^{\mathrm{in}}_\alpha$ is the space of delta-forms on X_α defined in Part I, Section 3.8. The space of "out" spaces is defined similarly.

Both spaces should be viewed as subspaces of the spaces of equivariant distributions on $\widetilde{X}$, as explained in Part I. The definition of these distributions requires regularization. The effect of this regularization is that the "big" space of states is non-canonically isomorphic to the direct sum of the subspaces $\widetilde{\mathcal{H}}^{\mathrm{in}}_\alpha$, but has a canonical filtration whose successive quotients are isomorphic to $\widetilde{\mathcal{H}}^{\mathrm{in}}_\alpha$. In particular, the Hamiltonian is not diagonalizable, but has Jordan blocks, by the same mechanism as the one described in Part I.

The space of "out" states is defined similarly, as a successive extension of spaces of delta-forms supported at the strata $\widetilde{X}^{\alpha,\mu}, \alpha \in A, \mu \in S_\alpha$.

Remark 3.1. In the case when X is a finite-dimensional Kähler manifold, equipped with a $\mathbb{C}^\times$-action with a non-empty set of fixed points (which has been our assumption in this paper), we always have $H_1(X,\mathbb{Z}) = 0$ and so *any* closed one-form is exact (see [22]). Therefore we do not need to consider the possibility that β is not exact, and so the above discussion appears to be superfluous. But for *infinite-dimensional* manifolds, such as the loop space LX and the space of connections on $\mathbb{S}^3$ which we consider below in the context of four-dimensional Yang–Mills theory, this is no longer true, so the above discussion will be useful in these cases.

Note, however, that "toy models" of Morse–Novikov quantum mechanics may already be constructed on finite-dimensional manifolds. But we need to consider a real manifold X with $H_1(X,\mathbb{Q}) \neq 0$. We have already considered above the new phenomena which occur in the case of Morse–Novikov quantum mechanics in the example of $X = \mathbb{S}^1$. In this more general situation we may still take the limit $\lambda \to \infty$ of the corresponding quantum mechanical model, and most of our results obtained in the case of Kähler manifolds will carry over to this case. In particular, the space of "in" states of such a model will be isomorphic to the subspace of equivariant states in $\prod_{\alpha,\mu} \widetilde{\mathcal{H}}_{\alpha,\mu}$, where $\widetilde{\mathcal{H}}_{\alpha,\mu}$ is the space of delta-forms on the descending manifolds $\widetilde{X}_{\alpha,\mu} \subset \widetilde{X}$ of the Morse–Novikov function. Note however that these strata are no longer isomorphic to $\mathbb{C}^n$, and so we obviously do not have a holomorphic factorization of the corresponding spaces of delta-forms. In realistic non-simply connected finite-dimensional situation one encounters various combinations of the two basic situations we have seen in the case of the circle: the discrete real spectrum with Jordan blocks and the discrete imaginary spectrum with periodic functions.

3.7. **Non-isolated critical points.** Up to now we have considered quantum mechanics on a non-simply connected manifold X with a Morse–Novikov function f defined on a covering $\widetilde{X}$ of X. The assumption that f is a Morse–Novikov function means that its critical points are isolated and non-degenerate. In this section we discuss what happens when the fixed points are not isolated and f is a *Morse–Bott–Novikov function.* This is precisely the situation we encounter in two-dimensional sigma models (where X is a loop space).

We have already discussed such quantum mechanical models in the case when X is simply-connected in Part I, Section 6.5. Let us recall briefly the results of our analysis. Thus, we are given a compact Kähler manifold X with a holomorphic $\mathbb{C}^\times$-action preserving the Kähler structure. We will assume that the set of fixed points of this action is non-empty. Then, according to [22], there exists a Morse–Bott function f such that its critical points are the fixed points of the $\mathbb{C}^\times$-action. Let $C_\alpha, \alpha \in A$, be the components of the fixed point set of the $\mathbb{C}^\times$-action on X (under our old assumptions, each C_α consisted of a single point). According to the results of [9, 11, 58], in this case X still has decompositions

$$X = \bigsqcup_{\alpha \in A} X_\alpha = \bigsqcup_{\alpha \in A} X^\alpha, \tag{3.37}$$

with the ascending and descending manifolds X_α and X^α defined in the same way as before. Each X_α is a $\mathbb{C}^\times$-equivariant holomorphic fibration over C_α. Each fiber is isomorphic to $\mathbb{C}^{n_\alpha}$, where n_α is the number of positive eigenvalues of the Hessian of f at the points of C_α. Moreover, locally over C_α, the bundle X_α is isomorphic to the subbundle N_α^+ of the normal bundle to $C_\alpha \subset X$ spanned by the eigenspaces of the Hessian of the function f with positive eigenvalues. In general, X_α does not have a natural structure of a vector bundle; in other words, the transition functions between local identifications of X_α with N_α^+ (or any other vector bundle) over open subsets of C_α may not be linear. However, we will assume in what follows that they are algebraic (that is, polynomial). This is the case when X is a projective algebraic variety (see [9]). Thus, we can speak of functions on X_α that are polynomial along the fibers of the projection $X_\alpha \to C_\alpha$.

Likewise, X^α is also a $\mathbb{C}^\times$-equivariant holomorphic bundle over C_α. Each fiber is isomorphic to $\mathbb{C}^{n-n_\alpha-\dim C_\alpha}$. Locally, the corresponding bundle over C_α is isomorphic to the subbundle N_α^- of the normal bundle to $C_\alpha \subset X$ spanned by the eigenspaces of the Hessian of the function f with negative eigenvalues. Again, we will assume that the transition functions between local identifications of X_α with N_α^- over open subsets of C_α are algebraic.

Consider, for example, the case of $X = \mathbb{CP}^2$ with the $\mathbb{C}^\times$-action corresponding to f given by the formula $(z_1 : z_2 : z_3) \mapsto (qz_1 : z_2 : z_3)$. Then the fixed point set has two components: the point $C_1 = (1 : 0 : 0)$ and the one-dimensional component $C_2 = \{(0 : z_2 : z_3)\}$ isomorphic to $\mathbb{CP}^1$. The corresponding strata X_1 and X_2 are the point $(1 : 0 : 0)$ and its complement, respectively. Note that X_2 is a line bundle over $C_2 = \mathbb{CP}^1$ isomorphic to $\mathcal{O}(1)$, which is also isomorphic to the normal bundle of $C_2 \subset \mathbb{CP}^2$. The strata X^1 and X^2 are the plane $\{(1 : u_1 : u_2)\}$ and $C_2 = \mathbb{CP}^1$, respectively.

The description of the spaces of "in" and "out" states of this model is similar to the one obtained previously in the Morse function case. Namely, $\mathcal{H}^{\text{in}}$ is isomorphic to the direct sum of the spaces $\mathcal{H}^{\text{in}}_\alpha, \alpha \in A$. Roughly speaking, each space $\mathcal{H}^{\text{in}}_\alpha$ is the space of L^2 differential forms on C_α extended in two ways: by polynomial differential forms in the bundle directions of X_α and by polynomials in the derivatives in the transversal directions to X_α in X. More precisely, the states in $\mathcal{H}_\alpha$ are L^2 sections of vector bundles over C_α. For example, in the case when $X = \mathbb{CP}^2$ we have $X_2 \simeq N_2^+$, which is the line bundle $\mathcal{O}(1)$ over $C_2 = \mathbb{CP}^1$. Then the purely bosonic part of $\mathcal{H}_2$ is the direct sum of the spaces of L^2 sections of the line bundles $\mathcal{O}(-1)^{\otimes n} \otimes \overline{\mathcal{O}(-1)}^{\otimes \overline{n}}, n, \overline{n} \geq 0$. To include the fermions, we need to add the corresponding spaces of differential forms, which are defined similarly.

In particular, the ground states, on which the Hamiltonian $\mathcal{L}_v$, where $v = \nabla f$, acts by zero, correspond to just the ordinary L^2 differential forms on C_α. Given such a form ω_α, let $\widetilde{\omega}_\alpha$ be its pull-back to X_α under the projection $X_\alpha \to C_\alpha$. Then $\widetilde{\omega}_\alpha$ defines a "delta-like" distribution supported on X_α, whose value on $\eta \in \Omega^\bullet(X)$ is equal to

$$\int_{X_\alpha} \widetilde{\omega}_\alpha \wedge \eta|_{X_\alpha}.$$

While these are the ground states of the model at $\lambda = \infty$, only those of them which correspond to harmonic differential forms $\omega_\alpha \in \Omega^\bullet(C_\alpha), \alpha \in A$, may be deformed to ground states for finite values of λ.

Other elements of $\mathcal{H}_\alpha$ are distributions obtained by applying to the distributions $\widetilde{\omega}_\alpha$ Lie derivatives in the transversal directions to X_α as well as multiplying them by differential forms on X_α which are polynomial along the fibers of the projection $X_\alpha \to C_\alpha$. The definition of these distributions requires a regularization similar to the one we used in the case of isolated critical points in Part I. Because of this regularization, we obtain non-trivial extensions between different spaces $\mathcal{H}^{\rm in}_\alpha$, and the action of the Hamiltonian is not diagonalizable.

The space of "out" states is defined in a similar fashion. As in the case of isolated critical points, we have a canonical pairing between the two spaces.

Now we consider the case of non-simply connected (but connected) X and a *Morse–Bott–Novikov function* f. The resulting picture is a combination of the case of Morse–Novikov function discussed in Section 3.6 and the case of Morse–Bott function discussed in this section. Namely, we have the decomposition (3.37) defined using the $\mathbb{C}^\times$-action corresponding to the vector field ξ, which is the holomorphic part of the gradient vector field $v = \nabla f = \xi + \overline{\xi}$. Let $\widetilde{X}_{\alpha,\mu}, \alpha \in A, \mu \in S_\alpha$ (resp., $\widetilde{X}^{\alpha,\mu}$), be the strata of the decomposition of the $H_1(X,\mathbb{Z})$-cover $\widetilde{X}$ into the union of ascending (resp., descending) manifolds of the function f. Here, as above, S_α is the $H_1(X,\mathbb{Z})$-torsor of components of the preimage of C_α in $\widetilde{X}$. The strata $\widetilde{X}_{\alpha,\mu}, \mu \in S_\alpha$, are the inverse images in $\widetilde{X}$ of the strata $X_\alpha \subset X$, and similarly for the strata $\widetilde{X}^{\alpha,\mu}$. Let $\widetilde{\mathcal{H}}^{\rm in}_{\alpha,\mu}$ be the space of "delta-forms" supported on $\widetilde{X}_{\alpha,\mu} \subset \widetilde{X}$, defined as in Part I, Section 3.8. We set

$$\widetilde{\mathcal{H}}^{\rm in}_\alpha = \prod_{\mu \in S_\alpha} \widetilde{\mathcal{H}}^{\rm in}_{\alpha,\mu}.$$

The space of states of the Morse–Bott–Novikov quantum mechanical model is the space of vectors

$$(\widetilde{\Psi}^{\rm in}_{\alpha,\mu}) \in \prod_{\alpha \in A, \mu \in S_\alpha} \widetilde{\mathcal{H}}^{\rm in}_{\alpha,\mu}$$

satisfying the τ-equivariance condition (3.36). As before, the spaces $\widetilde{\mathcal{H}}^{\rm in}_{\alpha,\mu}$ are defined as certain spaces of distributions on $\widetilde{X}$. Because of the regularization involved in the definition of these distributions, there are non-trivial extensions between them, and the "big" space of states $\bigoplus_{\alpha\in A}\widetilde{\mathcal{H}}^{\text{in}}_\alpha$ is really a successive extension of the spaces $\widetilde{\mathcal{H}}^{\text{in}}_{\alpha,\mu}$ (rather than a direct sum). Furthermore, the Hamiltonian is not diagonalizable, but has Jordan blocks, as before.

Thus, the structure we obtain is very similar to the case of isolated critical points. One essential difference is that we observe holomorphic factorization only in the fiber directions of the maps $X_\alpha \to C_\alpha$, but not along the manifolds C_α themselves.

4. Back to sigma models

We now apply the results of the previous section to the (type A twisted) two-dimensional sigma models in the limit $\overline{\tau} \to \infty$. As we explained in Section 2, such a model, with the target being a Kähler manifold X, may be viewed as a quantum mechanical model on the loop space X, corresponding to the Morse–Bott–Novikov function f given by formula (2.15). These are precisely the types of models considered in Section 3.

4.1. **Space of states.** According to the analysis of Section 3, in order to describe the space of states of our model we need to find the set of critical points of our Morse–Bott–Novikov function f on the loop space LX of a Kähler manifold X. This function is the Floer function given by formula (2.15).

In order to simplify the exposition, we will assume throughout this section that X is simply-connected, so that LX is connected (we will consider the case of a non-simply connected X – namely, a torus – in Section 4.6). Then the critical points of f are the constant maps. Thus, the set of critical points of f is the submanifold $X \subset LX$, so it has only one component. We need to describe the corresponding spaces $\widetilde{\mathcal{H}}_{\alpha,\mu} = \widetilde{\mathcal{H}}^{\text{in}}_{\alpha,\mu}$, in the notation of Section 3.7 (the spaces of "out" states may be described similarly). Here α takes only one value (since the manifold of critical points has only one component), so we will suppress this index. The other index μ runs over a torsor over $H_1(LX,\mathbb{Z}) = H_2(X,\mathbb{Z})$, which is by definition the set of components of the preimage $p^{-1}(X)$ of $X \subset LX$ in the $H_1(LX,\mathbb{Z})$-cover $p : \widetilde{LX} \to LX$. Since $\widetilde{LX}$ is described in terms of maps from a two-dimensional disc D to X (see Section 2.4), we actually have a canonical component in $p^{-1}(X)$. It consists of the (equivalence classes of) constant maps $D \to X$. Therefore this $H_2(X,\mathbb{Z})$-torsor has a canonical trivialization, and so the index μ really takes values in $H_2(X,\mathbb{Z})$. We therefore have the spaces $\widetilde{\mathcal{H}}_\mu, \mu \in H_2(X,\mathbb{Z})$. All of them are isomorphic canonically to each other, via the action of the group $H_2(X,\mathbb{Z})$ on the covering. Let us describe the structure of one of them; namely, $\widetilde{\mathcal{H}}_0$.

By analogy with the results in the finite-dimensional case (which were based on the semi-classical analysis of the wave-functions), $\widetilde{\mathcal{H}}_0$ is the space of "delta-forms" on the ascending submanifold $\widetilde{LX}_0 \subset \widetilde{LX}$. (More precisely, we will use a particular model for this space described below.) In terms of the description of $\widetilde{LX}$ as homotopy classes of maps $D \to X$, where D is a unit complex disc, given in Section 2.4, points of $\widetilde{LX}_0$ correspond to *holomorphic* maps $D \to X$. Indeed, the gradient flows of our Morse–Bott–Novikov function f correspond to the Cauchy–Riemann equations (2.13), (2.14) for the maps $D \to X$.

To give a more concrete description of the space $\widetilde{\mathcal{H}}_0$, we recall the explicit (local) coordinates $X^a_n, X^{\overline{a}}_n, n \in \mathbb{Z}$, on the loop space LX, the momenta $p'_{a,n}, p'_{\overline{a},n}, n \in \mathbb{Z}$, and the corresponding fermionic variables

$$\psi^a_n, \psi^{\overline{a}}_n, \pi_{a,n}, \pi_{\overline{a},n}, \qquad n \in \mathbb{Z},$$

introduced in Section 2.4.

Warning: to simplify notation, from now on we will denote $p'_{a,n}, p'_{\overline{a},n}$, by $p_{a,n}, p_{\overline{a},n}$, and similarly for the corresponding fields.

The OPE (2.9) give rise to the usual commutation relations

$$[p_{a,n}, X^b_m] = -i\delta^b_a \delta_{n,-m}, \qquad [\pi_{a,n}, \psi^b_m] = -i\delta^b_a \delta_{n,-m}, \tag{4.1}$$

$$[p_{\overline{a},n}, X^{\overline{b}}_m] = -i\delta^{\overline{b}}_{\overline{a}} \delta_{n,-m}, \qquad [\pi_{\overline{a},n}, \psi^{\overline{b}}_m] = -i\delta^{\overline{b}}_{\overline{a}} \delta_{n,-m}. \tag{4.2}$$

4.1.1. *The case of flat space.* Consider first the case when $X = \mathbb{C}^N$. Technically, it does not fall in the category of manifolds we consider, since it is not compact, but it is instructive to consider it because it provides a useful local model for the compact target manifolds. In this case holomorphic maps $D \to X$ are simply described by their Taylor series

$$\sum_{n \leq 0} X^a_n z^{-n}, \qquad a = 1, \ldots, N. \tag{4.3}$$

Hence we obtain a natural set of coordinates on the ascending manifold $\widetilde{LX}_0$ (actually, in this case there is no covering, but we will keep the tilde in the notation); namely, X^a_n and $X^{\overline{a}}_n$, where $a = 1, \ldots, N; n \leq 0$. Therefore, following the discussion in Section 3.7, it is natural to define the space $\widetilde{\mathcal{H}}_0$ of "delta-forms" supported on $\widetilde{LX}_0$ as the tensor product of three spaces: the space of L^2 differential forms on $X = \mathbb{C}^N$ realized in terms of the zero modes $X^a_0, X^{\overline{a}}_0$ and $\psi^a_0, \psi^{\overline{a}}_0$; the space of polynomial functions in the remaining coordinates on $\widetilde{LX}_0$, $X^a_n, \psi^a_n, n < 0$, and their complex conjugates; and the space of polynomial functions in the "derivatives" in the transversal directions, $p_{a,n}, \pi_{a,n}, n < 0$, and their complex conjugates.

The space $\widetilde{\mathcal{H}}_0$ should be compared with the usual Fock representation of the free $\beta\gamma$-bc system described by the superalgebra with the commutation relations (4.1) and (4.2). It is the tensor product of the chiral Fock representation $\mathcal{F}_0$ generated by the vacuum vector $|0\rangle$ satisfying

$$X^a_n|0\rangle = \psi^a_n|0\rangle = 0, \quad n > 0, \qquad p_{a,m}|0\rangle = \pi_{a,m}|0\rangle = 0, \quad m \geq 0,$$

and its anti-chiral counterpart $\overline{\mathcal{F}}_0$ generated by the vector $\overline{|0\rangle}$ satisfying analogous relations with respect to half of the the anti-holomorphic generators. Therefore

$$\mathcal{F}_0 = \mathbb{C}[X^a_n, p_{a,m}]_{n \leq 0, m < 0} \otimes \Lambda[\psi^a_n, \pi_{a,m}]_{n \leq 0, m < 0} \cdot |0\rangle, \tag{4.4}$$

$$\overline{\mathcal{F}}_0 = \mathbb{C}[X^{\overline{a}}_n, p_{\overline{a},m}]_{n \leq 0, m < 0} \otimes \Lambda[\psi^{\overline{a}}_n, \pi_{\overline{a},m}]_{n \leq 0, m < 0} \cdot \overline{|0\rangle}. \tag{4.5}$$

The tensor product $\mathcal{F}_0 \otimes \overline{\mathcal{F}}_0$ coincides with $\widetilde{\mathcal{H}}_0$, except for the zero mode part. In the case of $\widetilde{\mathcal{H}}_0$ we have the space of L^2 differential forms in the zero modes, whereas in the case of $\mathcal{F}_0 \otimes \overline{\mathcal{F}}_0$ we have the space of polynomials in the zero modes. Thus, $\widetilde{\mathcal{H}}_0$

may be viewed as an L^2 version of the tensor product of the Fock representations of the chiral and anti-chiral free $\beta\gamma$-bc systems.

4.1.2. *General Kähler manifolds.* For a general Kähler target manifold X, the space $\widetilde{\mathcal{H}}_0$ may be described in similar terms. Morally, this should be the space of "semi-infinite" delta-forms supported on the stratum $\widetilde{LX}_0 \subset \widetilde{LX}$, which consists of holomorphic maps from the unit disc D to X. Let us choose a covering of X by coordinate patches $X = \cup_\beta U_\beta$, which each patch U_β isomorphic to an open analytic subset of $\mathbb{C}^N$, providing us with complex coordinates $X^a, a = 1, \ldots, N$. Suppose that a map $\Phi : D \to X$ sends $0 \in D$ to a point in U_β. Then the restriction of Φ to a smaller disc $D' \subset D$ will also land in U_β. Therefore we obtain a map $D' \to \mathbb{C}^N$, which we may expand in Taylor series, using the coordinates X^a, as in formula (4.3). Thus, we obtain a set of coordinates $X^a_n, n \leq 0$, on the space of maps $\Phi : D \to X$ such that $0 \mapsto U_\beta$. These really capture the jet of the holomorphic map Φ at the origin $0 \in D$.[5]

On the other hand, a general loop $\gamma : \mathbb{S}^1 \to X$, which is not the boundary value of a holomorphic map $D \to X$, may be expanded in the Fourier series in both positive and negative directions, as in formula (2.10),

$$X^a(\sigma) = \sum_{n \in \mathbb{Z}} X^a_n e^{-in\sigma},$$

where σ is the coordinate on the circle to which the coordinate z on the disc used in formula (4.3) is related by the formula $e^{i\sigma}$ on the disc boundary $|z| = 1$. Therefore $X^a_n, n > 0$, give us coordinates in the transversal directions to $\widetilde{LX}_0 \subset \widetilde{LX}$.

Note that the differentials of the coordinates $X^a_n, n \neq 0$, and their complex conjugates may be viewed as the fiberwise coordinates on the normal bundle to $X \subset \widetilde{LX}$ restricted to $U_\beta \subset X$. According to formula (2.12), the eigenspaces of the Hessian of the Floer function with positive eigenvalues correspond to $X^a_n, n < 0$, and the eigenspaces of the Hessian of the Floer function with negative eigenvalues correspond to $X^a_n, n > 0$.

This discussion suggests the following model for the space of semi-infinite delta-forms supported on $\widetilde{LX}_0 \subset \widetilde{LX}$.

Let ψ^a_n be the fermionic counterparts of the coordinates X^a_n corresponding to the patch U_β, and let $p_{a,n}, \pi_{a,n}$ be the corresponding momenta variables. For each patch U_β we have the space of states of the free field theory described in the same way as in Section 4.1.1, except that we need to replace the L^2 condition by the smoothness condition. The reason for this is the following. When we considered the sigma model with the target $\mathbb{C}^N$, the L^2 condition was imposed as the condition specifying the behavior of the wave-functions at infinity. In the case of a compact Kähler manifold X the L^2 condition is is replaced by the gluing condition on overlaps of different patches.

These spaces should be glued on the overlaps $U_\beta \cap U_\gamma$. In other words, on the overlap of any two patches $U_\beta \cap U_\gamma$ we must define a transition function between the spaces of states attached to them. Then these spaces would form a vector bundle over X,

[5]It is certainly possible that under the map Φ the entire disc D does not land in any given patch U_β. However, the values of the coordinates X^a_n, corresponding to a given patch U_β, on such maps are well-defined as long as $\Phi(0) \in U_\beta$.

and a state in $\widetilde{\mathcal{H}}_0$ would by definition be a global section of this vector bundle. More concretely, it would be represented by a collection of local states on the patches, which agree on the overlaps.

Describing the transition functions for this vector bundle amounts to describing how the generating fields $X^a(z), p_a(z)$, etc., transform under general changes of coordinates on the target manifold $X^a = f^a(\{\widetilde{X}^b\})$. The existence of consistent transition functions is by no means obvious. In fact, in a purely bosonic version of the theory they may not exist. This has to do with an anomaly which has been studied extensively in recent years (see [40, 32, 6, 56, 44]). However, in the supersymmetric models that we are considering now the anomaly cancels and the transformation formulas do exist. They are given by the following formulas from [40]:

$$
\begin{aligned}
\widetilde{X}^\mu(z) &= g^\mu(X^\mu(z)), \\
\widetilde{p}_\mu(z) &= :\frac{\partial f^\nu}{\partial \widetilde{X}^\mu} p_\nu: \; + \; \frac{\partial^2 f^\lambda}{\partial \widetilde{X}^\mu \partial \widetilde{X}^\alpha} \frac{\partial g^\alpha}{\partial X^\nu} :\pi_\lambda \psi^\nu:, \\
\widetilde{\psi}^\mu &= \frac{\partial g^\mu}{\partial X^\nu} \psi^\nu, \\
\widetilde{\pi}_\mu &= \frac{\partial f^\nu}{\partial \widetilde{X}^\mu} \pi_\nu.
\end{aligned}
\tag{4.6}
$$

Here we let the Greek indices $\mu, \nu, ...$ take both holomorphic and anti-holomorphic values, e.g., $\mu = a, \overline{a}$, etc., and we set $f^{\overline{a}} = \overline{f^a}$. We denote $\widetilde{X}^\mu = g^\mu(\{X^\nu\})$ the inverse change of variables to $\widetilde{X}^\nu = f^\nu(\{\widetilde{X}^\mu\})$. These formulas are checked by an explicit computation (see [40]).

Thus, we obtain here the smooth (rather than holomorphic) version of the chiral de Rham complex, as introduced in [40] and reviewed more recently in [8]. To avoid the terminological confusion with the actual chiral de Rham complex of X involving only holomorphic variables (which we consider below in Section 4.5), we will call $\widetilde{\mathcal{H}}_0$ the *chiral-anti-chiral de Rham complex*.

The transformation formulas for the fermions ψ^μ, π_μ simply mean that they transform as sections of the tangent and cotangent bundles to X, respectively, as expected. The only surprise is the second term in the formula for the transformation of the momenta p_μ. This is due to the fact that the momenta variables we use here are the transformed variables denoted p'_μ at the end of Section 2.4. Had we used the original momenta p_μ, we would not have this term. But then the action would be given by formula (2.5) rather than (2.7), which involves the covariant derivatives (with respect to the Levi-Civita connection) rather than the ordinary derivative, with respect to the local coordinates X^μ. Therefore there would be non-trivial OPEs between the momenta variables themselves. We have avoided this by redefining the momenta $p_\mu \mapsto p'_\mu$ in formula (2.6) (and changing the notation for p'_μ back to p_μ) at the cost of introducing an inhomogeneous second term in the transformation formula (4.6) for the p_μ's (see [37] for a discussion of this point).

Formulas (4.6), when rewritten in terms of the Fourier coefficients of the fields, give rise to the transition functions of the bundles of spaces of states. It is clear from these formulas that this vector bundle is a successive extensions of tensor products of symmetric (for bosonic variables) and exterior (for fermionic variables) powers of the tangent and cotangent bundles on X and their complex conjugates (see [40] for more details). This completes the definition of $\widetilde{\mathcal{H}}_0$.

We now define, as in Section 3.7, the "big" space of states as the direct product

$$\widetilde{\mathcal{H}} = \prod_{\mu \in H_2(X,\mathbb{Z})} \widetilde{\mathcal{H}}_\mu,$$

with $\widetilde{\mathcal{H}}_\mu$ being the space of "delta-forms" on $\widetilde{LX}$ supported on the ascending manifold $\widetilde{LX}_\mu$ of the component of the set of critical points of the Floer function f labeled by μ. These spaces are canonically isomorphic to $\widetilde{\mathcal{H}}_0$, described above. The big space of states should be realized as a space of distributions on $\widetilde{LX}$. For this reason, as we have explained in detail in Part I and in Section 3 in the quantum mechanical setting, this space is non-canonically isomorphic to the above direct product. Canonically, we only have a filtration corresponding to the closure relations among the strata $\widetilde{LX}_\mu \subset \widetilde{LX}$. The associated graded pieces are isomorphic to $\widetilde{\mathcal{H}}_\mu$ as in the quantum mechanical models discussed above.

4.1.3. *Definition of the space of states.* We can now describe the space of states of our sigma model (in the infinite radius limit $\overline{\tau} \to -i\infty$, as defined in Section 2). Such a state is, by definition, a collection

$$(\widetilde{\Psi}_\mu) \in \prod_{\mu \in H_2(X,\mathbb{Z})} \widetilde{\mathcal{H}}_\mu$$

satisfying the τ-equivariance condition

$$\widetilde{\Psi}_{\mu+\gamma} = e^{\int_\gamma \tau}\ \widetilde{\Psi}_\mu, \qquad \gamma \in H_2(X,\mathbb{Z}). \tag{4.7}$$

Therefore we see that Ψ_0 determines the remaining Ψ_μ, and so the space of states is isomorphic to $\widetilde{\mathcal{H}}_0$. However, this isomorphism is non-canonical because the direct product decomposition of the "big" space of states is non-canonical.

The fact that the space of states of the sigma model in the infinite radius limit is not canonically isomorphic to chiral-anti-chiral de Rham complex $\widetilde{\mathcal{H}}_0$ is due to the instanton effects. What it really means is that the identification of $\widetilde{\mathcal{H}}_0$ with the space of states is only valid *perturbatively*, that is, without the instantons. As we will see below (and as we have already seen in quantum mechanics in Part I), the *instanton corrections* occur precisely because of the intricate structure of extensions between the spaces of delta-forms $\widetilde{\mathcal{H}}_\mu$. This leads to the non-diagonalizability of the Hamiltonian, divergence of the correlation functions, logarithms in the operator product expansion and other interesting phenomena. We will discuss this in more detail in Section 6.

4.2. **The sigma model in the infinite radius limit as a logarithmic CFT.** As in the quantum mechanical case discussed in detail in Part I, the action of the Hamiltonian of the two-dimensional sigma model can be read off the correlation functions. We will now use this information to show that this Hamiltonian is non-diagonalizable and that the sigma model in the infinite radius limit is in fact a *logarithmic* conformal field theory.

The Hamiltonian on each graded piece $\widetilde{\mathcal{H}}_\mu$ of the "big" space of states is equal to $L_0 + \overline{L}_0$, where L_0 is the 0th mode of the stress tensor $T(z)$, generating the Virasoro algebra with central charge 0, and $\overline{L}_0$ is its anti-holomorphic counterpart. In the free $\beta\gamma$-bc conformal field theory corresponding to each patch U_β these fields are given by the usual formulas

$$T(z) = i : (p_a(z)\partial_z X^a(z) + \pi_a(z)\partial_z \psi^a(z)) : , \tag{4.8}$$

$$\overline{T}(\overline{z}) = i : \left(p_{\overline{a}}(\overline{z})\partial_{\overline{z}} X^{\overline{a}}(\overline{z}) + \pi_{\overline{a}}(\overline{z})\partial_{\overline{z}} \psi^{\overline{a}}(\overline{z})\right) : \ . \tag{4.9}$$

Thus, the Hamiltonian is diagonalizable on each graded piece $\widetilde{\mathcal{H}}_\mu$. However, the fact that the big space of states is a successive extension of these pieces, rather than a direct product, opens the door to potential non-diagonalizability of the Hamiltonian of the sigma model. In other words, the true Hamiltonian may have Jordan blocks, as we saw in the quantum mechanical models in Part I. In fact, we will compute sample correlation functions below, and this will confirm that this is the case in sigma models as well. In addition, we will compute explicitly the nilpotent corrections to the Hamiltonian causing the Jordan blocks in the case of the target manifold $\mathbb{P}^1$ in Section 7.5 (more precisely, we will do this for the sigma model on $\mathbb{P}^1$ in a background gauge field).

What kind of Jordan block structure should we expect to see in the action of the Hamiltonian on the space of states of the sigma model? Let us identify the space of states of our model with $\widetilde{\mathcal{H}}_0$ using the τ-equivariance condition (4.7). The nilpotent entries of the Hamiltonian would bump $\Psi_0 \in \widetilde{\mathcal{H}}_0$ to some $\Psi' \in \widetilde{\mathcal{H}}_\gamma \simeq \widetilde{\mathcal{H}}_0$ corresponding to a stratum $\widetilde{LX}_\gamma, \gamma \in H_2(X, \mathbb{Z})$ in the closure of $\widetilde{LX}_0$. We will call such γ negative, because they have negative relative dimension compared to $\widetilde{LX}_0$. By the τ-equivariance condition, this Ψ' corresponds to $\Psi'' = e^{-\int_\gamma \tau}\Psi' \in \widetilde{\mathcal{H}}_0$. Thus, the nilpotent entries of the Hamiltonian will necessarily contain factors of the form $e^{-\int_\gamma \tau}$ for negative γ.

The Jordan block nature of the Hamiltonian implies that the sigma model in the limit $\overline{\tau} \to \infty$ is a logarithmic conformal field theory (LCFT). Note that the logarithmic corrections to the Virasoro generators have non-perturbative character: they are caused directly by the instantons!

The reason why we get Jordan blocks is the absence of anti-instantons in our model at $\overline{\tau} = -i\infty$. If they were present, the anti-instantons would contribute a small matrix element under the diagonal in the Hamiltonian making it diagonalizable.

4.3. **The observables.** The observables of our model are obtained by the state–operator correspondence from vectors in the space of states. Consider as a toy model the case of $X = \mathbb{C}^N$. Then, as we discussed above, the space of states is essentially the tensor product of the Fock representations of the chiral and anti-chiral $\beta\gamma$-bc systems. The corresponding fields are therefore normally ordered products of the basic fields $X^i(z), p_i(z)$, etc., their derivatives and complex conjugates. For a general Kähler target manifold X we have a sheaf of spaces of states which locally look like the spaces of states of the free theory. Therefore the states, as well as the fields, may be described as collections of states (or fields) in these free theories which agree on overlaps.

For example, the zero modes generate a subspace in the space of states which is isomorphic to the space $\Omega^\bullet(X)$ of differential forms on X. The corresponding fields are locally generated by the fields $X^a(z)$ and $\psi^a(z)$: a globally defined smooth differential form which locally, on a particular chart in X, is given by formula

$$\omega = \omega_{b_1,\ldots,b_m;\bar{b}_1,\ldots,\overline{b_{\overline{m}}}}(X^a, X^{\overline{a}})dX^{b_1} \wedge \ldots \wedge dX^{b_m} \wedge dX^{\bar{b}_1} \wedge \ldots \wedge dX^{\overline{b_{\overline{m}}}}$$

gives rise to the observable, whose restriction to this patch is

$$\mathcal{O}_\omega = \omega_{b_1,\ldots,b_m;\bar{b}_1,\ldots,\overline{b_{\overline{m}}}}\left(X^a(z), X^{\overline{a}}(\overline{z})\right)\psi^{b_1}(z)\ldots\psi^{b_m}(z)\psi^{\bar{b}_1}(\overline{z})\ldots\psi^{\overline{b_{\overline{m}}}}(\overline{z}).$$

These are called the *evaluation observables.* However, there are many more observables in the theory.

4.4. **Jet-evaluation observables.** First of all, we have an obvious generalization of the evaluation observables involving higher derivatives of the fields X^μ, ψ^μ. To define them, let JX be the space of ∞-jets of holomorphic maps from a small complex disc D to X. Such a map is defined by its Taylor series

$$X^a(z) = \sum_{n\leq 0} X_n z^{-n},$$

or equivalently, by the values of its derivatives $\partial_z^n X^a(z)$ at the origin in D (with respect to some coordinates on an open subset U_β in X which contains the image of origin). These are formal power series, with no convergence condition assumed. We have a natural forgetful map $JX \to X$, whose fibers are complex affine spaces with the coordinates $X_n^a, n < 0$. We will refer to JX as the *jet space* of X. It also goes by the name *jet scheme* (see, e.g., [24], Sections 9.4.4 and 11.3.3).

Next, let Σ be a smooth algebraic curve. We introduce the bundle $\mathcal{J}X$ of jet spaces over Σ whose fiber at $p \in \Sigma$ is the space J_pX of jets of holomorphic maps to X from a small complex disc D_p *around the point* p. A more precise definition is as follows: let $\operatorname{Aut} O$ be the group of jets coordinate changes

$$z \mapsto \rho(z) = a_1 z + a_2 z^2 + \ldots,$$

where $a_1 \neq 0$, and $a_n, n > 1$, are arbitrary complex numbers. (These are formal coordinate changes, so we do not assume that the series converges anywhere.) This group acts on JX as follows: $X^a(z) \mapsto X^a(\rho(z))$. On the other hand, we have a natural principal $\operatorname{Aut} O$-bundle $\mathcal{A}ut_\Sigma$ on Σ, whose fiber $\mathcal{A}ut_p$ at $p \in X$ is the space of jets of holomorphic local coordinates at p (see, e.g., [24], Ch. 6, for the precise definition). If t_p is one such jet of coordinates, then $\rho(t_x)$ is another, for any $\rho \in \operatorname{Aut} O$, and for any two jets of coordinates, t_p', t_p'', there exists a unique $\rho \in AutO$ such that $t_p'' = \rho(t_p')$ (so that $\mathcal{A}ut_p$ is an $\operatorname{Aut} O$-torsor). Now we define $\mathcal{J}X$ as the associated bundle

$$\mathcal{J}X := \mathcal{A}ut_\Sigma \underset{\operatorname{Aut} O}{\times} JX.$$

Then the fiber of $\mathcal{J}X$ at $p \in \Sigma$ is indeed the space of jets of maps $D_p \to X$. If we choose a particular coordinate z at $p \in \Sigma$, we identify D_p with D (the "coordinatized" disc), and hence the fiber J_pX of $\mathcal{J}X$ at p with JX. But the above definition of $\mathcal{J}X$ is independent of any such choices.

Next, let $\Omega^\bullet_{\mathrm{vert}}(\mathcal{J}X)$ be the sheaf of vertical differential forms on the bundle $\mathcal{J}X \to X$ (here "vertical" means that we consider differential forms only in the fiber directions of this bundle). By definition, a *jet-evaluation observable* is a smooth global section of $\Omega^\bullet_{\mathrm{vert}}(\mathcal{J}X)$. Locally, over an open subset $\mathcal{U}$ in Σ with a coordinate z, the bundle $\mathcal{J}X$ may be trivialized: $\mathcal{J}X|_{\mathcal{U}} \simeq \mathcal{U} \times JX$ and so a vertical differential form is just a function on $\mathcal{U}$ with values in differential forms on JX. Choosing an open subset $U_\beta \subset X$ with holomorphic coordinates $\{X^a\}$, we can write such a form of degree $(p, \overline{p})$ as follows

$$\begin{aligned}\Omega = \mathbb{A}_{IJKL\ldots}(X^a(z), \overline{X}^{\overline{a}}(\overline{z})) dX_0^{i_1} \wedge \ldots dX_0^{i_{m_1}} \wedge dX_{-1}^{j_1} \wedge \ldots \wedge dX_{-1}^{j_{m_2}} \\ \wedge dX_{-2}^{k_1} \wedge \ldots \wedge dX_{-2}^{k_{m_3}} \wedge \ldots \wedge d\overline{X}^{\overline{i}_1} \wedge \ldots \wedge d\overline{X}_{-2}^{\overline{k}_{\overline{m}_3}} \ldots,\end{aligned} \tag{4.10}$$

where the jet-scheme JX is coordinatized by:

$$X^a(z) = X_0^a + X_{-1}^a z + X_{-2}^a z^2 + \ldots,$$

and $m = m_1 + m_2 + \ldots$, $\overline{m} = \overline{m}_1 + \overline{m}_2 + \ldots$.

To the form Ω we then associate the operator

$$\mathcal{O}_{\mathbb{A}} = \mathbb{A}_{IJKL\ldots}(X^a(t), \overline{X}^{\overline{a}}(\overline{t}))\psi^{i_1} \ldots \psi^{i_{m_1}} \partial\psi^{j_1} \ldots \partial\psi^{j_{m_2}} \partial^2\psi^{k_1} \ldots \overline{\psi}^{\overline{i}_1} \ldots, \tag{4.11}$$

where $I = i_1 i_2 \ldots i_{m_1}$, $J = j_1 j_2 \ldots j_{m_2}$, $\ldots$, $K = k_1 k_2 \ldots k_{m_3}$, etc.

Introduce the following notation. For a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$ we denote $n = \ell(\lambda)$,

$$|\lambda| = \sum_i \lambda_i,$$

and write

$$\mathcal{D}^\lambda \mathbf{X} = \prod_{i=1}^{\ell(\lambda)} \frac{1}{\lambda_i!} \partial_z^{\lambda_i} X, \qquad \overline{\mathcal{D}}^\lambda \overline{\mathbf{X}} = \prod_{i=1}^{\ell(\lambda)} \frac{1}{\lambda_i!} \partial_{\overline{z}}^{\lambda_i} \overline{X}.$$

For instance,

$$\mathcal{D}^{1^3}\mathbf{X} = (\partial X)^3 \ , \qquad \mathcal{D}^{2,1}\mathbf{X} = \frac{1}{2}\partial^2 X \partial X \ , \qquad \mathcal{D}^3\mathbf{X} = \frac{1}{6}\partial^3 X,$$

$$\overline{\mathcal{D}}^{1^3}\overline{\mathbf{X}} = (\overline{\partial X})^3 \ , \qquad \overline{\mathcal{D}}^{2,1}\overline{\mathbf{X}} = \frac{1}{2}\overline{\partial}^2 \overline{X \partial X} \ , \qquad \overline{\mathcal{D}}^3\overline{\mathbf{X}} = \frac{1}{6}\overline{\partial}^3 \overline{X},$$

and so on. For the target space of complex dimension d we introduce operators labeled by the colored partitions: $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$: the bosonic ones

$$\mathcal{D}^{\vec{\lambda}}\mathbf{X} = \prod_{a=1}^{d} \prod_{i=1}^{\ell(\lambda^{(a)})} \frac{1}{\lambda_i^{(a)}!} \partial_z^{\lambda_i^{(a)}} X^a \ ,$$

and the fermionic ones:

$$\mathcal{D}^{\vec{\lambda}}\mathbf{\Psi} = \prod_{a=1}^{d} \prod_{i=1}^{n_a} \frac{1}{\left(\lambda_i^{(a)} - i + n_a - 1\right)!} \partial_z^{\lambda_i^{(a)} - i + n_a - 1} \psi^a \ ,$$

where $n_a = \ell(\lambda^{(a)})$ is the number of rows in the Young diagram of the partition $\lambda^{(a)}$. Analogously, we define the complex conjugate operators $\overline{\mathcal{D}}^{\vec{\lambda}}\overline{\mathbf{X}}, \overline{\mathcal{D}}^{\vec{\lambda}}\overline{\mathbf{\Psi}}$.

Then we formally expand:

$$\mathbb{A}_{IJKL\ldots}(X^a(t), \overline{X}^{\overline{a}}(\overline{t})) = \sum_{\vec{\lambda},\vec{\mu}} \mathbb{A}_{IJKL\ldots|\vec{\lambda}\vec{\mu}}(X^a_0, \overline{X}^{\overline{a}}_0)\mathcal{D}^{\vec{\lambda}}\mathbf{X}\overline{\mathcal{D}}^{\vec{\mu}}\overline{\mathbf{X}}. \tag{4.12}$$

Similarly, the fermionic content of the operator (4.11) may be repackaged using colored partitions:

$$\mathcal{O}_{\mathbb{A}} = \sum_{\vec{\lambda},\vec{\mu},\vec{\nu},\vec{\rho}} \mathbb{A}_{\vec{\lambda}\vec{\mu}\vec{\nu}\vec{\rho}}(X^a_0, \overline{X}^a_0)\mathcal{D}^{\vec{\lambda}}\mathbf{X}\overline{\mathcal{D}}^{\vec{\nu}}\overline{\mathbf{X}}\mathcal{D}^{\vec{\mu}}\mathbf{\Psi}\overline{\mathcal{D}}^{\vec{\rho}}\overline{\mathbf{\Psi}}. \tag{4.13}$$

On the overlaps $U_\beta \cap U_\gamma$ we have transition functions defined explicitly by formulas (4.6). It is easy to see that these are precisely the (classical) transition functions on the jet bundle $\mathcal{J}X$ defined above.

Since these observables do not depend on the momenta variables $p_a(z), \pi_a(z)$ and their complex conjugates, they are "classical" in the sense that in perturbation theory no normal ordering (or any other kind of regularization) is necessary to define them. (We will see below that non-perturbatively even these observables require regularization.) They transform in the same way as the classical jets (see formula (4.6)), without any quantum correction terms. Hence this class of observables is the easiest to study (beyond the ordinary evaluation observables).

Remark 4.1. Note that we have a tautological map $JX \to X$ corresponding to evaluating a jet of maps $D \to X$ at $0 \in D$, $X^a(z) \mapsto X^a(0)$. Likewise, we have a map $\mathcal{J}X \to \Sigma \times X$, defined in the same way pointwise. Any differential form on X gives rise to a differential form on the product $\Sigma \times X$ (constant along the first factor), and, by pull-back, on $\mathcal{J}X$. The corresponding observables are the ordinary evaluation observables; they depend on the fields X^μ, ψ^μ, but not on their derivatives. Likewise, we may consider the space $J^N X$ of N-jets of maps $D \to X$; these are determined by the first $(N-1)$ derivatives of $X^a(z)$ at the origin. Let $\mathcal{J}^N X$ be the corresponding bundle over Σ. We have natural forgetful maps $JX \to J^N X$ and $\mathcal{J}X \to \mathcal{J}^N X$ (in fact, JX and $\mathcal{J}X$ are the inverse limits of $J^N X$ and $\mathcal{J}^N X$, respectively, as $N \to \infty$). The jet-evaluation observables that depend only on the first $(N-1)$ derivatives of X^μ and ψ^μ correspond precisely to the differential forms on $\mathcal{J}X$ obtained by pull-back from $\mathcal{J}^N X$. $\square$

In addition to the jet-evaluation observables, which do not depend on the momenta variables p_a, π_a and their complex conjugates, there are also observables that do depend on them. For instance, each global vector field on X which locally reads as

$$v = v^b(X^a, X^{\overline{a}})\frac{\partial}{\partial X^b} + v^{\overline{b}}(X^a, X^{\overline{a}})\frac{\partial}{\partial X^{\overline{b}}}$$

gives rise to the observable

$$
\begin{aligned}
v(z) = {} & {:}v^b(X^a(z), X^{\overline{a}}(\overline{z}))p_b(z){:} \\
& + {:}\frac{\partial v^b}{\partial X^c}(X^a(z), X^{\overline{a}}(\overline{z}))\pi_b(z)\psi^c(z){:} + {:}\frac{\partial v^b}{\partial X^{\overline{c}}}(X^a(z), X^{\overline{a}}(\overline{z}))\pi_b(z)\psi^{\overline{c}}(z){:} \\
& + {:}v^{\overline{b}}(X^a(z), X^{\overline{a}}(\overline{z}))p_{\overline{b}}(\overline{z}){:} \\
& + {:}\frac{\partial v^{\overline{b}}}{\partial X^c}(X^a(z), X^{\overline{a}}(\overline{z}))\pi_{\overline{b}}(z)\psi^c(z){:} + {:}\frac{\partial v^{\overline{b}}}{\partial X^{\overline{c}}}(X^a(z), X^{\overline{a}}(\overline{z}))\pi_{\overline{b}}(z)\psi^{\overline{c}}(z){:}\,,
\end{aligned}
\tag{4.14}
$$

which corresponds to the action of the Lie derivative $\mathcal{L}_v$ on the differential forms.

In particular, to purely holomorphic differential forms or vector fields correspond purely chiral fields (i.e., the ones annihilated by $\partial_{\overline{z}}$). Yet more general fields may be obtained from global differential operators on X. If these operators are holomorphic, then we obtain fields from the chiral algebra of our model, also known as (the space of global section of) the chiral de Rham complex. We will discuss it in more detail in Section 4.5.

4.5. **Chiral algebra of the sigma model and chiral de Rham complex.** Now consider the chiral algebra of the sigma model in the infinite radius limit. Because of the state–operator correspondence, it is isomorphic to the space of states of the model which are annihilated by the operator $\overline{L}_{-1}$ (a Fourier coefficient of the field $\overline{T}(\overline{z})$ given by formula (4.9)) which corresponds to the derivative $\partial_{\overline{z}}$ (the chiral states). Locally, for each coordinate patch $U_\beta \subset X$ isomorphic to $\mathbb{C}^N$, we have the space of states of the free theory on $\mathbb{C}^N$ (more precisely, its version in which the L^2 condition on the zero modes is replaced by the smoothness condition, see Section 4.1). Its subspace of states annihilated by the operator $\overline{L}_{-1}$ is therefore isomorphic to the Fock module $\mathcal{F}$ of the free *chiral* $\beta\gamma$-*bc* system on U_β, as defined in Section 4.1. A global chiral state is therefore a collection of local chiral states Ψ_β, that is, elements of the chiral Fock module $\mathcal{F}$ corresponding to the patch U_β, which are compatible on the overlaps $U_\beta \cap U_\gamma$.

Formulas (4.6) for the transformation of chiral fields under holomorphic changes of coordinates coincide with the formulas given in [40]. Therefore we arrive at the definition of the *chiral de Rham complex* from [40] (see also [24], Sect. 18.5). Thus, we find that *the chiral algebra of the sigma model in the infinite radius limit is the space of global sections over* X *of the chiral de Rham complex*, as was previously observed in [25] (see also [37, 56]).

However, in contrast to most of the mathematical literature, we are not interested in the chiral algebra *per se*. Rather, we are interested in the full quantum field theory in the infinite radius limit $\overline{\tau} \to \infty$, in which the chiral and anti-chiral sectors are combined in a non-trivial way. Perturbatively, the space of states is isomorphic to the chiral-anti-chiral de Rham complex discussed above. The inclusion of the instantons requires that we take into account the non-trivial self-gluings of this space, realized as the space of delta-forms on semi-infinite strata in $\widetilde{LX}$. This changes the structure of the space of states and the correlation functions and leads to the logarithmic mixing of states and operators discussed in Section 5 below.

We now consider some explicit examples of sigma models. As our first example we take up the case of an elliptic curve $\mathbb{T} = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. Though this is essentially a free field theory, it is instructive to see how the familiar description of the space of states at finite radius changes when we take the infinite radius limit $\overline{\tau} \to \infty$. (In Section 5.5 we will compute the bosonic part of the partition function of this model and show how it can be obtained as a limit of the well-known partition function of the free bosonic field theory compactified on a torus.) We will then consider the target manifold $\mathbb{CP}^1$ and relate the description of the space of states given above with the one obtained in [25] using "holomortex operators".

4.6. **The case of a torus.** Consider the supersymmetric sigma model with the target manifold elliptic curve $\mathbb{T} = \mathbb{C}/2\pi(\mathbb{Z}+\mathbb{Z}T)$, where T is in the upper-half plane. Note that in our discussion above we had assumed that the target manifold is simply-connected. The case at hand is different, as $\pi_1(\mathbb{T}) = \mathbb{Z}^2$. This means that the loop space $L\mathbb{T}$ has components labeled by $\mathbb{Z}^2$, which we will denote by $L\mathbb{T}_\lambda, \lambda \in \mathbb{Z}^2$. In addition, the Floer function is not single-valued on any of these components, and so it is necessary to consider an $H_2(\mathbb{T}, \mathbb{Z}) = \mathbb{Z}$-covering of each of them. This covering is defined as follows: for each component $L\mathbb{T}_\lambda$ we have to choose a loop $\gamma^0_\lambda : \mathbb{S}^1 \to \mathbb{T}$ which is in the homotopy class λ. Then points of the covering $\widetilde{L\mathbb{T}}_\lambda$ are by definition pairs $(\gamma, [\widetilde{\gamma}])$, where $\gamma \in L\mathbb{T}_\lambda$ and $[\widetilde{\gamma}]$ is the equivalence class of maps $\widetilde{\gamma} : \mathbb{S} \times [0, 1] \to \mathbb{T}$ such that

$$\widetilde{\gamma}(\mathbb{S}^1 \times 0) = \gamma^\lambda_0 , \qquad \widetilde{\gamma}(\mathbb{S}^1 \times 1) = \gamma,$$

modulo the equivalence relation identifying any two maps $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ satisfying these boundary conditions whose difference in $H_2(\mathbb{T}, \mathbb{Z})$ is equal to 0. Then the Floer function (2.15) lifts to a single-valued function on $\widetilde{L\mathbb{T}}_\lambda$ (compare with the general construction in Section 3.2).

Note that for $\lambda = 0$ we may choose as the initial loop γ^0_0 any constant map and then the definition of $\widetilde{L\mathbb{T}}_0$ is the same as discussed above: we consider equivalence classes of maps from the disc D to $\mathbb{T}$. But for $\lambda \neq 0$ there are no maps $D \to \mathbb{T}$ which are in the homotopy class λ on the boundary. Instead, points of the covering are represented by maps from the cylinder $\mathbb{S}^1 \times [0, 1]$ with prescribed image on one boundary circle. This description enables us to identify each component $L\mathbb{T}_\lambda$ with the product of $\mathbb{T}$, corresponding to the zero mode, and an infinite-dimensional vector space corresponding to other modes (see formula (4.16) below). For $\lambda = 0$, our Floer function becomes a Bott-Morse function on $\widetilde{L\mathbb{T}}_0$; its critical points are the constant maps $D \to \mathbb{T}$ and their translates by the group $\mathbb{Z}$ of deck transformations of the covering. But on all other components this function has *no critical points*. We are therefore rather in the situation of a circle $\mathbb{R}/2\pi\mathbb{Z}$, equipped with the multi-valued function $x \mapsto \mu x$ considered in Section 3.4. More precisely, we have that kind of function along the first factor of the decomposition $\mathbb{T} \times \mathbb{C}^\infty$, and along the second factor we are in the situation of the target manifold $X = \mathbb{C}$. Therefore the space of states also decomposes into the tensor product of the space corresponding to the zero mode, which exhibits the same kind of spectrum as in Section 3.4, and the space of delta-forms, or, more concretely, the tensor product of Fock representations of the chiral and anti-chiral $\beta\gamma$-bc systems (see Section 4.1), without the zero modes.

Let us describe this space of states more explicitly. First of all, the states supported on different components of $L\mathbb{T}$ are distinguished by different winding numbers, as in the case of the sigma model at the finite radius. Therefore we obtain a decomposition of the space of states[6] $\mathcal{H}$ into sectors corresponding to different winding numbers

$$\mathcal{H} = \bigoplus_{m,n\in\mathbb{Z}} \mathcal{H}_{m,n}. \tag{4.15}$$

To describe the sectors $\mathcal{H}_{m,n}$, we expand a general smooth map from the circle with coordinate σ, where $\sigma \sim \sigma + 2\pi$, to $\mathbb{T}$ in Fourier series

$$X(\sigma) = \omega\sigma + X_0 + \sum_{n\neq 0} X_n e^{-in\sigma}, \tag{4.16}$$

where ω is the *winding operator*, which takes values in the lattice $\mathbb{Z} + \mathbb{Z}T$. It acts on $\mathcal{H}_{m,n}$ by multiplication by $m + nT$. The zero mode X_0 is a periodic variable taking values in $\mathbb{T} = \mathbb{C}/2\pi(\mathbb{Z} + \mathbb{Z}T)$, and the remaining modes take arbitrary complex values.

The space $\mathcal{H}_{m,n}$ is generated by a vector $|m,n\rangle$, which is annihilated by $X_k, \psi_k, k > 0$, $p_k, \pi_k, k \geq 0$, and their complex conjugates. Thus, we have

$$\mathcal{H}_{m,n} = \Omega(X_0, \overline{X}_0, \psi_0, \overline{\psi}_0) \otimes \mathbb{C}[X_k, p_k, \overline{X}_k, \overline{p}_k]_{k<0} \otimes \Lambda[\psi_k, \pi_k, \overline{\psi}_k, \overline{\pi}_k]_{k<0} \cdot |m,n\rangle, \tag{4.17}$$

where $\Omega(X_0, \overline{X}_0, \psi_0, \overline{\psi}_0)$ is the space of differential forms on $\mathbb{T}$ realized in terms of the zero modes $X_0, \overline{X}_0, \psi_0, \overline{\psi}_0$. It is spanned by the monomials

$$e^{ir(X_0\overline{T} - \overline{X}_0 T)/(\overline{T}-T) + is(X_0 - \overline{X}_0)/(T-\overline{T})} \psi_0^p \overline{\psi}_0^{\overline{p}}, \qquad r, s \in \mathbb{Z}, \quad p, \overline{p} = 0, 1. \tag{4.18}$$

Let us discuss the state–operator correspondence. The fields $\Psi_{m,n}(z,\overline{z})$ corresponding to the winding states $|m,n\rangle$ are versions of the *holomortex operators*[7] introduced in [25] in the case of the target manifold $X = \mathbb{C}^\times$. These are analogues of the well-known vortex operators responsible for the winding in the sigma model at finite radius. They satisfy the OPE

$$X(z)\Psi_{m,n}(w,\overline{w}) = (m + nT)\log(z - w)\Psi_{m,n}(w,\overline{w}) + \dots,$$
$$\overline{X}(\overline{z})\Psi_{m,n}(w,\overline{w}) = (m + n\overline{T})\log(\overline{z} - \overline{w})\Psi_{m,n}(w,\overline{w}) + \dots$$

Using this OPE, we find the fields $\Psi_{m,n}(z,\overline{z})$ explicitly:

$$\Psi_{m,n}(z,\overline{z}) = e^{i\int^z \left((m+nT)P + (m+n\overline{T})\overline{P}\right)}, \tag{4.19}$$

where

$$e^{\int^z (\alpha P + \beta\overline{P})} := \exp\left(\alpha \int_{z_0}^z p(w)dw + \beta \int_{z_0}^z \overline{p}(\overline{w})d\overline{w}\right).$$

This formula *a priori* depends on the choice of the point z_0 and the contour of integration. However, under a correlation function this ambiguity disappears because of the "charge conservation" condition: the total winding should be equal to zero. If this condition is satisfied, then we can pair the integrals in the exponent to obtain a linear combination of integrals between the points of insertion of these holomortex operators

[6] Here, as before, we discuss the space of "in" states; the structure of the space of "out" states is similar.

[7] This terminology is a shorthand for "holomorphic vortex".

(see [25], Section 2.2, for more details). A more precise algebraic construction of these fields as operators acting on $\mathcal{H}$ is obtained in the same way as in [25], Section 5.1.

To obtain the fields corresponding to other states in $\mathcal{H}$, we use the description of $\mathcal{H}$ given in formulas (4.17) and (4.18). This description gives us a natural basis of monomials. We then replace the variables X_k and $\overline{X}_k$ by $\frac{1}{(-k)!}\partial_z^{-k}X(z)$ and $\frac{1}{(-k)!}\partial_{\overline{z}}^{-k}\overline{X}(\overline{z})$, respectively, p_k and $\overline{p}_k$ by $\frac{1}{(-k-1)!}\partial_z^{-k-1}p(z)$ and $\frac{1}{(-k-1)!}\partial_{\overline{z}}^{-k-1}\overline{p}(\overline{z})$, respectively, and similarly for the fermionic variables. The normally ordered product of these fields is the field associated to a given monomial in the space of states.

4.7. **The case of $\mathbb{P}^1$.** Now we consider the supersymmetric sigma model with the target manifold $\mathbb{P}^1 = \mathbb{CP}^1$. We can describe this model (in the infinite radius limit) explicitly in the case of a special B-field; namely,

$$\frac{1}{2}\tau(\delta_0^{(2)} + \delta_\infty^{(2)}), \tag{4.20}$$

where $\delta_0^{(2)}$ and $\delta_\infty^{(2)}$ are the delta-like two-forms on $\mathbb{P}^1$ supported at two fixed points, 0 and ∞, and τ is a complex parameter. We choose the overall factor $\frac{1}{2}$ so that the integral of the B-field is equal to τ. More generally, we could take $\tau_1\delta_0^{(2)} + \tau_2\delta_\infty^{(2)}$, where $\tau_1 + \tau_2 = \tau$.

The states of our model have a path integral interpretation where we integrate over maps from the two-dimensional disc D of radius 1 to $\mathbb{P}^1$. More precisely, the states are represented by the path integrals of this form

$$\Phi(\gamma) = \int\limits_{\Phi: D\to\mathbb{P}^1, \Phi|_{\mathbb{S}^1}=\gamma} \mathcal{O}_1(z_1)\dots\mathcal{O}_n(z_n)e^{-S}, \tag{4.21}$$

where $\mathcal{O}_1,\dots,\mathcal{O}_n$ are some observables, and $z_1,\dots,z_n$ are their positions on D. As we have stressed before, in the infinite radius limit the path integral localizes on the holomorphic maps $D \to \mathbb{P}^1$. Such a map may be described as a meromorphic function on D. Let $w_1^+,\dots,w_m^+$ be the zeros of this map and $w_1^-,\dots,w_n^-$ the poles. Each zero and pole give us a factor of $q^{1/2} = e^{\tau/2}$ in our path integral from the B-field (4.20).

Therefore we obtain that the result may be written as the following state of the theory with the target $\mathbb{C}^\times = \mathbb{P}^1\backslash\{0,\infty\}$:

$$\begin{aligned}\sum_{m,n}\frac{q^{m/2}}{m!}\frac{q^{n/2}}{n!}&\int\Psi^+(w_1^+)\dots\Psi^+(w_m^+)d^2w_1^+\dots d^2w_m^+ \\ &\cdot\int\Psi^-(w_1^-)\dots\Psi^-(w_n^-)d^2w_1^-\dots d^2w_n^- \cdot|A\rangle,\end{aligned} \tag{4.22}$$

where $|A\rangle$ is a state in the theory on $\mathbb{C}^\times$ and $\Psi^+(w), \Psi^-(w)$ are vertex operators in this theory corresponding to the insertion of zeros and poles.

Summing up (4.22), we obtain

$$\exp\left(q^{1/2}\int\left(\Psi^+(w)+\Psi^-(w)\right)d^2w\right)\cdot|A\rangle.$$

This is a state in the theory with the target manifold $\mathbb{C}^\times$ which is *deformed* by the operators $q^{1/2}\Psi^+(w)$ and $q^{1/2}\Psi^-(w)$. Thus, we obtain a free field realization of the two-dimensional sigma model with the target manifold $\mathbb{P}^1$ and with the B-field (4.20), as a deformation of the free field theory with the target $\mathbb{C}^\times$ by these operators. This realization was found in [25]. Now we see that this deformation naturally arises in the framework of the general formalism developed in the present paper.

The vertex operators $\Psi^\pm(w)$ have been determined explicitly in [25], Section 3.1: these are the *holomortex operators*

$$\Psi^\pm(z) = e^{\pm i\int^z(p(w)dw+\overline{p}(\overline{w})d\overline{w})}\pi(z)\overline{\pi}(\overline{z}), \tag{4.23}$$

where $p(z), \overline{p}(\overline{z})$ are the bosonic momenta variables corresponding to the realization of $\mathbb{C}^\times$ as $\mathbb{C}/2\pi i\mathbb{Z}$, and $\pi(z)$ and $\overline{\pi}(\overline{z})$ are the corresponding fermionic variables (a different sign in the exponential, as compared with [25], is due to the fact that the action we use here, and hence the corresponding OPE, differ by a sign from those used in [25]). Using these formulas, we can express correlation functions of the sigma model with the target $\mathbb{P}^1$ as multiple integrals of correlation functions of the free theory on $\mathbb{C}^\times$. Some examples of such integrals may be found in [25], and more examples are presented below in Section 6.7.

Since the sigma model on $\mathbb{P}^1$ may be realized as a deformation of a free conformal field theory by strictly marginal operators, we obtain that the Hamiltonian and the supercharges in the theory on $\mathbb{P}^1$ may be obtained from those of the free theory by some deformation. This deformation turns out to be "nilpotent", in the sense that the diagonalizable Hamiltonian of the free theory is deformed by an upper triangular matrix giving rise to Jordan blocks. In quantum mechanics this mechanism was discussed in detail in Part I. In the case of the sigma model on $\mathbb{P}^1$ in the background of a non-trivial $\mathbb{C}^\times$-gauge field an analogous derivation of the Hamiltonian will be given below in Section 7.5.

These results may be generalized from $\mathbb{P}^1$ to other toric varieties, along the lines of [25].

5. Correlation functions

From the Lagrangian point of view, the correlation functions are represented by the path integral

$$\langle \mathcal{O}_1(p_1)\dots\mathcal{O}_n(p_n)\rangle = \int\limits_{\Phi:\Sigma\to X} \mathcal{O}_1(p_1)\dots\mathcal{O}_n(p_n)e^{-S},$$

where S is the action (2.7). As discussed above (see Part I, Sections 2.4–2.6, for the quantum mechanical version), this path integral localizes on *holomorphic maps* $\Phi:\Sigma\to X$, and we obtain a sum over the topological types of such maps, which correspond to the homology class of the image of Σ under Φ in $H_2(X,\mathbb{Z})$,

$$\sum_{\beta\in H_2(X,\mathbb{Z})} e^{-\int_\beta\tau} \int\limits_{\mathcal{M}_\Sigma(X,\beta)} \mathcal{O}_1(p_1)\dots\mathcal{O}_n(p_n) \tag{5.1}$$

(of course, only positive elements of $H_2(X,\mathbb{Z})$ with respect to the Kähler form will give rise to non-trivial contributions). Here $\mathcal{M}_\Sigma(X,\beta)$ denotes the moduli space of holomorphic maps Φ from the *parametrized* Riemann surface Σ to X of degree β, i.e., such that $[\Phi(\Sigma)] = \beta$. In what follows we will often assume that there are sufficiently many marked points $p_1, \dots, p_n$ on Σ, so that the marked curve has no continuous automorphisms (this means that $n \geq 3$ if Σ has genus 0 and $n \geq 1$ if Σ has genus 1). In this case we can interpret $\mathcal{M}_{\Sigma,(p_i)}(X, \beta)$ as the fiber of the map $\pi_{g,n}$ discussed below, which will imply that the integral converges.

It is customary to choose a basis $\beta_1, \dots, \beta_N$ of $H_2(X, \mathbb{Z})$. Then this sum becomes a power series in $q_i = e^{-\int_{\beta_i} \tau}, i = 1, \dots, N$.

In this section we consider these correlation functions in more details and compute some explicit examples.

5.1. **Gromov–Witten invariants.** In the case when $\mathcal{O}_i, i = 1, \dots, n$ are evaluation observables corresponding to closed smooth differential forms $\omega_i, i = 1, \dots, n$, on X, these integrals are special cases of the *Gromov–Witten invariants*. More precisely, for each point $p_i \in \Sigma, i = 1, \dots, n$, we have the evaluation map

$$\mathrm{ev}_{p_i} : \mathcal{M}_\Sigma(X, \beta) \to X. \tag{5.2}$$

Now, for a given collection of differential forms $\omega_1, \dots, \omega_n$ on X, the β-term in the sum (5.1) is given by the integral

$$\int\limits_{\mathcal{M}_\Sigma(X,\beta)} \mathrm{ev}^*_{p_1}(\omega_1) \wedge \dots \wedge \mathrm{ev}^*_{p_n}(\omega_n). \tag{5.3}$$

Then, even though the moduli space $\mathcal{M}_\Sigma(X, \beta)$ is not compact, the integral (5.3) is convergent for smooth differential forms ω_i on X, which is assumed to be compact. This follows from the fact that the evaluation maps extend to the Kontsevich compactification (the moduli space of stable maps), as we discuss below.

More general Gromov–Witten invariants are constructed as follows. Suppose that $(\Sigma, (p_i))$ does not admit any continuous automorphisms. Let $\mathcal{M}_{g,n}(X, \beta)$ be the moduli space (more precisely, Deligne–Mumford stack) of the data $(\Sigma, (p_i), \Phi)$, where Σ is a genus g Riemann surface, $p_1, \dots, p_n$ are distinct marked points on Σ, and $\Phi : \Sigma \to X$ is a holomorphic map. Then we have a projection $\pi_{g,n} : \mathcal{M}_{g,n}(X, \beta) \to \mathcal{M}_{g,n}$, where $\mathcal{M}_{g,n}$ is the moduli space (or Deligne–Mumford stack) of genus g curves with n marked points. $\mathcal{M}_\Sigma(X, \beta)$ is nothing but the fiber of $\pi_{g,n}$ at $(\Sigma, (p_i)) \in \mathcal{M}_{g,n}$. We have the evaluation maps

$$\mathrm{ev}_i : \mathcal{M}_{g,n}(X, \beta) \to X, \tag{5.4}$$

corresponding to evaluation at the ith point p_i (so that $\mathrm{ev}_{p_i} = \mathrm{ev}_i |_{\mathcal{M}_{g,n} \times p_i}$). The general Gromov-Witten invariants are obtained by taking the push-forward

$$\pi_{g,n*}(\mathrm{ev}^*_1(\omega_1) \wedge \dots \wedge \mathrm{ev}^*_n(\omega_n)), \tag{5.5}$$

which is a differential form on $\mathcal{M}_{g,n}$. In particular, (5.3) occurs as a special case when the degree of this differential form is equal to zero, so we obtain a function on $\mathcal{M}_{g,n}$. Then its value at $(\Sigma, (p_i)) \in \mathcal{M}_{g,n}$ is given by (5.3). More general observables give rise to differential forms of positive degree on $\mathcal{M}_{g,n}$.

The fibers of the map $\pi_{g,n}$ are non-compact, hence *a priori* the integrals obtained via this push-forward are not well-defined. To show that they are, we replace $\mathcal{M}_{g,n}(X, \beta)$ by its compactification, the Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$ and $\mathcal{M}_{g,n}$ by its Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$. The map $\pi_{g,n}$ extends to a map

$$\overline{\pi}_{g,n} : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}. \tag{5.6}$$

which is already proper (has compact fibers). The evaluation maps ev_i also extend to $\overline{\mathcal{M}}_{g,n}(X, \beta)$. Then the Gromov–Witten invariants may be defined by formula (5.5) with $\pi_{g,n}$ replaced by $\overline{\pi}_{g,n}$. Since $\overline{\pi}_{g,n}$ is proper, we see that these invariants are well-defined. Hence the original integrals (5.1) are also well-defined (for smooth differential forms ω_i).

These are the correlation functions of what is often referred to as the "sigma model coupled to gravity", with the observables being the "cohomological descendants" of the evaluation observables.

5.2. **BPS vs. non-BPS.** Among the observables, an important role is played by the *topological*, or *BPS observables*. These are the observables annihilated by the total supercharge $\mathcal{Q} + \overline{\mathcal{Q}}$ of the model. The supercharges $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ locally act on fields by the formulas

$$\mathcal{Q} \cdot \mathcal{O}(w, \overline{w}) = \left[\int ip_a(z)\psi^a(z)dz, \mathcal{O}(w, \overline{w}) \right],$$
$$\overline{\mathcal{Q}} \cdot \mathcal{O}(w, \overline{w}) = \left[\int ip_{\overline{a}}(\overline{z})\psi^{\overline{a}}(\overline{z})d\overline{z}, \mathcal{O}(w, \overline{w}) \right].$$

In particular, the supercharge acts on evaluation observables $\mathcal{O}_\omega$ corresponding to the differential forms ω as the de Rham differential d:

$$(\mathcal{Q} + \overline{\mathcal{Q}}) \cdot \mathcal{O}_\omega = \mathcal{O}_{d\omega}.$$

Therefore the BPS evaluation observables correspond to the closed differential forms on X.

In the above definition of the Gromov–Witten invariants we considered the BPS observables corresponding to closed differential forms $\omega_i, i = 1, \ldots, n$. However, *any* differential form ω on X gives rise to a legitimate observable in our theory, and the correlation functions of such observables are still given by the same integrals (5.3). The difference is, of course, that unlike the correlation functions of the BPS observables, the correlation functions of more general observables depend of $\overline{\tau}$, so this answer is correct only at $\overline{\tau} = -i\infty$.

Our goal in this paper is to go beyond the topological sector of the sigma model and consider the correlation functions of non-BPS observables. The reasons for doing this have already been explained in Part I and in the Introduction to this Part. Here we want to stress that if we only consider the BPS observables, we will not be able to gain any insights into the structure of the space of states of our theory beyond the ground states.

Indeed, from the Hamiltonian perspective, when Σ is the cylinder $\mathbb{S} \times \mathbb{R}$, the correlation function is given by the formula

$$\langle \mathcal{O}_1(p_1) \dots \mathcal{O}_n(p_n) \rangle = \langle 0 | \mathcal{O}_n e^{(t_{n-1}-t_n)H} \dots e^{(t_1-t_2)H} \mathcal{O}_1 | 0 \rangle, \tag{5.7}$$

Here t_i denotes the time coordinate of the point p_i (along the $\mathbb{R}$ factor of Σ), so that $|z_i| = e^{t_i}$, and we assume that $t_1 < t_2 < \dots < t_n$. Thus, in principle, we could derive information about the spectrum of the Hamiltonian and its diagonalizability by analyzing the correlation functions. For example, the appearance of terms of the form $(t_i - t_{i+1})e^{N(t_i - t_{i+1})}$ (which we will observe below), but not $(t_i - t_{i+1})^m e^{N(t_i - t_{i+1})}, m > 1$, means that the Hamiltonian H has a Jordan block of size 2 with the generalized eigenvalue N.

However, BPS observable are not suitable for this purpose. Indeed, since the vacuum state is annihilated by $\mathcal{Q}$, any correlation function of BPS observables (which commute with $\mathcal{Q}$) is automatically equal to 0 as soon as one of the observables is $\mathcal{Q}$-exact, that is, equal to the commutator of another observable and $\mathcal{Q}$. This means that the correlation functions of BPS observables only depend on their $\mathcal{Q}$-cohomology classes. One can modify any BPS observable by $\mathcal{Q}$-exact terms so as to make it commute with $\mathcal{Q}$ and $\mathcal{Q}^*$, where $[\mathcal{Q}, \mathcal{Q}^*]_+ = H$. Such a representative transforms a vacuum state, which is annihilated by $\mathcal{Q}, \mathcal{Q}^*$ to a state, which is again annihilated by $\mathcal{Q}$ and $\mathcal{Q}^*$, and hence by their anti-commutator H. Therefore no excited states on which H acts non-trivially, can arise in formula (5.7). Hence we do not learn anything about the spectrum of the model. In contrast, non-BPS observables transform ground states to excited states, and, as we will see below, we can learn a lot about the structure of the space of states from their correlation functions.

In addition, considering non-BPS observables allows us to bring into play some important $\mathcal{Q}$-exact observables, which are "invisible" in the BPS sector.

Examples are the observables corresponding to Lie derivatives with respect to vector fields on X given by formula (4.14). These observables are $\mathcal{Q}$-exact, as follows from the Cartan formula $\mathcal{L}_v = \{d, \imath_v\}$ for the Lie derivative $\mathcal{L}_v$.

This means that if we insert the observable $v(z)$ into a correlation function of BPS observables, then the result will always be zero. But these observables, and more general observables of this type corresponding to differential operators on X, play a very important role in the full theory. Indeed, on a Kähler manifold we often have a large Lie algebra of global holomorphic vector fields, and the corresponding Lie derivatives will belong to the chiral algebra of our theory (see Section 4.5 below). Hence they give rise to non-trivial Ward identities which impose relations between correlation functions in our model. But in order to obtain non-trivial correlation functions involving these operators we must include non-BPS observables.

Let us summarize: the correlation functions of the supersymmetric sigma model in the infinite radius limit are expressed in terms of integrals over finite-dimensional moduli spaces of holomorphic maps. Our goal is to use them to obtain information about the space of states of the theory and the action of the Hamiltonian. We are particularly interested in the appearance of logarithms in the correlation functions (when they are written in terms of coordinates z_i on the worldsheet $\mathbb{P}^1$), which indicate that the Hamiltonian is not diagonalizable.

However, the correlation functions of the BPS observables which have been almost exclusively studied in the literature up to now (and which correspond to the Gromov-Witten invariants) do not contain logarithms. In order to observe the appearance of logarithms, we must consider non-BPS observables. We note that the Hamiltonian is also diagonalizable on all purely chiral (and anti-chiral) states; thus, the chiral algebra of the theory, which we will discuss in more detail in Section 4.5 below, is free of logarithms.

5.3. **A simple example of non-BPS correlation function.** As our first example, we consider the target manifold $X = \mathbb{CP}^1$ and the worldsheet $\Sigma = \mathbb{P}^1$. Then the instantons are holomorphic maps from the parametrized $\mathbb{P}^1$ to $\mathbb{P}^1$. The moduli spaces are labeled by non-negative integers in this case corresponding to the degree of such a map. The moduli space of degree d instantons $\mathcal{M}_d = \mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1, 1)$ (in the notation of Section 5.1) has complex dimension $2d+1$. We consider the case when $d = 1$. Then the corresponding moduli space $\mathcal{M}_1$ is isomorphic to $PGL_2(\mathbb{C})$. Consider the correlator of the following evaluation observables:

$$\langle \mathcal{O}_{\omega_0}(0)\mathcal{O}_{\omega_\infty}(\infty)\mathcal{O}_{\omega_{\mathrm{FS}}}(1)\mathcal{O}_h(z,\overline{z})\rangle_{d=1}, \tag{5.8}$$

where

$$\omega_0 = \delta^{(2)}(x)d^2x\ , \quad \omega_\infty = \delta^{(2)}\left(\frac{1}{x}\right)\frac{d^2x}{|x|^4}\ , \quad \omega_{\mathrm{FS}} = \frac{d^2x}{(1+|x|^2)^2},$$

$$h = \frac{1}{1+|x|^2}. \tag{5.9}$$

The two-forms are closed, and hence correspond to BPS observables, but the function h is not. Its inclusion allows us to observe the logarithmic effects.

The delta-function two-forms ω_0, ω_∞, supported at $x = 0$ and $x = \infty$, respectively, reduce the integration over $\mathcal{M}_1$ to that over the locus consisting of the holomorphic maps of the form

$$x(w) = Aw. \tag{5.10}$$

Thus, (5.8) is equal to:

$$q\int \frac{d^2A}{(1+|A|^2)^2}\frac{1}{1+z\overline{z}|A|^2} \propto -q\left(\frac{1}{1-z\overline{z}} + \frac{z\overline{z}\log(z\overline{z})}{(1-z\overline{z})^2}\right) \tag{5.11}$$

(see Part I for details on the computation of integrals of this type) where $q = e^{-\tau}$ is the instanton factor. The $z, \overline{z}$-dependence in (5.11) implies the logarithmic nature of the two-dimensional conformal theory, in the same way as in the case of the quantum mechanical models analyzed in Part I. We recall that from the Hamiltonian point of view the correlation function is represented as the matrix element of the form (5.7). Therefore the monomials $(z\overline{z})^n$ correspond to eigenstates of the Hamiltonian with the eigenvalue n, whereas the terms $\log(z\overline{z})\cdot(z\overline{z})^n$ correspond to a Jordan block of size 2 with the generalized eigenvalue n. Therefore, expanding the right hand side of (5.11) in powers of $z\overline{z}$ (here we assume that $|z| < 1$), we find that the spectrum of the Hamiltonian contains all non-negative integers as well as Jordan blocks of size two with the generalized eigenvalues equal to all positive integers.

When we move away from the point $\overline{\tau} = -i\infty$ (back to finite radius), anti-instantons start making contributions. As the result, the Jordan blocks deform to matrices with small non-zero entries below the diagonal, and the Hamiltonian becomes diagonalizable.

5.4. **Spectrum from the genus one correlation functions.** Next, we discuss the genus one correlation functions in the sigma model with the target $\mathbb{P}^1$. We want to use them to probe the spectrum of our theory, the way we probed the spectrum in the analogous quantum mechanical models in Part I (using factorization over intermediate states). Again, in order to detect the spectrum on the excited states, we need to throw in some non-BPS observables.

The simplest correlation function of this type is that of the evaluation observables with the periodic boundary conditions on the fermions. We keep the target space $X = \mathbb{P}^1$, and the worldsheet Σ is the elliptic curve, with the modular parameter $q = \exp(2\pi i\tau)$. Let z denote the linear coordinate on Σ, so that

$$z \sim z + m + n\tau, \qquad m, n \in \mathbb{Z}.$$

Let $\mathcal{O}_\omega$ denote the evaluation observable corresponding to a differential form ω on the target $\mathbb{P}^1$.

Let $\omega = \omega(X, \overline{X})dXd\overline{X} \in \Omega^2(\mathbb{P}^1)$, $f \in C^\infty(\mathbb{P}^1)$. We start with the five-point function

$$\left\langle \mathcal{O}_{\omega_0}(z_1)\mathcal{O}_{\omega_\infty}(z_2)\mathcal{O}_{\omega_\infty}(z_3)\mathcal{O}_\omega(z_4)\mathcal{O}_f(z_5) \right\rangle_q =$$

$$\begin{aligned}(5.12)\qquad &\int_{\mathbb{P}^1} \omega\left(A\frac{\vartheta_{11}(z_4-z_1)\vartheta(z_4-z_0)}{\vartheta_{11}(z_4-z_2)\vartheta(z_4-z_3)}, \bar{A}\overline{\frac{\vartheta_{11}(z_4-z_1)\vartheta(z_4-z_0)}{\vartheta_{11}(z_4-z_2)\vartheta(z_4-z_3)}}\right) \\ &\times\left|\frac{\vartheta_{11}(z_4-z_1)\vartheta(z_4-z_0)}{\vartheta_{11}(z_4-z_2)\vartheta(z_4-z_3)}\right|^2 \, d^2A \, \cdot \, f\left(A\frac{\vartheta_{11}(z_5-z_1)\vartheta(z_5-z_0)}{\vartheta_{11}(z_5-z_2)\vartheta(z_5-z_3)}\right),\end{aligned}$$

where we chose for simplicity some of the observables to correspond to the delta-forms supported at $0, \infty \in X$, as indicated by the subscripts in (5.12) (compare with formula (5.11)). These delta-forms force the holomorphic maps from Σ to X to be of the special form:

$$X(z) = A\frac{\vartheta_{11}(z-z_1)\vartheta(z-z_0)}{\vartheta_{11}(z-z_2)\vartheta(z-z_3)},$$

where $A \in \mathbb{P}^1$ is the remaining modulus, and $z_0 = z_2 + z_3 - z_1$. To illustrate our point about the logarithmic nature of the sigma model Hamiltonian we take the further simplifying limit, where $z_2, z_3 \to 0$. Let us also take for ω the usual Fubini-Study form. In this case the integral (5.12) simplifies to

$$\int_{\mathbb{P}^1} \frac{d^2x}{(1+|x|^2|q_{\text{eff}}|^2)^2} f(x)$$

where

$$q_{\text{eff}} = \frac{\wp(z_4) - \wp(z_1)}{\wp(z_5) - \wp(z_1)},$$

where

$$x = A(\wp(z_5) - \wp(z_1))$$

and $\wp$ is the Weierstrass' function. The result can be now expanded in powers of q, as q_{eff} has such an expansion, following from the q-expansion of the $\wp$-function, and will be of the form (5.11), i.e., with integer powers of q or with $\log(q)$ multiplying the integer powers of q.

On the other hand, the correlation function (5.12) can be written as a trace:

$$\mathrm{Tr}_{\mathcal{H}} \left((-1)^F \, q^H e^{-z_5 H} f e^{(z_5 - z_4)H} \mathcal{O}_\omega e^{(z_4 - z_1)H} \mathcal{O}_{\omega_0} e^{z_1 H} \mathcal{O}_{\omega_\infty} \mathcal{O}_{\omega_\infty} \right).$$

Thus the q-dependence of (5.4) gives us the information about the spectrum of conformal dimensions in our theory. It confirms the integrality of the spectrum and the existence of Jordan blocks in the action of the Hamiltonian in the sigma model with the target $\mathbb{P}^1$.

5.5. **Sigma model on the torus: from finite to infinite radius.** Consider the sigma model with the torus $\mathbb{T}$ as the target manifold discussed in Section 4.6. An attractive feature of this model is that the correlation functions may be computed exactly at both finite and infinite radius, and this can help us learn how the correlation functions behave in the limit of infinite radius, $\overline{\tau} \to \infty$. After all, eventually we would like to compute the (non-BPS) correlation functions in the sigma models at the finite radius using those in the infinite radius limit as the starting point, by some sort of perturbation theory.

The sigma model on the torus is also useful in that it allows us to see explicitly how the spectrum of a well-defined unitary theory acquires an imaginary part or an unbounded branch, in the $\overline{\tau} \to \infty$ limit, as we have argued in Section 3.4 in the context of quantum mechanics on the non-simply connected target manifolds.

5.5.1. *Finite radius Hamiltonian.* For the sake of generality consider the sigma model on the $2d$-dimensional torus $\mathbb{T}^{2d}$, with the translation invariant metric G_{mn} and the antisymmetric two-form B_{mn}.[8] In Sections 5.5.1–5.5.4 we will consider the purely bosonic theory. We will be interested in its partition function, which coincides with the partition function of the bosonic sector of the supersymmetric sigma model. Then in Section 5.5.5 we will add fermions.

The bosonic theory has the following action on $\Sigma = \mathbb{S}^1 \times I$:

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma G_{mn} \left(\partial_t X^m \partial_t X^n + \partial_\sigma X^m \partial_\sigma X^n \right) + i B_{mn} \left(\partial_t X^m \partial_\sigma X^n - \partial_\sigma X^m \partial_t X^n \right) d\sigma dt$$

where X^m, $m = 1, \ldots, 2d$, correspond to the linear coordinates on $\mathbb{T}^{2d}$ with the period 2π. We assume the worldsheet to have Euclidean metric $dt^2 + d\sigma^2$, $\sigma \sim \sigma + 2\pi$. Thus the path integral measure is

$$DX \; e^{-S}$$

[8] Our notation in this section is slightly different from that of the main body of the paper. In particular, what we called previously the field B is now $B/4\pi\alpha'$. This is done in order to make the notation consistent with the traditional physics notation; see e.g., [46].

The momentum conjugate to X^m is then given by

$$P_m = i\frac{\delta L}{\delta \partial_t X^m} = \frac{1}{2\pi\alpha'}\left(iG_{mn}\partial_t X^n - B_{mn}\partial_\sigma X^n\right),$$

and the Hamiltonian is

$$H = \frac{1}{4\pi\alpha'}\int_{\mathbb{S}^1} \mathrm{d}\sigma\; G^{mn}\left(2\pi\alpha' P_m + B_{mk}\partial_\sigma X^k\right)\left(2\pi\alpha' P_n + B_{nl}\partial_\sigma X^l\right) + \\ + G_{mn}\partial_\sigma X^m \partial_\sigma X^n = \int_{\mathbb{S}^1} \pi\alpha' G^{mn} P_m P_n + J^m_n P_m \partial_\sigma X^n + \frac{g_{mn}}{4\pi\alpha'}\partial_\sigma X^m \partial_\sigma X^n, \tag{5.13}$$

where

$$\begin{aligned} J^n_m &= G^{kn} B_{km} \ , \\ g_{mn} &= G_{mn} + B_{am} B_{bn} G^{ab} \ , \\ g &= (1 - J^2) G. \end{aligned}$$

At the level of zero modes, we have

$$X^m \sim X^m_0 + w^m \sigma \ ,$$

and the Hamiltonian (5.13) reduces to

$$H_0 = \frac{1}{4\pi}\left(\alpha' G^{mn} k_m k_n + 2J^m_n k_m w^n + \frac{g_{mn}}{\alpha'} w^m w^n\right),$$

where $k_m \in \mathbb{Z}$ is the integer eigenvalue of the momentum zero mode,

$$-i\frac{\partial}{\partial X^m_0} = k_m \ ,$$

while $w^m \in \mathbb{Z}$ is the winding number of the X^m coordinate.

The inclusion of the oscillators, that is, the harmonics $X^m_l e^{il\sigma}$ and $P_{m,l} e^{-i\sigma l}$ with $l \in \mathbb{Z}_{\neq 0}$, simply adds non-negative integers to the eigenvalues of the Hamiltonian (modulo the zero point energies). Indeed, the B-dependence of the Hamiltonian on the X^m_l, $P_{m,l} \sim -i\frac{\partial}{\partial X_{m,l}}$ modes can be eliminated by the gauge transformation of the wave-function

$$\Psi(X) \longrightarrow \prod_{l \neq 0} e^{\frac{1}{4\pi\alpha'} il B_{mn} X^m_l X^n_{-l}} \ \cdot \ \Psi(X),$$

mapping the Hamiltonian on the non-zero modes to that of a system of harmonic oscillators

$$\alpha' G^{mn} P_{m,l} P_{n,-l} + \frac{l^2}{\alpha'} G_{mn} X^m_l X^n_{-l}$$

with the eigenvalues

$$|l|\left(N_l + \frac{1}{2}\right) \ , \ N_l \in \mathbb{Z}_{\geq 0}. \tag{5.14}$$

5.5.2. *The limit $\alpha' \to 0$.* The infinite radius limit that we are interested in is $\alpha' \to 0$.[9] The non-zero modes are not sensitive to the value of α', as we see in (5.14). However, the zero mode Hamiltonian (5.5.1) has a finite limit when $\alpha' \to 0$ only if

$$g_{mn} \underset{\alpha' \to 0}{\longrightarrow} 0 \quad \Longleftrightarrow \quad J^2 \underset{\alpha' \to 0}{\longrightarrow} 1.$$

For a positive-definite metric G_{mn} and a real two-form B_{mn} the condition $J^2 = 1$ is impossible to fulfill.

However, we know that we should allow complex-valued B_{mn} for our limit to exist. In this case $J = i\mathcal{J} + \alpha'\delta J$, where $\mathcal{J}$ is a complex structure on $\mathbb{R}^{2n}$. Let us write,

$$B_{mn} = i\,\omega_{mn} + \alpha'\Im_{mn}, \tag{5.15}$$

$$\mathcal{J} = G^{-1}\omega\ ,\ \mathcal{J}^2 = -1. \tag{5.16}$$

Then we obtain

$$H_0 = i\mathcal{J}^n_m\left(k_n w^m + i\Im_{mn'}w^m w^{n'}\right)\ , \tag{5.17}$$

with the characteristic imaginary part, exactly like in formula (3.23) for the energy levels in the quantum mechanical model on the circle.

5.5.3. *The partition function.* Let us compute the partition function of the bosonic sector of the supersymmetric sigma model on the torus,[10] in the limit $\alpha' \to 0$,

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{bos}}} q^{L_0}\overline{q}^{\overline{L_0}}.$$

It factorizes as a product of the zero mode part and the oscillator contribution. We have:

$$H = L_0 + \overline{L}_0,$$

and

$$L_0 - \overline{L}_0 = 2\int_{\mathbb{S}^1} P_m\partial_\sigma X^m = 2k_m w^m + \text{oscillators}\,.$$

The partition function becomes:

$$Z(q,\overline{q}) = \frac{1}{|\eta(q)|^{2d}}\sum_{\vec{k},\vec{w}} q^{\frac{1}{2}(1+i\mathcal{J})\vec{k}\cdot\vec{w}}\,\overline{q}^{\frac{1}{2}(1-i\mathcal{J})\vec{k}\cdot\vec{w}}\ (q\overline{q})^{i\mathcal{J}\Im(\vec{w},\vec{w})} \tag{5.18}$$

From the path integral point of view, we expect that the correlation functions may be written as a sum over holomorphic maps. In the case at hand those are the maps from the worldsheet torus $E_\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ to the target torus $\mathbb{T}^{2d}$, endowed with the complex structure $\mathcal{J}$. A holomorphic map $E_\tau \to \mathbb{T}^{2d}$ exists if and only if there exist integral vectors $\vec{w}^\vee, \vec{w} \in \mathbb{Z}^d$ such that

$$\vec{w}^\vee\Im - \frac{1}{2}(1 + i\mathcal{J})\vec{w} = 0,$$
$$\vec{w}^\vee\overline{\Im} - \frac{1}{2}(1 - i\mathcal{J})\vec{w} = 0.$$

[9]This is the limit denoted by $\overline{\tau} \to -i\infty$ in the main body of the paper, but in this section and the next, τ denotes the complex modulus of the worldsheet torus rather than the coupling constant.

[10]The full supersymmetric partition function is in the topological sector and is equal to q times the Euler characteristic of $\mathbb{T}^{2d}$, that is, 0.

Hence we should be able to represent our partition function $Z(q,\overline{q}), q = \exp 2\pi i\tau$, as a sum of delta-functions corresponding to these constraints. We indeed obtain such an expression using the Poisson resummation formula:

$$Z(q,\overline{q}) \sim \frac{1}{|\eta(q)|^{2d}} \sum_{\vec{w}^{\vee},\vec{w}\in\mathbb{Z}^d} \delta^{(2d)}\left(\vec{w}^{\vee}\mathfrak{T} - \frac{1}{2}(1+i\mathfrak{I})\vec{w}, \vec{w}^{\vee}\overline{\mathfrak{T}} - \frac{1}{2}(1-i\mathfrak{I})\vec{w}\right) e^{-2\pi \mathrm{Im}\mathfrak{T} i\mathfrak{I}\mathfrak{T}(\vec{w},\vec{w})} . \tag{5.19}$$

5.5.4. *Comparison with the computation in the infinite radius limit.* Let us compare the limit (5.18) with the trace

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{bos}}} q^{L_0}\overline{q}^{\overline{L_0}},$$

computed directly in the infinite radius limit $\alpha' \to 0$. Here $\mathcal{H}_{\mathrm{bos}}$ is the bosonic part of the space of states in the infinite radius limit that we have described in Section 4.6. According to formulas (4.15) and (4.17), $\mathcal{H}_{\mathrm{bos}}$ is the tensor product of the oscillator part and the momentum/winding part. Formula (4.8) for the stress tensor shows that the oscillators contribute the factor $1/\eta(q)^2$. The momentum/winding part is spanned by the monomials of the form

$$e^{ir(X_0\overline{T}-\overline{X}_0T)/(\overline{T}-T)+is(X_0-\overline{X}_0)/(T-\overline{T})}\cdot e^{i\int\left((m+nT)P+(m+n\overline{T})\overline{P}\right)}, \qquad r,s,m,n\in\mathbb{Z}. \tag{5.20}$$

(the first factor corresponds to momentum, see formula (4.18), and the second factor corresponds to winding, see formula (4.19)). Formulas (4.8) and (4.16) show that L_0 and $\overline{L}_0$ act on this state by the formulas

$$L_0 = ip_0\omega, \qquad \overline{L}_0 = i\overline{p}_0\overline{\omega}.$$

In the simplest case when the B-field is zero, we have $p_0 = -i\frac{\partial}{\partial X_0}, \overline{p}_0 = -i\frac{\partial}{\partial \overline{X}_0}$, and so they act on these states by the formulas

$$L_0 = \frac{(s-r\overline{T})(m+nT)}{T-\overline{T}}, \qquad \overline{L}_0 = -\frac{(s-rT)(m+n\overline{T})}{T-\overline{T}}.$$

Therefore we find that

$$\mathrm{Tr}_{\mathcal{H}_{\mathrm{bos}}} q^{L_0}\overline{q}^{\overline{L}_0} = \frac{1}{\eta(q)^2}\sum_{m,n,r,s\in\mathbb{Z}} q^{(s-r\overline{T})(m+nT)/(T-\overline{T})}\overline{q}^{-(s-rT)(m+n\overline{T})/(T-\overline{T})}. \tag{5.21}$$

This agrees with formula (5.18) in the case when $d=1$ with the zero B-field. Indeed, in this case the complex structure of the torus is uniquely determined by the metric: $T = T_1 + iT_2$,

$$G_{11} = \frac{1}{T_2}\sqrt{G}\,,\; G_{12} = \frac{T_1}{T_2}\sqrt{G}\,,\; G_{22} = \frac{|T|^2}{T_2}\sqrt{G}. \tag{5.22}$$

The remaining moduli of the torus are the volume and the B-field, which combine into:

$$U = \frac{1}{\alpha'}\left(B_{12} + i\sqrt{G}\right). \tag{5.23}$$

We take a limit $\alpha' \to 0$ while keeping U fixed. It means that B_{12} is taken to be complex and that

$$\overline{U} = U - \frac{2i}{\alpha'}\sqrt{G} \to \infty. \tag{5.24}$$

When $U = 0$ we obtain formula (5.21). For non-zero U there is an additional factor $(q\overline{q})^{U|m+nT|^2/(T-\overline{T})}$ on the right hand side. This corresponds to a shift of the eigenvalues of L_0 and $\overline{L}_0$ on the states (5.20) that is caused by the following redefinition of $p_0, \overline{p}_0$ for non-zero B-field:

$$p_0 \mapsto p_0 - i\frac{U}{T-\overline{T}}\overline{\omega}, \qquad \overline{p}_0 \mapsto \overline{p}_0 - i\frac{U}{T-\overline{T}}\omega.$$

5.5.5. *Fermions.* The fermionic contribution is universal, it does not depend on the choice of the complex structure on $\mathbb{T}^{2d}$. The fermionic part of the action is

$$S_{\text{ferm}} = \int (\pi_a \overline{\partial}_{\overline{z}}\psi^a + \overline{\pi}_{\overline{a}}\partial_z\overline{\psi}^{\overline{a}})d^2z.$$

The $\mathcal{Q}$-symmetry depends on the choice of the complex structure. In other words, the sigma model has extended fermionic symmetry on $\mathbb{T}^{2d}$ for $d > 1$. We shall not discuss this any further.

6. Logarithmic mixing of jet-evaluation observables

The most striking illustration of the logarithmic nature of the two-dimensional sigma model is the behavior of the correlation functions of the jet-evaluation observables introduced in Section 4.4. In this section we discuss these correlation functions. They are also given by integrals over the moduli spaces of stable maps, but we will see that in general they diverge at the boundary divisors corresponding to “bubbles” on the worldsheet. These integrals require regularization, and the ambiguity of the regularization scheme introduces a non-trivial mixing of operators.

In other words, we find that each jet-evaluation observable $\mathcal{O}_{\mathbb{A}}$ introduced in Section 4.4 is only well-defined perturbatively. The definition of a true operator of the sigma model corresponding to it – and its correlation functions – requires regularization. Depending on the regularization scheme, we obtain *a priori* different operators, which differ from each other by a linear combination of other operators. Those are the logarithmic partners of $\mathcal{O}_{\mathbb{A}}$. Together with $\mathcal{O}_{\mathbb{A}}$, the logarithmic partners span a subspace in the space of operators – or the space of states, via the state–operator correspondence – on which the Hamiltonian acts as a Jordan block.

This logarithmic mixing is in agreement with the description of the space of states given in Section 4.1 as an extension of the spaces of delta-forms on the ascending manifolds in $\widetilde{LX}$. It is also analogous to the similar logarithmic phenomena that we have observed and explored in quantum mechanical models in Part I.

6.1. **Jet Gromov–Witten invariants.** Recall the jet bundle $\mathcal{J}X$ over Σ, whose fiber J_pX over $p \in \Sigma$ consists of jets of holomorphic maps from a disc D_p around the point p to X. Given a holomorphic map $\Phi : \Sigma \to X$, we obtain a point in J_pX; namely, the restriction $\Phi|_{D_p}$ of Φ to $D_p \subset \Sigma$. Thus, we obtain the following generalization of the map ev_p,

$$\mathcal{J}\,\mathrm{ev}_p : \mathcal{M}_\Sigma(X,\beta) \to \mathcal{J}X, \tag{6.1}$$

Now recall that jet-evaluation observables are sections of the bundle $\Omega^\bullet_{\mathrm{vert}}(\mathcal{J}X)$ of vertical differential forms on $\mathcal{J}X$. Given n such observables $\widehat{\omega}_1,\ldots,\widehat{\omega}_n$, we define their correlation function as the following integral generalizing (5.3):

$$\int_{\mathcal{M}_\Sigma(X,\beta)} \mathcal{J}\,\mathrm{ev}^*_{p_1}(\widehat{\omega}_1)\wedge\ldots\wedge \mathcal{J}\,\mathrm{ev}^*_{p_n}(\widehat{\omega}_n). \tag{6.2}$$

These are the correlation functions of the observables of the sigma model involving the fields $X^a(z),\psi^a(z)$ and their complex conjugates and all of their derivatives, but not the momenta variables $p_a(z),\pi_a(z)$ and their complex conjugates and derivatives.[11] These integrals are compatible with the integrals (5.3) in the following sense. Recall the tautological map $\mathcal{J}X \to X\times\Sigma$ from Section 4.4. If $\omega_1,\ldots,\omega_n$ are differential forms on X and $\widehat{\omega}_1,\ldots,\widehat{\omega}_n$ are their pull-backs to $\mathcal{J}X$ via the composite map $\mathcal{J}X \to X\times\Sigma \to X$, then the integral (6.2) is equal to (5.3). In other words, if the observables depend only on $X^a(z),\psi^a(z)$ and their complex conjugates, but not on their derivatives, then we obtain the same answer as before.

To make this formula more concrete, let us choose local holomorphic coordinates $z_1,\ldots,z_n$ at the points $p_1,\ldots,p_n$. Then the fiber of $\mathcal{J}X$ at p_i may be identified with the space JX of jets of holomorphic maps from the coordinatized disc D to X. Therefore we obtain a map

$$\mathrm{Jev}_{p_i} : \mathcal{M}_\Sigma(X,\beta) \to JX, \tag{6.3}$$

sending $\Phi:\Sigma\to X$ to $\Phi|_{D_{p_i}}$, written as a power series with respect to the coordinate z_i. Then we may define the corresponding correlation function by the same formula as (6.2), but with $\mathcal{J}\,\mathrm{ev}_{p_i}$ replaced by Jev_{p_i} for all $i=1,\ldots,n$.

As in the case of the ordinary Gromov–Witten invariants, there is a problem in the definition of these integrals as the moduli spaces $\mathcal{M}_\Sigma(X,\beta)$ are non-compact. In the case of the Gromov–Witten invariants this problem is cured by using their Kontsevich compactification. The boundary strata of this compactification are not maps from Σ to X, but maps to X from stable singular curves which are obtained by attaching to Σ additional components of genus 0 ("bubbles") with fewer than three marked points, so that they are collapsed under the map to the moduli space of stable pointed curves (see Figure 3). One can easily extend the evaluation maps ev_{p_i} to this compactification. But can we extend the jet-evaluation maps Jev_{p_i}?

[11] For a discussion of the correlation functions of the observables involving the momenta variables, see [27], Section 5.

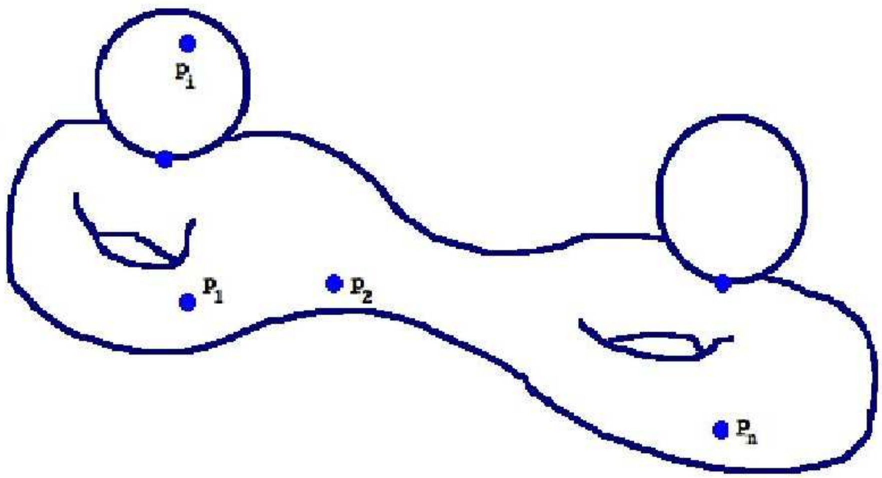

Figure 3. Stable curves obtained by attaching "bubbles" to the original curve.

6.2. **A genus zero example.** Let us consider the simplest example, the moduli space $\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1,1)$ of degree one maps from a genus 0 curve $\Sigma = \mathbb{P}^1$ with 3 marked points $0,1,\infty$ to $\mathbb{P}^1$. In this case we have $\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1,1) = \mathcal{M}_{0,3}(\mathbb{P}^1,1)$, in the notation of Section 5.1. This moduli space is isomorphic to the group PGL_2 of Möbius transformations on $\mathbb{P}^1$:

$$\Phi : z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta}. \tag{6.4}$$

We have an injective map

$$\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1,1) \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

corresponding to evaluating Φ at the three points $0,1,\infty$, whose image is the complement of the diagonals. Let X_1, X_2, X_3 be the three $\mathbb{P}^1$-valued functions on the moduli space $\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1,1)$, which are just the three evaluation maps. The Kontsevich compactification (the moduli space of stable maps)

$$\overline{\mathcal{M}}_{\mathbb{P}^1}(\mathbb{P}^1,1) = \overline{\mathcal{M}}_{0,3}(\mathbb{P}^1,1)$$

is just the the blow-up of $(\mathbb{P}^1)^3$ along the principal diagonal $X_1 = X_2 = X_3$.

In terms of the Möbius coordinates (6.4) on $\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1,1) \simeq PGL_2$, we have

$$X_1 = \frac{\beta}{\delta}, \qquad X_2 = \frac{\alpha+\beta}{\gamma+\delta}, \qquad X_3 = \frac{\alpha}{\gamma}.$$

These maps extend, tautologically, to the compactification. In addition to these evaluation maps $\mathrm{ev}_{p_i} = X_i$ on $\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1,1)$, we have their jet analogues Jev_{p_i}, which pick out not just the values, but also all the derivatives of Φ at $0, 1, \infty$ with respect to the local coordinates $z, z-1, z^{-1}$. We have:

$$\Phi(z) = X_1 + X_{31} \frac{zX_{12}/X_{23}}{1 + zX_{12}/X_{23}} \tag{6.5}$$

$$= X_3 + X_{13} \frac{z^{-1}X_{23}/X_{12}}{1 + z^{-1}X_{23}/X_{12}} \tag{6.6}$$

Therefore the first derivative of Φ at $z = \infty$ (with respect to z^{-1}) is equal to

$$-z^2 \frac{d\Phi}{dz}\bigg|_{z=\infty} = \frac{\alpha\delta - \beta\gamma}{\gamma^2} = \frac{(X_1 - X_3)(X_2 - X_3)}{X_1 - X_2}. \tag{6.7}$$

Thus, we see that this is a well-defined function on $\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1, 1)$, but it *cannot* be extended to the divisor in the Kontsevich compactification $\overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, 1)$ where $X_1 = X_2$! This divisor consists of stable maps to $\mathbb{P}^1$ from an unstable curve that is a union of two genus zero curves, one containing the points labeled $0, 1$, and the other containing one point ∞ (see Figure 4). Thus, the second component has two marked points (∞ and the connecting points between the two components), and therefore is unstable as a curve. As such, it has a continuous group of automorphisms, namely $\mathbb{C}^\times$. Because of this, the tangent space at the marked point ∞ is not well-defined as a vector space; only its quotient by $\mathbb{C}^\times$ is well-defined. Therefore the derivative of Φ at this point is not well-defined either, leading to the divergence in formula (6.7).

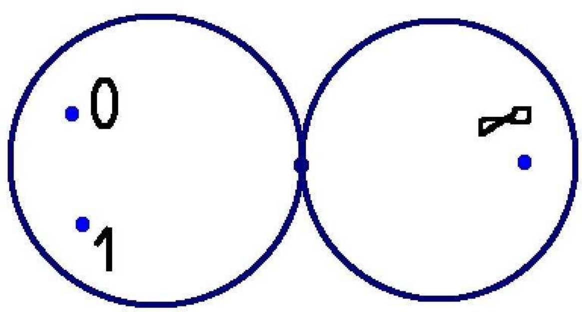

Figure 4. Degeneration of $\mathbb{P}^1$ with three points.

Likewise, we find that the first derivative of Φ at $z = 0$ (with respect to z) is equal to

$$\frac{(X_2 - X_1)(X_3 - X_1)}{X_3 - X_2},$$

and at $z = 1$ (also with respect to z) it is

$$\frac{(X_3 - X_2)(X_1 - X_2)}{X_3 - X_1}.$$

Thus, we observe the same kind of divergence of the derivative of Φ at a given point on the worldsheet $\mathbb{P}^1$ in the limit when the other two points are coming together. (Note that, on the contrary, the derivative becomes zero when our point of evaluation itself comes close to another point. In this case the two appear together on a stable component, on which the map Φ becomes constant, and hence has zero derivative.)

Thus, we encounter an important geometric phenomenon, which, as we will see in Sections 6.4 and 6.7, is responsible for the appearance of the logarithmic partners of operators of the sigma model and for the logarithmic terms in the OPE of jet-evaluation observables of the sigma model. The logarithmic nature of the sigma models in the infinite radius limit may therefore be traced to the fact that the jet evaluation map cannot be extended to the Kontsevich moduli space of stable maps.

6.2.1. *Higher derivatives.* For future use let us record here the expression for higher derivatives $\mathcal{D}^\lambda \mathbf{X} = \mathcal{D}^\lambda \Phi$ at $z = 0$ (we use the notation of Section 4.4)

$$\mathcal{D}^\lambda \mathbf{X} = (X_3 - X_1)^{\ell(\lambda)} \left(\frac{X_1 - X_2}{X_2 - X_3} \right)^{|\lambda|} \tag{6.8}$$

6.3. **Generalization to higher genus.** To understand better what is going on here, it is instructive to consider the more general setting of Gromov–Witten invariants described in Section 5.1, in which we are allowed to vary the pointed curve $(\Sigma, (p_i))$. Thus, we have the moduli space $\mathcal{M}_{g,n}(X, \beta)$ of triples $(\Sigma, (p_i), \Phi)$ and a morphism

$$\pi_{g,n} : \mathcal{M}_{g,n}(X, \beta) \to \mathcal{M}_{g,n},$$

as in Section 5.1. Now we define the moduli space $\mathcal{J}_{g,n}(X)$. This is a (non-trivial) bundle over $\mathcal{M}_{g,n}$, whose fiber over $(\Sigma, (p_i))$ is $J_{p_1} X \times \ldots \times J_{p_n} X$ (recall that $J_{p_i} X$ is the space of jets of holomorphic maps $D_{p_i} \to X$, where D_{p_i} is a small disc around p_i). We have the jet analogues of the maps (5.4),

$$\mathcal{J}\, \mathrm{ev}_i : \mathcal{M}_{g,n}(X, \beta) \to \mathcal{J}_{g,n}(X), \tag{6.9}$$

sending

$$(\Sigma, (p_i), \Phi) \mapsto \left(\Sigma, (p_i), \left(\Phi|_{D_{p_i}} \right) \right).$$

Using these maps, we define the *jet Gromov–Witten invariants* (which may be viewed as the correlation functions of the jet-evaluation observables of the sigma model with the target X coupled to gravity) by the formula

$$\pi_{g,n*}(\mathcal{J}\, \mathrm{ev}_1^*(\widehat{\omega}_1) \wedge \ldots \wedge \mathcal{J}\, \mathrm{ev}_n^*(\widehat{\omega}_n)), \tag{6.10}$$

of which (6.2) is a special case (when the degree of the resulting differential form on $\mathcal{M}_{g,n}$ is equal to zero). Here $\widehat{\omega}_i$ are vertical differential forms on $\mathcal{J}_{g,n}(X)$, with respect to the map $\mathcal{J}_{g,n}(X) \to \mathcal{M}_{g,n}$ (as defined in Section 4.4).

But here we again face the problem that the fibers of the map $\pi_{g,n}$ are not compact. To cure this problem, it is natural to use the Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$, as in Section 5.1. We also have the Deligne–Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable pointed curves $(\Sigma, (p_i))$ of genus g with n marked points, which is a compactification of $\mathcal{M}_{g,n}$. The map $\pi_{g,n}$ extends to a map (5.6) which has compact fibers. We then try to define the regularized integrals by the formula

$$\overline{\pi}_{g,n*}(\overline{\mathcal{J}}\,\mathrm{ev}_1^*(\widehat{\omega}_1) \wedge \ldots \wedge \overline{\mathcal{J}}\,\mathrm{ev}_n^*(\widehat{\omega}_n)). \tag{6.11}$$

However, to make sense of this formula, we need to extend the maps $\mathcal{J}\,\mathrm{ev}_i$ to the compactification $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

It is natural to try to extend $\mathcal{J}_{g,n}(X)$, which is a bundle over $\mathcal{M}_{g,n}$, to a bundle $\overline{\mathcal{J}}_{g,n}(X)$ over the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,n}$, and then extend the maps $\mathcal{J}\,\mathrm{ev}_i$.

In $\overline{\mathcal{M}}_{g,n}$ the marked points $p_1,\ldots,p_n$ are always smooth, and therefore the jet space $J_{p_i}X$ makes sense even if Σ becomes a stable singular curve. Hence the bundle $\mathcal{J}_{g,n}(X)$ over $\mathcal{M}_{g,n}$ extends to a bundle over $\overline{\mathcal{M}}_{g,n}$, which we denote by $\overline{\mathcal{J}}_{g,n}(X)$. Its fiber over a stable pointed curve $(\Sigma,(p_i))$ is $J_{p_1}X\times\ldots\times J_{p_n}X$ is again $(\Sigma,(p_i))$ is $J_{p_1}X\times\ldots\times J_{p_n}X$. We now wish to extend the maps (6.9) to maps

$$\overline{\mathcal{J}}\,\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{J}}_{g,n}(X). \tag{6.12}$$

However, here we encounter a problem. Namely, the points $(\Sigma,(p_i),\Phi)$ in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ may well correspond to unstable pointed curves $(\Sigma,(p_i))$. An example is a curve $\widetilde{\Sigma}$ which has two components, Σ_0 and Σ', where Σ_0 is a genus zero component containing exactly one marked point p_i. We denote by $\widetilde{p}_i$ the point of intersection of the two components (see Figure 5). In fact, curves of this type constitute one of the boundary divisors in $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

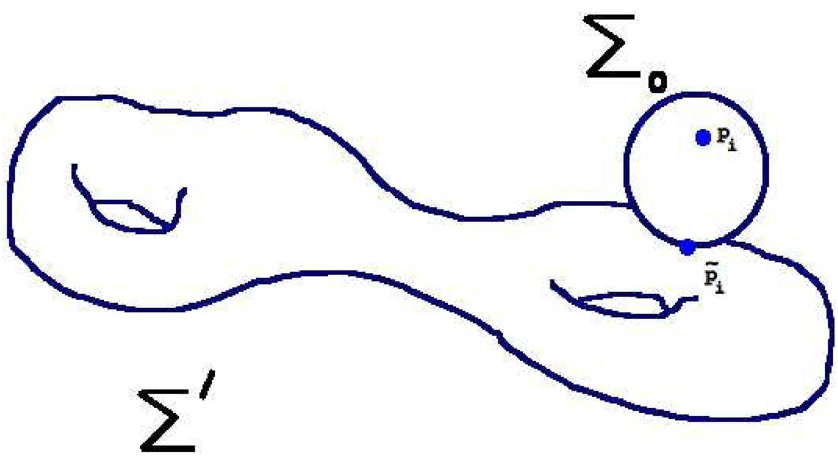

Figure 5. Stable curves corresponding to one of the divisors in $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

The restriction of Φ to Σ_0 should have a non-zero degree β_0; then the restriction of Φ to Σ' has degree $\beta' = \beta - \beta_0$. Now, the component Σ_0 is unstable, and therefore it is collapsed under the map $\overline{\pi}_{g,n}$. In other words, the image of $(\widetilde{\Sigma},(p_i),\Phi)$ is $(\Sigma',(p_1,\ldots,\widetilde{p}_i,\ldots,p_n)$, with the point $\widetilde{p}_i\in\Sigma'$ replacing p_i.

Recall first the situation with the ordinary evaluation maps ev_i. They take values in X, and hence they extend easily to $\overline{\mathcal{M}}_{g,n}(X,\beta)$: in the case of a singular curve $\widetilde{\Sigma}$

described above we just evaluate Φ at $p_i \in \Sigma_0$. We do not care that p_i "disappears" under the map $\overline{\pi}_{g,n}$ and is replaced by $\widetilde{p_i}$ (where the value of the map Φ would certainly be different in general), because we do not use the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ in the definition of ev_i (only in the definition of the integrals (6.11)). Since the evaluation maps extend to $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and the morphism $\overline{\pi}_{g,n}$ is proper, we see that the corresponding push-forward

$$\overline{\pi}_{g,n*}(\mathrm{ev}_1^*(\omega_1) \wedge \ldots \wedge \mathrm{ev}_n^*(\omega_n))$$

to $\overline{\mathcal{M}}_{g,n}$ is well-defined. Hence the integrals (6.2) are also well-defined.

The situation is different for the jet-evaluation maps. They take values not in X, but in a jet bundle $\mathcal{J}_{g,n}(X)$ over $\mathcal{M}_{g,n}$. Note that we have a forgetful map $t : \mathcal{J}_{g,n}(X) \to X^n \times \mathcal{M}_{g,n}$, which truncates the jets of maps $D_{p_i} \to X$ to their values at p_i, and $\mathrm{ev}_i = \mathrm{pr}_1 \circ t_i \circ \mathcal{J}\mathrm{ev}_i$; this factorization is the reason why the dependence on $\mathcal{M}_{g,n}$ disappears when we consider the ordinary evaluation maps. The jet bundle $\mathcal{J}_{g,n}(X)$ may be trivialized locally over $\mathcal{M}_{g,n}$ if we choose a family of local coordinates at the points p_i on Σ. Indeed, by definition, the value of the map $\mathcal{J}\mathrm{ev}_i$ on $(\Sigma, (p_i), \Phi)$ for a smooth Σ is the jet of the restriction of Φ to the disc D_{p_i}. Once we pick a local coordinate at p_i, we may view it as a point of $JX = \{D \to X\}$. Hence we may pull back differential forms on JX to $\mathcal{M}_{g,n}$.

Now, when Σ degenerates to a singular curve described above, we would like to consider the restriction of Φ to the disc D_{p_i}, where the point p_i is now on the unstable bubble Σ_0. In order to be able to view it as a point of JX, we need to choose a coordinate at p_i. But the image of $(\Sigma_0 \cup \Sigma', (p_i), \Phi)$ in $\overline{\mathcal{M}}_{g,n}$ under $\overline{\pi}_{g,n}$ is $(\Sigma', (p_1, \ldots, \widetilde{p_i}, \ldots, p_n)$. It does not know anything about p_i! The point p_i has disappeared under the map $\overline{\pi}_{g,n}$ and instead we now have the point $\widetilde{p_i}$ on the other component Σ'. A local coordinate at p_i has nothing to do with a local coordinate at $\widetilde{p_i}$, and in any case the jet of Φ at p_i is completely independent of the jet at $\widetilde{p_i}$, viewed as a point of Σ'.

Thus, we see that we cannot extend the map $\mathcal{J}\mathrm{ev}_i$ to a map (6.12). Of course, we could try to use instead of $\overline{\mathcal{M}}_{g,n}$ another moduli space (or stack), $\widetilde{\mathcal{M}}_{g,n}$, in which the component Σ_0 and the point p_i are included. For instance, we could take as $\widetilde{\mathcal{M}}_{g,n}$ the moduli stack of prestable curves. Then unstable bubbles such as Σ_0, with two marked points, would be allowed. The problem is that this unstable component Σ_0 has the group of automorphisms $\mathbb{C}^\times$. This group naturally acts on the corresponding space $J_{p_i}X$ of jets of maps $D_{p_i} \to X$, and as the result we are only able to identify $J_{p_i}X$ with JX up to this action. In other words, the restriction of Φ to D_{p_i} only gives us a well-defined element JX modulo the action of $\mathbb{C}^\times$. Thus, the only observables that make sense in this case are the differential forms that are invariant under this $\mathbb{C}^\times$-action. These are just the evaluation observables. They may be pulled back to $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and we obtain the usual Gromov–Witten invariants. But the pull-back of more general differential forms to $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is not well-defined.

This indicates that the differential forms $\mathcal{J}\mathrm{ev}_i^*(\widehat{\omega}_i)$ on $\mathcal{M}_{g,n}(X,\beta)$ (which are well-defined before the compactification) may have singularities when we try to extend them to the boundary divisors in $\overline{\mathcal{M}}_{g,n}(X,\beta)$. This is indeed the case for $g=0, n=3$, as we have seen above. In general, the result will be similar. Thus, we see that these differential forms have *poles* along the boundary divisors in $\overline{\mathcal{M}}_{g,n}(X,\beta)$. (Their residues could potentially have poles on the codimension two boundary strata in $\overline{\mathcal{M}}_{g,n}(X,\beta)$, and so on.)

Let us summarize: in order to define the correlation functions of jet-evaluation observables, we need to regularize integrals of the form (6.11). However, the standard prescription, which works well for the ordinary evaluation observables, cannot be applied. The reason is that the jet-evaluation maps $\mathcal{J}\,\mathrm{ev}_i$, well-defined in $\mathcal{M}_{g,n}(X,\beta)$ do not extend to the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps. Therefore we obtain integrals of differential forms on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ with poles on the compactification divisor. To compute the correlation functions, we need to regularize these integrals. This is directly related to the fact that the naive definition of the operators corresponding to the jet-evaluation observables (in terms of the $\beta\gamma$-bc-fields) requires regularization, a phenomenon familiar to us from the study of quantum mechanical models in Part I. This regularization is not unique and leads to a mixing of operators (and states, via the state–operator correspondence) with their logarithmic partners.

In other words, to obtain a true operator of the non-perturbative sigma model, we need to take a jet-evaluation observable in its perturbative definition *together* with a consistent set of regularization rules for all of its correlation functions. Changing the regularization rules will mean adding to this operator its logarithmic partners.

Thus, divergence of the integrals expressing the correlation functions of the jet-evaluation observables and potential ambiguity of their regularization are important manifestations of the logarithmic nature of the sigma models in the infinite radius limit. In Sections 6.4–6.7 we will present explicit examples of this regularization, which indicate the existence of a rich and interesting structure underlying the correlation functions of the jet-evaluation observables in the two-dimensional sigma model.

6.4. **Regularization of correlation functions.** The discussion of the previous two sections implies that the naive three-point correlation functions of jet-evaluation observables on $\mathbb{P}^1$ *diverge* and require regularization. In this section we explain how to implement this regularization and what it means for the operators of the sigma model.

Consider, for example, the correlation function of the evaluation observables $\omega_1\psi\overline{\psi}$, $\omega_2\psi\overline{\psi}$, placed at the points $0,1$, and the jet-evaluation observable $\omega_3\psi\overline{\psi}\partial X\overline{\partial X}$ placed at ∞. Using formula (6.7), we obtain that the corresponding correlation function is given by the integral

$$(6.13)\qquad q\int\limits_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}\prod_{\alpha=1}^{3}\omega_\alpha(X_\alpha,\overline{X}_\alpha)d^2X_1\wedge d^2X_2\wedge d^2X_3\cdot\frac{|X_1-X_3|^2|X_2-X_3|^2}{|X_1-X_2|^2}$$

(it corresponds to the instanton number 1, for dimensional reasons, hence the overall factor q). This integral diverges when when $X_1\to X_2$. Writing

$$X_2=X+\xi/2,\qquad X_1=X-\xi/2,$$

we obtain that the divergent part may be approximated by the integral

$$q \int\limits_{\mathbb{P}^1 \times \mathbb{P}^1 \times U} \omega_1(X,\overline{X})\omega_2(X,\overline{X})\omega_3(Y,\overline{Y}) d^2X \wedge d^2Y \; |X-Y|^4 \; \frac{d^2\xi}{|\xi|^2}, \tag{6.14}$$

where U is a small neighborhood of 0 in the ξ-plane. This integral has logarithmic divergence in ξ.

We will now explain why this divergence is not really surprising. In fact, we have observed and studied similar divergences in analogous quantum mechanical models in Part I. In those models it was the definition of the distributions corresponding to the states that required regularization. For instance, in the supersymmetric quantum mechanical model on $\mathbb{P}^1$ we have states of the form $X^n \overline{X}^{\overline{n}}, n \geq 0, \overline{n} \geq 0$, where X is a coordinate on $\mathbb{P}^1$. The matrix element of an evaluation observable $\omega(X,\overline{X})d^2X$, which is a differential form on $\mathbb{P}^1$, between this state and the co-vacuum is given by the integral

$$\int \omega(X,\overline{X}) X^n \overline{X}^{\overline{n}} d^2X. \tag{6.15}$$

These integrals diverge in general, and we regularize them by the corresponding *partie finie* (also known as the *Epstein–Glaser regularization* [17]). It is obtained by integrating over the domain $|X| < \epsilon^{-1}$, viewing the integral as a function in ϵ and picking the constant term in this function. This regularization is best interpreted as a way to extend the tempered distribution corresponding to $X^n \overline{X}^{\overline{n}}$ to an ordinary distribution on $\mathbb{P}^1$.

However, the resulting distribution is not canonical. Indeed, our prescription involved the choice of a coordinate on $\mathbb{P}^1$. But such a coordinate is only unique up to a scalar. So we could choose, say, 2ϵ as a small parameter, instead of ϵ. Then the answer will change, because of the presence of logarithmic dependence in ϵ in the integral over the domain $|X| < \epsilon^{-1}$. We will therefore pick up an extra term, which is a multiple of the derivative of the delta-function at ∞, $\partial_X^{n-1}\partial_{\overline{X}}^{\overline{n}-1}\delta_\infty^{(2)}$. This is the *logarithmic partner* of the state $X^n \overline{X}^{\overline{n}}$.

In other words, the distribution corresponding to $X^n \overline{X}^{\overline{n}}$ is only well-defined up to an addition of a multiple of $\partial_X^{n-1}\partial_{\overline{X}}^{\overline{n}-1}\delta_\infty^{(2)}$. We obtain that the space of "in" states (more precisely, its subspace consisting of differential forms of degree 0) is a non-trivial extension of the space of polynomials $\mathbb{C}[X,\overline{X}]$ by the space of delta-like distributions $\mathbb{C}[\partial_X, \partial_{\overline{X}}] \cdot \delta_\infty^{(2)}$ at $\infty \in \mathbb{P}^1$. Furthermore, the Hamiltonian is not diagonalizable on this space, but has Jordan block structure, mixing $X^n \overline{X}^{\overline{n}}$ and $\partial_X^{n-1}\partial_{\overline{X}}^{\overline{n}-1}\delta_\infty^{(2)}$. (Note that there is no mixing involving the vacuum state 1, corresponding to $n = \overline{n} = 0$, and to the purely chiral and anti-chiral states, with $\overline{n} = 0$ and $n = 0$, respectively.)

Our Morse function on $\mathbb{P}^1$ has only two critical points, and hence $\mathbb{P}^1$ decomposes into the union of two strata: $\mathbb{C}_0 = \mathbb{P}^1 \backslash \infty$ and the point ∞. They give rise to the two sectors in the space of "in" states. This is why states have at most one logarithmic partner and the maximal size of the Jordan block appearing in the Hamiltonian is two. For manifolds of higher dimension there will be more sectors and hence more logarithmic partners in general (see Part I for details).

We are now observing a similar phenomenon in the two-dimensional sigma model. Indeed, one possible Hamiltonian interpretation of (6.13) is as the matrix element

$$\langle\omega_1(X,\overline{X})\psi\overline{\psi} \mid \omega_2(X,\overline{X})\psi\overline{\psi} \mid \omega_3(X,\overline{X})\psi\overline{\psi}\partial X\overline{\partial X}\rangle \tag{6.16}$$

of the evaluation observable $\omega_2\psi\overline{\psi}$ between the excited state corresponding to

$$\omega_3(X,\overline{X})\psi\overline{\psi}\partial X\overline{\partial X}$$

and the co-vacuum state corresponding to $\omega_1(X,\overline{X})d^2X$. Note that in contrast to one-dimensional quantum mechanical models, in the two-dimensional theory we now have the state–operator correspondence. We may then interpret the same integral as the matrix element

$$\langle\omega_2(X,\overline{X})\psi\overline{\psi} \mid \omega_3(X,\overline{X})\psi\overline{\psi}\partial X\overline{\partial X} \mid \omega_1(X,\overline{X})\psi\overline{\psi}\rangle, \tag{6.17}$$

in which differential forms ω_1 and ω_2 are interpreted as vacuum and co-vacuum states, and $\omega_3(X,\overline{X})\psi\overline{\psi}\partial X\overline{\partial X}d^2X$ is interpreted as an operator placed at the point 1. However, it is the interpretation (6.16) that is closest to the quantum mechanical matrix element (6.15).

In light of the above discussion of the quantum mechanical models, it should not come as a surprise that the integral (6.16) has logarithmic divergence. It is quite similar to the logarithmic divergence of the integral (6.15), which, as we have seen above, is responsible for the appearance of logarithmic partners.

Hence we follow the same strategy as in quantum mechanics and *define* the matrix element (6.16) (or the matrix element (6.17)) as the *partie finie* regularization of the integral (6.16). In other words, we cut out the part of the domain of integration (which is the product of three copies of $\mathbb{P}^1$) which is within the radius ϵ of the diagonal $X_1 = X_2$ with respect to the Fubini-Study metric. (In principle, here we could choose an arbitrary metric; the ambiguity of this choice is one of the reasons that the resulting regularized integral is not canonically defined.) We then evaluate the integral as a function of ϵ. One can show that as a function in ϵ it may be uniquely represented in the form

$$C_0 + \sum_{i>0} C_i\epsilon^{-i} + C_{\log}\log\epsilon + o(1), \tag{6.18}$$

where the C_i's and $C_{\log}$ are some numbers (see [36], pp. 70-71). The *partie finie* of the above integral as the constant coefficient C_0 obtained after discarding the terms with negative powers of ϵ and $\log\epsilon$ in the integral (6.16) and taking the limit $\epsilon \to 0$.

Just as in the quantum mechanical case, this definition is non-canonical. It depends on the choices we have made (such as the metric on $\mathbb{P}^1$). However, formula (6.13) shows that the discrepancy for two different regularization schemes (with varying ω_1, ω_2) may be represented as a multiple of the integral

$$q\int_{\mathbb{P}^1} \omega_1(X,\overline{X})\omega_2(X,\overline{X}) \cdot {}^{\ell}\omega_3(X,\overline{X})\psi\overline{\psi}, \tag{6.19}$$

where, by definition,

$$ {}^{\ell}\omega(X,\overline{X}) = \int_{\mathbb{P}^1} |Y-X|^4\ \omega(Y,\overline{Y})d^2Y. \tag{6.20} $$

Note that ω_3 is a coefficient in front of a $(2,2)$-differential on $\mathbb{P}^1$, and the map (6.20) is a canonical map from the space of $(2,2)$-differentials on $\mathbb{P}^1$ to the space of $(-1,-1)$-differentials, so that the integral (6.19) is well-defined. The integral kernel in Section 6.20 is the inverse square of the absolute value squared of the so-called prime form.

We now interpret (6.19) as the following matrix element:

$$ q\langle\ \omega_1(X,\overline{X})\psi\overline{\psi}\mid\omega_2(X,\overline{X})\psi\overline{\psi}\mid{}^{\ell}\omega_3\pi\overline{\pi}\ \rangle\ . \tag{6.21} $$

Here we have introduced the fermionic fields $\pi,\overline{\pi}$ in order to reduce the total fermionic number (which counts the degree of the differential form on the moduli space of holomorphic maps) from 3 of the original matrix element (6.16) to 1. Hence we now integrate over the moduli space of holomorphic maps from to $\mathbb{P}^1$ of degree 0 (that is, the constant maps), which is one-dimensional.

6.5. **Logarithmic partners.** The upshot of this calculation is that

the state ${}^{\ell}\omega_3(X,\overline{X})\pi\overline{\pi}$ *is the logarithmic partner of* $\omega_3(X,\overline{X})\psi\overline{\psi}\partial X\overline{\partial X}$,

just like $\partial_X^{n-1}\partial_{\overline{X}}^{\overline{n}-1}\delta_\infty^{(2)}$ is the logarithmic partner of the state $X^n\overline{X}^{\overline{n}}$ in quantum mechanics! Likewise, the operator ${}^{\ell}\omega_3\pi\overline{\pi}(z,\overline{z})$ is the logarithmic partner of the operator $\omega_3\psi\overline{\psi}\partial X\overline{\partial X}(z,\overline{z})$.

Here it is useful to recall the structure of the space of states of the two-dimensional sigma model on $\mathbb{P}^1$ from Section 4.1. The "big" space of states $\widetilde{\mathcal{H}}$ has a filtration $\widetilde{\mathcal{H}}_{\geq i}, i\in\mathbb{Z}$, where $\widetilde{\mathcal{H}}_{\geq i}\simeq\widetilde{\mathcal{H}}_{\geq j}$ for all i,j and $\widetilde{\mathcal{H}}_{\geq i}/\widetilde{\mathcal{H}}_{\geq(i+1)}\simeq\widetilde{\mathcal{H}}_i$, the space of delta-forms on the ascending manifold $(\widetilde{L\mathbb{P}^1})_i$ corresponding to the ith preimage of the critical set $\mathbb{P}^1\subset L\mathbb{P}^1$ of constant loops. We have a "deck transformation" map sending $\widetilde{\mathcal{H}}_m$ to $\widetilde{\mathcal{H}}_{m+1}$ isomorphically. A physical state Ψ is a vector in $\widetilde{\mathcal{H}}$ which is an eigenvector of this transformation with the eigenvalue q.

The space $\widetilde{\mathcal{H}}$ may be identified with a direct product

$$ \widetilde{\mathcal{H}} = \prod_{m\in\mathbb{Z}}\widetilde{\mathcal{H}}_m, \tag{6.22} $$

but not canonically. Let us choose such an identification, in such a way that the deck transformation identifies all the $\widetilde{\mathcal{H}}_m$ with each other. Then it is convenient to identify all of them with $\widetilde{\mathcal{H}}_0$ and to write

$$ \widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_0\otimes\mathbb{C}[[T,T^{-1}]], $$

where T is a formal variable. A vector in this space is then represented by a sum

$$ \Psi = \sum_{m\in\mathbb{Z}}\widetilde{\Psi}_m T^m, \qquad \widetilde{\Psi}_m\in\widetilde{\mathcal{H}}_0 $$

(without any convergence condition!). The deck transformation acts by multiplication by T. Hence it sends this vector to

$$\sum_{m\in\mathbb{Z}} \widetilde{\Psi}_m T^{m+1}.$$

The eigenvector condition on Ψ then becomes

$$\widetilde{\Psi}_m = q\,\widetilde{\Psi}_{m+1}, \qquad m \in \mathbb{Z}. \tag{6.23}$$

By this condition, $\widetilde{\Psi}_0$ determines the remaining $\widetilde{\Psi}_m$, and so the space of states becomes isomorphic to $\widetilde{\mathcal{H}}_0$. This is in fact how we defined it in Section 4.1. However, this definition is non-canonical because to obtain it we need to choose the direct product decomposition (6.22) of the "big" space of states $\widetilde{\mathcal{H}}$, which is non-canonical.

6.5.1. *Toy model.* It might be helpful to illustrate the difference between the space $\widetilde{\mathcal{H}}_0$ and the true space of states by the following elementary example. Let V be a vector space and suppose that we are given an extension

$$0 \to V \to V_2 \to V \to 0. \tag{6.24}$$

Of course, any extension of vector spaces can be split, but if they carry additional structures (such as OPE, correlation functions, etc., in our case), then there may not be a splitting respecting these structures. We can iterate this extension and construct new extensions

$$0 \to V_n \to V_{n+1} \to V \to 0, \qquad n > 2.$$

The space V_n has a filtration $0 \subset V = V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = V$ such that the consecutive quotients are all isomorphic to V. Let us label the filtration on V_{2n} as follows: $V_{\geq i} = V_{n-i}, i = -n, \ldots, n$. Next, we take the limit of V_{2n} when $n \to \infty$ in a way compatible with this filtration. This means that to pass from V_{2n} to V_{2n+2} we "glue" one V to V_{2n} as a subspace and another V as a quotient. Then the limit V_∞ has a filtration $V_{\geq i}, i \in \mathbb{Z}$, such that $V_{\geq i}/V_{\geq (i+1)} \simeq V$. (This is the analogue of $\widetilde{\mathcal{H}}$.) Moreover, by construction, we have a canonical identification of $V_{\geq i}$ with $V_{\geq j}$ for all $i, j \in \mathbb{Z}$. Thus, we have a canonical "shift" operator S which maps $V_{\geq i}$ to $V_{\geq (i+1)}$ isomorphically.

Now let $\overline{V}$ be the subspace of vectors $v \in V_\infty$ which are eigenvectors of this transformation with eigenvalue $q \in \mathbb{C}^\times$, $S(v) = qv$. (This is the analogue of our space of states.) If we choose a splitting of the exact sequence (6.24), then we identify $V_n \simeq V^{\oplus n}$ and hence identify V_∞ with the direct product of infinitely many copies of V, labeled by the integers, splitting the filtration $(V_{\geq i})$. Then the space of eigenvectors of S as above may be identified with any of these copies of V, for instance, the one labeled by $0 \in \mathbb{Z}$. Thus, we obtain an isomorphism $V \simeq \overline{V}$. But this isomorphism is not canonical! Indeed, if we choose a different splitting of (6.24), that is, a different isomorphism $V_2 \simeq V \oplus V$, then the lift to $V \oplus V$ of a vector from the quotient V will be equal to $v + M(v)$ for some linear operator $M : V \to V$. Therefore a vector of $\overline{V}$ which appears as $v \in V$ under the previous identification, will appear as the vector $\widetilde{v} = v + qM(v) + q^2M^2(v) + \ldots$ with respect to the new identification.

6.5.2. *Back to the space of states of the sigma model.* The fact that the space of states cannot be canonically identified with $\widetilde{\mathcal{H}}_0$ (which, we recall, is the chiral-anti-chiral de Rham complex of X) has an important consequence. When we represent an excited state of the theory in the form $\widetilde{\Psi}_0 = \omega\psi\overline{\psi}\partial X\overline{\partial X}$, say, tacitly choose a direct product decomposition of the form (6.22). Indeed, this expression is well-defined as a (semi-infinite) delta-form on the ascending manifold $(\widetilde{L\mathbb{P}^1})_0$. But it is not intrinsically defined as a state in the sigma model with the target $\mathbb{P}^1$, because of there are non-trivial extensions of this space by the space of delta-forms of $(\widetilde{L\mathbb{P}^1})_1$ and other ascending manifolds in the closure of $(\widetilde{L\mathbb{P}^1})_0$.

This is similar to writing an excited state in the quantum mechanical model on $\mathbb{P}^1$ as a monomial $X^n\overline{X}^{\overline{n}}$ on the big cell $\mathbb{C}_0$. It is really well-defined as a state in the model on $\mathbb{C}_0$, but not in the model on $\mathbb{P}^1$, where it is actually mixed with its logarithmic partner $\partial_X^{n-1}\partial_{\overline{X}}^{\overline{n}-1}\delta^{(2)}_\infty$. (Note, however, that the vacuum states, such as the monomial 1 in the quantum mechanical mode, or the states $\omega(X,\overline{X})\psi\overline{\psi}$ in sigma model, are intrinsically defined, as they are not mixed with anything.) The analogue of this in the two-dimensional sigma model is the statement that the general states corresponding to delta-forms on a particular ascending manifold of the Morse–Bott–Novikov–Floer function, are mixed with the states corresponding to delta-forms on other ascending manifolds which appear in its closure.

Therefore we should not be surprised that the state with

$$\widetilde{\Psi}_0 = \omega\psi\overline{\psi}\partial X\overline{\partial X} \in \widetilde{\mathcal{H}}_0 \subset \widetilde{\mathcal{H}},$$

which is a delta-form on the ascending manifold $(\widetilde{L\mathbb{P}^1})_0$, is mixed with the state

$$\widetilde{\Psi}'_1 \otimes T = {}^{\ell}\omega\pi\overline{\pi} \otimes T \in \widetilde{\mathcal{H}}_0 \otimes T \subset \widetilde{\mathcal{H}}.$$

This is now a delta-form on the ascending manifold $(\widetilde{L\mathbb{P}^1})_1$, which lies in the closure of $(\widetilde{L\mathbb{P}^1})_0$. Now we can use the equivariance condition (6.23) to interpret this state as a state in $\widetilde{\mathcal{H}}_0$; namely, we identify $\widetilde{\Psi}'_1$ with

$$\widetilde{\Psi}'_0 = q\ {}^{\ell}\omega\pi\overline{\pi} \in \widetilde{\mathcal{H}}_0.$$

Note that the appearance of q here, from the equivariance condition (6.23), matches its appearance in the definition of logarithmic partners: in formula (6.21) we have to introduce q by hand, because we express the residue of a divergent integral over the moduli space of holomorphic maps of degree 1 (corresponding to the matrix element (6.16)) in terms of an integral over the moduli space of holomorphic maps of degree 0.

In this example we have only one logarithmic partner. More generally, there may be more of them, with higher powers of q (this is similar to what happens in quantum mechanics).

6.5.3. *Strata in $\widetilde{L\mathbb{P}^1}$ and boundary divisors in $\overline{\mathcal{M}}_\Sigma(X,\beta)$.* We note that the structure of the ascending manifolds in $\widetilde{L\mathbb{P}^1}$ exhibits an analogue of the "bubbling phenomenon" discussed at the end of Section 6.2, which was ultimately responsible for the singularity of the integrals such as (6.13). To see that, recall the interpretation of the covering

$\widetilde{L\mathbb{P}^1}$ of $L\mathbb{P}^1$ from Section 2.4 in which points of $\widetilde{L\mathbb{P}^1}$ are realized as equivalence classes of (continuous) maps $\widetilde{\gamma}: D \to X$, where D is a unit disc. In this interpretation the ascending manifold $(\widetilde{L\mathbb{P}^1})_0$ is realized as the subset of equivalence classes of *holomorphic maps* $\widetilde{\gamma}: D \to X$, whereas the ascending manifold in its closure, $(\widetilde{L\mathbb{P}^1})_1$, may be realized as the set of equivalence classes of holomorphic maps in which a sphere "bubbles out" at the origin of the disc (see Figure 6). This is in agreement with the description of the boundary strata in $\overline{\mathcal{M}}_\Sigma(X,\beta)$ as the maps from a curve Σ in which a sphere "bubbles out" at the point p.

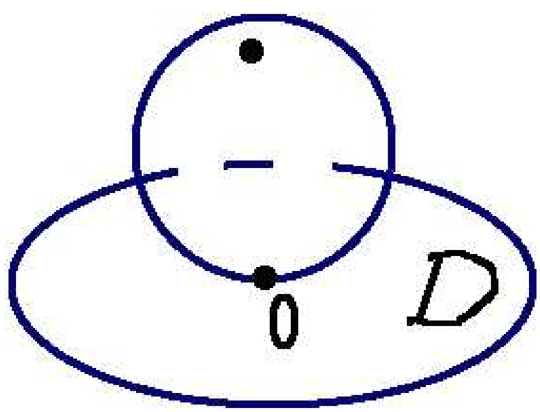

Figure 6. Disc with a "bubble".

The two pictures, the local one with $\widetilde{L\mathbb{P}^1}$, and the global one with $\overline{\mathcal{M}}_\Sigma(X,\beta)$, are actually closely related. Indeed, let us cut a small disc D around the point p from Σ. Let $\overline{\Sigma}$ be the closure of $\Sigma\backslash D$. Then D and $\overline{\Sigma}$ intersect along their boundary circles and general holomorphic maps $\Sigma \to X$ may be described as loops $\mathbb{S}^1 \to X$ which are simultaneous the boundary values of holomorphic maps $D \to X$ and $\overline{\Sigma} \to X$ (see Figure 7). Thus, we may identify the moduli space $\mathcal{M}_\Sigma(X,\beta)$ of holomorphic maps $\Sigma \to X$ (before the compactification) with the intersection of the space of boundary values of holomorphic maps $D \to X$ and the space of boundary values of holomorphic maps $\overline{\Sigma} \to X$, viewed as subsets in $L\mathbb{P}^1$ (see [31]).

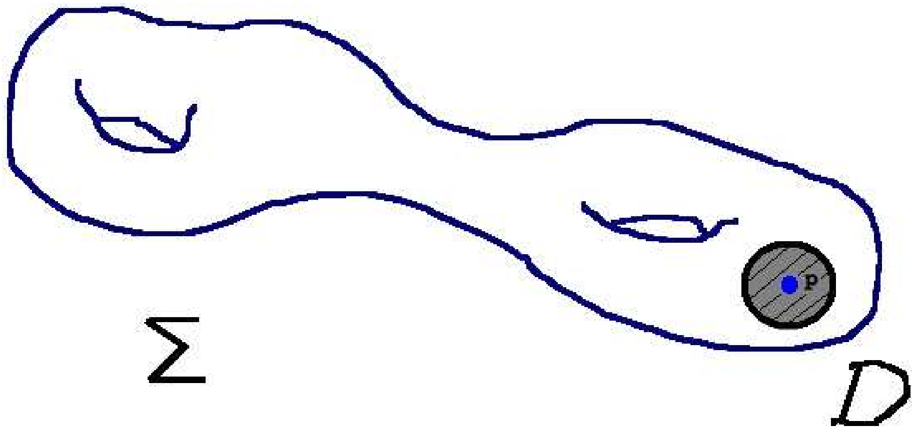

Figure 7. The curve Σ as the union of a disc D and its complement $\overline{\Sigma}$.

These subsets isomorphically lift to the covering $\widetilde{L\mathbb{P}^1}$, with the first subset lifting to $(\widetilde{L\mathbb{P}^1})_0$. The divisors in the stable map compactification $\overline{\mathcal{M}}_\Sigma(X,\beta)$ of $\mathcal{M}_\Sigma(X,\beta)$ then correspond to the intersection of the ascending manifolds $(\widetilde{L\mathbb{P}^1})_n, 0 < n \leq \beta$, which lie in the closure of $(\widetilde{L\mathbb{P}^1})_0$, with the space of boundary values of holomorphic maps $\overline{\Sigma} \to X$. This is why we have such parallelism between the patterns of gluing the boundary strata in $\widetilde{L\mathbb{P}^1}$ and in $\overline{\mathcal{M}}_\Sigma(X,\beta)$.

This way the loci of singularities of the correlation functions of the jet-evaluation observables may be directly linked to the structure of the closures of the ascending manifolds in $\widetilde{L\mathbb{P}^1}$ and hence to the non-trivial extensions in the space of operators of the sigma model which, as we have seen above, ultimately lead to the appearance of logarithmic partners.

Remark 6.1. Note that there is a difference between the two patterns: in the local picture, on $\widetilde{L\mathbb{P}^1}$, we do not fix the map of the bubbled sphere to $\mathbb{P}^1$, only its degree, whereas in the global picture, on $\overline{\mathcal{M}}_\Sigma(X,\beta)$, we do fix it, up to rescalings. For this reason the boundary strata in $\overline{\mathcal{M}}_\Sigma(X,\beta)$ have codimension one, whereas the corresponding codimension of boundary strata in $\overline{\mathcal{M}}_\Sigma(X,\beta)$ grows linearly with the degree of the map from the bubbled sphere. Note also that in $\overline{\mathcal{M}}_\Sigma(X,\beta)$ there are other boundary divisors corresponding to spheres that bubble out away from the point p. They do not have analogues in $\widetilde{L\mathbb{P}^1}$.[12]

6.6. **General case.** We now investigate the general phenomenon of logarithmic mixing of operators. In the formulas below we indicate the passage from a given operator to its logarithmic partner by an arrow. The first example considered above is

$$A(X,\overline{X})\psi\partial X\overline{\psi\partial X} \longrightarrow qB_A(X,\overline{X})\pi\overline{\pi},$$

[12]We thank A. Givental for a useful discussion of these issues.

where

$$B_A(X) = \int_{\mathbb{P}^1} A(Y)|X-Y|^4 d^2Y.$$

Similarly, we obtain

$$A(X,\overline{X})\psi\partial\psi\overline{\psi\partial\psi} \longrightarrow qB_A(X,\overline{X})p\overline{p},$$

Consider next the case of purely bosonic operators. Suppose we have two partitions λ and μ with $|\lambda| = |\mu| = 3$. Then, using formula (6.8), we obtain

$$C_{\lambda\mu}(X,\overline{X})\mathcal{D}^{\lambda}\mathbf{X}\overline{\mathcal{D}}^{\mu}\overline{\mathbf{X}} \longrightarrow q\cdot B_{C_{\lambda\mu}}\pi\partial\pi\overline{\pi}\overline{\partial}\overline{\pi}, \tag{6.25}$$

where the fermionic content of the log-partner is uniquely fixed by the dimension and the ghost number (the $\pi\psi$ charge) considerations, and

$$B_{C_{\lambda\mu}}(X) = 4\int_{\mathbb{P}^1} d^2Y\, C_{\lambda\mu}(Y)|X-Y|^2(X-Y)^{\ell(\lambda)}(\overline{X}-\overline{Y})^{\ell(\mu)}\ . \tag{6.26}$$

The more general operator mixing for jet-evaluation observables may be described similarly. Suppose that we wish to determine whether an operator A, of perturbative conformal dimension $(\Delta,\overline{\Delta})$ is log-mixed with operators ${}^{\ell}A^{(r)}$, so that

$$Q^{L_0}\overline{Q}^{\overline{L}_0}(A) = Q^{\Delta}\overline{Q}^{\overline{\Delta}}\cdot\left(A + \log|Q|^2\left[{}^{\ell}A^{(1)}\right] + \ldots + \frac{1}{k!}\left(\log|Q|^2\right)^k\left[{}^{\ell}A^{(k)}\right]\right), \tag{6.27}$$

where we allow for multiple log-partners. Then for a test operator B with the same perturbative conformal dimension we should have

$$\langle A(z,\overline{z})B(0)\rangle = z^{-2\Delta}\overline{z}^{-2\overline{\Delta}}\left(G_{AB} + \sum_{r=1}^{k}\frac{(\log|z|^2)^r}{r!}G_{{}^{\ell}A^{(r)},B}\right), \tag{6.28}$$

where $G_{\tilde{A}B}$ is the perturbative Zamolodchikov metric. Therefore we could try to find ${}^{\ell}A^{(r)}$ by computing the two-point functions $\langle A(z,\overline{z})B(0)\rangle$ and interpreting the logarithmic terms as the two-point functions $\langle {}^{\ell}A^{(r)}(z,\overline{z})B(0)\rangle$. As before, all of these correlation functions may be computed in terms of (regularized) integrals over the moduli spaces of holomorphic maps. This leads to a kind of bootstrap, which in principle should enable us to compute the logarithmic partners by a recursive procedure. The first term in this recursion corresponds to taking the perturbative part in $\langle {}^{\ell}A^{(r)}(z,\overline{z})B(0)\rangle$. Let us discuss it in more detail in the case when $r=1$. Let us write

$${}^{\ell}A^{(1)} = \sum_{m>0} {}^{\ell}A^{(1)}_m q^m.$$

It is useful to pass to the logarithmic coordinate $x = \log X, \psi^x = \psi^X/X$ and write

$$x = x_0 + \sum_{i=1}^{m}\log\left(\frac{1-w_i^+z}{1-w_i^-z}\right),$$

$$\psi = dx_0 + \sum_{i=1}^{m}\left(-\frac{dw_i^+z}{1-w_i^+z} + \frac{dw_i^-z}{1-w_i^-z}\right) \tag{6.29}$$

in the sector with the instanton number m. If two operators A and B are jet-evaluation observables, that is, depend only on x, ψ and their derivatives, then the correlation function $\langle A(\infty)B(0)\rangle$ is given by the sum

$$\sum_{m \in \mathbb{Z}_+} q^m \int\limits_{\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1,m)} A \wedge B,$$

where $\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1, m)$ is the moduli space of degree m holomorphic maps from a parametrized worldsheet $\mathbb{P}^1$ to the target $\mathbb{P}^1$. Let us assume for simplicity that if $B(0)$ is not a jet-evaluation observable, then $\langle A(\infty)B(0)\rangle = 0$, so we may restrict ourselves to observables of this type.

Now, the moduli space $\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1, m)$ is acted upon by the group $\mathbb{C}^\times$ which preserves the points 0 and ∞, and the correlation function is non-trivial only if the dimensions of A and B coincide. Therefore the integrand will be $\mathbb{C}^\times$-invariant, hence the integral will be divergent, due to the volume of $\mathbb{C}^\times$. We interpret this divergence as the $\log|z|^2$ term in formula (6.28). Hence we can identify the prefactor with the two-point function of ${}^\ell A_m^{(1)}$ and B. Thus, the correlator of the log-partner ${}^\ell A^{(1)}$ of A and B should be equal to the sum of the integrals of the differential forms corresponding to A and B over the quotients of $\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1, m), m > 0$, by $\mathbb{C}^\times$.

The quotient $\mathcal{M}_{\mathbb{P}^1}(\mathbb{P}^1, m)/\mathbb{C}^\times$ may be compactified to the moduli space of stable maps with two marked points, $\overline{\mathcal{M}_{0,2}(\mathbb{P}^1, m)}$, which is mapped by the evaluation maps $\mathrm{ev}_0 \times \mathrm{ev}_\infty$ to $\mathbb{P}^1 \times \mathbb{P}^1$. Let us denote the pre-image of the point $(e^x, e^y) \in \mathbb{P}^1 \times \mathbb{P}^1$ by $\mathcal{M}(x, y; m)$. This space is a particular compactification of the space of m-tuples of pairs (w_i^+, w_i^-), $i = 1, \ldots, m$, which obey:

$$\prod_{i=1}^m \frac{w_i^+}{w_i^-} = e^{x-y} \tag{6.30}$$

modulo the $\mathbb{C}^\times$-action:

$$(w_i^+, w_i^-) \mapsto (tw_i^+, tw_i^-)\ , t \in \mathbb{C}^\times \tag{6.31}$$

Then we compute the integral over this moduli space (using formula (6.8)) and interpret the result as the perturbative correlation function of ${}^\ell A_m^{(1)}$ and B. As the result, we obtain the following expression for ${}^\ell A_m^{(1)}$:

$${}^\ell A_m^{(1)}(x) = \sum_B B^\vee(x) \int_{\mathbb{P}^1} A(y) \int_{\mathcal{M}(x,y;m)} \Omega_{x,y}(A, B; m) \wedge \overline{\Omega}_{\overline{x},\overline{y}}(A, B; m), \tag{6.32}$$

where $\Omega_{x,y}(A, B; m)$ is a meromorphic top degree form on $\mathcal{M}(x, y; m)$ constructed out of A and B. We sum over the space of all local operators using some basis B and the dual basis $B^\vee$ with respect to the perturbative Zamolodchikov metric.

For example, if (in the notation of Section 4.4)

$$A = A_{\lambda\mu}(x, \overline{x})\mathcal{D}^\lambda \mathbf{X} \overline{\mathcal{D}}^\mu \overline{\mathbf{X}},$$

$$B = B_{\lambda'\mu';\nu,\kappa}(x, \overline{x})\mathcal{D}^{\lambda'} \mathbf{X} \overline{\mathcal{D}}^{\mu'} \overline{\mathbf{X}}\, \psi\partial^{\nu_1-1}\psi\partial^{\nu_2-2}\psi \ldots \overline{\psi}\overline{\partial}^{\kappa_1-1}\overline{\psi} \ldots\ ,$$

then

$$\Omega_{x,y}(A,B;m) = \iota_E \frac{d^m w^+ \wedge d^m w^-}{d\log\left(\prod_i w_i^+/w_i^-\right)}$$
$$\prod_k \left[\sum_i (w_i^+)^{\lambda_k} - (w_i^-)^{\lambda_k}\right]\left[\sum_i (w_i^+)^{-\lambda'_k} - (w_i^-)^{-\lambda'_k}\right]$$
$$\times\ \mathrm{Det}\|\,(w_i^+)^{-\nu_k+k}\ (w_i^-)^{-\nu_k+k}\,\|_{1\le i\le m\, 1\le k\le 2m}, \tag{6.33}$$

where ι_E is the contraction with the Euler vector field $E = w_i^+\partial_{w_i^+} + w_i^-\partial_{w_i^-}$.

Formula (6.33) can be also derived using the free field realization with the help of the holomortex operators $\Psi^\pm$ discussed in Section 4.7, along the lines of the OPE calculation in Section 6.7.7.

The above integrals may further diverge. Their regularization, in turn, will give rise to terms with higher power of logarithms in $\langle A(z,\overline{z})B(0)\rangle$, which we can use to recursively compute $A_m^{(r)}$ with $r > 1$. We will discuss this in more detail in the follow-up paper [28].

6.7. **Operator product expansion.** An important feature of quantum field theory is the operator product expansion (OPE) which gives rise to an algebraic structure on the space of fields. The OPE may be described in especially nice terms in two-dimensional conformal field theories (CFT), where the expansion may be written in terms of the rational functions of the form $(z-w)^n(\overline{z}-\overline{w})^{\overline{n}}$. In logarithmic CFT the expansion also involves terms with the logarithms $\log(z-w), \log(\overline{z}-\overline{w})$. Our analysis of the two-dimensional sigma models in the infinite radius limit shows that they are logarithmic CFTs. Therefore it is natural to ask whether the OPE in these models may be computed and logarithmic terms be observed explicitly. In this section we will address this question in the case of the target manifold $\mathbb{P}^1$ (this is the simplest non-trivial case). We will show that there are instanton corrections to the OPE which do involve logarithmic terms. However, these corrections only appear when we consider observables involving both chiral and anti-chiral fields. The chiral algebra of the model (which we have determined to be the global chiral de Rham complex in Section 4.5) is free of logarithms.[13]

6.7.1. *Instanton corrections to OPE: factorization approach.* Our goal is to understand the instanton corrections to the OPE in the sigma model on $\mathbb{P}^1$ in the infinite radius limit. We will compute the OPE by studying the factorization of the four-point correlation functions in a particular channel.

We start with the correlation function of dimension zero observables, inserted at the points z_1, z_2, z_3, z_4 on the worldsheet $\mathbb{P}^1$, and consider the channel where $z_1 \to z_2, z_3 \to z_4$ (see Figure 8).

[13]This is similar to the fact that in quantum mechanical models considered in Part I the subspaces of purely chiral and anti-chiral states have bases (such as the monomial bases $X^n, n \ge 0$, and $\overline{X}^{\overline{n}}, \overline{n} \ge 0$, in the $\mathbb{P}^1$ model) of true eigenvectors of the Hamiltonians; in other words, there are no Jordan blocks in the Hamiltonian on the chiral and anti-chiral states.

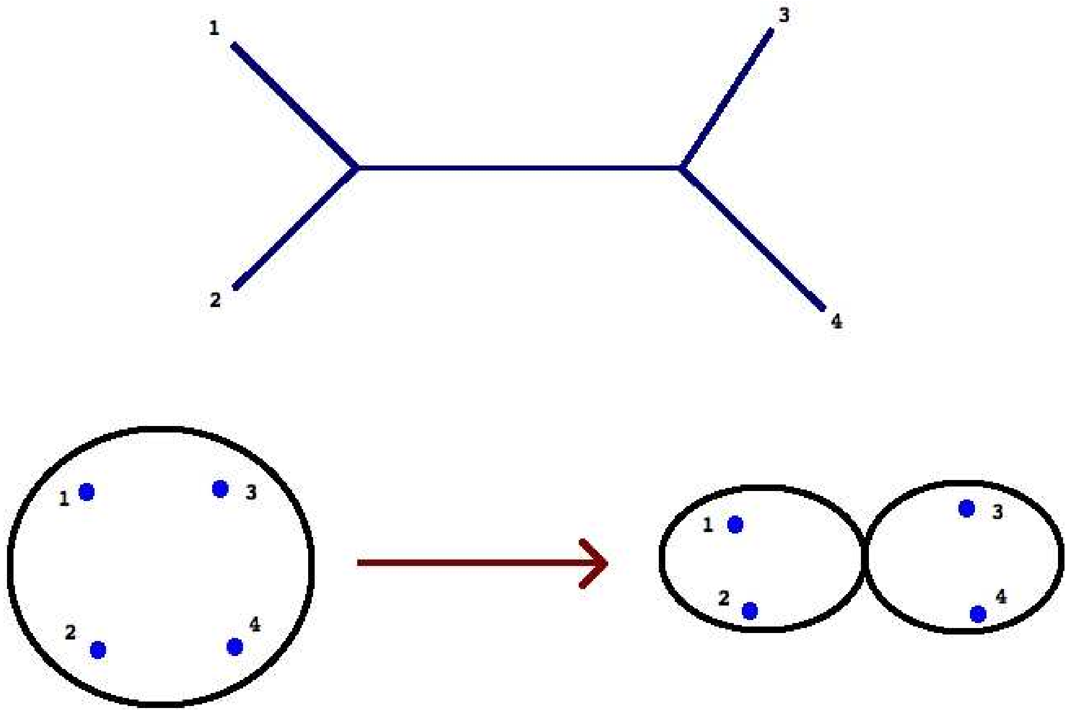

Figure 8. The degeneration of $\mathbb{P}^1$ with four points that we consider.

In principle, one could also try to do this by analyzing the three-point functions in the limit when $z_1 \to z_2$. However, the PGL_2-invariance of the correlation functions of dimension zero operators eliminates any free parameters in the three-point functions, and therefore it is difficult to analyze the corresponding limit. In the case of four-point functions we have one free parameter which we can use to study the limit.

Let $\omega_i \in \Omega^2(\mathbb{P}^1)$, $i = 1, 2, 3$, be three smooth two-forms on $\mathbb{P}^1$, and f a smooth function on $\mathbb{P}^1$). Consider the four-point function

$$\left\langle \mathcal{O}_{\omega_1}(z_1)\mathcal{O}_{\omega_2}(z_2)\mathcal{O}_{\omega_3}(z_3)\mathcal{O}_f(z_4) \right\rangle. \tag{6.34}$$

where $\mathcal{O}_\omega(z) = \omega(X(z), \overline{X}(\overline{z}))\psi\overline{\psi}(z, \overline{z})$, and $\mathcal{O}_f(z) = f(X(z), \overline{X}(\overline{z}))$. The complex dimension of the moduli space of holomorphic maps Φ from $\mathbb{P}^1$ (with at least three marked points) to $\mathbb{P}^1$ of degree d is $2d+1$. Therefore the only non-zero contribution to this correlation function will come from the component with $d = 1$, which is isomorphic to PGL_2 and which we compactify to $(\mathbb{P}^1)^3$ by considering the values of Φ at z_1, z_2, z_3, denoted by X_1, X_2, X_3 (see Section 6.2)[14]. The value at z_4 is determined by these, and

[14] Recall from Section 6.2 that the stable map compactification $\overline{\mathcal{M}_{0,3}(\mathbb{P}^1, 1)}$ of this moduli space is the blow-up of $(\mathbb{P}^1)^3$ along the principal diagonal $X_1 = X_2 = X_3$; thus, it differs from $(\mathbb{P}^1)^3$ by a measure zero subset.

is equal to

$$\Phi(z_4) = \frac{\lambda X_1 X_{23} + X_3 X_{12}}{X_{12} + \lambda X_{23}} = X_3 + \frac{\lambda X_{23}/X_{12}}{1 + \lambda X_{23}/X_{12}} X_{13}, \tag{6.35}$$

where

$$\lambda = \frac{z_{43} z_{21}}{z_{41} z_{23}}. \tag{6.36}$$

Here and below we use the notation $z_{ij} = z_i - z_j$.

The correlation function is therefore equal to

$$q \int_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1} \prod_{\alpha=1}^{3} \omega_\alpha(X_\alpha, \overline{X}_\alpha) d^2 X_1 \wedge d^2 X_2 \wedge d^2 X_3 \ f\left(\frac{\lambda X_1 X_{23} + X_3 X_{12}}{X_{12} + \lambda X_{23}}\right), \tag{6.37}$$

where $q = e^{-\tau}$ is the instanton parameter. This integral converges for smooth f and $\omega_\alpha = \omega_\alpha(X, \overline{X}) d^2 X$.

We now want to study the asymptotics of (6.37) in the limit $z_2 \to z_1$, $z_3 \to z_4$. In an ordinary (non-logarithmic) CFT we expect to have the following OPE (recall that conformal dimensions of the evaluation observables are equal to zero):

$$\mathcal{O}_{\omega_1}(z_1, \overline{z}_1)\ \mathcal{O}_{\omega_2}(z_2, \overline{z}_2) = \sum_A A(z_1, \overline{z}_1) z_{12}^{\Delta_A} \overline{z}_{12}^{\overline{\Delta}_A}, \tag{6.38}$$

$$\mathcal{O}_{\omega_3}(z_3, \overline{z}_3)\ \mathcal{O}_f(z_4, \overline{z}_4) = \sum_B B(z_4, \overline{z}_4) z_{34}^{\Delta_B} \overline{z}_{34}^{\overline{\Delta}_B}, \tag{6.39}$$

where $A(z_1, \overline{z}_1)$ denotes a field of conformal dimensions $(\Delta_A, \overline{\Delta_A})$, and similarly for $B(z_4, \overline{z}_4)$. (Recall that the conformal dimension of $\mathcal{O}_\omega$ is $(0,0)$.)

In the ordinary CFT we would have the following expansion of (6.37):

$$\begin{aligned} &\Big\langle \mathcal{O}_{\omega_1}(z_1)\mathcal{O}_{\omega_2}(z_2)\mathcal{O}_{\omega_3}(z_3)\mathcal{O}_f(z_4) \Big\rangle \to \sum_{A,B} z_{12}^{\Delta_A} \overline{z}_{12}^{\overline{\Delta}_A} z_{34}^{\Delta_B} \overline{z}_{34}^{\overline{\Delta}_B} \Big\langle \mathcal{O}_A(z_1)\mathcal{O}_B(z_4) \Big\rangle = \\ &\qquad \sum_{A,B} \left(\frac{z_{12}}{z_{41}}\right)^{\Delta_A} \left(\frac{\overline{z}_{12}}{\overline{z}_{41}}\right)^{\overline{\Delta}_A} \left(\frac{z_{34}}{z_{41}}\right)^{\Delta_B} \left(\frac{\overline{z}_{34}}{\overline{z}_{41}}\right)^{\overline{\Delta}_B} G_{AB}. \end{aligned} \tag{6.40}$$

Here G_{AB} is the Zamolodchikov metric, and $G_{AB} \neq 0$ only for $\Delta_A = \Delta_B, \overline{\Delta}_A = \overline{\Delta}_B$.

Let us write

$$\lambda = \frac{z_{43} z_{21}}{z_{41}(-z_{41} + z_{43} + z_{21})} = -\sum_{a,b=0}^{\infty} \frac{(a+b)!}{a!b!} \left(\frac{z_{43}}{z_{41}}\right)^{a+1} \left(\frac{z_{21}}{z_{41}}\right)^{b+1}. \tag{6.41}$$

If we formally expand (6.37) as a power series in λ near $\lambda = 0$, the coefficients should give us the OPE (6.38), (6.39).

6.7.2. *Instanton corrections to the chiral part of the operator product expansion.* The lowest order term in λ is the term λ^0. It is equal to the constant

$$q\int_{\mathbb{P}^1}\omega_1\int_{\mathbb{P}^1}\omega_2\int_{\mathbb{P}^1}f\omega_3, \tag{6.42}$$

which corresponds to

$$A(z_1,\overline{z}_1)=\left(\int_{\mathbb{P}^1}\omega_1\int_{\mathbb{P}^1}\omega_2\right)\mathbf{1}\ , \qquad \Delta_A=0,$$

$$B(z_4,\overline{z}_4)=f(X(z_4),\overline{X}(\overline{z}_4))\omega_3(X(z_4),\overline{X}(\overline{z}_4))\psi(z_4)\overline{\psi}(\overline{z}_4)\ , \qquad \Delta_B=0,$$

in (6.38), (6.39). This is in fact the term responsible for the quantum cohomology of $\mathbb{P}^1$.

The terms in the OPE expansion of $\mathcal{O}_{\omega_1}$ and $\mathcal{O}_{\omega_2}$ which are analytic in λ near $\lambda=0$ come from the terms of dimension $(k,0)$ which appear in front of λ^k. By expanding the integrand in (6.37) in λ, we obtain the following terms proportional to λ^k:

$$\begin{aligned} &q\,\lambda^k\sum_l C^{(k,l)}\int_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}\omega_1(X_1,\overline{X}_1)\omega_2(X_2,\overline{X}_2)d^2X_1\wedge d^2X_2\\ &\times\left(\frac{X_2-X_3}{X_1-X_2}\right)^k(X_3-X_1)^l\,\partial^l_{X_3}f(X_3,\overline{X}_3)\omega_3(X_3,\overline{X}_3)d^2X_3, \end{aligned} \tag{6.43}$$

where

$$C_f^{(k,l)}=\frac{(-1)^{k-l}(k-1)!}{l!(l-1)!(k-l)!}.$$

The integrals in (6.43) are conditionally convergent, for $k>1$. We shall assume that the prescription of the integration over the angles first is employed. The (k,l) term in (6.43) can be interpreted in the following way. Perturbatively, the OPE $\mathcal{O}_f(z,\overline{z})\mathcal{O}_\omega(0)$ we has the following terms analytic in z:

$$\begin{aligned} &\mathcal{O}_f(z,\overline{z})\mathcal{O}_\omega(0)\sim\sum_\mu z^{|\mu|}\mathcal{O}_{\omega f^{(\mu,0)}}(0)\\ &\mathcal{O}_{\omega f^{(\mu,0)}}=\omega(X,\overline{X})\mathcal{D}^\mu\mathbf{X}\,\partial_X^{\ell(\mu)}f(X,\overline{X})\psi\overline{\psi} \end{aligned} \tag{6.44}$$

(in the notation of Section 4.4). The operators $\mathcal{O}_{\omega f^{(\mu,0)}}$ are examples of the jet-evaluation observables. The three-point function $\langle\mathcal{O}_{\omega_1}(z_1)\ \mathcal{O}_{\omega_2}(z_2)\ \mathcal{O}_{\omega f^{(\mu,0)}}(z_3)\rangle$ is saturated by the one-instanton contribution, which, according to formula (6.8), is given by the integral (6.43), where $k=|\mu|, l=\ell(\mu)$. We interpret this as the appearance of the following terms in the OPE:

$$\mathcal{O}_{\omega_1}(z,\overline{z})\ \mathcal{O}_{\omega_2}(0)\sim q\sum_\mu z^{|\mu|}\mathcal{O}_{12,\mu}(0), \tag{6.45}$$

where

$$\mathcal{O}_{12,\mu} = :C_{12,\mu}(X)\prod_{i=1}^{\ell(\mu)}(-1)^{\mu_i-1}\frac{\mu_i!}{(2\mu_i-1)!}\partial_z^{\mu_i-1}p: + \text{terms with fermions},$$

$$C_{12,\mu}(X) = \int_{\mathbb{P}^1\times\mathbb{P}^1}\frac{(X-X_1)^{\ell(\mu)}(X-X_2)^{|\mu|}}{(X_1-X_2)^{|\mu|}}\omega_1(X_1,\overline{X}_1)\omega(X_2,\overline{X}_2)d^2X_1\wedge d^2X_2.$$

Indeed, the two-point function of $\mathcal{O}_{12,\mu}$ and $\mathcal{O}_{\omega f^{(\mu,0)}}$ is equal to the integral (6.43) with $k = |\mu|, l = \ell(\mu)$. To fix the fermionic terms in $\mathcal{O}_{12,\mu}$, we need to compute more two-point functions of this type.

To give an example of the fermionic terms arising in $\mathcal{O}_{12,\mu}$, we consider the following OPE, obtained in a similar way:

$$\mathcal{O}_{\omega_1}(z,\overline{z})\,(\omega_2(X,\overline{X})\partial_z X\overline{\psi})(0) \sim q\,\pi V_{12}(X)(0) + \ldots, \tag{6.46}$$

where we have a holomorphic vector field $V_{12}\partial_X$ given by the formula

$$V_{12}(X) = \int_{\mathbb{P}^1\times\mathbb{P}^1}\frac{(X-X_1)(X-X_2)}{X_1-X_2}\,\omega_1(X_1,\overline{X}_1)\omega_2(X_2,\overline{X}_2)d^2X_1\wedge d^2X_2\,. \tag{6.47}$$

By applying the supercharge $\mathcal{Q} = \int p(z)\psi(z)dz$ to both sides of (6.46), we obtain that the z-term in (6.45), corresponding to $\mu = (1)$, is equal to

$$[\mathcal{Q}, q\pi V_{12}(X)] = q\left(:C_{12,(1)}(X)p: + \partial_X C_{12,(1)}(X){:}\,\psi\pi:\right),$$

which corresponds to the Lie derivative by the vector field $V_{12}(X)\partial_X$ (see Section 4.4). Thus, we have

$$\mathcal{O}_{12,(1)} = :C_{12,(1)}(X)p: + \partial_X C_{12,(1)}(X){:}\,\psi\pi:.$$

The second term involves the fermionic combination $:\psi\pi:$. Similar terms arise in front of higher powers of z in (6.45).

6.7.3. *Logarithmic terms in the OPE.* A truly interesting term in the expansion of (6.37) is

$$\lambda\overline{\lambda} \approx \left|z_{12}z_{34}z_{41}^{-2}\right|^2, \tag{6.48}$$

which is equal to

$$\lambda\overline{\lambda}\int\limits_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}\prod_{\alpha=1}^{3}\omega_\alpha(X_\alpha,\overline{X}_\alpha)\;\partial_X\overline{\partial}_{\overline{X}}f(X_3,\overline{X}_3)\;\frac{|X_1-X_3|^2|X_2-X_3|^2}{|X_1-X_2|^2}\cdot \tag{6.49}$$
$$\cdot\, d^2X_1\wedge d^2X_2\wedge d^2X_3.$$

This integral is very similar to the integral (6.13) studied above, and, like that integral, it is also divergent. A troublesome region is $X_1 \approx X_2$. To extract the divergent part of (6.49), we write (similarly to the calculation in Section 6.4)

$$X_1 = X - \xi/2, \qquad X_2 = X + \xi/2.$$

Then the divergent part of (6.49) may be approximated by the integral

$$\lambda\overline{\lambda} \int_{\mathbb{P}^1_1 \times \mathbb{P}^1_3 \times U} \omega_1(X_1, \overline{X}_1)\omega_2(X_1, \overline{X}_1) \, \left(\omega_3 \partial_X \overline{\partial}_{\overline{X}} f\right)(X_3, \overline{X}_3)|X_1 - X_3|^4 \cdot \\ \cdot \, d^2X_1 \wedge d^2X_3 \wedge \frac{d^2\xi}{|\xi|^2}, \tag{6.50}$$

where U is a small neighborhood of 0 in the ξ-plane. The integral (6.50) has logarithmic divergence in ξ. This means that the naive model (6.38), (6.39) for the OPE that we had assumed above is incorrect. What is going on here?

6.7.4. *Expansion of integrals.* We need to pause for a moment and try to understand the analytic phenomenon that we have just encountered. The integral (6.37) is well-defined for all values of λ. Yet, when we expand the integrand in λ near $\lambda = 0$, we obtain divergent integrals. The reason is that we are trying to switch the order in which we take two different limits: one corresponds to expansion in Taylor series, and the other is the limit $R \to \infty$ we take by evaluating our integral over the region $|X_i| < R$ in each of the three $\mathbb{P}^1$. To understand this better, we consider as a toy model, the integral

$$\int_0^\infty \frac{dt}{(1+\lambda t)(1+t)^2}, \qquad \lambda \in \mathbb{R}. \tag{6.51}$$

It converges for all non-negative values of λ. However, the integrals of the coefficients in the expansion of the integrand in λ are divergent. The explanation is that the true expansion of (6.51) in λ actually contains $\log \lambda$. Indeed, the exact answer is

$$-\frac{1}{1+\lambda} + \frac{\lambda \log \lambda}{1-\lambda^2}.$$

When we try to expand it in the neighborhood of $\lambda = 0$, we find, in addition to a power series in λ (corresponding to the first term), a power series times $\log \lambda$.

On the other hand, we may first expand the integrand in (6.51) in a power series in λ and then integrate the terms of this expansion. Then we obtain integrals of the form

$$\int_0^\infty \frac{t^n dt}{(1+t^2)}, \qquad n \geq 0, \tag{6.52}$$

which diverge for $n > 0$, just like the integral (6.49). We would like to relate the *partie finie* regularization of the integrals (6.52) and the logarithmic terms in the expansion of (6.51).

Consider the following more general situation. Let $f^\lambda(t)$ be an analytic function in λ with the Taylor series expansion

$$f^\lambda(t) = \sum_{n \geq 0} f_n(t)\lambda^n,$$

Suppose that we have the following expansion:

$$\int_0^\infty f^\lambda(t)dt = \sum_{n \geq 0} (C_n \lambda^n + C_{\log,n}\lambda^n \log \lambda)$$

in a small neighborhood of $\lambda = 0$. On the other hand, we have (see [36], pp. 70-71)

$$\int_0^{1/\epsilon} f_n(t)dt = C'_n\epsilon^0 + C'_{\log,n}\log\epsilon + \sum_{i>0} C'_{i,n}\epsilon^{-i} + o(1). \tag{6.53}$$

Empirical evidence from calculations with (6.51) and similar integrals suggests the following conjecture:

$$C_n = C'_n, \qquad C_{\log,n} = C'_{\log,n}. \tag{6.54}$$

It might be well-known to specialists in analysis, but we were unable to locate it in the literature.

6.7.5. *Back to the integral* (6.49). We have found that the $|\lambda|^2$-term in the expansion of this integral is divergent. This means that our model (6.38), (6.39) for the OPE was oversimplified, and in fact, in addition to the power terms in $z_{ij}, \overline{z}_{ij}$, there are logarithmic terms. Indeed, in view of our conjecture (6.54), we should expect the term with $|\lambda|^2$ as well as $|\lambda|^2\log|\lambda|^2$. Recalling (6.48), we find that the expansion of $|\lambda|^2\log|\lambda|^2$ will contain the terms $|z_{12}|^2|z_{34}|^2\log|z_{12}|^2$ as well as $|z_{12}|^2|z_{34}|^2\log|z_{34}|^2$. Note that since we consider the limit $z_{12}, z_{34} \to 0$, with finite z_{14}, we are not expanding in terms of z_{14}. Such an expansion will be relevant when we consider the factorization of the four-point function (6.34) in a different channel.

Te appearance of the term $|z_{12}|^2|z_{34}|^2\log|z_{12}|^2$ means that we should include the term with $|z_{12}|^2\log|z_{12}|^2$ in formula (6.38). According to the above conjecture, to find this term, we need to introduce the "cut-off" $|\xi| > \epsilon$ in the integral (6.49). Then the ϵ^0-term in the corresponding expansion of the form (6.53) should give us the correlation function

$$\langle A(z_1,\overline{z}_1)B(z_3,\overline{z}_3)\rangle$$

where A is the $|z_{12}|^2$-term in the OPE (6.38) and B is the $|z_{34}|^2$-term in the OPE (6.38). These are in fact the terms in the perturbative OPEs of our operators, so we have

$$A = \omega_1(X,\overline{X})\omega_2(X,\overline{X})\psi\partial\psi\overline{\psi\partial\psi}$$

and

$$B = \omega_3(X,\overline{X})\partial_X\overline{\partial}_{\overline{X}}f(X,\overline{X})\partial X\overline{\partial X}\psi\overline{\psi}, \tag{6.55}$$

On the other hand, we interpret the $\log\epsilon$-term in the expansion of (6.49) as the correlation function

$$\langle\mathbb{A}(z_1,\overline{z}_1)B(z_3,\overline{z}_3)\rangle,$$

where now $\mathbb{A}$ is the $|z_{12}|^2\log|z_{12}|^2$-term in the modified OPE (6.38), and B is given by formula (6.55). We reproduce this two-point function if we set

$$\mathbb{A} = q\ {}^{\ell}A(X,\overline{X})p\overline{p},$$

where ${}^{\ell}(\omega_1\omega_2)$ is given by formula (6.20),

$${}^{\ell}(\omega_1\omega_2)(X,\overline{X}) = \int_{\mathbb{P}^1} |Y-X|^4\ (\omega_1\omega_2)(Y,\overline{Y})d^2Y. \tag{6.56}$$

This is precisely the logarithmic partner of $A(z,\overline{z})$ that we have already found in Section 6.5. (This is not surprising because the two computations, one in Section 6.5, involving three-point functions, and another one here, involving four-point functions, are essentially equivalent.)

Thus, though naively the operator $A(z,\overline{z})$ appears to be a primary field of conformal dimension $(1,1)$, it is in fact a logarithmic primary, with the log-partner $\mathbb{A}(z,\overline{z})$ of exact conformal dimension $(1,1)$. This logarithmic partner appears as the coefficient in front of the logarithmic term $|z_{12}|^2 \log|z_{12}|$ in the OPE (6.38).

Likewise, the appearance of the term $|z_{12}|^2|z_{34}|^2 \log|z_{12}|^2$ in the integral (6.49) implies that there is a logarithmic term $|z_{34}|^2 \log|z_{34}|^2$ in the OPE (6.39). It is nothing but the logarithmic partner of the field $B(z,\overline{z})$ appearing in formula (6.55),

$$\mathbb{B} = {}^{\ell}(\omega \partial_X \partial_{\overline{X}} f)\pi\overline{\pi},$$

which we had also found previously in Section 6.5.

Let us summarize: we have found the following terms in the operator product expansions (up to some numeric factors):

$$\begin{aligned}(\omega_1(X,\overline{X})\psi\overline{\psi})(z,\overline{z})\ (\omega_2(X,\overline{X})\psi\overline{\psi})(w,\overline{w}) &\sim q\int_{\mathbb{P}^1}\omega_1 d^2X \int_{\mathbb{P}^1}\omega_2 d^2X\cdot\mathbf{1}\\ &+ (\omega_1(X,\overline{X})\omega_2(X,\overline{X})\psi\partial\psi\overline{\psi\partial\psi})(w,\overline{w})|z-w|^2\\ &+ q\ ({}^{\ell}(\omega_1\omega_2)p\overline{p})(w,\overline{w})|z-w|^2\log|z-w|^2+\dots,\end{aligned} \tag{6.57}$$

$$\begin{aligned}&(\omega(X,\overline{X})\psi\overline{\psi})(z,\overline{z})\ f(X,\overline{X})(w,\overline{w}) \sim\\ &\qquad(\omega(X,\overline{X})\partial_X\overline{\partial}_{\overline{X}} f(X,\overline{X})\partial X\overline{\partial X}\psi\overline{\psi})(w,\overline{w})|z-w|^2\\ &\qquad+q\ ({}^{\ell}(\omega\partial_X\partial_{\overline{X}}f)\pi\overline{\pi})(w,\overline{w})|z-w|^2\log|z-w|^2+\dots\end{aligned} \tag{6.58}$$

Of course, there are many other terms on the right hand sides of these formulas, but in principle, they may all be computed in the same way as above.

In a similar way, we compute another example, when both observables are functions on $\mathbb{P}^1$:

$$\begin{aligned}f(X,\overline{X})(z,\overline{z})g(X,\overline{X})(w,\overline{w}) &\sim (fg(X,\overline{X}))(w,\overline{w})\\ &+|z-w|^6\log|z-w|^2\ (B_{f,g}(X,\overline{X})\pi\partial\pi\overline{\pi\partial\pi})(w,\overline{w})+\dots,\end{aligned} \tag{6.59}$$

where the coefficient function $B_{f,g}(X,\overline{X})$ is given by the integral transform:

$$\begin{aligned}&B_{f,g}(X,\overline{X}) =\\ &\int_{\mathbb{P}^1} d^2Y\ \mathrm{Coeff}_{y^3\overline{y}^3}\left[f(Y+y,\overline{Y}+\overline{y})g(Y-y,\overline{Y}-\overline{y})\left|(Y-X)^2-y^2\right|^4\right].\end{aligned} \tag{6.60}$$

To compute the last line, we use the following correlation function:

$$\begin{aligned}&\langle A(X)\psi\partial\psi\partial^2\psi\overline{\psi\partial\psi\partial^2\psi}(z)\ f(X(z_1))\ g(X(z_2))\rangle =\\ &4\left|\frac{z_1-z_2}{(z_1-z)(z_2-z)}\right|^6\int_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1} d^2X_1d^2X_2d^2X_3\ f(X_1)g(X_2)A(X_3)\ \left|\frac{X_{13}X_{23}}{X_{12}^2}\right|^4.\end{aligned} \tag{6.61}$$

Near the diagonal $X_1 = X_2$ the integral in (6.61) diverges in a power-like fashion:

$$\int \frac{d^2v}{|v|^8} \times \int_{\mathbb{P}^1\times\mathbb{P}^1} f(X)g(X)A(Y)|X-Y|^8 d^2X\, d^2Y. \tag{6.62}$$

(The integral (6.61) also looks divergent at infinity in X_1, X_2, but this is an illusion.) In order to extract the logarithmic term in the OPE (6.59), we have to expand the integrand in (6.61) in X_{12} to get the term with logarithmic divergence. This gives the integral (6.60).

6.7.6. *Analogy with quantum mechanics.* There is a precise analogy between the behavior of the correlation functions of the sigma models observed above and the correlation functions in quantum mechanical models studied in Part I.

These correlation functions are best understood in terms of the moduli spaces of stable maps and their quantum mechanical analogues, compactified moduli spaces of gradient trajectories.

In the quantum mechanical case (on $\mathbb{P}^1$), we considered in Section 5.2 of Part I the two-point functions of the form

$$_\infty\langle\widehat{\omega}(t_1)\widehat{F}(t_2)\rangle_0 = \int_{\mathbb{P}^1} \omega F(qz, q\overline{z}), \qquad q = e^{t_2-t_1}. \tag{6.63}$$

Here $\widehat{\omega}$ and $\widehat{F}$ are observables corresponding to a two-form and a function on $\mathbb{P}^1$, respectively. The integral is over the moduli space of gradient trajectories connecting the point $z = 0$ ("north pole") and the point $z = \infty$ ("south pole") on $\mathbb{P}^1$. (This moduli space may be compactified to $\mathbb{P}^1$.) We consider the expansion of (6.63) when $t_1 \gg t_2$, which corresponds to the limit $q \to 0$.

In order to understand this limit, it is instructive to look at the moduli space $\mathcal{M}_{0,\infty,2}$ of triples $(t_1, t_2, \Phi(t))$, where $t_1 > t_2$ are points on the affine line $\mathbb{R}$, and $\Phi(t) : \mathbb{R} \to \mathbb{P}^1$ is a gradient trajectory such that $\Phi(-\infty) = 0, \Phi(+\infty) = \infty$, modulo the diagonal action of $\mathbb{R}$,

$$(t_1, t_2, \Phi(t)) \to (t_1 + u, t_2 + u, \Phi(t + u)), \qquad u \in \mathbb{R}$$

(see Section 2.6 of Part I). This moduli space is compactified to $\overline{\mathcal{M}}_{0,\infty,2}$ in a way similar to the stable map compactification for the moduli space of holomorphic maps (see [12]).

We have a map $\mathcal{M}_{0,\infty,2} \to \mathbb{R}_{>0}$ taking $(t_1, t_2, \Phi(t))$ to $t_1 - t_2$. Then the moduli space over which we integrate in (6.63) is just the fiber of this map over $t_1 - t_2$.

We also compactify $\mathbb{R}_{>0}$, which is isomorphic to the open interval $(0, 1)$, by the closed interval $[0, 1]$, adding points at $t = 0$ and $t = +\infty$. Then we have a map $\overline{\mathcal{M}}_{0,\infty,2} \to [0, 1]$. The limit of the integral (6.63) when $t_1 - t_2 \to +\infty$ (which corresponds to the point $1 \in [0, 1]$) may be described in terms of integration over the fiber of the latter map over $1 \in [0, 1]$.

The result is the identity (5.6) presented in Section 5.2 of Part I. The expansion of the integral (6.63) as $q \to 0$ is expressed as the sum of three terms: the first one involves the expansion of the function f in the Taylor series around the point $0 \in \mathbb{P}^1$, the second one involves the expansion of the two-form ω in the Taylor series around the point $\infty \in \mathbb{P}^1$ – these are power series in q (which converge on a small disc in the q-plane), and the third term is the logarithmic correction which involves $\log(q)$. It balances out the first two terms and depends on the kind of regularization used in the first two terms.

Now, the four-point function (6.37) may be analyzed in a similar way. Here we have the moduli space of stable maps $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^1, 1)$, mapping to the Deligne–Mumford moduli space $\overline{\mathcal{M}}_{0,4}$, which is isomorphic to $\mathbb{P}^1$ via the double ratio λ given by formula (6.36). The integral (6.37) is over a smooth fiber of this map at some $\lambda \neq 0, 1, \infty$ (recall that this fiber is the blow-up of $(\mathbb{P}^1)^3$ at the main diagonal, so the integral is the same as over $(\mathbb{P}^1)^3$). We are interested in the expansion of this integral at $\lambda = 0$. Hence it should be described in terms of the fiber of the map $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^1, 1) \to \overline{\mathcal{M}}_{0,4}$ at $\lambda = 0$.

This is in a complete analogy with the quantum mechanical situation discussed above. Indeed, the expansion of (6.63) near $q = 0$ is similar to the expansion of (6.37) near $\lambda = 0$. The map $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^1, 1) \to \overline{\mathcal{M}}_{0,4}$ is the analogue of the map $\overline{\mathcal{M}}_{0,\infty,2} \to [0, 1]$. In the quantum mechanical case, (regularized) integrals over the fiber of the latter map over $1 \in [0, 1]$ (corresponding to $q = 0$) may be used to obtain the expansion of (6.63) leading up to the identity (5.6) of Part I. Likewise, the expansion of (6.37) near $\lambda = 0$ may be understood in terms of (regularized) integrals over the fiber of the map $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^1, 1) \to \overline{\mathcal{M}}_{0,4}$ at $0 \in \mathbb{P}^1 = \overline{\mathcal{M}}_{0,4}$ (which corresponds to $\lambda = 0$). This gives rise to an identity which is is analogous to the quantum mechanical identity (5.6) of Part I and describes completely the OPEs $\mathcal{O}_{\omega_1}(z_1)\mathcal{O}_{\omega_2}(z_2)$ and $\mathcal{O}_{\omega_3}(z_3)\mathcal{O}_f(z_4)$. We have focused above on the logarithmic terms of these OPEs, which are obtained from the logarithmic divergences of the integrals over the fiber at $\lambda = 0$. We will discuss the full identity and the full OPEs that it describes in the follow-up paper [28].

6.7.7. *Computation of the instanton corrections using the holomortex operators.* In Section 4.7 we have reviewed the description [25] of the sigma model with the target $\mathbb{P}^1$ as a deformation of a free field theory by the holomortex operators. This description gives us an alternative way for computing the correlation functions and the OPE in the sigma model on $\mathbb{P}^1$. Here we show how this works on the example of the OPE of two simplest evaluation observables. Let f be a smooth function on $\mathbb{P}^1$, and ω a smooth two-form on $\mathbb{P}^1$. They correspond to the dimension zero observables,

$$\mathcal{O}_f(z, \overline{z}) = f(x(z), \overline{x}(\overline{z})) \ , \ \mathcal{O}_\omega(z, \overline{z}) = \omega(x(z), \overline{x}(\overline{z}))\psi(z)\overline{\psi}(\overline{z}).$$

Here we use the logarithmic variables $x, \overline{x}$ on $\mathbb{P}^1$ and the corresponding variables p, ψ, π and their complex conjugates.

We wish to calculate their OPE

$$\mathcal{O}_f(z, \overline{z})\mathcal{O}_\omega(0, 0),$$

using the holomortex description of the sigma model on $\mathbb{P}^1$.

Let us compute the one-instanton correction to this OPE and try to reproduce the last term in (6.58), containing the logarithm. One-instanton correction corresponds to including one holomortex operator of each kind, $\Psi^+(w_+, \overline{w}_+)$ and $\Psi^-(w_-, \overline{w}_-)$, given by formula (4.23), and integrating over their positions, w_+ and w_- (note that according to the definition, $\Psi_\pm$ transforms as a $(1, 1)$-form on the worldsheet, so that this integration is intrinsically defined). Thus, we need to study the integral

$$(6.64)\qquad q\int d^2w_+d^2w_-\left(\mathcal{O}_f(z,\overline{z})\mathcal{O}_\omega(0,0)\Psi^+(w_+,\overline{w}_+)\Psi^-(w_-,\overline{w}_-)\right),$$

where

$$\Psi^{\pm}(z)=e^{\pm i\int^z P}\pi(z)\overline{\pi}(z),\qquad P=p(w)dw+\overline{p}(\overline{w})d\overline{w}.$$

Inside the brackets, we take the ordinary, that is, *perturbative* OPE of the operators involved in the framework of the free field theory. (The instanton corrections come about due to the insertion and integration of the holomortex operators.)

The expression in brackets is given by the formula

$$(6.65)\qquad f(x(z),\overline{x}(\overline{z}))\omega(x(0),\overline{x}(0))\psi(0)\overline{\psi}(0)e^{i\int_{w_-}^{w_+}P}\pi(w_+)\overline{\pi}(\overline{w}_+)\pi(w_-)\overline{\pi}(\overline{w}_-).$$

We have the OPE

$$x(z)e^{i\int_{w_-}^{w_+}P}=\log\left(\frac{z-w_+}{z-w_-}\right)e^{i\int_{w_-}^{w_+}P}+{:}x(z)e^{i\int_{w_-}^{w_+}P}{:}\,,$$

and similarly for $\overline{x}(\overline{z})$. Therefore for any function $f(x(z),\overline{x}(\overline{z}))$ we have

$$f(x(z),\overline{x}(\overline{z}))e^{i\int_{w_-}^{w_+}P}={:}f(\widetilde{x}(z),\overline{\widetilde{x}}(\overline{z}))e^{i\int_{w_-}^{w_+}P}{:}\,,$$

where

$$\widetilde{x}(z)=x(z)+\log\left(\frac{z-w_+}{z-w_-}\right),$$
$$\overline{\widetilde{x}}(\overline{z})=\overline{x}(\overline{z})+\log\left(\frac{\overline{z}-\overline{w}_+}{\overline{z}-\overline{w}_-}\right).$$

The fermionic part of (6.65) is given by the formula

$$\psi(0)\overline{\psi}(0)\pi(w_+)\overline{\pi}(\overline{w}_+)\pi(w_-)\overline{\pi}(\overline{w}_-)\sim\left|\frac{1}{w_+}-\frac{1}{w_-}\right|^2\pi(0)\overline{\pi}(0).$$

Thus, (6.65) is equal to

$${:}f(\widetilde{x}(z),\overline{\widetilde{x}}(\overline{z}))\omega(\widetilde{x}(0),\overline{\widetilde{x}}(0))e^{i\int_{w_-}^{w_+}P}{:}\left|\frac{1}{w_+}-\frac{1}{w_-}\right|^2\pi(0)\overline{\pi}(0).$$

Now we need to expand $f(\widetilde{x}(z),\overline{\widetilde{x}}(\overline{z}))$ at $z=0$. Since we wish to reproduce the last term in (6.58), we need to take the $|z|^2$ coefficient in the expansion, which is equal to

$$(\partial_x\partial_{\overline{x}}f)(\widetilde{x}(0),\overline{\widetilde{x}}(0))\left.\frac{\partial\widetilde{x}}{\partial z}\frac{\partial\overline{\widetilde{x}}}{\partial\overline{z}}\right|_{z=0}=(\partial_x\partial_{\overline{x}}f)(\widetilde{x}(0),\overline{\widetilde{x}}(0))\left|\frac{1}{w_+}-\frac{1}{w_-}\right|^2$$

plus terms that are less singular (in the variable a introduced below) as $w_+,w_-\to 0$, which do not contribute to the divergent part of the integral that we are trying to calculate; hence we ignore them in this computation.

Substituting this back into the integral (6.64), we obtain that the naive $|z|^2$-term in the OPE is given by the integral

$$q|z|^2\pi(0)\overline{\pi}(0)\int d^2w_+d^2w_-\left|\frac{1}{w_+}-\frac{1}{w_-}\right|^4{:}\omega(\widetilde{x}(0),\overline{\widetilde{x}}(0))(\partial_x\partial_{\overline{x}}f)(\widetilde{x}(0),\overline{\widetilde{x}}(0))e^{i\int_{w_-}^{w_+}P}{:}\,.$$

Let us set $u = e^{x(0)}, v = e^{\widetilde{x}(0)}$ and make the following change of integration variables (note that u is fixed in the integral):

$$(w_+, w_-) \mapsto \left(a = w_+ w_- \, , \; v = u \frac{w_+}{w_-} \right) \, .$$

Accordingly, we now write everything in terms of the algebraic coordinate $X = e^x$ on $\mathbb{P}^1$ and use the corresponding fermionic variables, which we will denote by $\psi', \overline{\psi}', \pi', \overline{\pi}'$. Note that we have $\pi'(0)\overline{\pi}'(0) = \frac{1}{|u|^2}\pi(0)\overline{\pi}(0)$ and that

$$\omega(\widetilde{x}(0), \overline{\widetilde{x}}(0)) d^2\widetilde{x}(0) = \omega(\widetilde{x}(0), \overline{\widetilde{x}}(0)) \frac{d^2 v}{|v|^2}.$$

Therefore in terms of our new variables we obtain the following integral:

$$q|z|^2 \pi'(0)\overline{\pi}'(0) \int \frac{d^2 a}{|a|^2} \, d^2 v |u - v|^4 \, {:}\omega(v, \overline{v})(\partial_v \partial_{\overline{v}} f)(v, \overline{v}) e^{i \int_{av/u}^{au/v} P}{:} \, .$$

This integral diverges as $a \to 0$. As we explained before, the $|z|^2 \log |z|^2$ term in the OPE should be equal to the prefactor in front of the divergent term $\frac{d^2 a}{|a|^2}$. This prefactor is equal to (note that in the limit $a \to 0$ the exponential operator drops out)

$$q\pi'(0)\overline{\pi}'(0) \int d^2 v |u - v|^4 \omega(v, \overline{v})(\partial\overline{\partial} f)(v, \overline{v}) = q \, ({}^{\ell}(\omega \partial\overline{\partial} f)\pi'\overline{\pi}')(0),$$

which coincides with the last term in formula (6.58). Thus, we have reproduced a one-instanton logarithmic correction to the OPE using the holomortex calculus! In a similar way one can reproduce the logarithmic term in the OPE (6.57) (for the derivation of the first term in (6.57), see [25], Section 3.4).

7. Gauged sigma models

The sigma models presented above can be treated, in a certain restricted sense, as the quantum mechanical models on the loop space. However, the Floer function f, as we have discussed, is only-well defined on a universal cover of the loop space. In addition, it is not a Morse–Novikov function, for its critical points are not isolated. It is, however, a Morse–Bott–Novikov function. In this section we shall consider a deformation of the standard sigma model, which can be described using the Morse–Novikov functions, i.e., the functions on the covering space with isolated critical points. This deformation is the sigma model in a background gauge field (we will see below for which gauge group). Note that in this deformation we do not integrate over the gauge field, so it only plays a classical role. But at the end of the section we will also comment on the models obtained by integrating over the gauge field.

7.1. **Quantum mechanical models.** As before, we first look at the corresponding quantum mechanical models. Let us start with the following simple observation concerning deformations. In ordinary quantum mechanics on some phase space $\mathcal{P}$ the first order action has the form

$$\int p dq - H(p, q) dt \tag{7.1}$$

The pdq part of the action is can be written, invariantly, as $\int \phi^* d^{-1}\omega$, where $\phi : I \to \mathcal{P}$ is the map of the worldline to the phase space. This term in the action does not require additional geometric structures on the worldline. The term $\int H(p,q)dt$, which is responsible for the non-trivial dynamics of the system, contains a one-form dt. There are several ways to interpret dt in (7.1). We can view it as the ein-bien, i.e. defining a metric $g = dt^2$ on I, or as a gauge field $A = dt$. In the evolution operator $\exp\left(-T\hat{H}\right)$, the time interval T will be interpreted as a length of the worldline in the first approach, or as the holonomy of the gauge field in the second. In passing from one to two dimensions both interpretations, with the metric and with the gauge field, prove useful. Moreover, they become in a sense related.

Consider now the deformation problem. Suppose we wish to deform the Hamiltonian $H(p,q)$ by adding to it a small perturbation $H \to H + \epsilon h$. The action (7.1) becomes

$$\int pdq - H(p,q)dt - \epsilon h(p,q)dt. \tag{7.2}$$

Let us assume that the deformation forms a first class system with the original Hamiltonian, i.e., the Poisson bracket $\{H, h\}$ is a linear combination of H and h. More generally let us assume that the Hamiltonian of the model is a linear combination of Hamiltonians H^a forming a representation R of a Lie algebra $\mathfrak{g} = \mathrm{Lie}G$:

$$[\hat{H}^a, \hat{H}^b] = f^{ab}_c \hat{H}^c,$$

or, classically,

$$\{H^a, H^b\} = f^{ab}_c H^c.$$

In this case it is natural to couple H^a to a $\mathfrak{g}$-valued gauge field:

$$S = \int pdq - A_a(t)H^a(p,q)dt. \tag{7.3}$$

The Lie algebra $\mathfrak{g}$ acts in the phase space $\mathcal{P}$ by Poisson vector fields, and we assume that this action integrates to the action of the Lie group G.

The evolution operator can be interpreted as the element $P\exp\int_0^T Adt$ of the Lie group G, taken in the representation R in the Hilbert space of the model. The path integral is invariant under the gauge transformations:

$$A \mapsto gAg^{-1} + gdg^{-1}\ ,\ g : I \to G\ ,\ \text{s.t. } g(0) = g(T) = 1\ . \tag{7.4}$$

This is shown by performing a change of integration variables

$$(p(t), q(t)) \mapsto g(t)(p(t), q(t)).$$

The main example of Part I, the quantum mechanics on $\mathbb{P}^1$ (i.e., $\mathcal{P} = T^*\mathbb{P}^1$) dealt with the abelian $\mathfrak{g} = \mathbb{R}^2 = \mathrm{Lie}\mathbb{C}^\times$. The gauge field in this case has two components which can be normalized as: $A = (\alpha dt, \beta dt)$, the first component couples to the gradient vector field $\frac{1}{2}\left(X\partial_X + \overline{X}\partial_{\overline{X}}\right)$, the second component couples to the Hamiltonian vector field $\frac{i}{2}\left(X\partial_X - \overline{X}\partial_{\overline{X}}\right)$. The time T evolution operator is equal to: $q^L\overline{q}^{\overline{L}}$, where $L = X\partial_X, \overline{L} = \overline{X}\partial_{\overline{X}}$, and $q = e^{-\frac{\alpha T}{2} + i\frac{\beta T}{2}}, \overline{q} = e^{-\frac{\alpha T}{2} - i\frac{\beta T}{2}}$. We see that the evolution operator (and correlation functions) are invariant with respect to the transformations $\beta \mapsto \beta +$

$2\pi\frac{n}{T}$, for $n \in \mathbb{Z}$. The conceptual reason for this invariance is the gauge transformation (7.4),

$$g(t) = e^{\frac{2\pi i n t}{T}} \, . \tag{7.5}$$

Clearly, α, or, more precisely, αT, is a gauge invariant.

7.2. **Deformations of sigma models.** Now let us move on to the sigma model case. First of all, we wish to describe invariantly the analogue of the α,β-system above. The phase space corresponding to the sigma model with the target space X and the worldsheet $\Sigma = \mathbb{S}^1 \times \mathbb{R}^1$ is $\mathcal{P} = T^*LX$, the cotangent bundle to the loop space. The coordinates are $\left(X^a(\sigma), \overline{X}^{\overline{a}}(\sigma)\right)$, the X-valued functions on $\mathbb{S}^1$, and the momenta are the one-forms $\left(P_{a\sigma}(\sigma), \overline{P}_{\overline{a}\sigma}(\sigma)\right)$. The Hamiltonian is

$$\begin{aligned} H_{\alpha,\beta} &= \alpha \int_{\mathbb{S}^1} \left(P_{a\sigma} i\partial_\sigma X^a - \overline{P}_{\overline{a}\sigma} i \partial_\sigma \overline{X}^{\overline{a}} \right) d\sigma \\ &+ \beta \int_{\mathbb{S}^1} \left(P_{a\sigma} \partial_\sigma X^a + \overline{P}_{\overline{a}\sigma} \partial_\sigma \overline{X}^{\overline{a}} \right) d\sigma. \end{aligned} \tag{7.6}$$

Now let us write the analogue of the action (7.2):

$$S = \int_{\mathbb{R}^1} dt \left[\int_{\mathbb{S}^1} \left(P_{a\sigma} \partial_t X^a + \overline{P}_{\overline{a}\sigma} \partial_t \overline{X}^{\overline{a}} \right) d\sigma - H_{\alpha,\beta} \right] \tag{7.7}$$

$$= \int_{\mathbb{R}^1 \times \mathbb{S}^1} dt d\sigma \left[P_{a\sigma} \left(\partial_t X^a - i\alpha \partial_\sigma X^a - \beta \partial_\sigma X^a \right) + \overline{P}_{\overline{a}\sigma} \left(\partial_t \overline{X}^{\overline{a}} + i\alpha \partial_\sigma \overline{X}^{\overline{a}} - \beta \partial_\sigma \overline{X}^{\overline{a}} \right) \right].$$

Using the complex coordinate $z = \alpha t - i\left(\sigma + \beta t\right)$ on Σ, the metric

$$g = dz d\overline{z} = \alpha^2 dt^2 + \left(d\sigma + \beta dt\right)^2 \tag{7.8}$$

and the redefinitions $P_{a\sigma} \to -\frac{i}{2} p_{az}, \overline{P}_{\overline{a}\sigma} \to -\frac{i}{2} \overline{p}_{\overline{a}\overline{z}}$, we show that up to the topological terms, the action (7.7) becomes the bosonic part of the action (2.5). The parameters α, β have been traded for the metric on the worldsheet Σ. Note that β is both a part of the two-dimensional metric and a component of a one-dimensional gauge field, which is nothing but the Kaluza-Klein gauge field corresponding to the compactification from two to one dimension. Since we study the finite time evolution $t \in I = [0,T]$, one can perform the coordinate transformation $\sigma \to \sigma + 2\pi\frac{nt}{T}$, $n \in \mathbb{Z}$, which is trivial at the endpoints. This amounts to acting on β via $\beta \mapsto \beta + 2\pi\frac{n}{T}$, which is a gauge transformation (7.5).

In what follows we shall denote by $H_{\alpha,\beta}$ the full Hamiltonian corresponding to (2.5), including the fermions.

Now suppose the target space X of our sigma model is the Kähler manifold with the holomorphic $\mathbb{C}^\times$-action, so that the corresponding $U(1)$ subgroup acts isometrically. We can then define a $\mathbb{C}^\times$-action on the loop space with the isolated fixed points. These fixed points are the constant loops which land at the fixed points of the $\mathbb{C}^\times$-action in X.

In this case we can deform the $H_{\alpha,\beta}$ Hamiltonian by the one generating the $\mathbb{C}^\times$-action on the target space. Let us discuss the Lagrangian aspects of this deformation.

Let v be the holomorphic vector field on X, generating a holomorphic $\mathbb{C}^{\times}$-action. A holomorphic vector field can be multiplied by an arbitrary complex constant $v \to \mu v$. When this vector field generates $\mathbb{C}^{\times}$ we can fix its normalization by requiring that $i\,(v - \overline{v})$ generates $U(1) = \mathbb{R}/2\pi\mathbb{Z}$.

Consider the two-dimensional sigma model with the target space X, on the worldsheet $\Sigma = \mathbb{R} \times \mathbb{S}^1$, with the following action, generalizing that in (7.7), (2.5):

$$\begin{aligned} S = \int_{\Sigma} & \left(-ip_a\left(\partial_{\overline{z}}X^a + \mu v^a\right) - ip_{\overline{a}}\left(\partial_z\overline{X^a} + \overline{\mu v^a}\right) + \right. \\ & + i\pi_a\left(D_{\overline{z}}\psi^a + \mu D_b v^a\psi^b\right) + i\pi_{\overline{a}}\left(D_z\psi^{\overline{a}} + \overline{\mu}D_{\overline{b}}\overline{v}^{\overline{a}}\psi^{\overline{b}}\right)\Big)\, d^2z + \int_{\Sigma}\tau_{a\overline{b}}dX^a \wedge d\overline{X^b}, \end{aligned} \tag{7.9}$$

where $\mu \in \mathbb{C}$ is a complex constant. The action (7.9) describes the quantum mechanics on the loop space LX with the Hamiltonian $H_{\alpha,\beta,\gamma,\delta} = H_{\alpha,\beta} + h_{\widetilde{\alpha},\widetilde{\beta}}$, where the Hamiltonian $h_{\widetilde{\alpha},\widetilde{\beta}}$ is given by:

$$\begin{aligned} h_{\widetilde{\alpha},\widetilde{\beta}} = \widetilde{\alpha}\int_{\mathbb{S}^1} & \left(P_{a\sigma}v^a(X(\sigma)) + \overline{P}_{\overline{a}\sigma}\overline{v}^{\overline{a}}(\overline{X}(\sigma))\right) d\sigma + \\ & i\widetilde{\beta}\int_{\mathbb{S}^1}\left(P_{a\sigma}v^a(X(\sigma)) - \overline{P}_{\overline{a}\sigma}\overline{v}^{\overline{a}}(\overline{X}(\sigma))\right) d\sigma, \end{aligned} \tag{7.10}$$

where $\widetilde{\alpha}, \widetilde{\beta} \in \mathbb{R}$, $\mu = \widetilde{\alpha} + i\widetilde{\beta}$.

The equations of motion are the following modification of the equations of holomorphic maps:

$$\partial_{\overline{z}}X^a + \mu v^a = 0. \tag{7.11}$$

The path integral localizes on the moduli space of solutions of these equations, in the same way as in the ordinary sigma model.

Let us recapitulate. The loop space LX for Kahler X with $U(1)$-isometry has a natural $G = \mathbb{C}^{\times} \times \mathbb{C}^{\times}$-action. The first $\mathbb{C}^{\times}$ has been already exploited, its $U(1)$ part is the rotation of the loops. The second $\mathbb{C}^{\times}$ translates the loops in target space using the target space $\mathbb{C}^{\times}$-action. Thus, the quantum mechanical Hamiltonian (7.10) on LX has four parameters, $(\alpha, \beta, \widetilde{\alpha}, \widetilde{\beta})$. The action

$$S = \int P \cdot \partial_t X \ dt d\sigma - \left(H_{\alpha,\beta} + h_{\widetilde{\alpha},\widetilde{\beta}}\right) dt,$$

given by the expression (7.9), is well-defined on the cylinder $\mathbb{S}^1 \times \mathbb{R}^1$. However, we wish to study sigma models on more general worldsheets. To this end we need to find a way to write the v-dependent couplings in (7.9) is a covariant way. Since the vector field v generates a symmmetry of the target space, it is natural to couple it to a gauge field on the worldsheet. In other words, we replace

$$\left(\partial_{\overline{z}}X^a + \mu v^a\right)$$

by

$$\nabla_{\overline{z}}X^a := \partial_{\overline{z}}X^a + A_{\overline{z}}v^a,$$

where $A_{\overline{z}}$ is the $(0,1)$-component of the gauge field. Since we don't take a quotient of X by the action $\mathbb{C}^{\times}$, the gauge field is to be viewed as a background field, just like the worldsheet complex structure:

$$S = -\int_{\Sigma} d^2z \, \left(ip_a \overline{\nabla}_{\overline{z}} X^a + ip_{\overline{a}} \nabla_z \overline{X^a} \right) \; + \; \text{fermions} \; + \int_{\Sigma} \tau_{a\overline{b}} dX^a \wedge d\overline{X^b}. \tag{7.12}$$

7.3. **Gauge group.** What is the gauge group corresponding to the action (7.9)? Is it $\mathbb{C}^{\times}$, $\mathbb{R}$, or $U(1)$? In other words, what are the gauge transformations of $A = A_z dz + A_{\overline{z}} d\overline{z}$ leaving invariant the correlation functions in the theory with the action (7.9)?

We are dealing with the twisted supersymmetric model. That model has a non-anomalous $U(1)_L \times U(1)_R$ symmetry,

$$A_{\overline{z}} \mapsto A_{\overline{z}} + \overline{\partial}_{\overline{z}} f \ , \ A_z = A^*_{\overline{z}} \mapsto A_z + \partial_z \overline{f} \ , \tag{7.13}$$

for complex valued f. The transformation (7.13) is accompanied by the transformation $X \mapsto \exp(f \cdot v) X$, which is a $z, \overline{z}$-dependent diffeomorphism of the target space. [15].

Thus, on a compact Riemann surface, the correlation functions in our model are functions on the Jacobian, $\mathrm{Jac}(\Sigma)$, which we identify with the quotient of the space of $(0,1)$-forms $A_{\overline{z}}$ by the action of the group of $\mathbb{C}^{\times}$-gauge transformations:

$$A_{\overline{z}} \mapsto A_{\overline{z}} + \overline{\partial}_{\overline{z}} f \tag{7.14}$$

Note that (7.13) is not a $\mathbb{C}^{\times}$-gauge transformation of the full connection $A = A_z dz + A_{\overline{z}} d\overline{z}$. Indeed, the $\mathbb{C}^{\times}$-gauge transformation would transform $A_z \mapsto A_z + \partial_z f$, in addition to (7.14), in variance with (7.13). That transformation cannot be accompanied, for general complex valued f, by a diffeomorphism of the target space and does not, therefore, define a symmetry of the theory. In fact, by using (7.13) we can make A into a unitary flat connection. So, in the end, the sigma model is naturally coupled to the $U(1)$-flat connection. This result seems surprising and a little bit counterintuitive.

In order to understand it better, let us look at one example. We take $X = \mathbb{CP}^1$, and $\Sigma = T^2$, the torus which is the quotient of the complex line by the lattice: $z \sim z + 2\pi i \,(m + n\tau)$, $m, n \in \mathbb{Z}$, where

$$\tau = (\beta + i\alpha) \frac{T}{2\pi}. \tag{7.15}$$

The vector field v on X is our friend $X\partial_X + \overline{X}\partial_{\overline{X}}$. The equations of motion read

$$\partial_{\overline{z}} X + A_{\overline{z}} X = 0. \tag{7.16}$$

We may, and will, assume that $A_{\overline{z}} = \mu \in \mathbb{C}$ is a constant. Let us perform a gauge transformation (in fact, we should rather call it a change of variables in the path integral):

$$X(z, \overline{z}) \mapsto e^{\overline{\mathbf{m}} z + \mathbf{m} \overline{z}} X(z, \overline{z}), \tag{7.17}$$

[15]Note that in the purely bosonic sigma models, which we shall study in Part III, the $\mathbb{C}^{\times}$-gauge invariance is broken generically by anomaly down to the $U(1)$ gauge invariance. The issue here is the definition of the measure in the path integral. In the bosonic model the chiral measure is defined using a holomorphic top form which may be not invariant under the $\mathbb{C}^{\times}$-action. Under the $U(1)$-action the holomorphic top form is multiplied by the phase, which may cancel the similar phase of the conjugate anti-chiral measure.

accompanied by the change $\mu \mapsto \mu - \mathbf{m}$. The periodicity of the $\mathbb{C}^\times$-valued function requires:

$$2\pi i\,(\overline{\mathbf{m}} - \mathbf{m}) = 2\pi i m, \quad m \in \mathbb{Z},$$

$$2\pi i\,(\overline{\mathbf{m}}\tau - \mathbf{m}\overline{\tau}) = 2\pi i n, \quad n \in \mathbb{Z}, \tag{7.18}$$

which gives

$$\mathbf{m} = \frac{n - m\tau}{\tau - \overline{\tau}}, \qquad m, n \in \mathbb{Z}. \tag{7.19}$$

So we see that the moduli space of physically inequivalent μ-parameters is an elliptic curve; in particular, it is compact. This is in striking contrast with the situation in quantum mechanics.

Indeed, the quantum mechanical analogue of the partition function of the sigma model on the torus is the calculation of the path integral on the circle, i.e., the trace of the evolution operator:

$$Z(T, \alpha, \beta) = \mathrm{Tr}\; e^{-T\left(\alpha H_{\mathbb{R}^\times} + \beta H_{U(1)}\right)}. \tag{7.20}$$

We denote by $H_{\mathbb{R}^\times}$ the Hamiltonian corresponding to the gradient vector field $V = -\nabla f$, and by $H_{U(1)}$ the Hamiltonian corresponding to the Hamiltonian vector field $U = \omega^{-1} df$. As the latter generates an action of $U(1)$, the partition function (7.20) is invariant under the shifts $\beta \to \beta + n$ for $n \in \mathbb{Z}$. This is analogous to the n-dependent shifts in (7.19). However, the gradient vector field generates an action of the group $\mathbb{R}^\times$, therefore no periodicity in the T-dependence[16] is expected. The two-dimensional result (7.19), however, predicts a periodicity of a new kind: $T \to T + m$, $m \in \mathbb{Z}$. It is now time to explain the origin of this mysterious symmetry.

The truth of the matter is the existence of a novel symmetry of the loop space LX, sometimes referred to as the *spectral flow*. It is related to the $U(1)$ action on X. Namely, we define an action of $\mathbb{Z}$ on LX by the formula

$$x(\sigma) \mapsto \; g_n(x(\sigma)) = {}^n x(\sigma) = e^{in\sigma} \cdot x(\sigma), \tag{7.21}$$

where $e^{i\vartheta} \cdot x$ denotes the action of the element $e^{i\vartheta} \in U(1)$ on $x \in X$.

Now, let U_1, U_2 be the vector fields on LX generating the compact subgroup $U(1) \times U(1) \subset G = \mathbb{C}^\times \times \mathbb{C}^\times$, U_1 being the rotation of the loop, U_2 coming from the $U(1)$ action on X. Let V_1, V_2 denote the generators of $\mathbb{R}^\times \times \mathbb{R}^\times \subset G$. We claim that

$$g_n^* V_1 = V_1 + nV_2\;, \qquad g_n^* U_1 = U_1 + nU_2.$$

It then follows that

$$g_n^{-1} H_{\alpha,\beta}\, g_n = H_{\alpha,\beta} + n h_{\alpha,\beta} \tag{7.22}$$

at the quantum level. Thus, the trace of the evolution operator has an additional $\mathbb{Z}$-symmetry, leading up to two integer parameters in (7.19).

[16]One should not confuse the periodicity of the analytic continuation of $Z(T, \alpha, \beta)$ which may occur due to the integrality of the spectrum of $H_{\mathbb{R}^\times}$ – here we are talking about the real values of T.

As a side remark, note that we may define the space LX_ϑ of twisted loops which satisfy

$$x(\sigma) \in LX_\vartheta\,,\ x(\sigma + 2\pi) = e^{i\vartheta} \cdot x(\sigma). \tag{7.23}$$

Then, (7.21) generalizes to the action of $\mathbb{R}$ on $\cup_{\vartheta \in \mathbb{S}^1} LX_\vartheta$, covering the standard action of $\mathbb{R}$ on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$: $x(\sigma) \mapsto {}^{\vartheta}x(\sigma) = e^{i\vartheta\sigma} \cdot x(\sigma)$. While we do not really need these spaces here, they give rise to the twisted sectors in the space of states in the models obtained by gauging away the group $U(1)$ which will be discussed in Section 7.6.

We conclude this section with one further remark. One may wonder why the coupling to the vector fields V_1, U_1 is most naturally described using the two-dimensional metric while the coupling to the vector fields V_2, U_2 is described using the two-dimensional gauge field. The reason is that V_1, U_1 are actually a part of a much larger Lie algebra acting on the loop space, the complexification of the Virasoro algebra (with trivial central charge in our supersymmetric case), while V_2, U_2 are the generators of the Cartan subalgebra of the affine current algebra $\hat{u}(1)_L \times \hat{u}(1)_R$ (with the trivial level). The former usually couples to the two-dimensional metric while the latter to the two-dimensional gauge field.

7.4. **Singularities of the gauge fields.** We shall now show that we have to allow the gauge fields with singularities. Indeed, take our starting example, $\Sigma = \mathbb{R}^1 \times \mathbb{S}^1$. We can interpret (7.9) as (7.12) with the particular gauge field $A = dt$. If we compactify the cylinder by adding two points, we get a two-sphere $\mathbb{S}^2$, with the coordinate $z = e^{-t+i\sigma}$ (note that this z differs from the coordinate z used earlier in this section). The points we have added are $z = 0$ and $z = \infty$. The gauge field

$$A = -2dt = \left(\frac{dz}{z} + \frac{d\overline{z}}{\overline{z}} \right) \tag{7.24}$$

has two singular points, at $z = 0$ and $z = \infty$. We can generalize (7.24) to

$$A = \left(\alpha \frac{dz}{z} + \overline{\alpha} \frac{d\overline{z}}{\overline{z}} \right), \qquad \alpha \in \mathbb{C}. \tag{7.25}$$

The action (7.12) forces the map $X(z, \overline{z})$ to obey:

$$\overline{\partial}_{\overline{z}} X^a + \frac{\overline{\alpha}}{\overline{z}}\, v^a(X) = 0. \tag{7.26}$$

The condition of non-singularity near $z = 0$ or $z = \infty$ implies that

$$v^a\left(X\left(z=0\right)\right) = 0, \qquad v^a\left(X\left(z=\infty\right)\right) = 0.$$

Thus the map $X(z, \overline{z})$ sends the points $z = 0$ and $z = \infty$ to the fixed points of $\mathbb{C}^\times$-action on X. Consider the component $\mathcal{M}_{a,b;d}$ of the moduli space of such maps, for which $X\left(z=0\right) = a$, $X\left(z=\infty\right) = b$. In addition the component of the moduli space is labeled by $d = \left[X(\mathbb{S}^2)\right] \in H_2(X, \mathfrak{Z})$. The dimension of the component $\mathcal{M}_{a,b;d}$ is equal to

$$\dim \mathcal{M}_{a,b;d} = \mathfrak{n}_-(b) - \mathfrak{n}_+(a) + \int_d c_1(X), \tag{7.27}$$

where $c_1(X)$ is the first Chern class of X, and $\mathfrak{n}_\pm$ are the positive and negative Morse indices of the critical points a, b. In order to understand (7.27) we should interpret the dimension of $\mathcal{M}_{a,b;d}$ as the index of $\overline{\partial}$-operator, acting on the space of linearized maps.

As we have already discussed, the points a, b, viewed as constant loops landing at a, b, should be viewed as the critical points of Morse–Novikov function on the loop space LX obtained by deformation of the Morse–Bott–Novikov function (2.15),

$$f(\widetilde{\gamma}) = \int_D \widetilde{\gamma}^*(\omega_K) - \int_{\mathbb{S}^1} \gamma^*(H)dt, \tag{7.28}$$

where H is the Morse function on X, whose gradient vector field is v; it coincides with the Hamiltonian of the $U(1)$ action (here we use the notation introduced in Section 2.4). This function is well-defined on the universal cover $\widetilde{LX}$ of LX. Its critical points are the preimages in $\widetilde{LX}$ of the constant maps $\gamma : \mathbb{S}^1 \to \mathbb{P}^1$ landing at the critical points of H.

7.5. **Sigma model on $\mathbb{P}^1$ in the background of a gauge field.** We consider as an example of the models studied above the supersymmetric sigma model with the target $\mathbb{P}^1$ in the infinite radius limit, coupled to the vector field $v = X\partial_X + \overline{X}\partial_{\overline{X}}$, where X is the algebraic coordinate on $\mathbb{P}^1$ (as studied in Part I) and the connection (7.25), where we will assume that α is real, and $-1 < \alpha < 0$. As discussed above, this model may be recast as a quantum mechanical model on the loop space $\mathbb{P}^1$ with the Morse–Bott–Novikov function (2.15) deformed to the Morse–Novikov function (7.28). The equation of gradient trajectory is now (7.26), which we rewrite as follows:

$$\partial_{\overline{z}} X + \frac{\alpha}{\overline{z}} X = 0. \tag{7.29}$$

Let us describe the space of "in" states of this model using the general results of Section 3.6. Let $\widetilde{L\mathbb{P}^1}_0$ (resp., $\widetilde{L\mathbb{P}^1}_\infty$) be the ascending manifolds in $\widetilde{L\mathbb{P}^1}$ consisting of (homotopy classes of) maps $D \to \mathbb{P}^1$ satisfying (7.29) and such that $0 \in D \mapsto 0 \in \mathbb{P}^1$ (resp., $\infty \in \mathbb{P}^1$). We denote by $\widetilde{L\mathbb{P}^1}_{0,m}$ and $\widetilde{L\mathbb{P}^1}_{\infty,m}, m \in \mathbb{Z}$, their translates with respect to the group $\pi_1(L\mathbb{P}^1) = \mathbb{Z}$ acting on $\widetilde{L\mathbb{P}^1}$. Let $\widetilde{\mathcal{H}}_{0,m}$ and $\widetilde{\mathcal{H}}_{\infty,m}, m \in \mathbb{Z}$, be the spaces of semi-infinite delta-forms supported on these ascending manifolds. Introduce the big space of states

$$\widetilde{\mathcal{H}} = \prod_{m\in\mathbb{Z}} \widetilde{\mathcal{H}}_{0,m} \oplus \prod_{m\in\mathbb{Z}} \widetilde{\mathcal{H}}_{\infty,m}. \tag{7.30}$$

The space $\mathcal{H}$ of "in" states of our model is *non-canonically* isomorphic to the space of vectors

$$(\Psi_{0,m}, \Psi_{\infty,m})_{m\in\mathbb{Z}} \in \widetilde{\mathcal{H}}$$

satisfying the equivariance condition (3.36),

$$\widetilde{\Psi}_{0,m} = q\widetilde{\Psi}_{0,m+1}, \qquad \widetilde{\Psi}_{\infty,m} = q\widetilde{\Psi}_{\infty,m+1}. \tag{7.31}$$

This condition determines all $\Psi_{0,m}, \Psi_{\infty,m}, m \in \mathbb{Z}$, from $\Psi_{0,0}, \Psi_{\infty,0}$. Therefore we may identify

$$\mathcal{H} \simeq \widetilde{\mathcal{H}}_{0,0} \oplus \widetilde{\mathcal{H}}_{\infty,0}.$$

However, there are non-trivial extensions at play here, similar to those described in Section 6.5, leading up to the non-diagonalizability of the Hamiltonian and the logarithmic mixing of operators.

Let us now describe the spaces $\widetilde{\mathcal{H}}_{0,0}$ and $\widetilde{\mathcal{H}}_{\infty,0}$ more precisely.

7.5.1. *The space* $\widetilde{\mathcal{H}}_{0,0}$. Recall that $\widetilde{\mathcal{H}}_{0,0}$ is the space of semi-infinite delta-forms supported on the maps $D \to \mathbb{P}^1$ satisfying (7.29) and such that $0 \in D \mapsto 0 \in \mathbb{P}^1$. Maps of this form may be written as follows:

$$X(z,\overline{z}) = |z|^{-\alpha} \sum_{n \le 0} \gamma_n z^{-n}, \tag{7.32}$$

where the series converges on a disc of small radius (recall that $-1 < \alpha < 0$). Hence we obtain natural coordinates $\gamma_n, n \le 0$, on $\widetilde{L\mathbb{P}^1}_{0,0}$. The transversal coordinates are $\gamma_n, n > 0$.

The space $\widetilde{\mathcal{H}}_{0,0}$ may be modeled on the Fock representation of the chiral-anti-chiral $\beta\gamma$-bc system associated with the open subset $\mathbb{C}_0 = \mathbb{P}^1 \backslash \infty$ of $\mathbb{P}^1$. It is generated by the fields

$$\beta(z) = \sum_{n \in \mathbb{Z}} \beta_n z^{-n-1}, \qquad \gamma(z) = \sum_{n \in \mathbb{Z}} \gamma_n z^{-n},$$
$$b(z) = \sum_{n \in \mathbb{Z}} b_n z^{-n-1}, \qquad c(z) = \sum_{n \in \mathbb{Z}} c_n z^{-n},$$

and their complex conjugates. We have the standard OPEs

$$\begin{aligned} \gamma(z)\beta(w) &= \frac{1}{z-w} + :\gamma(z)\beta(w):\,, \\ c(z)b(w) &= \frac{1}{z-w} + :c(z)b(w):\,. \end{aligned} \tag{7.33}$$

We set $\widetilde{\mathcal{H}}_{0,0} = \mathcal{F}_0 \otimes \overline{\mathcal{F}}_0$, where $\mathcal{F}_0$ is the Fock representation of the chiral $\beta\gamma$-bc system, generated by a vacuum vector $|0\rangle$ such that

$$\gamma_n|0\rangle = c_n|0\rangle = 0, \quad n > 0, \qquad \beta_m|0\rangle = b_m|0\rangle = 0, \quad m \ge 0.$$

Therefore we have

$$\mathcal{F}_0 = \mathbb{C}[\gamma_n, \beta_m]_{n \le 0, m < 0} \otimes \Lambda[c_n, b_m]_{n \le 0, m<0} \cdot |0\rangle.$$

The anti-chiral counterpart $\overline{\mathcal{F}}_0$ is generated by the vector $\overline{|0\rangle}$ satisfying analogous relations with respect to the anti-chiral generators.

7.5.2. *Digression: chiral de Rham complex of* $\mathbb{P}^1$. Likewise, we have the Fock representation $\mathcal{F}_\infty$ of the chiral $\beta\gamma$-bc system associated with the open subset $\mathbb{C}_\infty = \mathbb{P}^1 \backslash 0$ of $\mathbb{P}^1$. It is generated by the vacuum vector $\widetilde{|0\rangle}$ under the action of the Fourier coefficients of the fields $\widetilde{\beta}(z), \widetilde{\gamma}(z), \widetilde{b}(z), \widetilde{c}(z)$, satisfying the same relations as above, so that

$$\mathcal{F}_\infty = \mathbb{C}[\widetilde{\gamma}_n, \widetilde{\beta}_m]_{n \le 0, m<0} \otimes \Lambda[\widetilde{c}_n, \widetilde{b}_m]_{n \le 0, m < 0} \cdot \widetilde{|0\rangle}.$$

$\overline{\mathcal{F}}_\infty$ is its anti-chiral counterpart.

The two open sets, $\mathbb{C}_0$ and $\mathbb{C}_\infty$, overlap over $\mathbb{C}^\times$. On this overlap we also have the chiral algebra which we denote by $\mathcal{F}^\times$ and we have two embeddings

$$\begin{multline*} \mathcal{F}_0 = \mathbb{C}[\gamma_n, \beta_m]_{n\leq 0, m<0} \otimes \Lambda[c_n, b_m]_{n\leq 0, m<0} \cdot |0\rangle \hookrightarrow \\ \mathbb{C}[\gamma_0^{-1}] \otimes \mathbb{C}[\gamma_n, \beta_m]_{n<0, m<0} \otimes \Lambda[c_n, b_m]_{n\leq 0, m<0} \cdot |0\rangle = \mathcal{F}^\times, \end{multline*} \tag{7.34}$$

$$\begin{multline*} \mathcal{F}_0 = \mathbb{C}[\widetilde{\gamma}_n, \widetilde{\beta}_m]_{n\leq 0, m<0} \otimes \Lambda[\widetilde{c}_n, \widetilde{b}_m]_{n\leq 0, m<0} \cdot |0\rangle \hookrightarrow \\ \mathbb{C}[\widetilde{\gamma}_0^{-1}] \otimes \mathbb{C}[\widetilde{\gamma}_n, \widetilde{\beta}_m]_{n<0, m<0} \otimes \Lambda[\widetilde{c}_n, \widetilde{b}_m]_{n\leq 0, m<0} \cdot |0\rangle = \mathcal{F}^\times, \end{multline*} \tag{7.35}$$

The general formulas (4.6) give us transformation formulas between the fields with and without tildes. We have

$$\widetilde{\gamma}(z) = \gamma(z)^{-1} = \gamma_0^{-1} \frac{1}{1 + \gamma_0^{-1} \sum_{n\neq 0} \gamma_n z^{-n}}, \qquad \widetilde{c}(z) = -\gamma(z)^{-2} c(z),$$

$$\widetilde{\beta}(z) = -{:}\gamma(z)^2 \beta(z){:} - 2\gamma(z){:}b(z)c(z){:}\,, \qquad \widetilde{b}(z) = \gamma(z)^2 b(z).$$

These formulas give rise to an identification of two versions of $\mathcal{F}^\times$, one in (7.34) and the other in its counterpart (7.35) with the tildes. This in turn allows us to "glue" together $\mathcal{F}_0$ and $\mathcal{F}_\infty$ giving us the usual definition of the chiral de Rham complex on $\mathbb{P}^1$ [40].

The cohomologies of the chiral de Rham complex on $\mathbb{P}^1$ (in the Cech realization) are equal to the cohomologies of the two-step complex

$$\mathcal{F}_0 \oplus \mathcal{F}_\infty \to \mathcal{F}^\times, \tag{7.36}$$

with the differential being the sum of the embeddings (7.34) and (7.35).

7.5.3. *The space* $\widetilde{\mathcal{H}}_\infty$. Naively, one might expect that the space $\widetilde{\mathcal{H}}_\infty$ is equal to $\mathcal{F}_\infty \otimes \overline{\mathcal{F}}_\infty$. But equation (7.29) written with respect to the coordinate $Y = X^{-1}$ around the point $\infty \in \mathbb{P}^1$ has the form

$$\partial_{\overline{z}} Y - \frac{\alpha}{\overline{z}} Y = 0.$$

A solution $Y(z, \overline{z})$ of this equation such that $Y(0,0) = \infty$ has the form

$$Y(z, \overline{z}) = |z|^\alpha \sum_{n<0} \widetilde{\gamma}_n z^{-n}.$$

Thus, unlike the solution (7.32), it involves only the negative modes of $\widetilde{\gamma}(z)$, not the zero mode. Therefore we obtain that $\widetilde{\mathcal{H}}_\infty = \mathcal{F}^1_\infty \otimes \overline{\mathcal{F}}^1_\infty$, where $\mathcal{F}^1_\infty$ is generated by a vacuum vector $\widetilde{|1\rangle}$ satisfying

$$\widetilde{\gamma}_n \widetilde{|1\rangle} = \widetilde{c}_n \widetilde{|1\rangle} = 0, \quad n \geq 0, \qquad \widetilde{\beta}_m \widetilde{|1\rangle} = \widetilde{b}_m \widetilde{|1\rangle} = 0, \quad m > 0. \tag{7.37}$$

Hence we have

$$\mathcal{F}^1_\infty = \mathbb{C}[\widetilde{\gamma}_n, \widetilde{\beta}_m]_{n<0, m\leq 0} \otimes \Lambda[\widetilde{c}_n, \widetilde{b}_m]_{n<0, m\leq 0} \cdot \widetilde{|1\rangle}.$$

Thus, $\mathcal{F}^1_\infty$ is different from $\mathcal{F}_\infty$; in fact, we have a short exact sequence

$$0 \to \mathcal{F}_\infty \to \mathcal{F}^\times \to \mathcal{F}^1_\infty \to 0, \tag{7.38}$$

so that we have

$$\mathcal{F}^1_\infty \simeq \mathcal{F}^\times/\mathcal{F}_\infty.$$

The anti-chiral counterpart $\overline{\mathcal{F}}^1_\infty$ is defined in a similar way.

7.5.4. *Space of states.* We conclude that the space of states of our model is non-canonically isomorphic to the direct sum

$$(\mathcal{F}_0 \otimes \overline{\mathcal{F}}_0) \oplus (\mathcal{F}^1_\infty \otimes \overline{\mathcal{F}}^1_\infty). \tag{7.39}$$

However, the canonical structure is more complicated.

Let us look again at the ascending manifolds of the Morse–Novikov function. There is one of them in each complex "semi-infinite" dimension, and the structure of their closures is the following: the closure of $\widetilde{L\mathbb{P}^1}_{0,n}$ contains $\widetilde{L\mathbb{P}^1}_{\infty,n}$ (as a complex codimension one stratum), and the closure of $\widetilde{L\mathbb{P}^1}_{\infty,n}$ contains $\widetilde{L\mathbb{P}^1}_{0,n+1}$ (also in codimension one). This leads to non-trivial extensions between the spaces of delta-forms supported on consecutive strata; namely, there are extensions

$$0 \to \widetilde{\mathcal{H}}_{\infty,n} \to ? \to \widetilde{\mathcal{H}}_{0,n} \to 0,$$

$$0 \to \widetilde{\mathcal{H}}_{0,n+1} \to ? \to \widetilde{\mathcal{H}}_{\infty,n} \to 0,$$

which are obtained similarly to the the extensions of the spaces of delta-forms in the quantum mechanical models discussed in detail in Part I.

Thus, the structure of the big space of states $\widetilde{\mathcal{H}}$ is more complicated: instead of the direct sum decomposition (7.30), we have a space that has a canonical filtration such that the alternating consecutive quotients are isomorphic to $\widetilde{\mathcal{H}}_{0,n}$ and $\widetilde{\mathcal{H}}_{\infty,n}, n \in \mathbb{Z}$. The space of states of our model is then obtained by imposing the equivariance condition (7.31). The non-triviality of the above extensions leads to the non-diagonalizability of the Hamiltonian, as we will see in the next subsection.

We remark that the above semi-infinite stratification of $\widetilde{L\mathbb{P}^1}$ (and $\widetilde{LX}$ for a general flag variety X) and the corresponding spaces of holomorphic delta-forms have been considered in [19]. These spaces are representations of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ with level 0, which are closely related to the Wakimoto modules. In the $\mathcal{N} = (0,2)$ supersymmetric version the level 0 algebra $\widehat{\mathfrak{g}}$ gets replaced by the $\widehat{\mathfrak{g}}$ at the critical level $-h^\vee$ (see [19, 23]). The corresponding models are closely related to the geometric Langlands correspondence. This will be discussed in Part III of this article.

7.5.5. *Action of the Hamiltonian.* We now compute the action of the Hamiltonian using as the prototype the formulas obtained in the quantum mechanical models in Part I.

According to the results of Section 4.8 of Part I, if we choose an identification of the space of states with the direct sum of spaces of delta-forms on the ascending manifolds X_α of the Morse function, then the naive Hamiltonian (which is the Lie derivative with respect to the gradient vector field) will get a correction term given by the formula (up to some constant factors, which we will ignore)

$$\sum_{\alpha,\beta} \delta_{\alpha\beta} \otimes \overline{\delta}_{\alpha\beta}. \tag{7.40}$$

Here the summation is over pairs of adjacent strata such that X_β is a codimension one stratum in the closure of X_α, and $\delta_{\alpha\beta}$ and $\overline{\delta}_{\alpha\beta}$ are the holomorphic and anti-holomorphic *Grothendieck–Cousin* (GC) operators acting between the spaces of holomorphic and anti-holomorphic delta-forms on the two strata. Their definition is spelled out in Section 4.8 of Part I.

Formula (7.40) suggests that if we identify the space of states of our model with the direct sum (7.39), then the naive Hamiltonian $L_0 + \overline{L}_0$ will get a correction term equal to

$$\delta_{(0,0),(\infty,0)} \otimes \overline{\delta}_{(0,0),(\infty,0)} + \delta_{(\infty,0),(0,1)} \otimes \overline{\delta}_{(\infty,0),(0,1)}, \tag{7.41}$$

where the δ's are the infinite-dimensional analogues of the GC operators, which we will now compute explicitly.

We start with the first summand. Its chiral factor, the operator

$$\delta_{(0,0),(\infty,0)} : \mathcal{F}_0 \to \mathcal{F}^1_\infty, \tag{7.42}$$

is supposed to take a holomorphic delta-form supported on the stratum $\widetilde{L\mathbb{P}^1}_{0,0}$ and extract its "polar part" along the codimension one stratum $\widetilde{L\mathbb{P}^1}_{\infty,0}$, as in Section 4.8 of Part I. Therefore this operator is equal to the composition

$$\mathcal{F}_0 \hookrightarrow \mathcal{F}^\times \twoheadrightarrow \mathcal{F}^1_\infty,$$

where the first map is just the embedding (7.34), and the second map is the map in the short exact sequence (7.38). According to (7.38), the latter map is an operator $S_\infty : \mathcal{F}^\times \to \mathcal{F}^1_\infty$ whose kernel is equal to $\mathcal{F}_\infty$. (The operator $\overline{\delta}_{(0,0),(\infty,0)}$ is defined similarly.) We now need to find the operator S_∞.

7.5.6. *Friedan–Martinec–Shenker bosonization.* We will show that the operator S_∞ arises naturally from the Friedan–Martinec–Shenker (FMS) bosonization [30], which is a realization of the chiral $\beta\gamma$-system in terms of scalar fields. Recall that we are presently considering the $\beta\gamma$-bc-system $\mathcal{F}_\infty$, in which the generating fields have tildes. But to simplify our notation, we will omit the tildes in the formulas below.

Introduce two chiral fields, $u(z)$ and $v(z)$ with the OPEs

$$u(z)u(w) = -\log(z-w) + {:}u(z)v(w){:}\,,$$
$$v(z)v(w) = \log(z-w) + {:}v(z)v(w){:}\,.$$

The FMS bosonization formulas look as follows:

$$\gamma(z) = e^{u(z)+v(z)}, \qquad \beta(z) = -{:}\partial_z v(z) e^{-u(z)-v(z)}{:}\,. \tag{7.43}$$

It is easy to check that the fields $\beta(z)$ and $\gamma(z)$ satisfy the OPE (7.33).

Let us denote the (purely bosonic) $\beta\gamma$-chiral algebra by $\mathcal{F}_{\infty,\mathrm{bos}}$. Let L be the (Minkowski) lattice chiral algebra L, generated from $\{e^{nu+mv}, n, m \in \mathbb{Z}\}$ under the action of $\partial_z u(z), \partial_z v(z)$ and their derivatives.

Formulas (7.43) give rise to a homomorphism $\mathcal{F}_{\infty,\mathrm{bos}} \to L$. However, unlike the well-known bosonization of fermions, this is *not* an isomorphism. Nevertheless, let us invert $\gamma(z)$, i.e., consider the bigger chiral algebra $\mathcal{F}^\times_{\mathrm{bos}}$ generated by $\gamma(z)^{\pm 1}$, $\beta(z)$ and their derivatives. Let L_0 be the subalgebra of the chiral algebra L, generated by the one-dimensional sublattice

$$\{e^{n(u+v)}, n \in \mathbb{Z}\} \subset \{e^{nu+mv}, n, m \in \mathbb{Z}\}$$

under the action of $\partial_z u(z), \partial_z v(z)$ and their derivatives. Then

$$\mathcal{F}^{\times}_{\mathrm{bos}} \simeq L_0$$

(see [20] for the proof). In particular, $\mathcal{F}^{\times}_{\mathrm{bos}}$ and $\mathcal{F}_{\infty,\mathrm{bos}}$ act on L.

Now let $L_1 \subset L$ be the subspace of L generated by $\{e^{nu+(n+1)v}, n \in \mathbb{Z}\}$. Then the vector $e^v \in L_1$ satisfies the relations (7.37). Thus, we obtain an embedding $\mathcal{F}^1_{\infty,\mathrm{bos}} \hookrightarrow L_1$.

Consider the *screening operator*

$$S = \int e^{v(z)} dz : L_0 \to L_1.$$

It was proved in [20] that

$$\begin{aligned} \operatorname{Ker} S &= \mathcal{F}_{\infty,\mathrm{bos}} \subset \mathcal{F}^{\times}_{\mathrm{bos}} = L_0, \\ \operatorname{Im} S &= \mathcal{F}^1_{\infty,\mathrm{bos}} \subset L_1. \end{aligned}$$

Now let us add the fermionic bc-system $\mathcal{F}_{\mathrm{ferm}}$ generated by $b(z)$ and $c(z)$ and consider the full chiral $\beta\gamma$-bc-system $\mathcal{F}_\infty$ as above. We have

$$\mathcal{F}^{\times} = \mathcal{F}^{\times}_{\mathrm{bos}} \otimes \mathcal{F}_{\mathrm{ferm}} = L_0 \otimes \mathcal{F}_{\mathrm{ferm}}, \qquad \mathcal{F}_\infty = \mathcal{F}_{\infty,\mathrm{bos}} \otimes \mathcal{F}_{\mathrm{ferm}}.$$

Consider the operator

$$S_\infty = \int e^{v(z)} dz \otimes 1 : L_0 \otimes \mathcal{F}_{\mathrm{ferm}} \to L_1 \otimes \mathcal{F}_{\mathrm{ferm}}.$$

Then (note that the fermionic parts in $\mathcal{F}_\infty$ and $\mathcal{F}^1_\infty$ are the same)

$$\begin{aligned} \operatorname{Ker} S_\infty &= \mathcal{F}_{\infty,\mathrm{bos}} \otimes \mathcal{F}_{\mathrm{ferm}} = \mathcal{F}_\infty, \\ \operatorname{Im} S_\infty &= \mathcal{F}^1_{\infty,\mathrm{bos}} \otimes \mathcal{F}_{\mathrm{ferm}} = \mathcal{F}^1_\infty. \end{aligned}$$

Therefore we may, and will, view it as an operator acting from $\mathcal{F}^{\times}$ to $\mathcal{F}^1_\infty$. Thus, we have found an operator $\mathcal{F}^{\times} \to \mathcal{F}^1_\infty$ whose kernel is equal to $\mathcal{F}_\infty$. Composing it with the embedding $\mathcal{F}_0 \hookrightarrow \mathcal{F}^{\times}$, we obtain the sought-after GC operator (7.42). We conclude that *the GC operator is equal to the FMS screening operator*!

7.5.7. *Mystery of the FMS bosonization revealed.* The FMS formulas have given us what we were looking for, but what is the geometric meaning of these formulas? To answer this question, let us look at the chiral algebra $\mathcal{F}^{\times}$ from the point of view of the logarithmic coordinate x on $\mathbb{C}^{\times}$ such that $\gamma = e^x$. We will denote the fields of the corresponding chiral de Rham complex by $x(z), p(z), \psi(z), \pi(z)$, and similarly for the anti-chiral analogue. We have the OPE formulas (2.9),

$$x(z)p(w) = \frac{i}{z-w} + {:}x(z)p(w){:}\,, \qquad \psi(z)\pi(w) = \frac{i}{z-w} + {:}\psi(z)\pi(w){:}\,.$$

Using the general transformation formulas (4.6) under the action of changes of variables, we find that

$$\gamma(z) = e^{x(z)}, \qquad \beta(z) = -i\left(:p(z)e^{-x(z)}: + :\psi(z)\pi(z)e^{-x(z)}:\right),$$
$$c(z) = e^{x(z)}\psi(z), \qquad b(z) = -ie^{-x(z)}\pi(z),$$

Next, we make the T–duality transform of [25], Section 2.3,

$$p(z) \mapsto \partial_z U(z),$$

where $U(z)$ has the following OPE with $x(w)$:

$$U(z)x(w) = -i\log(z-w) + :U(z)x(w):.$$

Then the above formulas become

$$\gamma(z) = e^{x(z)}, \qquad \beta(z) = -i\left(:\partial_z U(z)e^{-x(z)}: - :\psi(z)\pi(z)e^{-x(z)}:\right),$$
$$c(z) = e^{x(z)}\psi(z), \qquad b(z) = -ie^{-x(z)}\pi(z).$$

Finally, observe that if we introduce an additional field $\phi(z)$ with the OPE

$$\phi(z)\phi(w) = \frac{1}{z-w} + :\phi(z)\phi(w):,$$

and bosonize the fermions $\psi(z), \pi(z)$ by the usual formulas

$$\psi(z) = e^{\phi(z)}, \qquad \pi(z) = ie^{-\phi(z)}, \qquad \partial_z\phi(z) = -i:\psi(z)\pi(z):,$$

then we can rewrite the above formulas as follows:

$$\gamma(z) = e^{x(z)}, \qquad \beta(z) = :\partial_z(-iU(z) + \phi(z))e^{-x(z)}:,$$
$$c(z) = e^{x(z)+\phi(z)}, \qquad b(z) = e^{-x(z)-\phi(z)}.$$

In particular, if we set $u(z) = x(z) - iU(z) + \phi(z), v(z) = -\phi(z) + iU(z)$, then we recover the FMS formulas (7.43).

Thus, the FMS formulas can be explained by extending the $\beta\gamma$-system to its supersymmetric version and using the change of variables $\gamma = e^x$ in the corresponding chiral de Rham complex, followed by the T–duality of [25] (see also [10] for a closely related computation).

7.5.8. *FMS screenings and holomortex operators.* Now we can write the screening operator S_∞ in terms of the variables $x(z), p(z), \psi(z), \pi(z)$. We find that

$$e^v = e^{-\phi+iU} = -i\pi e^{i\int p dz}.$$

Actually, we have worked above with the chiral de Rham complex on the open subset $\mathbb{C}_0 \subset \mathbb{P}^1$, with the coordinate $\gamma = e^x$. But we need to make our computation on the open subset $\mathbb{C}_\infty$ with the coordinate $\widetilde{\gamma} = \gamma^{-1} = e^{-x}$. Therefore we need to replace $p \mapsto -p, \pi \to -\pi$ in the formulas, so that

$$e^v = i\pi e^{-i\int p dw}.$$

Finally, we find that

$$S_\infty = i\int \pi e^{-i\int p dw} dz.$$

This formula expresses the GC operator $\delta_{(0,0),(\infty,0)}$ of formula (7.42) in terms of the logarithmic coordinate on $\mathbb{P}^1$.

Likewise, we find that the anti-chiral operator $\overline{\delta}_{(0,0),(\infty,0)}$ is given by the formula $-i\int \overline{\pi} e^{-i\int \overline{p}d\overline{w}} d\overline{z}$. Hence the first correction term to the Hamiltonian is

$$\delta_{(0,0),(\infty,0)} \otimes \overline{\delta}_{(0,0),(\infty,0)} = -\int \pi(z) e^{-i\int p dw} dz \otimes \int \overline{\pi}(\overline{z}) e^{-i\int \overline{p}d\overline{w}} d\overline{z}.$$

We interpret the latter integral as the integral

$$(7.44) \quad -\fint \pi(z)\overline{\pi}(\overline{z}) e^{-i\int(p(w)dw+\overline{p}d\overline{w})}|z|^2 d\sigma := \\ -\left(\int_{|z|=\epsilon} \pi(z)\overline{\pi}(\overline{z}) e^{-i\int(p(w)dw+\overline{p}d\overline{w})}|z|^2 d\sigma\right)_{\epsilon^0},$$

where σ is the phase of z, regularized via the familiar Epstein–Glazer procedure. Namely, we retain the ϵ^0-term in the expansion of this integral as a function of ϵ.

Deformation of the Hamiltonian by this operator corresponds to the deformation of the action by the integral

$$\int \pi(z)\overline{\pi}(\overline{z}) e^{-i\int(p(w)dw+\overline{p}d\overline{w})} d^2z = \int \Psi^-(z,\overline{z}) d^2z,$$

which is one of the two holomortex operators introduced in [25] and in Section 4.7.

7.5.9. *The second holomortex.* We claim that the second correction term $\delta_{(\infty,0),(0,1)} \otimes \overline{\delta}_{(\infty,0),(0,1)}$ to the Hamiltonian (see (7.41)) corresponds to the second holomortex operator.

The chiral factor $\delta_{(\infty,0),(0,1)}$ is supposed to take a holomorphic delta-form supported on the stratum $\widetilde{L\mathbb{P}^1}_{\infty,0}$ and extract its "polar part" along the codimension one stratum $\widetilde{L\mathbb{P}^1}_{0,1}$. The result is a holomorphic delta-form on $\widetilde{L\mathbb{P}^1}_{0,1}$. The operator $\overline{\delta}_{(\infty,0),(0,1)}$ is its complex conjugate. Thus we obtain an operator $\widetilde{\mathcal{H}}_{(\infty,0)} \to \widetilde{\mathcal{H}}_{(0,1)}$. By the q-equivariance condition (7.31), $\widetilde{\mathcal{H}}_{(0,1)}$ is identified with $\widetilde{\mathcal{H}}_{(0,0)}$ up to multiplication by q. Therefore we obtain an operator

$$\widetilde{\mathcal{H}}_{(\infty,0)} \to \widetilde{\mathcal{H}}_{(0,0)}.$$

To compute this operator, we observe that we have a basic symmetry in our problem, reversing the two critical points 0 and ∞. It acts by $\gamma \mapsto \gamma^{-1} = \widetilde{\gamma}$, or $x \mapsto -x$. The operator $\delta_{(\infty,0),(0,1)} \otimes \overline{\delta}_{(\infty,0),(0,1)}$ may be obtained from $\delta_{(0,0),(\infty,0)} \otimes \overline{\delta}_{(0,0),(\infty,0)}$ by applying this change of variables, and multiplying by q (because of the equivariance condition). Thus, we find that

$$\delta_{(0,0),(\infty,1)} \otimes \overline{\delta}_{(0,0),(\infty,1)} = -q\int \pi(z) e^{i\int p dw} dz \otimes \int \overline{\pi}(\overline{z}) e^{i\int \overline{p}d\overline{w}} d\overline{z},$$

which we again interpret as the integral

$$(7.45) \qquad -q\fint \pi(z)\overline{\pi}(\overline{z}) e^{i\int(p(w)dw+\overline{p}d\overline{w})}|z|^2 d\sigma,$$

regularized, as above, via the Epstein–Glaser procedure.

We conclude that the naive Hamiltonian acquires the correction equal to the sum of (7.44) and (7.45). This corresponds to the deformation of the action of the free theory with the target $\mathbb{C}^\times$ by the holomortex operators

$$\int \pi(z)\overline{\pi}(\overline{z}) \left(e^{-i\int(p(w)dw+\overline{p}d\overline{w})} + e^{i\int(p(w)dw+\overline{p}d\overline{w})}\right) d^2z = \int (\Psi^-(z,\overline{z}) + q\Psi^+(z,\overline{z}))d^2z.$$

By rescaling the different summands in the space of states, we can rewrite the integrand as $q_1\Psi^+(z,\overline{z})+q_2\Psi^-(z,\overline{z})$, where $q_1q_2 = q$. If we choose $q_1 = q_2 = q^{1/2}$, then we obtain the action of the sigma model on $\mathbb{P}^1$ found in [25] and reviewed above in Section 4.7.

Let us summarize. We have adapted the quantum mechanical formulas from Part I to compute the correction terms to the Hamiltonian of the sigma model on $\mathbb{P}^1$ coupled to the vector field v. These correction terms are given by the infinite-dimensional analogues of the Grothendieck–Cousin operators corresponding to adjacent semi-infinite cells in $\widetilde{L\mathbb{P}^1}$. We have found that they may be expressed as the screenings arising in the FMS bosonization, which, in turn, coincide with the holomortex operators from [25] and Section 4.7. Thus, we obtain that the Hamiltonian of our model is equal to the deformation of the Hamiltonian of the free theory on $\mathbb{C}^\times$ by the holomortex operators, as expected.

7.5.10. *Cohomology of the supercharges.* We can also use the quantum mechanical formulas from Part I, Section 4.9, to derive the action of the supercharges of the sigma model on $\mathbb{P}^1$. The total supercharge splits as the sum of two terms $Q + \overline{Q}$, where (up to a factor of $-i$)

$$\begin{aligned} Q &= \int \psi p dz + \oint \left(qe^{i\int P} - e^{-i\int P}\right)\overline{\pi}d\overline{z}, \\ \overline{Q} &= \int \overline{\psi}\overline{p}d\overline{z} + \oint \left(qe^{i\int P} - e^{-i\int P}\right)\pi dz. \end{aligned}$$

These are the supercharges found in [25]. The cohomology of $Q + \overline{Q}$ is equal to the quantum cohomology of $\mathbb{P}^1$, that is, $H^0 = H^2 = \mathbb{C}$ and $H^i = 0$ for $i \neq 0, 2$. In [25] we have also computed the cohomology of the right-moving supercharge $\overline{Q}$ and found that we obtain the same answer (with the non-trivial cohomology occurring in degrees 0 and 1 with respect to the grading associated with $\overline{Q}$).

Finally, let us compute the cohomology of the perturbative version of $\overline{Q}$, that is, when we set $q = 0$. Then the term $q\oint e^{i\int P}\pi dz$ will disappear. The cohomology of the resulting complex reduces to the cohomology of the two-step complex

$$\mathcal{F}^\times \xrightarrow{S_\infty} \mathcal{F}^1_\infty.$$

Because of the exact sequence (7.38), this complex has the same cohomology as the complex (7.36), which is the Cech complex computing the cohomology of the chiral de Rham complex of $\mathbb{P}^1$. Thus, we conclude that the cohomology of the right-moving supercharge $\overline{Q}$ *in the perturbative regime* is isomorphic to the chiral de Rham complex of $\mathbb{P}^1$, in agreement with the prediction of [37, 56]. (The same result was obtained in [25] in a different way. See also [10, 18, 39, 33] for related work.) When we include the instantons, we turn on a non-zero parameter q and hence obtain an extra term $q\oint e^{i\int P}\pi dz$ in the differential. Because of this, the cohomology shrinks to the quantum cohomology of $\mathbb{P}^1$.

Similar results may be obtained for other toric varieties, along the lines of the above analysis and [25].

In Part III of this paper we will consider the $\mathcal{N} = (0, 2)$ supersymmetric sigma models in the infinite radius limit. (We have studied the corresponding quantum mechanical models in Part I, Section 6.4.) For the target manifold $\mathbb{P}^1$, we will find, by methods similar to the ones used above, that cohomology of the (right-moving) supercharge is, perturbatively, equal to the cohomology of the chiral algebra of differential operators of $\mathbb{P}^1$, the purely bosonic version of the chiral de Rham complex [32]. This is in agreement with [56]. When we include the instantons, the cohomology becomes identically zero, so this model has spontaneously broken supersymmetry. This agrees with the results of [57, 50] obtained by other methods. In Part III we will also obtain similar results for the $\mathcal{N} = (0, 2)$ supersymmetric sigma model on the flag manifolds of simple Lie groups, which, like the model on $\mathbb{P}^1$, possess affine Kac–Moody algebra symmetry of critical level $k = -h^\vee$ [19, 23].

7.6. **Gauging the Lie group symmetry.** Let X be a Kahler manifold with the isometric action of a Lie group G. We can gauge the infinite radius limit of the sigma model on X described above to obtain a sigma model on X/G (in the infinite radius limit). To this end we enlarge the set of fields by adding the gauge multiplet: $(A, \Psi, \phi, \overline{\phi}, \eta, \chi, H)$, on which the $\mathcal{Q}$-operator acts as follows:

$$
\begin{aligned}
&\mathcal{Q}A = \Psi \ , \ \mathcal{Q}\Psi = d_A\phi \ , \ \mathcal{Q}\phi = 0 \\
&\mathcal{Q}\overline{\phi} = \eta \ , \ \mathcal{Q}\eta = [\phi, \overline{\phi}] \\
&\mathcal{Q}\chi = H \ , \ \mathcal{Q}H = [\phi, \chi].
\end{aligned}
$$

We then write the gauged sigma model action as follows:

$$
\begin{aligned}
S = S_0 + \mathcal{Q}\cdot \Bigg(\int_\Sigma &\mathrm{tr}\left(\chi \cdot (F_A + \mu(x, \overline{x})\omega)\right) + \mathrm{tr}\left(\Psi \star d_A\overline{\phi}\right) + \mathrm{tr}\left(\eta[\phi, \overline{\phi}]\right) \\
&+ \pi_{iw}\left(\bar{\partial}_{\overline{w}}x^i + A^a_{\overline{w}}V^i_a(x)\right) + \overline{\pi}_{\overline{i}\overline{w}}\left(\partial_w \overline{x}^{\overline{i}} + A^a_w V^{\overline{i}}_a(\overline{x})\right) \\
&+ G^{i\overline{j}}\left(\pi_{iw}p_{\overline{j}\overline{w}} + p_{iw}\pi_{\overline{j}\overline{w}} + \Gamma\text{-terms}\right) + \mathrm{tr}\left(\chi H\right)\Bigg).
\end{aligned}
$$

The study of these models is beyond the scope of this paper (another model with gauge symmetry, the four-dimensional Yang–Mills theory, will be discussed in the next section). We note, however, that they may be analyzed following the same methods that we have used in the study of the ordinary sigma models in the infinite radius limit, in the preceding sections. In particular, the space of states of this model may be described as certain extensions of spaces of delta-forms on semi-infinite strata in the universal coverings of the loop space, as well as the twisted loop spaces, introduced in Section 7.3.

The correlation functions are given by integrals over the moduli spaces of twisted maps $\Sigma \to X$ corresponding to holomorphic principal G-bundles $\mathcal{P}$ over Σ (that is, holomorphic sections of the associated bundle $\mathcal{P} \underset{G}{\times} X$), satisfying a stability condition. In the case when the observables represent equivariant cohomology classes, we obtain gauge theory analogues of the Gromov–Witten invariants. (Another approach to these invariants, via K–theory, is presented in [29].) These comprise the topological sector of the gauged sigma model. More general correlation functions, such as the correlation functions of gauge theory analogues of the jet-evaluation observables introduced above, will be represented by divergent integrals over these moduli spaces. Their regularization will involve the logarithmic mixing of the type discussed in Section 6.

8. Four-dimensional gauge theory

In this section we briefly discuss the four-dimensional analogue of our constructions. The natural venue for the instanton physics is the place where they were originally found, namely the four dimensional gauge theory [7]. The approach of this paper, namely, the reduction to the supersymmetric quantum mechanics, the weak coupling limit after some redefinition of the wave-functions, and the extraction of the spectrum of the resulting Hamiltonian from the correlation functions, which typically reduce to finite-dimensional integrals, works in the four-dimensional case as well. However, there is yet another interesting twist of the story: the so-called equivariant Morse theory.

The plan of this section is the following. First, we discuss the gauged supersymmetric quantum mechanics. The analogue of the $\overline{\tau} \to \infty$ limit in this theory can be performed in several ways, and the one which is most relevant for the four-dimensional gauge theory will be reviewed. Then we briefly introduce the $\mathcal{N} = 2$ twisted superfields, and write the Lagrangian of the theory. In the limit $\overline{\tau} \to \infty$ the path integral becomes the sum of integrals over finite-dimensional moduli spaces of instantons, i.e., solutions to the anti-self-duality equations on the curvature of the gauge field. We discuss the analogue of the evaluation observables in this theory and conclude with the example of the instanton correlation function in the case of $G = SU(2)$ gauge theory, at the instanton charge one. We find the logarithmic dependence of the correlator on the positions of the operators, and conclude that the theory is a logarithmic conformal theory in four dimensions.

8.1. **Self-dual Yang–Mills theory.** The obvious analogue of our "weak coupling with instantons" limit in four dimensions is the so-called self-dual Yang–Mills theory. The action of (ordinary, i.e., non-supersymmetric) Yang–Mills theory, on a Riemannian manifold $\mathbf{M}^4$, i.e., the action of Euclidean theory, has the form:

$$S_{\mathrm{YM}} = \frac{1}{4g^2}\int_{\mathbf{M}^4} \mathrm{tr} F \wedge \star F - \frac{i\vartheta}{2\pi}\int_{\mathbf{M}^4} \mathrm{tr} F \wedge F. \tag{8.1}$$

Using the decomposition

$$F = F^+ + F^-, \qquad \star F^{\pm} = \pm F^{\pm}$$

of the curvature two-form, we can rewrite (8.1) as

$$S_{\mathrm{YM}} = \frac{i}{4\pi}\left(\tau \ \mathrm{tr}\|F^-\|^2 - \overline{\tau} \ \|F^+\|^2\right), \tag{8.2}$$

where

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2}$$

is the complexified coupling. Our limit consists of taking $\overline{\tau} \to \infty$ while keeping τ finite. In this limit the action (8.2) is not very useful. Instead, a first order action is more adequate:

$$S_{\mathrm{sdYM}} = -i \int_{\mathbf{M}^4} \mathrm{tr}\; H^+ \wedge F + \frac{\tau}{4\pi}\; \mathrm{tr} F \wedge F. \tag{8.3}$$

At this point we need to fix our normalization of the Killing form tr on the Lie algebra $\mathfrak{g}$ of the Lie group G. We normalize it so that the instanton charge

$$-\frac{1}{8\pi^2} \int_{\mathbf{M}^4} \mathrm{tr} F \wedge F$$

assumes integer values, and is a non-negative integer on the anti-self-dual connections.

8.1.1. *Supersymmetric Yang–Mills theory.* Let us now discuss the $\mathcal{N} = 2$ supersymmetric Yang–Mills theory [51]. We shall consider, as in the case of the sigma models, the twisted supersymmetry. This is done in order to have a naturally defined measure in the path integral. In addition to the gauge field A, there are the fermionic one-form ψ, self-dual two-form χ^+, the fermionic scalar η, and the pair of bosonic scalar fields ϕ and $\overline{\phi}$. All fields transform in the adjoint representation of the gauge group G. In the standard super-Yang–Mills theory instead of the integer spin fields ψ, χ^+, η one has a pair of Weyl fermions, $\lambda_{\alpha i}$, and their conjugates $\overline{\lambda}_{\dot{\alpha} i}$, $\alpha, \dot{\alpha}, i = 1, 2$. The bosonic symmetry group of the physical theory is the group $Spin(4) \times SU(2)_I$. The twisting consists of embedding $Spin(4)$ into this group is a non-standard way, under which $\overline{\lambda}_{\dot{\alpha} i} = \psi_\mu$, $\lambda_{\alpha i} = \chi^+ \oplus \eta$. In the standard theory the fields ϕ and $\overline{\phi}$ are complex conjugates of each other. In the applications of the twisted theory to the Donaldson theory it is much more natural to view ϕ and $\overline{\phi}$ as independent fields [54]. In the modern language this is a reflection of the fact that the mathematically better defined are the I-models, as opposed to the B-models which look more natural physically (see the discussion of this in the case of two-dimensional sigma models in [25]).

In order to proceed we need to know the normalizations of various terms in the action of super-Yang–Mills theory. One way to fix the normalization is to view the theory as the dimensional reduction of the six-dimensional minimal supersymmetric gauge theory. In six dimensions the action reads, schematically:

$$S_{\mathrm{6dSYM}} = \frac{1}{4g_6^2} \int \; \mathrm{tr}\left(F \wedge \star_6 F + \overline{\lambda} \not{D}_A \lambda \right).$$

Upon reduction, one gets the four-dimensional action:

$$S_{\mathrm{4dSYM}} = \frac{1}{4g^2}\int \mathrm{tr}\,\big(F\wedge\star F + D_A\phi\wedge\star D_A\overline{\phi} + \mathrm{vol}_g\ [\phi,\overline{\phi}]^2 \\ + \chi^+\wedge D_A^+\psi + \eta\wedge\star D_A^*\psi \\ + \phi[\chi^+,\chi^+] + \phi[\eta,\eta]\mathrm{vol_g} + \overline{\phi}[\psi,\star\psi]\big) \\ - \frac{i\vartheta}{8\pi^2}\int \mathrm{tr}F\wedge F, \tag{8.4}$$

where

$$\mathrm{vol_g} = \mathrm{g}^{\frac{1}{2}}d^4x = \star 1$$

is the metric volume-form. The last term in (8.4), the one with the theta-angle, can be turned on in four dimensions without breaking Lorentz invariance. In six dimensions this term requires an introduction of a background two-form (it is dual to the axion scalar in four dimensions). Introduce the auxiliary self-dual bosonic two-form field H^+, and rewrite the gauge kinetic term:

$$S_{\mathrm{4dSYM}} = \\ \frac{1}{g^2}\,\mathcal{Q}\cdot\int \mathrm{tr}\left(-i\chi^+\wedge F + \ \chi^+\wedge H^+ + \frac{1}{4}\psi\wedge\star D_A\overline{\phi} + \frac{1}{4}\mathrm{vol_g}\ \eta[\phi,\overline{\phi}]\right) - \frac{i\tau}{4\pi}\int \mathrm{tr}F\wedge F. \tag{8.5}$$

Here we have used the topological supercharge $\mathcal{Q}$, i.e., the supercharge $\varepsilon^{\alpha i}Q_{\alpha i}$ which becomes a scalar upon the twisting. It acts as follows:

$$\mathcal{Q}A = \psi\ ,\ \mathcal{Q}\psi = D_A\phi \\ \mathcal{Q}\chi^+ = H^+\ ,\ \mathcal{Q}H^+ = [\phi,\chi^+] \\ \mathcal{Q}\overline{\phi} = \eta\ ,\ \mathcal{Q}\eta = [\phi,\overline{\phi}]\ ,\ \mathcal{Q}\phi = 0. \tag{8.6}$$

8.1.2. *The weak coupling limit* $\overline{\tau}\to\infty$. As (8.5) stands, it is not suited for considering our limit $g\to 0$ with τ fixed. However, let us perform the following simple field redefinition:

$$\left(\overline{\phi},\eta,H^+,\chi^+\right) \mapsto \left(g^2\ \overline{\phi}, g^2\ \eta, g^2\ H^+, g^2\ \chi^+\right) \tag{8.7}$$

which keeps (8.6) intact. Now we are in position to take the $g\to 0$ limit. Indeed, with (8.7) the action (8.5) splits as:

$$S_{\mathrm{4dSYM}} = S_{\mathrm{ssdYM}} + g^2\ \mathcal{Q}\int \mathrm{tr}\left(\chi^+\wedge H^+ + \eta[\phi,\overline{\phi}]\ \mathrm{vol_g}\right), \tag{8.8}$$

where S_{ssdYM} is the action of the limit theory:

$$S_{\mathrm{ssdYM}} = \\ \int \mathrm{tr}\left(-iH^+\wedge F + i\chi^+\wedge D_A^+\psi + \eta\wedge\star D_A^*\psi + D_A\phi\wedge\star D_A\overline{\phi} + [\psi,\star\psi]\overline{\phi}\right) - \frac{i\tau}{4\pi}\int \mathrm{tr}F\wedge F. \tag{8.9}$$

The action (8.8) makes sense on any four-manifold $\mathbf{M}^4$. The twist made sure that on any $\mathbf{M}^4$ this theory has at least one fermionic symmetry, generated by $\mathcal{Q}$. The usual feature of such a theory is the $\mathcal{Q}$-exactness of the stress-energy tensor, which follows from the fact that all the metric dependence of (8.8) is contained in the $\mathcal{Q}$-exact terms in the Lagrangian.

The conformal invariance of (8.9) (even that of (8.8), at the classical level) is much less appreciated. To achieve it, let us assume that ϕ is a scalar, degree zero field, while $\overline{\phi}$ and η are half-densities, i.e., transform as section of $\mathrm{vol}_{\mathrm{g}}^{\frac{1}{2}}$, under the coordinate transformations. Then (8.9) can be rewritten, with explicit metric dependence, as (we use the standard notation $\mathrm{g} = \det(\mathrm{g}_{\mu\nu})$):

$$
\begin{aligned}
S_{\mathrm{ssdYM}} = \mathcal{Q} \int & -i \mathrm{g}^{\mu\mu'} \mathrm{g}^{\nu\nu'} \mathrm{g}^{\frac{1}{2}} \mathrm{tr}\left(\chi^{+}_{\mu\nu} F_{\mu'\nu'}\right) + \mathrm{g}^{\mu\nu} \mathrm{g}^{\frac{1}{4}} \mathrm{tr}\left(\psi_{\nu} D_{\mu} \overline{\phi}\right) \\
& - \frac{1}{4} \mathcal{Q} \int \mathrm{g}^{\mu\nu} \mathrm{g}^{\frac{1}{4}} \mathrm{tr}\left(\psi_{\mu} \overline{\phi}\right) \partial_{\nu} \log(\mathrm{g}) \\
& - \frac{i\tau}{4\pi} \int_{\mathbf{M}^4} \mathrm{tr} F \wedge F,
\end{aligned} \tag{8.10}
$$

where $D_{\mu} = \partial_{\mu} + [A_{\mu}, \cdot]$.

The path integral in the theory (8.3) localizes onto the anti-self-dual gauge field configurations:

$$F^{+}_{A} = 0. \tag{8.11}$$

We now wish to apply our techniques to the case of gauge theory. To this end we need to reformulate the theory as the quantum mechanics of the same type we encountered before. It turns out that in addition to the complication of the configuration space being non-simply connected, we have an additional feature – *equivariance*. We shall spend some time discussing the equivariant versions of Morse theory and the supersymmetric quantum mechanics.

8.1.3. *Four-dimensional gauge theory as quantum mechanics.* We can interpret the four-dimensional gauge theory as supersymmetric gauged quantum mechanics [3, 51]. Let us consider the four manifold of the form $\mathbf{M}^4 = \mathbb{R} \times M^3$, where M^3 is a compact three-dimensional manifold. Let t denote the coordinate along the $\mathbb{R}$ factor. Let us assume the four-dimensional metric to be of the product form:

$$g_{\mu\nu} dx^{\mu} dx^{\nu} = dt^2 + h_{ij}(\vec{x}) dx^i dx^j \tag{8.12}$$

where $\vec{x} = (x^1, x^2, x^3) = (x^i)$. The four-dimensional gauge field splits as:

$$A = A_t dt + a = A_t dt + a_i dx^i$$

Let $B_a = \star_3 F_a$ be the three-dimensional magnetic field, a one-form on M^3, valued in the adjoint bundle. In the gauge $A_t = 0$ the anti-self-duality equation (8.11)

$$F^{+}_{A} = 0 \Leftrightarrow \dot{a} + B_a = 0$$

can be interpreted as the gradient flow, with respect to the "Morse function" f given by the *Chern–Simons functional*

$$f = -\frac{1}{2} \int_{M^3} \mathrm{tr}\left(a da + \frac{2}{3} a^3\right), \tag{8.13}$$

if the metric on the space $\mathcal{A}$ of gauge fields on M^3 is defined with the help of the metric h_{ij}:

$$h_{aa} = \int_{M^3} \mathrm{tr}\ \delta a \wedge \star_3 \delta a = \int\ d^3x\ h^{\frac{1}{2}} h^{ij}\ \mathrm{tr} \delta a_i \delta a_j \tag{8.14}$$

This metric and the functional (8.13) are invariant under the gauge transformations from the group $\mathcal{G}_0$, the component of identity of the full gauge group. Thus f can be viewed as a Morse (or, if $\pi_1(M^3) \neq 0$, a Morse–Bott) function on the space $\hat{\mathcal{X}} = \mathcal{A}/\mathcal{G}_0$ of the gauge equivalence classes of connections on a principal G-bundle $\mathcal{P}$ on M^3, together with a choice of a path, up to homotopy, connecting the connection with the trivial one.

The actual symmetry group of the gauge theory is $\mathcal{G}$, and the actual configuration space is $\mathcal{X} = \hat{\mathcal{X}}/\Gamma$, $\Gamma = \pi_0(\mathcal{G}) = \mathcal{G}/\mathcal{G}_0$. For example, for $M^3 = \mathbb{S}^3$, and simple G, $\Gamma = \mathbb{Z}$.

An important subtlety is related to the singularities of $\mathcal{X}$ and $\hat{\mathcal{X}}$. These arise because the gauge group does not act freely on $\mathcal{A}$. For example, the trivial connection $a = 0$ is left fixed by the group G of constant gauge transformations, while generic a has a trivial stabilizer. In addition, there is the whole zoo of connections which have a stabilizer H, which is anywhere between G and the trivial one.

Gauge theory is supposed to give us a *definition* of the quantum mechanics on the space with singularities of this form. Of course, such a singularity may or may not be a serious issue. Consider, for example, quantum mechanics of a free particle on a group manifold G. Now let us impose as a gauge symmetry the adjoint action of G. At the level of wave-functions this is a very simple selection rule: only the $Ad(G)$-invariant functions on G should be kept, in other words, only the characters of the irreducible representations of G. Now, if we look at the quotient space $X = G/Ad(G) = T/W$, it has singularities, and the point $g = 1$, the point with the maximal stabilizer, is one of them. The vicinity of the point $a = 0$ in $\mathcal{X}$ is somewhat similar to the vicinity of the point $g = 1$ in X.

A safe way to avoid dealing with the singular quotients is to discuss the instanton equations in the gauge-covariant way. The equations (8.11), written in the form of the evolution equations, read:

$$\dot{a} = D_a A_t + B_a \tag{8.15}$$

We shall now discuss the general setup where the equations like (8.15) naturally appear.

8.1.4. *Equivariant integration on the space of paths.* We now study supersymmetric quantum mechanics on a smooth manifold X, possibly, infinite-dimensional. We assume that X is endowed with G-invariant function f, which has the Morse property in the directions transversal to G-orbits (so that it gives rise to an ordinary Morse function on X/G if this quotient exists and is smooth). The corresponding gradient vector field $v^\mu = h^{\mu\nu}\partial_\nu f$ commutes with the action of G. We define a generalization of the gradient trajectory: given a map $A_t : \mathbb{R} \to \mathfrak{g}$, consider the equation

$$\dot{x}^\mu(t) = v^\mu(x(t)) + \mathcal{V}^\mu_a(x(t)) A^a_t(t) \tag{8.16}$$

A solution of this equation gives rise to an ordinary gradient trajectory on X/G, if it exists. In general, we use this definition as a replacement of the notion of gradient trajectory.

The space $\mathcal{X}$ of pairs $(x(t), A_t(t))$ is acted upon by the group $\mathcal{G} = \mathrm{Maps}(\mathbb{R}, G)$:

$$g : (x(t), A_t(t)) \mapsto \left(g(t) \cdot x(t),\ g^{-1}(t)\partial_t g(t) + g^{-1}(t)A_t(t)g(t)\right)$$

Ideally, we would like to divide $\mathcal{X}$ by $\mathcal{G}$, however, the critical points may have stabilizers. A safer route is to use the equivariant integration theory developed below.

We arrive at the quantum mechanical model with the following structure. The configuration space of the theory is the space $\Pi TX \times \Pi T\mathfrak{g} \times \mathfrak{g}$. The functions on this space are the differential forms on $X \times \mathfrak{g}$, taking values in the space of function on the Lie algebra $\mathfrak{g}$. The coordinates on the Lie algebra $\mathfrak{g}$ are denoted by $\phi, \overline{\phi}$, and the coordinates on $\Pi T\mathfrak{g}$ are $(\overline{\phi}, \eta)$, $\eta = d\overline{\phi}$.

The supersymmetry generator $\mathcal{Q}$ acts as follows:

$$\begin{aligned} \mathcal{Q}x^\mu = \psi^\mu, &\qquad \mathcal{Q}\psi^\mu = \mathcal{V}^\mu(\phi) = \phi^a \mathcal{V}^\mu_a \\ \mathcal{Q}\overline{\phi} = \eta, &\qquad \mathcal{Q}\eta = [\phi, \overline{\phi}] \\ \mathcal{Q}\pi_\mu = p_\mu, &\qquad \mathcal{Q}p_\mu = \partial_\mu \mathcal{V}^\nu(\phi)\pi_\nu. \end{aligned} \tag{8.17}$$

This generator squares to the infinitesimal gauge transformation of G, generated by ϕ:

$$\mathcal{Q}^2 = \mathcal{L}_{\mathcal{V}(\phi)} + ad(\phi).$$

In order to ensure the nilpotent nature of $\mathcal{Q}$, one imposes the condition of the G-invariance:

$$\mathcal{H} = \left(\Omega^\bullet(X \times \mathfrak{g}) \otimes \mathrm{Fun}(\mathfrak{g})\right)^G.$$

In the Lagrangian approach this invariance is enforced with the help of the Lagrange multiplier A_t which can be interpreted as a one-dimensional gauge field taking values in $\mathfrak{g}$. Its superpartner Ψ_t obeys:

$$\mathcal{Q}A_t = \Psi_t, \qquad \mathcal{Q}\Psi_t = D_t\phi = \partial_t\phi + [A_t, \phi].$$

The Lagrangian of a first order theory (or, rather, $\overline{\tau} = \infty$ theory) looks as follows:

$$L = \mathcal{Q}\left(-i\pi_\mu\left(\dot{x}^\mu - \mathcal{V}^\mu(A_t) - v^\mu\right) + \mathrm{tr}\left(\Psi_t D_t \overline{\phi}\right) + h_{\mu\nu}\mathcal{V}^\mu(\overline{\phi})\psi^\nu\right). \tag{8.18}$$

Note that the Mathai-Quillen interpretation of the Lagrangian (8.18) can be deduced from the lectures [14] on the so-called *projection form.*

The finite radius theory differs from (8.18) by

$$\Delta L = \frac{1}{\lambda}\mathcal{Q}\left(h^{\mu\nu}\pi_\mu p'_\nu + \mathrm{tr}\left([\phi, \overline{\phi}]\eta\right)\right). \tag{8.19}$$

8.2. **Gauged quantum mechanics.** At this stage it is perhaps useful to recall a few facts about the gauged supersymmetric quantum mechanics and the $\overline{\tau} \to \infty$ limit in this context. The available reviews consider the zero-dimensional quantum field theories, i.e., integrals over the quotient spaces, e.g., [14]. We need to discuss a quantum mechanical version, corresponding to one-dimensional quantum field theories.

So let us consider the following more general situation. Let X be compact smooth manifold with the compact simple Lie group G action. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathcal{V} : \mathfrak{g} \to$

$Vect(X)$ be the corresponding homomorphism of the Lie algebras. Let (x^μ, ψ^μ) denote local bosonic and fermionic coordinates on ΠTX. Let $h = h_{\mu\nu} dx^\mu dx^\nu$ be a G-invariant metric on X. Sometimes it is convenient to introduce a basis $\mathbf{t}_a$ on $\mathfrak{g}$, and the corresponding structure constants:

$$[\mathbf{t}_a, \mathbf{t}_b] = f_{ab}^c \mathbf{t}_c$$

We shall assume this basis to be orthonormal with respect to the Killing form, which we denote by tr. We shall fix the normalization of tr later.

8.2.1. *Equivariant cohomology.* We first discuss a finite-dimensional integral. Suppose we wish to integrate differential forms over the quotient X/G, assuming it exists. The forms on X/G are ΠTG-invariant forms on X. In other words, these are G-invariant, horizontal forms on X, sometimes also called the *basic* forms:

$$\varpi \in \Omega^\bullet_{\mathrm{basic}}(X) \Leftrightarrow \ \mathcal{L}_{\mathcal{V}(\xi)} \varpi = 0, \ \iota_{\mathcal{V}(\xi)} \varpi = 0 \qquad \forall \xi \in \mathfrak{g} \tag{8.20}$$

If (8.20) holds, then ϖ is a pull-back of some differential form $\varpi' \in \Omega^\bullet(X/G)$, of the same degree. Sometimes the quotient X/G does not exist, because the group G may be acting with fixed points, etc. In these circumstances one should use another model of the de Rham complex of X/G. There are, in fact, several well-known models. For example, one can use the Weil or Cartan models of *equivariant cohomology* of X. We start by reviewing the Weil model. Instead of dividing X by G, and restricting the de Rham complex of X, one multiplies it by an acyclic complex, which models the de Rham complex of EG, a contractible space, on which G acts freely, and then imposes the condition of being basic, i.e., being a pull-back from $(X \times EG)/G$, the latter quotient (by the diagonal action of G) always being well-defined. The de Rham complex of EG is modeled by the Weil algebra $\mathcal{W}^\bullet(\mathfrak{g})$ of $\mathfrak{g}$, the space of functions on $\Pi T\mathfrak{g}$. The unusual feature of this space is that the odd coordinates c^a are viewed as one-forms (this is not really unusual), but the even, bosonic, coordinates ϕ^a, are viewed as two-forms. In other words, $\Pi T\mathfrak{g}$ is viewed not as a supermanifold, but rather as a graded manifold. The differential (which eventually migrates to our supercharge $\mathcal{Q}$) acts on

$$\Omega^\bullet(X) \otimes \mathcal{W}^\bullet(\mathfrak{g}) \tag{8.21}$$

as follows:

$$\mathcal{Q} = d_X + \delta,$$
$$\delta c = \phi - \frac{1}{2}[c, c]\,, \ \delta\phi = [\phi, c].$$

The cohomology of δ on $\mathcal{W}^\bullet(\mathfrak{g})$ is one-dimensional (in degree 0), as it should be since it represents the de Rham complex of the contractible space EG.

The action of the group ΠTG on $\Omega^\bullet(X) \otimes \mathcal{W}^\bullet(\mathfrak{g})$ is generated by the operators ι_a and $\mathcal{L}_a = \{\mathcal{Q}, \iota_a\}$:

$$\iota_a = \iota_{\mathcal{V}_a} + \frac{\partial}{\partial c^a}\ , \ \mathcal{L}_a = \mathcal{L}_{\mathcal{V}_a} + f_{ab}^d \left(\phi^d \frac{\partial}{\partial \phi^b} + c^d \frac{\partial}{\partial c^b} \right). \tag{8.22}$$

Of course, we are not interested in $\Omega^\bullet(X) \otimes \mathcal{W}^\bullet(\mathfrak{g})$, we need the space of basic forms

$$W^\bullet_G(X) = \left(\Omega^\bullet(X) \otimes \mathcal{W}^\bullet(\mathfrak{g})\right)_{\mathrm{basic}}\,,$$

which is defined as the subcomplex in $\Omega^\bullet(X)\otimes\mathcal{W}^\bullet(\mathfrak{g})$, annihilated by $\iota_a, \mathcal{L}_a$. The useful observation of J. Kalkman is that the first order differential equations $\iota_a \varpi(c,\phi) = 0$ can be solved by:

$$\varpi(c,\phi) = e^{-c^a \iota_{\mathcal{V}_a}} \alpha(\phi),$$

where

$$\alpha \in \Omega^\bullet(X) \otimes \mathrm{Fun}(\mathfrak{g}), \qquad \alpha(\phi) \in \Omega^\bullet(X).$$

The remaining equations $\mathcal{L}_a \varpi(c,\phi) = 0$ translate to the conditions of G-invariance on α:

$$g^*\alpha(\phi) = \alpha(g\phi g^{-1}), \qquad \forall g \in G.$$

Thus, the de Rham complex of $X_G = (X \times EG)/G$ is modeled on

$$\Omega^\bullet_G(X) = \left(\Omega^\bullet(X) \otimes \mathrm{Fun}(\mathfrak{g})\right)^G$$

with the differential $\mathcal{Q}$, which acts on α as follows:

$$\mathcal{Q}\alpha(\phi) = d_X \alpha(\phi) + \iota_{\mathcal{V}(\phi)}\alpha(\phi) \tag{8.23}$$

When X/G exists, the space X_G is a fiber bundle over X/G with the contractible fiber EG. Whether X/G exists or not, the space X_G is always a fiber bundle over BG, with the fiber X. In this way, one sees explicitly the structure of the $H^\bullet(BG) = (S^\bullet \mathfrak{g}^*)^G$-module, which is also clear from the Cartan description.

8.2.2. *Equivariant integration.* Given a form $\alpha \in \Omega^\bullet_G(X)$, in the case of free G-action, we can ask the following natural question: how does one produce a basic form $\beta \in \Omega^\bullet_{\mathrm{basic}}(X)$? If G acts freely, then one can find the so-called connection forms $\Theta^a \in \Omega^1(X)$, i.e., the forms which obey

$$\iota_{\mathcal{V}_a}\Theta^b = \delta^b_a, \qquad \mathcal{L}_{\mathcal{V}_a}\Theta^b = -f^b_{ac}\Theta^c. \tag{8.24}$$

The connection form is defined up to a shift by a section of $\Omega^1(X/G)\underset{G}{\times}\mathfrak{g}$. The connection Θ defines the curvature two-forms

$$F = d\Theta + \frac{1}{2}[\Theta,\Theta]\ ,\ F^a = d\Theta^a + \frac{1}{2} f^a_{bc}\Theta^b \wedge \Theta^c,$$

which are automatically horizontal, $\iota_{\mathcal{V}_a} F^b = 0$. As far as the G-action is concerned, the connection and curvature obey:

$$g^*F = g^{-1}Fg\ ,\ g^*\Theta = g^{-1}\Theta g.$$

Given $\alpha \in \Omega^\bullet_G(X)$, i.e., a G-equivariant map from $\mathfrak{g}$ to the differential forms on X, $\alpha(\phi) \in \Omega^\bullet(X)$, we define:

$$\beta = e^{-\Theta^a \iota_{\mathcal{V}_a}} \alpha\left(-F\right). \tag{8.25}$$

The formula (8.25) defines obviously a G-invariant form. It is also not difficult to convince oneself that β defined by (8.25) is horizontal, $\iota_{\mathcal{V}_a}\beta = 0$. One should pay attention to the ordering of the exponential, since the operations of contracting with $\mathcal{V}^a$ and multiplication by Θ^b do not commute, thanks to (8.24). Finally, note that $d\beta = 0$ if and only if $\mathcal{Q}\alpha = 0$.

Given $\beta \in \Omega^k(X)_{\mathrm{basic}}$, $k = \dim(X/G)$, one should compute an integral

$$I(\beta) = \int_{X/G} \beta$$

The problem is to compute $I(\beta)$ in terms of $\alpha \in \Omega^\bullet_G(X)$. This is done using the *projection form* [14].

As the result, we get the following form $\Gamma_\alpha \in \Omega^{\bullet + \dim G}(X)$:

$$\Gamma_\alpha = \frac{1}{\mathrm{Vol}(G)} \int d\phi d\overline{\phi} d\eta \; e^{-\eta_a \Theta^a - \overline{\phi}_a(\phi^a + F^a)} \; \alpha(\phi). \tag{8.26}$$

One checks that if X/G exists, then $I(\beta) = \int_X \Gamma_\alpha$. The observation of [14] is that the form Γ_α is not changed (in [14] the attention was only paid to the $\mathcal{Q}$-cohomology, but the argument can be recycled to show the actual form independence) if the canonically normalized connection (8.24) is replaced by a general $\mathfrak{g}$-valued one form $\Sigma = \Sigma_a \mathbf{t}^a$ on X obeying

$$\iota_{\mathcal{V}_a} \Sigma_b = H_{ab} \; , \; \mathcal{L}_{\mathcal{V}_a} \Sigma_b = -f^c_{ab} \Sigma_c, \tag{8.27}$$

where at every point $x \in X$ the matrix H_{ab} is a non-degenerate G-invariant, $\mathcal{L}_{\mathcal{V}_c} H_{ab} = 0$, bilinear pairing on $\mathfrak{g}_x$, the tangent space to the G-orbit, passing through the point x. The formula (8.26) gets modified to

$$\Gamma_\alpha = \frac{1}{\mathrm{Vol}(G)} \int \; d\phi d\overline{\phi} d\eta \; e^{-\eta^a \Sigma_a + \overline{\phi}^a \left(\phi^b H_{ab} + d\Sigma_a\right)} \; \alpha(\phi). \tag{8.28}$$

The relation to the previous formalism is achieved by writing $\Theta^a H_{ab} = \Sigma_b$, and by changing the variables: $\eta \to \eta + [\overline{\phi}, \Theta]$. Finally, one may study the one-parametric family of projection forms depending on a parameter λ (these forms *do care* about the representatives, only for $\mathcal{Q}$-closed forms α is the λ-dependence unobservable)

$$\Gamma_\alpha(\lambda) = \frac{1}{\mathrm{Vol}(G)} \int_{\mathfrak{g} \times \Pi T \mathfrak{g}} d\phi d\overline{\phi} d\eta \; e^{-i\eta^a \Sigma_a + i\overline{\phi}^a \left(\phi^b H_{ab} + d\Sigma_a\right) - \frac{1}{\lambda}\mathrm{tr}([\eta,\eta]\phi) - \frac{1}{\lambda}\mathrm{tr}[\phi,\overline{\phi}]^2} \; \alpha(\phi). \tag{8.29}$$

In the limit $\lambda \to \infty$ we get our original form Γ_α. In the limit $\lambda \to 0$ one gets an integral realization of the relation between the G-equivariant and the W-invariant part of the T-equivariant cohomology, where T and W are the maximal torus and the Weyl group of G, respectively. Indeed, we can use the G-invariance to reduce the integral over ϕ to the integral over $\mathfrak{t}/W$, $\mathfrak{t} = \mathrm{Lie}(T)$. When λ is sent to 0, the $\frac{1}{\lambda}\mathrm{tr}[\phi, \overline{\phi}]^2$ term in (8.29) dominates, and the integral over $\mathfrak{g}/\mathfrak{t}$ part of $\overline{\phi}$ and η becomes essentially Gaussian (outside of the discriminant in $\mathfrak{t}/W$):

$$\Gamma_\alpha(0) = \frac{1}{\mathrm{Vol}(T)|W|} \int_{\mathfrak{t} \times \Pi T \mathfrak{t}} d\phi d\overline{\phi} d\eta \; \Delta^2(\phi) \; e^{-i\eta^k \Sigma_k + i\overline{\phi}^k \left(\phi^j H_{kj} + d\Sigma_k\right)} \; \alpha(\phi), \tag{8.30}$$

where

$$\Delta^2(\phi) = \prod_{\text{roots of } \mathfrak{g}} \langle \mathrm{root}, \phi \rangle.$$

We close the section by giving some examples of equivariant forms. Let (X, ω) be a symplectic manifold with the G-action preserving the symplectic form ω. Suppose that the G-action is Hamiltonian and one can define the moment map $\mu : X \to \mathfrak{g}^*$, such that

$$\iota_{\mathcal{V}_a}\omega = -d\mu_a.$$

Then $\Omega(\phi) = \omega + \phi^a \mu_a$, as well as any functions of $\Omega(\phi)$ are equivariantly closed. Another source of equivariant forms comes from G-equivariant vector bundles $\mathcal{E}$ over X. Let us denote the fiber of $\mathcal{E}$ over a point $x \in X$ by $\mathcal{E}_x$. The G-action on X lifts to the action on $\mathcal{E}$. In particular, $g : \mathcal{E}_x \to \mathcal{E}_{g\cdot x}$. Given a connection ∇ on $\mathcal{E}$, i.e., a way to parallel transport the vectors in $\mathcal{E}_x$, the action of G on $\mathcal{E}$ can be decomposed into the parallel transport from x to $g \cdot x$ and the action of G on the vector space $\mathcal{E}_{g\cdot x}$: for any $\xi \in \mathfrak{g}$, and $\psi_x \in \mathcal{E}_x$, we have:

$$\xi \cdot \psi_x = \nabla_{\mathcal{V}(\xi)}\psi_x + R(\xi) \cdot \psi_x. \tag{8.31}$$

Let $F_\nabla = \nabla^2$ denote the curvature two-form. The bundle $\mathcal{E}$ defines the so-called equivariant Euler class $\mathrm{Euler}_{\mathcal{E}} \in H_G^{\mathrm{rk}\mathcal{E}}(X)$. This cohomology class has many useful representatives. A one-parametric family of such representatives is constructed given a section $s : X \to \mathcal{E}$. Schematically, it is given by the Mathai-Quillen form

$$\mathrm{Euler}_{\mathcal{E}}(u) = \int_{\Pi T\mathcal{E}^*_x} e^{ip\cdot s + \pi\cdot\nabla s - \frac{1}{2}u\langle p,p\rangle - \frac{1}{2}u\langle \pi, \left(\Gamma_{\mathcal{V}(\phi)} + R(\phi) + F_\nabla\right)\cdot \pi\rangle - \frac{1}{2}u\langle \pi, \Gamma_\mu dx^\mu p\rangle}, \tag{8.32}$$

where $\Gamma_{\mathcal{V}(\phi)} = \iota_{\mathcal{V}(\phi)}\Gamma$, and Γ is a connection one-form for ∇ on $\mathcal{E}$. The differential $\mathcal{Q}$ is made to act on the auxiliary fields (π, p), which are the fermionic and bosonic coordinates on $\Pi\mathcal{E}^*_x$ and $\mathcal{E}^*_x$ respectively, as follows:

$$\mathcal{Q}\pi_e = p_e - \Gamma^h_{e\mu}\pi_h dx^\mu,$$
$$\mathcal{Q}p_e = R(\phi)^h_e \pi_h - \Gamma^h_{e\mu} p_h dx^\mu + \frac{1}{2}F^h_{e\mu\nu}\pi_h dx^\mu \wedge dx^\nu - \Gamma^h_{e\mu}\mathcal{V}^\mu(\phi)\pi_h.$$

Finally, the simplest examples of the equivariant (and also equivariantly closed) forms are invariant functions on $\mathfrak{g}$, e.g., invariant polynomials $P(\phi) \in \mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{t}]^W$.

8.2.3. *The first glimpses of quantum mechanics.* So far we have discussed the integrals of differential forms, or equivariant differential forms. We have also mentioned the equivariant differential $\mathcal{Q}$.

There are several points of view on $\mathcal{Q}$ and the equivariant forms. One viewpoint treats the latter as the functions on some super (or graded) manifold, where $\mathcal{Q}$ acts as the odd vector field, obeying the non-trivial (for odd fields) *master* equation

$$\{\mathcal{Q}, \mathcal{Q}\} = 0.$$

Another point of view identifies the equivariant forms with the wave-functions of a supersymmetric quantum mechanics. In this approach $\mathcal{Q}$ is viewed as an operator acting in the space of states of the quantum mechanical system. One also needs to define a conjugate operator $\mathcal{Q}^*$. For example, for

$$\mathcal{Q} = \psi^\mu\left(p_\mu + \partial_\mu f\right) + \chi_\mu \mathcal{V}^\mu_a \phi^a + \mathrm{tr}\left(\eta\frac{\partial}{\partial\overline{\phi}} + [\phi, \overline{\phi}]\frac{\partial}{\partial\eta}\right)$$

we get:

$$\mathcal{Q}^* = \chi_\mu h^{\mu\nu}\left(p_\nu + \partial_\nu f\right) + h_{\mu\nu}\psi^\mu \mathcal{V}^\nu_a \overline{\phi}^a + \mathrm{tr}\left(\frac{\partial}{\partial\eta}\frac{\partial}{\partial\phi} + [\phi,\overline{\phi}]\eta\right).$$

Note that the definition of $\mathcal{Q}^*$ requires a metric on the Lie algebra and on X, whereas the definition of $\mathcal{Q}$ is metric-independent.

8.2.4. *Hamiltonian interpretation of gauged quantum mechanics.* We can now elaborate on the points of the section 8.2.3 and discuss the Hamiltonian interpretation of the theory (8.18).

Note that the quantum mechanics with action (8.19) contains, as a subsector, the quantum mechanics on the Lie algebra $\mathfrak{g}$. In fact, there are two versions of the theory. In one, which is mostly adopted in the conventional physical applications, the variables ϕ and $\overline{\phi}$ are treated as complex conjugates. In this case we are dealing with some kind of anharmonic oscillator on the complexification $\mathfrak{g}_{\mathbb{C}}$. In the second approach, which is more directly related to the Mathai-Quillen form and equivariant cohomology, the fields ϕ and $\overline{\phi}$ are independent real Lie algebra valued variables. In this case one is dealing with the indefinite oscillator on $(\dim\mathfrak{g}, \dim\mathfrak{g})$ signature space $\mathfrak{g}\oplus\mathfrak{g}$.

In either case the Hamiltonian of the resulting model looks as follows:

$$H = \mathcal{L}_v - \mathrm{tr}\frac{\partial^2}{\partial\phi\partial\overline{\phi}} + \overline{\phi}^b\left(H_{ab}\phi^a + d\Sigma_b\wedge + f_b^{ac}\frac{\partial^2}{\partial\eta^a\partial\eta^c}\right) + \eta^b\Sigma_b\wedge \tag{8.33}$$

where

$$H_{ab} = h_{\mu\nu}\mathcal{V}^\mu_a\mathcal{V}^\nu_b\ ,\ \Sigma_a = h_{\mu\nu}\mathcal{V}^\mu_a dx^\nu$$

and we have dropped the term

$$\delta H = \frac{1}{\lambda}\mathrm{tr}[\phi,\overline{\phi}]^2 + \frac{1}{\lambda}\ \mathrm{tr}\left(\eta[\phi,\eta]\right) \tag{8.34}$$

with which we may play in various ways. The simplest model appears to be with the term (8.34) dropped, i.e., with $\lambda = \infty$. In this case the fields ϕ and $\overline{\phi}$ enter at most linearly and can be integrated out. If, instead, we send λ to 0, then the anharmonic oscillator potential will force the Lie algebra $\mathfrak{g}$ variables to be confined near a maximal torus $\mathfrak{t}$, just like in the finite-dimensional example (8.30).

The standard approach to solving the model with the Hamiltonian (8.33) would be to use the Born-Oppenheimer approximation. It consists in first solving the harmonic (in the absence of (8.34)) oscillator in ϕ and $\overline{\phi}$, whose frequencies are given by the eigenvalues of the induced metric $H_{ab}(x)$. If the group G acts on X without fixed points, then the frequencies never go down to zero, and one can approximate the wave-functions by the ground state wave-functions in ϕ, $\overline{\phi}$ directions, times some forms on the X space.

However, in the vicinity of a fixed point of the G the metric $H_{ab}(x)$ becomes degenerate. Then the Born-Oppenheimer approximation can no longer be applied, and the theory becomes more intricate. In particular, a new "branch" develops, where the wave function does not exponentially decay when ϕ goes to infinity.

We shall not discuss the transition between branches. Some of the remarks on this problem (in the context of supersymmetric models with higher degree of supersymmetry) can be found in [55].

8.2.5. *Example of the group manifold.* Consider the example $X = G$ where the group G acts by the right multiplication.

The space of states of this model, viewed as the gauged quantum mechanics, is the space of differential forms on G tensored with the space of functions of ϕ and with the differential forms on another copy of $\mathfrak{g}$:

$$\Psi(g, dg, \phi, \overline{\phi}, d\overline{\phi}).$$

It is customary to denote $d\overline{\phi}$ by η, and it is convenient to denote by ψ the left-invariant form $g^{-1}dg$. Then the equivariant differential acts as follows:

$$\begin{aligned}
&\mathcal{Q}g = g\psi \ , \\
&\mathcal{Q}\psi = \phi - \frac{1}{2}[\psi, \psi], \\
&\mathcal{Q}\phi = 0, \\
&\mathcal{Q}\overline{\phi} = \eta, \\
&\mathcal{Q}\eta = [\phi, \overline{\phi}].
\end{aligned} \tag{8.35}$$

By passing to the G-invariant variables

$$\begin{aligned}
&A = dgg^{-1} \ , \qquad A^* = g\eta g^{-1} + [A, g\overline{\phi}g^{-1}], \\
&\Phi = g\phi g^{-1} - \frac{1}{2}[A, A] \ , \qquad \Phi^* = g\overline{\phi}g^{-1},
\end{aligned} \tag{8.36}$$

we map $\mathcal{Q}$ to the following simple differential operator:

$$\mathcal{Q} = \Phi\frac{\partial}{\partial A} + A^*\frac{\partial}{\partial \Phi^*}. \tag{8.37}$$

8.2.6. *Observables in gauge theory.* We now go back to the problem of our interest: the four-dimensional supersymmetric twisted gauge theory. The first question which we should address is what are the analogues of the evaluation observables in the gauge theory. In the case of the twisted supersymmetric sigma model the evaluation observables $\mathcal{O}(x, \psi)$ have many nice properties: i) they corresponded to the dimension zero operators; ii) the set of these operators is non-trivially acted upon by the supercharge $\mathcal{Q}$; iii) their definition did not require any short distance regularization perturbatively. If the condition i) is relaxed one gets the jet-evaluation observables. Finally, these observables have a clear geometric interpretation.

In gauge theory we must demand that the observables be gauge-invariant. The gauge theory observables, analogous to the operators $\mathcal{O}(x, \psi)$ in sigma model are the local gauge-invariant functionals $\mathcal{O}(A, \psi)$. These operators rarely obey the property i). However, they do obey ii) and iii). Geometrically, these correspond to the $\mathcal{G}_{\mathbf{M}^4}$-invariant differential forms on $\mathcal{A}^+_{\mathbf{M}^4}$, the space of anti-self-dual gauge fields. Indeed, the integral over H^+ field enforces the anti-self-duality condition, $F^+_A = 0$, and the integral over the gauge equivalence classes of A becomes an integral over the moduli space of instantons

$$\mathcal{M} = \mathcal{A}^+_{\mathbf{M}^4}/\mathcal{G}_{\mathbf{M}^4} \ , \tag{8.38}$$

where $\mathcal{A}^+_{\mathbf{M}^4}$ is the space of anti-self dual connections on $\mathbf{M}^4$ (satisfying equation (8.11)) and $\mathcal{G}_{\mathbf{M}^4}$ is the group of gauge transformations. Then the integration over χ^+, η makes ψ a one-form on $\mathcal{M}$, valued in one-forms on $\mathbf{M}^4$ and satisfying the equation

$$D_A^+\psi = 0 \ , \qquad D_A^*\psi = 0 \ . \tag{8.39}$$

The last equation $D_A^*\psi = 0$ can be interpreted as a gauge-fixing for the fermionic gauge symmetry $\delta_\varepsilon\psi = D_A\varepsilon$. In classifying the observables we can replace the condition of horizontality by the cohomology of the corresponding BRST operator, which acts as follows:

$$\delta\psi = D_A\phi. \tag{8.40}$$

Thus, we define the gauge-evaluation observables as the cohomology of the operator (8.40) acting on the gauge-invariant functionals $\mathcal{O}(A, \psi, \phi)$.

After the functional integration over $\overline{\phi}$, the field ϕ becomes a curvature two-form

$$\phi = \frac{1}{\Delta_A}[\psi, \star\psi], \tag{8.41}$$

where $\Delta_A = D_A D_A^* + D_A^* D_A$ is the gauge-covariant Laplacian (here acting on zero-forms).

8.2.7. *Universal connection.* According to [54], it is convenient to think of ϕ, ψ, F_A as of the three components of the universal curvature:

$$\mathbf{F} = \phi + \psi + F_A \tag{8.42}$$

of the universal connection on the universal principal G-bundle over $\mathcal{M} \times \mathbf{M}^4$. Let us review the construction of this connection.

Let $A(m) = A_\mu(x, m)dx^\mu$, $m \in \mathcal{M}$, $x \in \mathbf{M}^4$ be a family of anti-self-dual connections on $\mathbf{M}^4$, defined over some open domain $U \subset \mathcal{M}$, so that for each $m \in U$, $F^+_{A(m)} = 0$ and the gauge equivalence class obeys $[A(m)] = m$. Now let us study the m-dependence. Clearly, the derivative $\frac{\partial}{\partial m^i}A(m)$ obeys the linearized instanton equation,

$$D_A^+ \frac{\partial A(m)}{\partial m^i} = 0 \ .$$

Therefore it can be decomposed:

$$\frac{\partial A(m)}{\partial m^i} = \psi_i(m) + D_{A(m)}\varepsilon_i(m), \tag{8.43}$$

where the set of one-forms $\psi_i(m)$, $i = 1, \ldots, \mathrm{dim}\mathcal{M}$, obeys

$$D^+_{A(m)}\psi_i(m) = 0, \ D^*_{A(m)}\psi_i(m) = 0 \ .$$

More precisely, the set (ψ_i) forms a basis in H^1_m, which coincides with $T_m\mathcal{M}$ in a nice situation, where both H^0_m and H^2_m vanish. Assuming that this is the case, (ε_i) in (8.43) is the set of compensating gauge transformations,

$$\varepsilon_i(m) = \varepsilon_i(x, m),$$

which we can actually compute from (8.43) that

$$\varepsilon_i(m) = \frac{1}{\Delta_{A(m)}}\ D^*_{A(m)}\frac{\partial A(m)}{\partial m^i}.$$

In components:

$$\begin{aligned} &\psi_i(m) = \psi_{i\mu}(x,m)dx^\mu \\ &D_\mu\psi_{i\nu} - D_\nu\psi_{i\mu} + \frac{1}{2}\epsilon_{\mu\nu\mu'\nu'}\mathrm{g}^{\mu'\mu''}\mathrm{g}^{\nu\nu''}\mathrm{g}^{\frac{1}{2}}\ D_{\mu''}\psi_{i\nu''} = 0 \\ &D_\mu\left(\mathrm{g}^{\mu\nu}\mathrm{g}^{\frac{1}{2}}\ \psi_{i\nu}\right) = 0. \end{aligned} \tag{8.44}$$

By combining $A(m)$ and $\varepsilon(m)$,

$$\mathbf{A} = A(m) + \varepsilon_i(m)dm^i, \tag{8.45}$$

we arrive at (8.42) with

$$\begin{aligned} &\mathbf{F} = \mathbf{dA} + \frac{1}{2}[\mathbf{A},\mathbf{A}] = \phi + \psi + F_{A(m)} \\ &\psi = \psi_{i\mu}dm^i \wedge dx^\mu \\ &\phi = \frac{1}{2}\phi_{ij}dm^i \wedge dm^j \\ &\phi_{ij} = \frac{\partial}{\partial m^i}\varepsilon_j - \frac{\partial}{\partial m^j}\varepsilon_i + [\varepsilon_i,\varepsilon_j], \end{aligned} \tag{8.46}$$

and (8.41) follows:

$$\begin{aligned} \Delta_{A(m)}\phi_{ij} = \frac{\partial}{\partial m^{[i}}D^*_{A(m)}\frac{\partial A(m)}{\partial m^{j]}} + [D^*_{A(m)}\frac{\partial A(m)}{\partial m^{[i}},\epsilon_{j]}] - D^*_{A(m)}[\frac{\partial A(m)}{\partial m^{[i}},\epsilon_{j]}]- \\ [\frac{\partial A(m)}{\partial m^{[i}},\star D_{A(m)}\epsilon_{j]}] + [D_{A(m)}\epsilon_i,\star D_{A(m)}\epsilon_j] = [\psi_i,\star\psi_j]\ , \end{aligned}$$

as claimed.

The universal connection one-form $\mathbf{A}$ was defined over $U \subset \mathcal{M}$, starting with a section $A(m) : U \to \mathcal{A}^+_{\mathbf{M}^4}$. Now, we may have chosen a different section:

$$A(m)' = A(m)^{g(m)} = g(m)A(m)g^{-1}(m) + g(m)dg^{-1}(m)$$

which is related to $A(m)$ by the gauge transformation $g(m)$. It will change the connection one-form $\mathbf{A}$ by the corresponding gauge transformation. Hence the invariant observables, like the differential forms $\mathrm{tr}\mathbf{F}^l$, are well-defined forms on $\mathcal{M} \times \mathbf{M}^4$.

8.2.8. *Deformation complex, finite g^2, and the gauge theory.* The fermion kinetic term of the theory in the limit (8.9) is directly related to the Atiyah–Hitchin–Singer (AHS) complex, which is the instanton version of the deformation complex corresponding to the general moduli problem. Recall that the AHS complex is built given a solution A to the instanton equations, $F^+_A = 0$. Then:

$$0 \longrightarrow \Omega^0(\mathbf{M}^4) \otimes_{\mathcal{P}} \mathfrak{g} \xrightarrow{D_A} \Omega^1(\mathbf{M}^4) \otimes_{\mathcal{P}} \mathfrak{g} \xrightarrow{D^+_A} \Omega^{2,+}(\mathbf{M}^4) \otimes_{\mathcal{P}} \mathfrak{g} \longrightarrow 0 \tag{8.47}$$

The sequence (8.47) is indeed a complex, as $D^+_A \circ D_A = F^+_A = 0$. The cohomology of the AHS complex will be denoted by $H^i_{[A]}$. The AHS complex and its cohomology in particular characterize the vicinity of the point $[A]$ in the moduli space $\mathcal{M}$. First of all, if the only non-vanishing cohomology group of the AHS complex is $H^1_{[A]}$, then the moduli space $\mathcal{M}$ is smooth, and its tangent space at $[A]$ coincides with $H^1_{[A]}$. If, on the other hand, the zeroth cohomology $H^0_{[A]}$ of (8.47) is non-vanishing, then the moduli space is singular, as the point $[A]$ has a non-trivial stabilizer, $G_{[A]}$, whose Lie algebra is isomorphic to $H^0_{[A]}$. Finally, the group $H^2_{[A]}$ is an obstruction to the smoothness of $\mathcal{M}$. More precisely it is responsible for the possibility to extend the first order deformation of $[A]$, labeled by $H^1_{[A]}$, to the second order deformation. Indeed, suppose the first order deformation $a_{[1]}$, which is a solution to

$$D^+_A a_{[1]} = 0$$

up to the infinitesimal gauge transformation $a_{[1]} \sim a_{[1]} + D_A\varphi_{[1]}$, is given. Then the second order deformation $a_{[2]}$ has to be such that

$$D^+_A a_{[2]} + \frac{1}{2}[a_{[1]}, a_{[1]}] = 0$$

and it is defined up to the following transformations:

$$a_{[2]} \sim a_{[2]} + D_A\varphi_{[2]} + [a_{[1]}, \varphi_{[1]}] + \frac{1}{2}[D_A\varphi_{[1]}, \varphi_{[1]}].$$

Such $a_{[2]}$ can be found if and only if the image of $[a_{[1]}, a_{[1]}]$ in $H^2_{[A]}$ is zero. The map

$$K = [\ \cdot \overset{\wedge}{,}\ \cdot\ ] : H^1_{[A]} \times H^1_{[A]} \longrightarrow H^2_{[A]}$$

is called the *Kuranishi map*. There exists a theory which identifies the germ $\mathcal{M}_{[A]}$ of the moduli space near the point $[A]$ with the quotient of the preimage $K^{-1}(0) \subset H^1_{[A]}$ of zero in $H^1_{[A]}$ by the stabilizer $G_{[A]}$, $\mathcal{M}_{[A]} \approx K^{-1}(0)/G_{[A]}$.

The $g^2 \to 0$ limit of the gauge theory, as the infinite radius limit of the sigma model, has some subtleties, related to the existence of unwanted zero modes. In our derivation of the reduction of the path integral to the integral over the space $\mathcal{M}$ of instantons, we assumed that the integral over the fermionic fields χ^+ and η gave us the equations (8.39). In general, however, it may happen that the conjugate operators

$$D^{+,*}_A \oplus D_A : \left(\Omega^{2,+} \oplus \Omega^0\right) \otimes_{\mathcal{P}} \mathfrak{g} \to \Omega^1 \otimes_{\mathcal{P}} \mathfrak{g}$$

have non-vanishing kernel, which is clearly isomorphic to $H^2_m \oplus H^0_m$ (in this case the space of solutions to the equations (8.39) has the kernel which is strictly larger then the virtual tangent space $T_m\mathcal{M}$). Then the integral over these zero modes of χ^+ and η has to be regularized. Also, the fermionic zero modes χ^+_0, η_0 are accompanied by the bosonic zero modes $H^+_0, \overline{\phi}_0$, which might produce infinities unless properly regularized.

Finally, the formula (8.42) for the field ϕ now has to be modified. Indeed, since D_A has zero modes on scalars, the solution to the equation

$$\Delta_A\phi = [\psi, \star\psi]$$

now has to be written as

$$\phi = \phi_0 + \frac{1}{\Delta_A}[\psi, \star\psi],$$

where ϕ_0 solves $D_A\phi_0 = 0$.

Let us now turn on the coupling constant g^2, but only on the zero modes $H_0^+, \chi_0^+, \overline{\phi}_0, \eta_0$:

$$\Delta S = g^2\ \mathcal{Q} \int \mathrm{tr}\left(\chi_0^+ \wedge H_0^+ + \eta_0[\overline{\phi}_0, \phi_0]\right), \tag{8.48}$$

where the zero modes are the solutions of the equations

$$\begin{aligned} D_A\overline{\phi}_0 &= 0, \qquad D_A^{+,*}H_0^+ = 0 \\ D_A\eta_0 &= 0, \qquad D_A^{+,*}\chi_0^+ = 0. \end{aligned} \tag{8.49}$$

There is one more subtlety related to the fact that we now treat the fields $\overline{\phi}, \eta$ as half-densities, but we shall ignore it.

Note that the supercharge $\mathcal{Q}$ preserves the space of zero modes:

$$\begin{aligned} \mathcal{Q}\overline{\phi}_0 &= \eta_0\ , \qquad \mathcal{Q}\eta_0 = [\phi_0, \overline{\phi}_0] \\ \mathcal{Q}\chi_0^+ &= H_0^+\ , \qquad \mathcal{Q}H_0^+ = [\phi_0, \chi_0^+]. \end{aligned} \tag{8.50}$$

Thus the contribution of the vicinity of the point $m \in \mathcal{M}$ to the correlation function of some "evaluation observables" will be given by the integral

$$\int_{H_m^2} \mathrm{d}H_0^+ \mathrm{d}\chi_0^+\ e^{\int\ i\mathrm{tr}\left(H_0^+[a_{[1]},a_{[1]}]\right) - i\mathrm{tr}\left(\chi_0^+[a_{[1]},\psi_{[1]}]\right) - g^2\mathrm{tr}H_0^+H_0^+ - g^2\mathrm{tr}\chi_0^+[\phi_0,\chi_0^+]} \cdot \int_{H_m^0} \mathrm{d}\overline{\phi}_0\mathrm{d}\eta_0\ e^{\int \mathrm{tr}\eta_0[a_{[1]},\star\psi_{[1]}] - g^2\mathrm{tr}(\eta_0[\phi_0,\eta_0]) - g^2\mathrm{tr}[\phi_0,\overline{\phi}_0]^2}, \tag{8.51}$$

which we should view as the differential form on H_m^1 by decomposing the corresponding zero modes:

$$a_{[1]} = \sum_{i=1}^{\mathrm{rk}H_m^1} \psi_i\ \mathfrak{m}^i\ , \qquad \psi_{[1]} = \sum_{i=1}^{\mathrm{rk}H_m^1} \psi_i\ \mathrm{d}\mathfrak{m}^i$$

The differential form (8.51) is the smooth (for $g^2 > 0$) representative of the Poincare dual to the zero locus of the kuranishi map. The ϕ_0 dependence signifies the G_m-equivariant nature of the differential form, while the integral over H_m^0 in (8.51) gives the projection form, which is suited to define the integration theory over the quotient $K^{-1}(0)/G_m$. Now, it seems that even here we don't need to take $g^2 > 0$, as even in the $g^2 \to 0$ limit we get something reasonable: the Poincare dual gets represented by the delta form $\delta(K(\mathfrak{m}))$, and the projection form gives what we expect it to give. In fact, it depends on the degeneracy of the kuranishi map. In the nice situations, where it is sufficiently non-degenerate, the $g^2 = 0$ expression (8.51) defines a well-defined differential form on H_m^1. However this is not always the case, we give an example momentarily.

The usefulness of having $g^2 > 0$ at this point is the possibility of working with another representative of the Poincare dual of the zero locus of K; namely, the Euler class of the corresponding obstruction bundle $\cup_{m\in\mathcal{M}}H_m^2$ written in terms of the curvatures of the corresponding metric connections. This representation arises in the large g^2 limit.

In many applications involving the correlation functions of the BPS observables both representatives are equally good, and sometimes one is simpler to work with than another. In our story, we shall try to work with $g^2 = 0$ for as long as it is possible.

Remark 8.1. In the case of two-dimensional sigma model coupled to the two-dimensional topological gravity the discussion similar to the one we just had, is quite important in figuring out the contribution of the degree zero maps of higher genus Riemann surfaces. The analogous obstruction bundle is related, in this case, to the Hodge bundle on the moduli space of Riemann surfaces. Taking into account its contribution, i.e., the integrals of its Chern classes, to the Gromov-Witten invariants plays important rôle in the modern topological string theory. □

Likewise, the trivial connection, $A = 0$, which is unavoidable in any gauge theory in the sector with the trivial 't Hooft fluxes, comes with the whole host of non-trivial cohomology of the AHS complex:

$$H^i_{[0]} = H^i(\mathbf{M}^4, \mathbb{R}) \otimes \mathfrak{g}, \quad i = 0, 1,$$

$$H^2_{[0]} = H^{2,+}(\mathbf{M}^4, \mathbb{R}) \otimes \mathfrak{g}.$$

Moreover, in this case the stabilizer $G_0 = G$ coincides with the group G itself, while the kuranishi map is given by

$$K(\mathfrak{m}) = f^c_{ab}\mathfrak{m}^{ak}\mathfrak{m}^{bl}\omega_p \int_{\mathbf{M}^4} \omega^p \wedge \alpha_k \wedge \alpha_l,$$

where $(\omega_p)_{p=1,\dots,b_2^+}$ is the basis in the space of self-dual harmonic two-forms, and $(\alpha_l)_{l=1,\dots,b_1}$ is the basis in the space of harmonic one-forms.

Now, to give an example of the necessity of $g^2 > 0$ regularization, consider the case of a simply-connected manifold $\mathbf{M}^4$. In this case the kuranishi map corresponding to $A = 0$ is identically zero, so the expansion of the $H^+F^+ + \chi^+ D_A\psi$ term does not give us anything interesting. The integrals (8.51) can be evaluated, to produce

$$\left(\mathrm{Det}'_{\mathfrak{g}}\mathrm{ad}\phi_0\right)^{b_2^+ + b_0},$$

where the b_2^+ in the exponent comes from the H_0^+, χ_0^+ integral, while the origin of the b_0 contribution is the measure on the ϕ_0. The integral over η_0, $\overline{\phi}_0$ gives the determinants which cancel each other. The unfortunate subtlety is the prime in the determinant. The components of $H_0^+, \chi_0^+, \eta_0, \overline{\phi}_0$ which commute with ϕ_0 do not enter the g^2-dependent part of (8.51). These modes should be taken care of separately. This and further subtleties of the *stack* nature of the moduli space $\mathcal{M}$ are beyond the scope of this paper.

8.3. **Gauge theory and logarithms.** We shall now discuss the main problem of our interest – the appearance of logarithms in the correlation functions the four-dimensional gauge theory.

8.3.1. *Evaluation observables in gauge theory.* The correlation functions of the "evaluation observables", which we now take to be the gauge-invariant functionals of A, ψ and ϕ (they correspond to the equivariant forms on $\mathcal{A}$, the space of four-dimensional connections), reduce, in the $\overline{\tau} \to \infty$ limit, to the integrals over the moduli space of gauge instantons. We shall need some description of these moduli spaces. Since we wish to demonstrate the logarithmic nature of our theory, it suffices to consider the simplest case, the charge one instantons, for the gauge group $G = SU(2)$.

8.3.2. *From correlation functions to matrix elements.* First of all, we need to set up our calculation in such a way so as to be able to interpret it quantum mechanically. It is convenient to take as a four-manifold the four-sphere $\mathbf{M}^4 = \mathbb{S}^4$, which we can view as the one-point compactification of $\mathbb{R}^4 = \mathbb{C}^2 = \mathbb{H}^1$, a quaternionic line. Let us use the quaternionic notation: $\mathbf{v} = v^0 + v^1\mathbf{i} + v^2\mathbf{j} + v^3\mathbf{k} \in \mathbb{H}^1$, or, equivalently, the 2×2 matrix notation:

$$\mathbf{v} = \begin{pmatrix} v^0 + iv^1 & v^2 - iv^3 \\ -v^2 - iv^3 & v^0 - iv^1 \end{pmatrix} = \begin{pmatrix} w^1 & \overline{w}^2 \\ -w^2 & \overline{w}^1 \end{pmatrix} . \tag{8.52}$$

where $(w^1, w^2) \in \mathbb{C}^2$ represents the point $(v^0, v^1, v^2, v^3) \in \mathbb{R}^4$, upon some identification $\mathbb{R}^4 \simeq \mathbb{C}^2$. The standard round metric on $\mathbb{S}^4$,

$$ds^2 = \frac{d\mathbf{v}^2}{(1 + |\mathbf{v}|^2)^2}$$

is conformal to the metric on $\mathbb{R} \times \mathbb{S}^3$, with the points $\mathbf{v} = 0$ and $\mathbf{v} = \infty$ deleted,

$$ds^2 = \frac{1}{4\cosh^2(t)}\left(dt^2 + d\Omega_3^2\right),$$

where $t = \log|\mathbf{v}|$.

The path integral of the gauge theory on $\mathbb{S}^4$ can be interpreted as the vacuum matrix element in the quantum mechanics, where the space of states is obtained by quantizing the gauge fields on $\mathbb{S}^3$. The Hamiltonian of the theory is the generator of the t translations, $H = \partial_t$, which is the dilatation operator in the $\mathbf{v}$ coordinates.

We claimed that the path integral on $\mathbb{S}^4$ computes a vacuum matrix element. The precise vacuum states depend on the type of operators which are inserted at the points $\mathbf{v} = 0$ and $\mathbf{v} = \infty$, because these are the only points points fixed by the dilatation operator.

8.3.3. *ADHM construction.* The moduli space of charge one instantons on $\mathbb{S}^4$ is well-understood. For our quantum mechanical purposes we should actually consider the space of instantons which are located neither at $\mathbf{v} = 0$ nor at $\mathbf{v} = \infty$. For simplicity we take $G = SU(2)$ in what follows.

The instanton moduli are $m = (\mathbf{b}, \mathbf{m} = \rho g_2)$, and $(\mathbf{b}, \mathbf{m})$ is identified with $(\mathbf{b}, -\mathbf{m})$, so that $\mathcal{M}_{2,1} \simeq \mathbb{R}^4 \times \left(\mathbb{R}^4 \backslash \{0\}\right)/\mathbb{Z}_2$.

The charge one $SU(2)$ instantons on $\mathbb{S}^4$ are constructed with the help of a family of operators

$$\mathcal{D}^+ = \begin{pmatrix} b_0 - z_0 & b_1 - z_1 & I \\ -\overline{b}_1 + \overline{z}_1 & \overline{b}_0 - \overline{z}_0 & J^\dagger \end{pmatrix}$$

parametrized by the points $z \in \mathbb{C}^2 \approx \mathbb{R}^4 = \mathbb{S}^4 - \{\infty\}$. The ADHM equations imply that

$$\mathcal{D}^+ = \begin{pmatrix} rg_1 & \rho g_2 \end{pmatrix},$$

where $\det g_1 = \det g_2 = 1$,

$$g_1^\dagger g_1 = g_2^\dagger g_2 = \mathbf{1}_{2\times 2}, \qquad r^2 = \|b - z\|^2, \qquad \rho^2 = II^\dagger = J^\dagger J.$$

The "observation point" $\mathbf{z} = z_0 + z_1\mathbf{j} = \mathbf{b} - rg_1$.

The gauge field is written in terms of the normalized solution Ψ to the equation

$$\mathcal{D}^+\Psi = 0, \qquad \Psi^\dagger\Psi = 1_{2\times 2}$$

as

$$A = \Psi^\dagger d\Psi. \tag{8.53}$$

We point out an important feature of the ADHM construction. If we let d in formula (8.53) act not only on z but also on the moduli, then A becomes a universal connection, that is, a connection on the universal bundle over

$$\mathcal{M} \times \mathbb{S}^4,$$

whose restriction onto a fiber $m \times \mathbb{S}^4$ of projection to $\mathcal{M}$ gives the corresponding connection on $\mathbb{S}^4$. In our case

$$\Psi = \begin{pmatrix} \rho g_1^\dagger \\ -rg_2^\dagger \end{pmatrix} \frac{1}{\sqrt{r^2 + \rho^2}}$$

and

$$\mathbf{A} = x\theta_1 + (1-x)\theta_2,$$

where $\theta_i = g_i dg_i^\dagger$, and

$$x = \frac{\rho^2}{r^2 + \rho^2}.$$

The universal curvature $\mathbf{F} = d\mathbf{A} + \mathbf{A}^2$ and the corresponding $\operatorname{tr}\mathbf{F}^2$ invariant is given by

$$\operatorname{tr}\mathbf{F}^2 = d\left[x^2(3-2x)\right] \wedge \operatorname{tr}\,\theta_{12}^3, \tag{8.54}$$

where

$$\theta_{12} = (g_2^\dagger g_1) d(g_1^\dagger g_2).$$

We can rewrite (8.54) in a more suggestive form, using quaternions:

$$\operatorname{tr}\mathbf{F}^2 = \frac{d^4\mathbf{v}_z}{(1 + |\mathbf{v}_z|^2)^4}, \tag{8.55}$$

where

$$\mathbf{v}_z = \frac{1}{\rho} g_2^\dagger \cdot (\mathbf{b} - \mathbf{z}).$$

The geometry behind formula (8.55) can be found in [2].

8.3.4. *Gauge theory calculation.* The following correlation function is a good candidate for exhibiting the logarithmic structure of the four-dimensional gauge theory:

$$C_{\mathcal{O}\mathcal{O}\mathcal{S}}(x_1, x_2; x_3) = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{S}(x_3)\mathbf{GF}(\infty)\rangle \ , \tag{8.56}$$

where

$$\mathcal{O}(x) \;=\; \mathrm{tr}\,\phi^2(x) \quad , \quad \mathcal{S}(x) \;=\; \mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}(x),$$

and

$$\mathbf{GF}(\mathbf{y}) \sim \prod_a \delta(\phi^a(\mathbf{y}))\delta(\overline{\phi}^a(\mathbf{y}))\eta^a(\mathbf{y}) \times \prod_a \delta(\lambda^a(\mathbf{y}))\overline{c}^a(\mathbf{y})c^a(\mathbf{y})$$

is the operator which fixes the gauge transformations at the point $\mathbf{y}$ to be trivial (it is a suitably regularized product of the Faddeev-Popov ghosts and the delta functions $\delta(\phi^a)$, all taken at one point in $\mathbf{M}^4$).

The correlator (8.56) is saturated by the charge one instanton. Indeed, on the moduli space $\mathcal{M}_{2,1}$ of charge 1 instantons with the gauge group $SU(2)$ the operators $\mathrm{tr}\phi^2$ become four-forms, and $\mathrm{tr}F^2_{\mu\nu}$ – the density of the topological charge – a function. The dimension of the moduli space $\mathcal{M}^{\infty}_{N,k}$ of charge k instantons with the gauge group $SU(N)$, considered up to the gauge transformations, equal to the identity at one point (e.g., $\mathbf{y} = \infty$), is equal to $4Nk$.

Using the expressions (8.55) for the universal curvature invariants, computed with the help of the ADHM construction, we reduce (8.56) to the following integral:

$$C_{\mathcal{O}\mathcal{O}\mathcal{S}}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3) = \frac{1}{|\mathbf{x}_{12}|^4}\ \mathcal{C}\left(\frac{\overline{\mathbf{x}}_{12}\cdot(\mathbf{x}_{13}+\mathbf{x}_{23})}{|\mathbf{x}_{12}|^2}\right), \tag{8.57}$$

where

$$\mathcal{C}(\mathbf{q}) = \int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{d^4\mathbf{v}_1}{(1+|\mathbf{v}_1|^2)^4}\frac{d^4\mathbf{v}_2}{(1+|\mathbf{v}_2|^2)^4}\frac{|\mathbf{v}_-|^4}{(1+|\mathbf{v}_+ - \mathbf{v}_-\cdot\mathbf{q}|^2)^4} \tag{8.58}$$

with

$$\mathbf{v}_{\pm} = \frac{\mathbf{v}_1 \pm \mathbf{v}_2}{2},$$

and we use quaternionic notation.

Formula (8.57) is very suggestive in that it resembles the two-dimensional holomortex formalism (see Section 4.7 and Section 6.7.7). Indeed, it looks like the correlation function of two holomortex operators, inserted at the points $\mathbf{v}_1$ and $\mathbf{v}_2$. This suggests that the four-dimensional Yang-Mills theory may be defined as a deformation of a much simpler model by analogues of the two-dimensional holomortex operators. The precise definition of these operators is beyond the scope of the present paper. We will only remark that to observe this holomortex structure of the correlation functions it is important to go beyond the topological sector of the Yang-Mills theory.

The integral (8.57) is invariant under the $SU(2)$ rotations, $\mathbf{q} \mapsto u\mathbf{q}\overline{u}$, $u\overline{u} = 1$. Thus, we can assume that $\mathbf{q} = q \in \mathbb{C}$, or, in matrix form,

$$\mathbf{q} = \begin{pmatrix} q & 0 \\ 0 & \overline{q} \end{pmatrix} \ .$$

Without any detailed calculations it is clear that $\mathcal{C}(q)$ has singularities when $q \to -1, +1, \infty$. These limits correspond to the situations, where the operator $\mathcal{S}$ hits one of the operators $\mathcal{O}$, or goes away to infinity.

The correlation function (8.57) can also be rewritten in the following symmetric form:

$$C_{\mathcal{O}\mathcal{O}\mathcal{S}}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3) = |\mathbf{x}_{12}|^4 \; \Gamma\left(\mathbf{x}_{12}^2, \mathbf{x}_{13}^2, \mathbf{x}_{23}^2\right), \tag{8.59}$$

$$\Gamma(a_{12}, a_{13}, a_{23}) \; = \int_{\mathbb{R}^3_{\geq 0}} \frac{d^3t\,(t_1t_2t_3)^3\, e^{-(t_1+t_2+t_3)}}{(t_1t_2a_{12} + t_1t_3a_{13} + t_2t_3a_{23})^4}.$$

Note that the expression in the denominator in (8.59) can be rewritten using

$$\frac{t_1t_2|\mathbf{x}_{12}|^2 + t_1t_3|\mathbf{x}_{13}|^2 + t_2t_3|\mathbf{x}_{23}|^2}{t_1+t_2+t_3} = \sum_{i=1}^{3} t_i\,|\mathbf{x}_i - \mathbf{x}_t|^2,$$

where

$$\mathbf{x}_t = \frac{t_1\mathbf{x}_1 + t_2\mathbf{x}_2 + t_3\mathbf{x}_3}{t_1+t_2+t_3}.$$

Therefore the integral (8.59) can be written as the integral over the plane triangle with the vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ (see Figure 9).

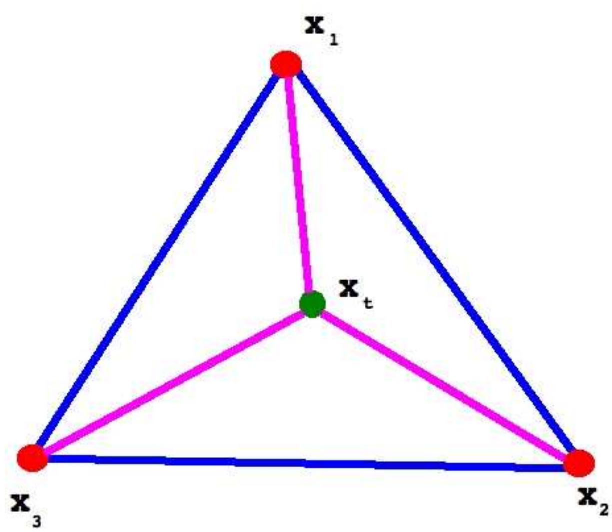

Figure 9. The triangle.

Let us introduce the kinematic variables

$$s_2 = |\mathbf{x}_{12}|/|\mathbf{x}_{23}| \;,\; s_3 = |\mathbf{x}_{13}|/|\mathbf{x}_{23}|, \tag{8.60}$$

which obey the usual triangle inequalities

$$|1 - s_2| \leq s_3 \leq 1 + s_2.$$

The integrals (8.57),(8.59) can be reduced to the one-dimensional integral

$$\mathcal{C}(\mathbf{q}) \ = \int_{-\infty}^{+\infty} \mathrm{d}\vartheta \, \rho \left(s_2^2 + s_3^2 + 2 s_2 s_3 \cosh\vartheta \right) \ , \tag{8.61}$$

where the function

$$\rho(m) = \frac{m^2+6m+3}{(m-1)^6}\log(m) - \frac{10m^2+19m+1}{3m(m-1)^5} \tag{8.62}$$

interpolates between

$$\rho(m) = \frac{1}{60} - \frac{4}{105}(m-1) + \frac{5}{84}(m-1)^2 + \dots$$

for $m \to 1$, and

$$\rho(m) = \left(\log(m) - \frac{10}{3} \right) \frac{1}{m^4} + 12 \left(\log(m) - \frac{23}{12} \right) \frac{1}{m^5} + \dots$$

for $m \to \infty$. The argument of the function ρ in (8.61) is bounded below by

$$m_{\min} = (s_2 + s_3)^2 \geq 1$$

(the equality is achieved only when the point $\mathbf{x}_3$ is on the line connecting $\mathbf{x}_1$ and $\mathbf{x}_2$). Therefore the function ρ is finite for all values of ϑ, and the integral (8.61) converges for all $\mathbf{x}_i$, $i = 1, 2, 3$.

Now we interpret the correlation function (8.56) as the vacuum matrix element

$$\langle \mathrm{vac} | \ \mathcal{O} \, e^{i\varphi J} \, e^{-TH} \, \mathcal{S} \ | \mathrm{vac}_{\mathcal{O}} \rangle. \tag{8.63}$$

Here the Hamiltonian H generates the radial evolution, with the position $\mathbf{x}_1$ of one of the operators $\mathcal{O}$ being the origin, J is one of the $SO(4)$ generators

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ,$$

in the notation of (8.52), and φ is the angular distance between the location $\mathbf{z}$ of $\mathcal{S}$ and that of the second $\mathcal{O}$, $\mathbf{x}_2$, when viewed from the point $\mathbf{x}_1$:

$$e^{i\varphi} = \frac{1-q}{|1-q|} = \frac{\overline{\mathbf{x}}_{12} \cdot \mathbf{x}_{32}}{|\mathbf{x}_{12}||\mathbf{x}_{32}|} \ .$$

Finally, the "time" T is related to $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$ as follows:

$$e^T = \frac{|\mathbf{x}_{12}|}{|\mathbf{x}_{32}|} = s_2 \ .$$

The precise analytic expression for the correlation function (8.59) is rather complicated. The function $\rho(m)$ decays sufficiently fast for large $m \gg m_{\max} \sim 4$. Therefore the typical range of ϑ which contributes to the integral (8.61) is $\sim \log\left(\frac{C}{s_2 s_3}\right)$ where C is a numerical constant of the order one. This is why we expect (8.61) to contain only the simple logarithms $\log(s_2)$, $\log(s_3)$. This is a natural result for the one-instanton correlation function, as we have argued in the case of quantum mechanics.

The upshot of this calculation is that the logarithms do appear in the correlation functions of this model. This signifies the logarithmic nature of the four-dimensional conformal theory which we obtain in the $\overline{\tau} \to \infty$ limit of the twisted $\mathcal{N} = 2$ super-Yang–Mills theory.

8.3.5. *Operator product expansion and logarithmic partners.* The next natural step in our program is analogous to the computation of the instanton corrections to the OPE in two-dimensional sigma models in Section 6.7. Let us investigate the asymptotics of the correlation function $\mathcal{C}_{\mathcal{O}\mathcal{O}\mathcal{S}}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3)$ in the limit $\mathbf{x}_2 \to \mathbf{x}_1$. The naive asymptotics of the integral (8.58) as $\mathbf{q} \to \infty$ is a logarithmically divergent integral

$$\mathcal{C}(\mathbf{q}) \sim \frac{1}{|\mathbf{q}|^4} \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{d^4\mathbf{v}_1}{(1+|\mathbf{v}_1|^2)^4} \frac{d^4\mathbf{v}_2}{(1+|\mathbf{v}_2|^2)^4} \frac{1}{|\mathbf{v}_-|^4},$$

which is actually cut off at $|\mathbf{v}_-| \sim |\mathbf{q}|^{-1}$, so that

$$\mathcal{C}_{\mathcal{O}\mathcal{O}\mathcal{S}}(\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3) \sim \frac{1}{|\mathbf{x}_{23}|^4} \log \frac{|\mathbf{x}_{12}|}{|\mathbf{x}_{23}|}\ , \qquad \mathbf{x}_2 \to \mathbf{x}_1.$$

We interpret this as the one-instanton correction to the OPE:

$$\mathrm{tr}\phi^2(\mathbf{x}_1)\,\mathrm{tr}\phi^2(\mathbf{x}_2) \sim \Lambda_{\mathrm{QCD}}\,\log|\mathbf{x}_{12}|\,\mathrm{tr}(H^+)^2(\mathbf{x}_2) + \ldots$$

This formula is analogous to the logarithmic terms in formulas (6.57) and (6.58) obtained in sigma models. Continuing along these lines, we can find the analogues of the jet-evaluation observables in the four-dimensional gauge theory and observe the logarithmic mixing of these observables. We plan to study this in more detail in a follow-up paper.

9. Conclusions

In this paper we have studied (twisted) supersymmetric two-dimensional sigma models and four-dimensional gauge theories in the $\overline{\tau} \to \infty$ limit. We have used the quantum mechanical models considered in Part I as a prototype. A special feature of these models is that the path integral localizes on the finite-dimensional moduli spaces of instanton configurations (holomorphic maps in two dimensions and anti-self-dual connections in four dimensions). This gives us good control of the correlation functions and enables us to describe rather explicitly the spaces of states and the spectra of these models. However, to do this we must go beyond the topological sector and consider observables that are not annihilated by the supercharge.

In two dimensions, we identify a large class of observables of this type which we call *jet-evaluation observables*. These are differential forms on the jet space of the target manifold. They generalize the familiar evaluation observables in that they depend not only on the value of a holomorphic map in the target manifold, but also on its derivatives. Their correlation functions are given by integrals over the moduli spaces of holomorphic maps, generalizing the Gromov–Witten invariants. However, in contrast to the Gromov–Witten invariants, these integrals generally diverge on the boundary divisors of the moduli spaces of holomorphic maps and need to be regularized. Their regularization is not canonical, reflecting what we call *logarithmic mixing* of operators (and states) of the sigma model. This is analogous to the appearance of the logarithmic structures in the quantum mechanical models which we studied in Part I. We have presented here many explicit examples of the correlation functions, OPE and logarithmic mixing in two-dimensional sigma models, particularly, in the case of the target manifold $\mathbb{P}^1$. We have also revisited the holomortex description of the latter model introduced in [25] and shown that it may be used to effectively reproduce these results.

Our conclusion is that the twisted $\mathcal{N} = (2, 2)$ supersymmetric sigma models are logarithmic conformal field theories, with central charge $c = 0$. Such logarithmic CFTs have been extensively studied in the literature recently, see, e.g., [34, 38, 35, 47, 16, 41] and references therein. These models are quite interesting for both theoretical reasons and for their applications to condensed matter. Here we consider a new class of models of this type which have many attractive features: they are defined geometrically (as sigma models) and their correlation functions are computed explicitly as (regularized) integrals over moduli spaces of holomorphic maps. We hope that further understanding of these models will be beneficial for the investigation of logarithmic CFTs as well as their physical applications.

In Part III of this paper we will study sigma models with less supersymmetry, such as the $\mathcal{N} = (0, 2)$ and purely bosonic sigma models. These models are more difficult to analyze because the definition of the measure in the path integral becomes more problematic, which leads to various anomalies that we avoid in the $\mathcal{N} = (2, 2)$ models. In particular, these models are not conformally invariant unless the target manifold is Calabi-Yau. Nevertheless, they may still possess non-trivial chiral algebras of symmetries, such as an affine Kac–Moody algebra of critical level $k = -h^\vee$ in the case when the target manifold is a flag manifold of a simple Lie groups. The latter models have applications to the geometric Langlands correspondence. In addition, the models with the target manifold $\mathbb{P}^1$ (and probably other flag manifolds as well) admit an analogue of the holomortex description, similar to the one discussed above.

We have also studied the twisted four-dimensional supersymmetric Yang–Mills theory. The $\overline{\tau} \to \infty$ limit of this model may be studied along the same lines as above, with the added complication that we need to take into account equivariance with respect to the gauge transformations. However, much of the same structure that we have observed in one- and two-dimensional models also appears in four dimensions. In particular, we have computed some sample correlation functions (in the one instanton sector for the group $SU(2)$) which exhibit the same logarithmic behavior that we have seen in lower dimensions. These correlation functions are given by integrals over the moduli spaces of anti-self-dual connections, which are described explicitly by the ADHM construction. By careful analysis of these correlation functions we obtain logarithmic terms in the OPE of the analogues of jet-evaluation observables. Thus, we find the same kind of logarithmic mixing of operators that we have seen in two-dimensional sigma models. We believe that further investigation of these phenomena will lead to better understanding of the four-dimensional gauge theory beyond its topological sector.

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