

Pre-Calabi-Yau algebras and $\xi\partial$ -calculus on higher cyclic Hochschild cohomology

Natalia Iyudu, Maxim Kontsevich

Abstract

We formulate the notion of pre-Calabi-Yau structure via the higher cyclic Hochschild complex and study its cohomology. A small quasi-isomorphic subcomplex in higher cyclic Hochschild complex gives rise to the graphical calculus of $\xi\partial$ -monomials. Developing this calculus we are able to give a nice combinatorial formulation of the Lie structure on the corresponding Lie subalgebra. Then using basis of $\xi\partial$ -monomials and employing elements of Gröbner bases theory we prove homological purity of the higher cyclic Hochschild complex and as a consequence obtain L_∞ -formality. This construction in particular allows an easy interpretation of a pre-Calabi-Yau structure as a noncommutative Poisson structure. We give an explicit formula showing how the double Poisson algebra introduced in [25] appears as a particular part of a pre-Calabi-Yau structure. This result holds for any associative algebra A and emphasizes the special role of the fourth component of a pre-Calabi-Yau structure in this respect.

MSC: 16A22, 16S37, 16Y99, 16G99, 16W10, 17B63 **Keywords:** A-infinity structure, pre-Calabi-Yau algebra, inner product, cyclic invariance, graded pre-Lie algebra, necklace bracket, Maurer-Cartan equation, Poisson structure, double Poisson bracket, Hochschild (co)homology, L-infinity structure, formality.

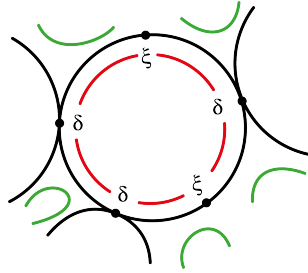
1 Introduction

The notion of pre-Calabi-Yau algebra appeared independently and more or less at the same time in [14], [15], [19]. This structure is present in many different areas, including topology of compact manifolds with boundary, algebraic geometry, symplectic geometry. For example, Fano varieties are endowed with a pre-Calabi-Yau structure, open Calabi-Yau manifolds have this structure, from the HMS conjecture it is expected that the Fukaya wrapped category of an open symplectic manifold have a pre-Calabi-Yau structure, etc. The important feature of this structure are degree shifts, so it captures some essential moments present in the study of shifted structures. The definition of the pre-Calabi-Yau structure can be given via the higher cyclic Hochschild complex (in this shape it becomes applicable to infinite dimensional algebras as well, especially, to algebras with infinite dimensional graded components). Namely, pre-Calabi-Yau structure is a solution of the Maurer-Cartan equation with respect to generalized necklace bracket in the higher cyclic Hochschild complex.

We study here the higher cyclic Hochschild complex (for definition see section 3), its homological and Lie structure. One of the tools is to introduce on this complex a calculus of noncommutative cyclic words with labels. We start with the free associative algebra $A = \mathbb{K}\langle X \rangle$ with finite number

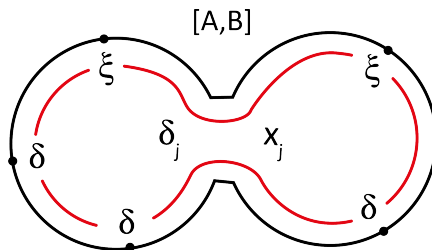
of generators $X = \{x_1, \dots, x_r\}$ and 'labels' $\partial_1, \dots, \partial_r, \xi$. Elements of free algebra $\mathbb{K}\langle X \rangle$ are written cyclically on the circle and separated by labels. These generalised cyclic monomials with labels, which we call $\xi\partial$ -monomials, represent operations on tuples of monomials from A and form a convenient basis in the higher cyclic Hochschild complex. Whenever we are dealing with the higher cyclic Hochschild complex itself, without embracing in further word combinatorics, we can speak of an arbitrary formally smooth algebra A in the sense of Cuntz and Quillen [4].

To deal with the higher cyclic Hochschild complex $C^\bullet(A)$ we choose a small subcomplex $\xi^\bullet$, quasi-isomorphic to the whole complex. We specify in section 5 a particular embedding of the subcomplex $\xi^\bullet$ into $C^\bullet(A)$ by choosing a basis of $\xi\partial$ -monomials in $\xi^\bullet$ and describing the operation in $C^\bullet(A)$ corresponding to a given $\xi\partial$ -monomial. The operation is schematically shown in the following picture.



Here the black arches are input monomials from A and green arches are output monomials. We suppose orientation is clockwise everywhere, in particular, outputs are to be read from according to this orientation. In the above picture we see the $\xi\partial$ -monomial which encodes an operation $\Phi : A^{\otimes 3} \rightarrow A^{\otimes 5}$.

We describe the generalised necklace bracket which endows the higher cyclic Hochschild complex with a graded Lie algebra structure. In section 6 we show how this bracket works in terms of $\xi\partial$ -monomials. By this we not only prove that small subcomplex $\zeta_A^{(\bullet)}$ is a Lie subalgebra in $\mathfrak{g} = (C_A^{(\bullet)}(A), [\cdot, \cdot]_{gen.neckl})$, but also give a concrete combinatorial formula for this bracket via $\xi\partial$ -monomials. We prove that the bracket $[A, B]$ of two $\xi\partial$ -monomials A and B is obtained from the initial ones according the rule $[A, B] = A \circ B - B \circ A$, where $A \circ B$ described as follows. It is a linear combination of $\xi\partial$ -monomials obtained from $\xi\partial$ -monomials representing A and B by all possible gluings of ∂_j and x_j as shown in the picture.



Namely, we glue all ∂_j from $\xi\partial$ -monomial A to a corresponding x_j from B , then cut at the place of gluing, and open up to obtain one new $\xi\partial$ -monomial (x_j and ∂_j disappear, all remaining monomials are read off according to clockwise orientation).

The choice of this basis in higher cyclic Hochschild complex allows, among other things, an easy interpretation of pre-Calabi-Yau structure as a noncommutative Poisson structure. Namely, the $\xi\partial$ -monomial produce an obvious formal analogue of polyvector field which in turn create a Poisson structure on the representation space of A , via kind of Schouten bracket. This phenomena was observed in many particular situations, so hopefully the abstract formulation of what is going on in terms of higher cyclic Hochschild complex and formal calculus of $\xi\partial$ -monomials on it refines the understanding and can be applied to even wider variety of situations.

We discuss in section 4 how the double Poisson bracket invented by Van den Bergh [25] as a structure which induce a Poisson bracket on representation space of algebra, appear as a part of pre-Calabi-Yau structure. In [8] we gave a detailed proof of the following fact. Any pre-Calabi-Yau structure with $m_4 = 0$ on arbitrary associative algebra gives rise to a double Poisson bracket according to the formula [8]:

$$(*) \quad \langle g \otimes f, \{\{b, a\}\} \rangle := \langle m_3(a, f, b), g \rangle,$$

Moreover, an arbitrary double Poisson bracket can be obtained from pre-Calabi-Yau structure of special type, with only second and third multiplications m_2 and m_3 present. We comment here on main idea behind this proof from the point of view of the definition of pre-Calabi-Yau structure via higher cyclic Hochschild complex. The special role of the forth component m_4 of pre-Calabi-Yau structure is that in this case the precise isomorphism between these two structures can be constructed, without any correcting terms.

In section 7 we concentrate on homological properties of the higher cyclic Hochschild complex and prove its homological purity. We again use the small quasi-isomorphic subcomplex $\zeta_A^\bullet$, introduced in section 5. From the expression of the differential in the whole dualised bar complex $C_A^{(\bullet)}(A)$, which we spell out in section 5, we get a differential in $\zeta = \zeta_{\mathbb{K}}^{(\bullet)}$.

While the elements of higher cyclic Hochschild complex are defined as elements of $\text{Hom}(A^{\otimes m}, A^{\otimes N})^{\mathbb{Z}_N}$, invariant under $\mathbb{Z}_N$ -action, our homology calculations are reduced to a related non $\mathbb{Z}_N$ invariant complex $\bar{\zeta}$, corresponding to operations with fixed point. This is possible since the differential commutes with the cyclisation procedure (see lemma 7.12).

The complex with the fixed point $\bar{\zeta} = \oplus \bar{\zeta}_m^k$, where $\bar{\zeta}^k = \{ \text{monomials } u \in \mathbb{K}\langle \xi, x_i, \partial_i \rangle, \text{ starting from } \xi \text{ or } \partial_i, \text{ such that } \deg_{\partial} u = k \}_{\mathbb{K}}, \deg_{\partial, \xi} u = m \}_{\mathbb{K}}$ has natural bigrading by ∂ -degree, and by degree with respect to ξ and ∂_i th, $i = 1, r$, which we call *weight*. Essential for our considerations will be *cohomological grading* by ξ -degree: $\bar{\zeta} = \oplus \bar{\zeta}(l)$, where $\bar{\zeta}(l) = \oplus_{m-k=l} \bar{\zeta}_m^k$.

Theorem 1.1. *The homology of the complex $\bar{\zeta} = \oplus \bar{\zeta}_m^k$ is sitting in the diagonal $k = m$. Consequently, the complex $\bar{\zeta} = \oplus \bar{\zeta}(l)$, $\bar{\zeta}(l) = \oplus_{m-k=l} \bar{\zeta}_m^k$ is pure, that is its homology is sitting only in the last place of the complex $\bar{\zeta}$ with respect to cohomological grading by ξ -degree. Homological purity hence holds for the higher cyclic Hochschild complex $\bar{\zeta}$.*

This purity result is obtained via use of the Gröbner bases theory in the ideals of free algebra and the basis of $\xi\partial$ -monomials. As a consequence of purity result we are able to deduce L_∞ -formality for this complex.

2 General definitions

The typical example of an algebra in this paper is a free associative algebra $A = \langle x_1, \dots, x_r \rangle$, the most noncommutative algebra possible. We develop elements of noncommutative geometry based on this algebra following the spirit of [16, 13]. For example, we adopt the ideology introduced and developed in this paper, which says that noncommutative structure should manifest as a corresponding commutative structure on representation spaces. Sometimes we consider instead of free algebra *formally smooth* algebras in a sense of J. Cuntz and D. Quillen [4]:

Definition 2.1. An algebra A is formally smooth (=quasi-free) if and only if it satisfies one of the following equivalent properties:

- (1) (Lifting property for nilpotent extensions) for any algebra B , a two-sided nilpotent ideal $I \in B(I = BIB, I^n = 0 \text{ for } n \gg 0)$, and for any algebra homomorphism $f : A \rightarrow B/I$, there exists an algebra homomorphism $\tilde{f} : A \rightarrow B$ such that $f = pr_{B \rightarrow B/I} \circ \tilde{f}$ is a natural projection.
- (2) $Ext_{A-mod-A}^2(A, M) = 0$ for any bimodule $M \in A-mod-A$.
- (3) The A -bimodule $\Omega_A^1 = \text{Ker}(m_A : A \otimes A \rightarrow A)$ is projective.

We denote here by $A-mod-A$ the category of all A -bimodules, which is the same as A^e -modules, that is modules over $A^e = A \otimes A^{op}$. We consider mainly Homs of A -bimodules or $A^{\otimes N}$ -bimodules which we denote $\text{Hom}_{A-mod-A}$ or $\text{Hom}_{A^{\otimes N}-mod-A^{\otimes N}}$ respectively.

To give a definition of pre-Calabi-Yau structure as it was originally defined in [14], [15], [19] we need to start with the definition of A_∞ -algebra, or *strong homotopy associative algebra* introduced by Stasheff [21].

In fact, there are two accepted conventions of grading of an A_∞ -algebra. They differ by a shift in numeration of graded components. In one convention, we call it *shifted convention*, each operation has degree 1. While the other, which we call a *naive convention* is determined by making the binary operation of degree 0, hence the degrees of operations m_n of arity n become $2 - n$. If the degree of element x in *naive convention* is $\deg x = |x|$, then shifted degree in $A^{sh} = A[1]$, which fall into *shifted convention*, will be $\deg^{sh} x = |x|'$, where $|x|' = |x| - 1$, since $x \in A^i = A[1]^{i+1}$.

The formulae for the graded Lie bracket, Maurer-Cartan equations and cyclic invariance of the inner form are different in different conventions. We mainly use the *shifted convention*, but sometimes need the *naive convention* as well.

Let A be a $\mathbb{Z}$ graded vector space $A = \bigoplus_{n \in \mathbb{Z}} A_n$, and $C^l(A, A)$ be Hochschild cochains $C^l(A, A) = \text{Hom}(A[1]^{\otimes l}, A[1])$, for $l \geq 0$, $C^\bullet(A, A) = \prod_{k \geq 1} C^k(A, A)$.

On $C^\bullet(A, A)[1]$ there is a natural structure of graded pre-Lie algebra, defined via composition:

$$\circ : C^{l_1}(A, A) \otimes C^{l_2}(A, A) \rightarrow C^{l_1+l_2-1}(A, A) :$$

$$f \circ g(a_1 \otimes \dots \otimes a_{l_1+l_2-1}) =$$

$$\sum (-1)^{|g| \sum_{j=1}^{i-1} |a_j|} f(a_1 \otimes \dots \otimes a_{i-1} \otimes g(a_1 \otimes \dots \otimes a_{i+l_2+1}) \otimes \dots \otimes a_{l_1+l_2-1})$$

The operation $\circ$ defined in this way does satisfy the graded right-symmetric identity:

$$(f, g, h) = (-1)^{|g||h|} (f, h, g)$$

where

$$(f, g, h) = (f \circ g) \circ h - f \circ (g \circ h).$$

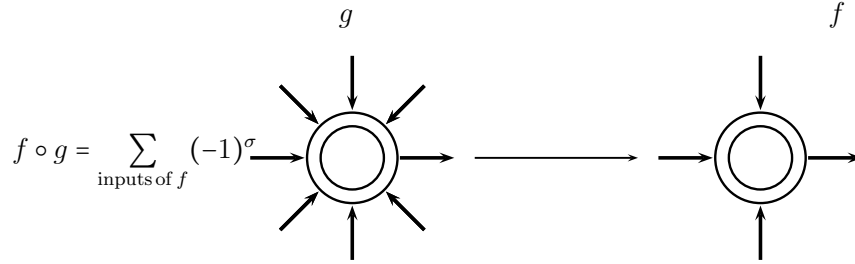
As it was shown in [6] the graded commutator on a graded pre-Lie algebra defines a graded Lie algebra structure.

Thus the Gerstenhaber bracket $[-, -]_G$:

$$[f, g]_G = f \circ g - (-1)^{|f||g|} g \circ f$$

makes $C^\bullet(A)$ into a graded Lie algebra. Equipped with the derivation $d = \text{ad } m_2$, $(C^\bullet(A), m_2)$ becomes a DGLA, which is a Hochschild cohomological complex.

Graphically the corresponding composition can be depicted as follows.



With respect to the Gerstenhaber bracket $[-, -]_G$ we have the Maurer-Cartan equation

$$[m^{(1)}, m^{(1)}]_G = \sum_{p+q=k+1} \sum_{i=1}^{p-1} (-1)^\varepsilon m_p(x_1, \dots, x_{i-1}, m_q(x_j, \dots, x_{i+q-1}), \dots, x_k) = 0, \quad (2.1)$$

where

$$\varepsilon = |x_1|' + \dots + |x_{i-1}|', \quad |x_i|' = |x_i| - 1 = \deg x_i - 1$$

The Maurer-Cartan in *naive convention* is:

$$[m^{(1)}, m^{(1)}] = \sum_{p+q=k+1} \sum_{i=1}^{p-1} (-1)^\varepsilon m_p(x_1, \dots, x_{i-1}, m_q(x_j, \dots, x_{i+q-1}), \dots, x_k) = 0, \quad (2.2)$$

where

$$\varepsilon = i(q+1) + q(|x_1| + \dots + |x_{i-1}|),$$

Definition 2.2. An element $m^{(1)} \in C^\bullet(A, A)[1]$ which satisfies the Maurer-Cartan equation $[m^{(1)}, m^{(1)}]_G$ with respect to the Gerstenhaber bracket $[-, -]_G$ is called an A_∞ -structure on A .

Equivalently, it can be formulated in a more compact way as a coderivation on the coalgebra of the bar complex of A .

In particular, for example, associative algebra with zero derivation $A = (A, m = m_2^{(1)})$ is an A_∞ -algebra. The component of the Maurer-Cartan equation of arity 3, says that the binary operation of this structure, the multiplication m_2 is associative:

$$(ab)c - a(bc) = dm_3(a, b, c) + (-1)^\sigma m_3(da, b, c) + (-1)^\sigma m_3(a, db, c) + (-1)^\sigma m_3(a, b, dc)$$

We can give now definition of pre-Calabi-Yau structure (in *shifted convention*).

Definition 2.3. A d-pre-Calabi-Yau structure on a finite dimensional A_∞ -algebra A is

- (I). an A_∞ -structure on $A \oplus A^*[1-d]$,
- (II). cyclic invariant with respect to natural non-degenerate pairing on $A \oplus A^*[1-d]$, meaning:

$$\langle m_n(\alpha_1, \dots, \alpha_n), \alpha_{n+1} \rangle = (-1)^{|\alpha_1|'(|\alpha_2|'+\dots+|\alpha_{n+1}|')} \langle m_n(\alpha_2, \dots, \alpha_{n+1}), \alpha_1 \rangle$$

where the inner form $\langle, \rangle$ on $A \oplus A^*$ is defined naturally as $\langle (a, f), (b, g) \rangle = f(b) + (-1)^{|g|'|a|'} g(a)$ for $a, b \in A, f, g \in A^*$

- (III) and such that A is an A_∞ -subalgebra in $A \oplus A^*[1-d]$.

The signs in this definition written in shifted convention are assigned according to the Koszul rule, which is, by the way, not quite the case for *naive convention*, where the cyclic invariance with respect to the natural non-degenerate pairing on $A \oplus A^*[1-d]$, from (II) sounds:

$$\langle m_n(\alpha_1, \dots, \alpha_n), \alpha_{n+1} \rangle = (-1)^{n+|\alpha_1|'(|\alpha_2|'+\dots+|\alpha_{n+1}|')} \langle m_n(\alpha_2, \dots, \alpha_{n+1}), \alpha_1 \rangle$$

The appearance of the arity n , which influence the sign in this formula, does not really fit with the Koszul rule, this is the feature of the *naive convention*, and this is why it is more convenient to work with the *shifted convention*.

The cyclic invariance condition and inner form symmetricity in *shifted convention* look like:

$$\langle m_n(\alpha_1, \dots, \alpha_n), \alpha_{n+1} \rangle = (-1)^{|\alpha_1|'(|\alpha_2|'+\dots+|\alpha_{n+1}|')} \langle m_n(\alpha_2, \dots, \alpha_{n+1}), \alpha_1 \rangle \quad (2.3)$$

$$\langle x, y \rangle = -(-1)^{|x|'|y|'} \langle y, x \rangle \quad (2.4)$$

The most simple example of pre-Calabi-Yau structure demonstrates that this structure does exist on any associative algebra. Namely, the structure of associative algebra on A can be extended to the associative structure on $A \oplus A^*[1-d]$ in such a way, that the natural inner form is (graded)cyclic with respect to this multiplication. This amounts to the following fact: for any A -bimodule M the associative multiplication on $A \oplus M$ is given by $(a+f)(b+g) = ab + af + gb$. In this simplest situation both structures on A and on $A \oplus A^*$ are in fact associative algebras. More examples one can find in [7], [3].

Note that the notion of pre-Calabi-Yau algebra introduced in [15], [19], [22], as an A_∞ -atructure on $A \oplus A^*$, uses the fact that A is finite dimensional, since there is no natural grading on the dual algebra $A^* = \text{Hom}(A, \mathbb{K})$, induced from the grading on A in infinite dimensional case. The general definition via higher cyclic Hochschild complex, suitable for infinite dimensional algebra was given in [15], [14], and we will use it here. It is equivalent to the definition, where the $\text{Hom}(A, \mathbb{K})$ considered as graded Hom : $A^* = \oplus (A_n)^* = \underline{\text{Hom}}(A, \mathbb{K})$, in case the graded components are finite dimensional.

3 Higher cyclic Hochschild complex

First, we should define *higher cyclic Hochschild cochains* and *generalised necklace bracket*.

Definition 3.1. For $N \geq 1$ the space of N -higher cyclic Hochschild cochains is defined as

$$C_{cycl}^{(N)}(A) := \prod_{r_1, \dots, r_N \geq 0} \text{Hom}_{A^{\otimes N} \text{-mod-} A^{\otimes N}} \left(\bigotimes_{i=1}^N A^{\otimes r_i}, A_{cycl}^{\otimes N} \right)^{\mathbb{Z}_N},$$

With differential coming from dualised bar complex of $A^{\otimes N}$ -bimodules. The $A^{\otimes N}$ -bimodule structure on $A_{cycl}^{\otimes N}$ is defined as follows: for any $x_1 \otimes \dots \otimes x_N \in A_{cycl}^{\otimes N}$ and elements $a_1 \otimes \dots \otimes a_N, b_1 \otimes \dots \otimes b_N \in A^{\otimes N}$,

$$(a_1 \otimes \dots \otimes a_N) \bullet (x_1 \otimes \dots \otimes x_N) \bullet (b_1 \otimes \dots \otimes b_N) = a_1 x_1 b_2 \otimes \dots \otimes a_N x_N b_1.$$

The symbol $Hom(A^{\otimes r}, A^{\otimes N})^{\mathbb{Z}_N}$ means that we take only elements of Hom which are 'invariant' with respect to obvious $\mathbb{Z}_N$ -action.

Denote by $C_{cycl}^{(\bullet)}(A) = \prod_{N \geq 1} C_{cycl}^{(N)}(A)$ the space of all higher cyclic Hochschild cochains. Further throughout the paper we omit subscript *cycl*, when it is clear that we are in cyclic situation, then we write just $C^{(\bullet)}(A) = \prod_{N \geq 1} C^{(N)}(A)$. The complex obviously can also be considered as product of corresponding $\mathbb{K}$ -module morphisms or as a collection of N operations with one output each.

The corresponding differential is written precisely in section 5.

Let us comment on the meaning of $Hom(\bigotimes_{i=1}^N A^{\otimes r_i}, A^{\otimes N})^{\mathbb{Z}_N}$. An element of $Hom(\bigotimes_{i=1}^N A^{\otimes r_i}, A^{\otimes N})$ can be thought of as a collection of N operations from $A^{\otimes r_i}$ to A with one fixed point: the operation from which we start, when we move along the circle on which all operations are situated. The cyclicity requirement on the higher cyclic Hochschild complex says that together with each operation with fixed point we have a sum of the same operations with all possible fixed points. It means that operation is actually symmetrized by cyclic permutations of outputs. In other words, it can be expressed in the following way. Consider an obvious action of $\mathbb{Z}_N$ on $Hom(\bigotimes_{i=1}^N A^{\otimes r_i}, A^{\otimes N})$, which cyclically permutes N outputs (together with N participating operations). We require that elements of the complex are 'invariant' under $\mathbb{Z}_N$ -action, that is the higher cyclic Hochschild complex is formed by $Hom(A[1]^{\otimes r}, A[1]_{cycl}^{\otimes N})^{\mathbb{Z}_N}$.

The space of all higher cyclic Hochschild cochains denoted $C_{\mathbb{K}}^{(\bullet)}(A)$ or $C_A^{(\bullet)}(A)$, depending on whether $\mathbb{K}$ -module Homs or $A^{\otimes N}$ -bimodule Homs are involved, sometimes we omit the $\mathbb{K}$ when it is clear from the context.

Note, that $C_{\mathbb{K}}^{(1)}(A)$ is the space of usual Hochschild cochains.

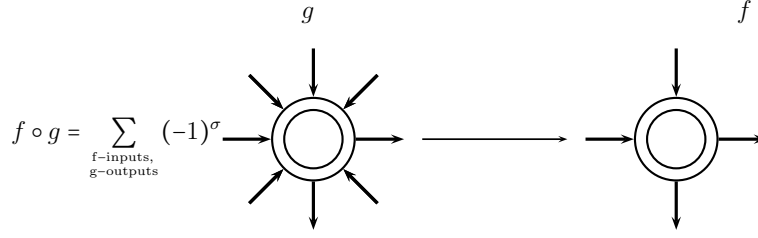
Now we start to define a Lie bracket on the complex, and at this point it becomes important which shifts of the grading on A we chose, so we consider the higher cyclic Hochschild complex with the following shifts:

$$C_{cycl}^{(N)}(A) := \prod_{r_1, \dots, r_N \geq 0} Hom_{A^{\otimes N} \text{-mod-} A^{\otimes N}}(\bigotimes_{i=1}^N A^{\otimes r_i}, A[1]_{cycl}^{\otimes N})^{\mathbb{Z}_N}.$$

Definition 3.2. The *generalized necklace bracket* between two elements $f, g \in C^{(N)}(A)$ is given as $[f, g]_{gen.neckl} = f \circ g - (-1)^\sigma g \circ f$, where composition $f \circ g$ consists of inserting all outputs of g to all inputs from f with signs assigned according to the Koszul rule.

Note that the point is not fixed in the elements of our complex, thus generalized necklace bracket produces also operations without a fixed points, that is cyclicly invariant operations. We can think of it as of insertion of operations with fixed points according to the rule, and then 'symmetrizing the result' by taking each resulting operation with all possible fixed points to the output.

The composition for the generalised necklace bracket can be graphically depicted as follows:



Since the defined above composition $f \circ g$ makes $C^{(\bullet)}$ into a graded pre-Lie algebra, the generalized necklace bracket obtained from it as a graded commutator, makes $C^{(\bullet)}$ into a graded Lie algebra. We denote it by $\mathbf{g} = (C^{(\bullet)}(A, A[1]), [\cdot, \cdot]_{gen.neckl})$.

Let us denote by $C^{(N,d)} \subset C^N(A)$: $C^{(N,d)} = C^{(N,dN \pmod{2})}$, where $C^{(N,0)}$ are cochains invariant under $\mathbb{Z}_N$ -action, and $C^{(N,1)}$ are anti-invariant cochains.

Definition 3.3. The d-pre-Calabi-Yau structure on A is an element from the space of higher cyclic Hochschild cochains

$$C^{(N,d)} \subset C^{(\bullet)}(A) = \prod Hom_{A^{\otimes N} \text{-mod-} A^{\otimes N}} \left(\bigotimes_{i=1}^N A^{\otimes r_i}, A_{cycl}^{\otimes N} \right)^{\mathbb{Z}_N},$$

$m = \sum_{N \geq 0} m^{(N)}$, $m^{(N)} \in C^{(N,d)}(A)$, i.e. 'invariant' with respect to $\mathbb{Z}_N$ action element of degree $(d-2)(N-2) - \sum r_i$ and a solution to the Maurer-Cartan equation $[m, m]_{gen.neckl} = 0$ with respect to generalised necklace bracket.

Any such solution makes $C^{(\bullet)}(A)$ into a DGLA with the differential $\text{ad } m$. Indeed, $\text{ad } m$ become a derivative, that is $(\text{ad } m)^2 = 0$: $[[x, m], m] = 0$ for any x if m is a solution of M equation: $[m, m] = 0$, since $[\cdot, \cdot]$ satisfies a Jacobi identity.

Note that in the above definition if we would want to have operations of degrees not dependant of degrees of inputs/outputs as solutions of Maurer-Cartan equations, we would use the following shifted version of the higher cyclic Hochschild complex:

$$C^{(N)} = Hom_{A^{\otimes N} \text{-mod-} A^{\otimes N}} \left(\bigotimes_{i=1}^N (\text{Hom } A[1]^{\otimes r_i}, A_{cycl}^{\otimes N}[2-d]) \right)^{\mathbb{Z}_N},$$

For the sake of simplicity and clarity we in many occasions consider in this text the grading, where A is sitting in the zero component: $A_0 = A$. This prompts us to deal with 2-pre-Calabi-Yau structures.

4 Double Poisson bracket and Maurer-Cartan equation

In this section we discuss a bijective correspondence between particular part of pre-Calabi-Yau structure and the structure of double Poisson bracket invented by Van den Bergh [25] as a structure which produces the Poisson bracket on representation spaces.

Remind, that double Poisson bracket defined as a map $\{\{\cdot, \cdot\}\} : A \otimes A \rightarrow A \otimes A$ satisfying the following axioms:

Anti-symmetry:

$$\{\{a, b\}\} = -\{\{b, a\}\}^{op} \quad (4.1)$$

Here $\{\{b, a\}\}^{op}$ means the twist in the tensor product, i.e. if $\{\{b, a\}\} = \sum_i b_i \otimes c_i$, then $\{\{b, a\}\}^{op} = \sum_i c_i \otimes b_i$.

Double Leibniz:

$$\{\{a, bc\}\} = b\{\{a, c\}\} + \{\{a, b\}\}c \quad (4.2)$$

and double Jacobi identity:

$$\{\{a, \{\{b, c\}\}\}\}_L + \tau_{(123)}\{\{b\{\{c, a\}\}\}\}_L + \tau_{(132)}\{\{c\{\{a, b\}\}\}\}_L \quad (4.3)$$

Here for $a \in A \otimes A \otimes A$, and $\sigma \in S_3$

$$\tau_\sigma(a) = a_{\sigma^{-1}(1)} \otimes a_{\sigma^{-1}(2)} \otimes a_{\sigma^{-1}(3)}.$$

The $\{\{\}\}_L$ defined as

$$\{\{b, a_1 \otimes \dots \otimes a_n\}\}_L = \{\{b, a_1\}\} \otimes a_1 \otimes \dots \otimes a_n$$

The connection between the two structures is described by the following theorem.

Theorem 4.1. *Let we have A_∞ -structure on $(A \oplus A^*, m = \sum_{i=2, i \neq 4}^\infty m_i^{(1)})$. Define the bracket by the formula*

$$(*) \quad \langle g \otimes f, \{\{b, a\}\} \rangle := \langle m_3(a, f, b), g \rangle,$$

where $a, b \in A$, $f, g \in A^*$ and $m_3(a, f, b) = c \in A$ corresponds to the component of solution to the Maurer-Cartan $m_3: A \times A^* \times A \rightarrow A$ corresponding to the cyclic tensor $A \otimes A^* \otimes A \otimes A^*$. Then this bracket does satisfy all axioms of the double Poisson algebra.

Moreover, pre-Calabi-Yau structures corresponding to the cyclic tensor $A \otimes A^* \otimes A \otimes A^*$ with $m_i = 0, i \geq 4$ are in the bijective correspondence defined by $(*)$ with the double Poisson brackets for an arbitrary associative algebra A .

The detailed proof of this theorem, taking into account signs and other details, was given in [8] in terms of definition 2.3 of pre-Calabi-Yau structure. We translate here main idea of this proof, using definition 3.3 via higher cyclic Hochschild complex. It looks very transparent this way, which emphasises another advantage of this definition.

In terms of definition 3.3 the Maurer-Cartan equation on 'invariant' with respect to action of cyclic group elements from the higher cyclic Hochschild complex of particular kind, described in the theorem look like:

$$\left[\begin{array}{c} \downarrow \\ \circlearrowleft m_3 \\ \uparrow \end{array} + \begin{array}{c} \downarrow \\ \circlearrowleft m_2 \\ \swarrow \end{array} , \begin{array}{c} \downarrow \\ \circlearrowleft m_3 \\ \uparrow \end{array} + \begin{array}{c} \downarrow \\ \circlearrowleft m_2 \\ \swarrow \end{array} \right] = 0$$

Hence from the Maurer-Cartan the following equations follow:

$$\begin{array}{c}
\begin{array}{ccc}
\begin{array}{c} \downarrow \\ \circlearrowleft m_3 \end{array} & \begin{array}{c} \rightarrow \\ \circlearrowright m_2 \end{array} & \\
\begin{array}{c} \uparrow \\ \circlearrowright m_3 \end{array} & &
\end{array}
+
\begin{array}{ccc}
\begin{array}{c} \downarrow \\ \circlearrowleft m_3 \end{array} & \begin{array}{c} \rightarrow \\ \circlearrowright m_2 \end{array} & \\
\begin{array}{c} \uparrow \\ \circlearrowright m_3 \end{array} & &
\end{array}
+ \\
\\
\begin{array}{ccc}
& \begin{array}{c} \swarrow \\ \downarrow \\ \searrow \\ \circlearrowleft m_2 \end{array} & \\
& \downarrow & \\
\begin{array}{c} \leftarrow \\ \circlearrowleft m_3 \end{array} & & \begin{array}{c} \rightarrow \\ \circlearrowright m_3 \end{array}
\end{array}
=0
\end{array}$$

$$\left(\begin{array}{ccc}
& \begin{array}{c} \uparrow \\ \downarrow \\ \circlearrowleft m_3 \end{array} & \begin{array}{c} \uparrow \\ \downarrow \\ \circlearrowleft m_3 \end{array} \\
\begin{array}{c} \leftarrow \\ \circlearrowleft m_3 \end{array} & & \begin{array}{c} \rightarrow \\ \circlearrowright m_3 \end{array}
\end{array} \right) \stackrel{Z_3}{=} 0$$

These two clearly correspond to Leibnitz and Jacobi identities respectively. The thing to be checked now related to the following fact. The element of the higher cyclic Hochschild complex with two inputs and two outputs canonically corresponds (via the pairing on $A \oplus A^*$) to the maps $D : A \times A \rightarrow A \times A$ and $M : A \times A^* \times A \rightarrow A$. This correspondence defined however only up to an arbitrary permutation of terms A . To establish an isomorphism between the two structures, we then need to choose appropriately this correspondence, which is done by formula (*). After that axioms of double Poisson bracket can be checked, taking into account signs. Moreover we ensure that no other axioms appear from the Maurer-Cartan equation in case of the structure $(A \oplus A^*, m_2 + m_3)$, hence all double Poisson brackets can be obtained from these structures, that is the map defined by (*) is a surjection. This means that structures of mentioned type are indeed in a bijective correspondence with the double Poisson brackets.

5 Small subcomplex in the higher cyclic Hochschild complex

We consider now a subcomplex ζ of the higher cyclic Hochschild complex, which we define as follows. Take a quotient complex

$$R_{\min} = 0 \rightarrow \Omega \rightarrow A \otimes A \rightarrow A \rightarrow 0$$

of the bar complex (considered as a complex of A -bimodules). Namely, $R_{\min} = \mathcal{B}/\mathcal{F}$, where $\mathcal{B}$ is the bar complex

$$\mathcal{B} = \dots A \otimes A \otimes A \xrightarrow{D_3} A \otimes A \rightarrow A \rightarrow 0$$

denote its usual differential by D_A or just D , when it is clear that we are talking about complex of A -bimodules.

Let $\mathcal{F}$ be the subcomplex generated by $A^{\otimes k}$ with $k \geq 4$ and $\ker D_3$. i.e. $\mathcal{F} = \oplus A^{\otimes \geq 4} \oplus \ker D_3$. Note that $\Omega = A \otimes A \otimes A / \ker D_3$ is isomorphic to the kernel of the multiplication map $\mu : A \otimes A \rightarrow A$. We equip $R_{\min}$ with the grading for which $R_{-1} = \Omega$ and $R_0 = A \otimes A$. Thus we have a resolution $R_{\min} \in \text{Compl}(A^e - \text{mod})$ of a diagonal bimodule A .

Then we consider N th tensor power of $R_{\min}$:

$$R_{\min}^{\otimes N} \in \text{Compl}((A^e)^{\otimes N} - \text{mod}),$$

and dualise it by taking Hom to an $A^{\otimes N}$ -bimodule $A_{\text{cycl}}^{\otimes N}$ with the defined above structure.

$$\text{Hom}_{(A^e)^{\otimes N}}(R_{\min}^{\otimes N}, A_{\text{cycl}}^{\otimes N}) =: \zeta^{(N)}.$$

For $N = 1$ applying the functor $\text{Hom}_{A^e}(-, A)$ to $R_{\min} \in \text{Compl}(A^e - \text{mod})$ we get a subcomplex $\zeta = \text{Ann}(\mathcal{F}) = \text{Hom}_{A^e}(R_{\min}, A)$ of the usual Hochschild complex $C^\bullet(A, A) = \text{Hom}_{A^e}(\mathcal{B}, A)$:

$$C^\bullet(A, A) \supset \text{Hom}_{A^e}(R_{\min}, A)$$

where

$$\zeta = \text{Ann}(\mathcal{F}) = \{\Phi \in C^\bullet(A, A)[1] : \Phi(h) = 0 \text{ for } h \in \mathcal{F}\}.$$

Thus

$$\zeta = \text{Ann}(\mathcal{F}) = \{\text{chains in } \text{Hom}_{A^e}(\mathcal{B}, A), \text{ turning } \mathcal{F} \text{ into } 0\}$$

$$= \{\Phi(a_1 \otimes \dots \otimes a_n), \text{ s.t. } \Phi(a_1 \otimes \dots \otimes a_n) = 0, n > 3 \text{ and}$$

$$\Phi(a_1 \otimes a_2 \otimes a_3) = 0 \text{ iff } a_1 \otimes a_2 \otimes a_3 \in \ker D_3 = \text{Im} D_4\}.$$

That is, $\Phi \in \text{Hom}_{A^e}(A^{\otimes 3}, A)$ is in ζ if and only if it is an A -bimodule derivation, that is satisfies the Leibnitz rule:

$$\Phi(a_1 \otimes a_2 a_3 \otimes a_4) = \Phi(a_1 \otimes a_2 \otimes a_3 a_4) - \Phi(a_1 a_2 \otimes a_3 \otimes a_4).$$

Note that $\text{Hom}_{A^e}(A \otimes A, A)$ is naturally isomorphic to A , while $\text{Hom}_{A^e}(\Omega, A)$ is naturally identified with $\text{Der}_{A^e}(A^{\otimes 3}, A)$, which interprets $\zeta = \text{Hom}_{A^e}(R_{\min}, A)$:

$$0 \leftarrow \text{Hom}_{A^e}(\Omega, A) \xleftarrow{D_3^*} \text{Hom}_{A^e}(A \otimes A, A) \xleftarrow{D_2^*} \text{Hom}_{A^e}(A, A) \leftarrow 0$$

as

$$0 \xleftarrow{D_3^*} \text{Der}_{A^e}(A^{\otimes 3}, A) \xleftarrow{D_2^*} A \leftarrow \mathbb{K}.$$

We can pass from Hom_{A^e} to $\text{Hom}_{\mathbb{K}}$, and since

$$\text{Hom}_{A^e}(A \otimes A, A) \simeq A, \quad \text{Hom}_{A^e}(A^{\otimes n+2}, A) \simeq \text{Hom}_{\mathbb{K}}(A^{\otimes n}, A), \quad \text{Hom}_{A^e}(\Omega, A) \simeq \text{Der}_{\mathbb{K}} \subset \text{Hom}_{\mathbb{K}}(A, A)$$

we have an isomorphic complex over $\mathbb{K}$:

$$\zeta_{\mathbb{K}} : \quad \mathcal{E} = 0 \leftarrow \text{Der}_{\mathbb{K}} A \xleftarrow{d_2^*} A \leftarrow 0$$

where $\text{Der}_{\mathbb{K}} A$ is the space of usual derivations from $\text{Hom}_{\mathbb{K}}(A, A)$.

We explain here in more details what we do in case of arbitrary N . We take N th tensor power of small complex R_{min} and dualize it by $\text{Hom}_{A^{\otimes N} \text{-mod-} A^{\otimes N}}(-, A_{cycl}^{\otimes N})$.

The structure of $A^{\otimes N}$ module on $\mathcal{B}$ is natural.

Now $R_{min}^{\otimes N} = (\mathcal{B}/\mathcal{F})^{\otimes N} = \mathcal{B}^{\otimes N}/\mathcal{J}$ where $\mathcal{J} = \mathcal{F} \otimes \mathcal{B}^{\otimes(N-1)} + \mathcal{B} \otimes \mathcal{F} \otimes \mathcal{B}^{\otimes(N-2)} + \dots + \mathcal{B}^{\otimes(N-1)} \otimes \mathcal{F}$.

Here we need to check of course that $\mathcal{J}$ is a submodule in $A^{\otimes N}$ -bimodule $\mathcal{B}^{\otimes N}$.

Thus $\zeta_A^{(N)} = \text{Ann } \mathcal{J} = \text{Hom}_{A^{\otimes N} \text{-mod-} A^{\otimes N}}(R_{min}^{\otimes N}, A_{cycl}^{\otimes N})^{\mathbb{Z}_N} \subset \text{Hom}_{A^{\otimes N} \text{-mod-} A^{\otimes N}}(\mathcal{B}^{\otimes N}, A_{cycl}^{\otimes N})^{\mathbb{Z}_N}$.

We can describe this annihilator as

$$\zeta^{(N)} = \text{Ann } \mathcal{J} = \{\Phi \in \text{Hom}_{A^{\otimes N} \text{-mod-} A^{\otimes N}}(\mathcal{B}^{\otimes N}, A_{cycl}^{\otimes N})^{\mathbb{Z}_N} \mid$$

$$\Phi(\mathcal{B}^{\otimes r} \otimes \mathcal{F} \otimes \mathcal{B}^{\otimes s}) = 0, \quad \forall r + s = N - 1\}.$$

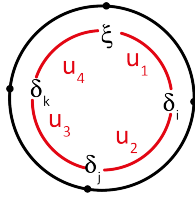
This means $\zeta^{(N)} = \text{Ann } \mathcal{J}$ formed by those element of vector space

$$E^{\otimes N} = \bigcap_{s+r=N-1} \{\mathcal{B}^{\otimes r} \otimes \text{Ann } \mathcal{F} \otimes \mathcal{B}^{\otimes s}\},$$

which are $A^{\otimes N}$ -bimodule morphisms.

This leads us to the description of the small complex $\zeta^{(N)}$ in terms of the appropriately chosen basis.

Starting from this place, when we choose a basis, we will deal with a free algebra A , in stead of just formally smooth algebra. The obvious free basis in Ω_{A-A} consists of $dx_i = 1 \otimes x_i - x_i \otimes 1$, and denote the basis of free A -bimodule $A \otimes A$ by ξ . Let us denote the elements of dual bases in $\text{Hom}_{A-A}(\Omega, A)$ and $\text{Hom}_{A-A}(A \otimes A, A)$ respectively by ∂_i and ξ^* : $\partial_i(dx_j) = \delta_{ij}1$, $\xi^*(\xi) = 1$. Corresponding bases of $\text{Hom}_{\mathbb{K}}(\Omega, A)$ and $\text{Hom}_{\mathbb{K}}(A \otimes A, A)$ are $\{\partial_i u, u \in \langle X \rangle\}$ and $\{\xi u, u \in \langle X \rangle\}$ respectively. The basis of the complex $\zeta_{\mathbb{K}}^{(N)} = \text{Hom}_{\mathbb{K}}(R_{min}^{\otimes N}, A_{cycl}^{\otimes N})$ thus consists of cyclic monomials on ξ^* , ∂_i and x_i which we depict as follows. We will further write just ξ in stead of ξ^* .



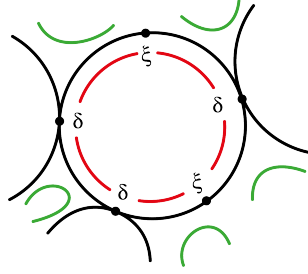
Pic.1

This monomial, which we call $\xi\partial$ -monomial corresponds to the following operation, i.e. element of the higher cyclic Hochschild complex $\text{Hom}_{\mathbb{K}}(A^{\otimes n}, A^{\otimes k})$, where n is the ∂ -degree of $\xi\partial$ -monomial μ and k is its ∂, ξ -degree.

Let X -monomials $u_1, \dots, u_n$ be an input of Φ_{μ} . The output will be a linear combination of tuples of monomials from A colored green in the following picture. All circles in the picture are oriented clockwise, so one can read outputs following the orientation of the circles. The sum in the linear

combination is over all 'intersections' of variables x_i from the input monomials (black) with ∂_i in the $\xi\partial$ -monomial μ .

The $\xi\partial$ -monomial depicted below represent an operation $\Phi : A^{\otimes 3} \rightarrow A^{\otimes 5}$.



Pic.2

In terms of the above $\xi\partial$ -basis we now describe differentials $D_A^* = D^*$ and $D_{\mathbb{K}}^* = d^*$ in dualized complexes.

Let us spell out first the usual differential D_A on one copy of the bar complex $\mathcal{B}$:

$$D_A(u_1 \otimes \dots \otimes u_n) = u_1 u_2 \otimes \dots \otimes u_n - u_1 \otimes u_2 u_3 \otimes \dots \otimes u_n + \dots + (-1)^n u_1 \otimes \dots \otimes u_{n-1} u_n.$$

After we dualise this complex by $\text{Hom}_{A\text{-mod-}A}(-, A)$, we get a usual dual differential D_A^* :

$$(D_A^* f)(u_1 \otimes \dots \otimes u_{n+1}) = f(u_1 u_2 \otimes \dots \otimes u_{n+1}) - f(u_1 \otimes u_2 u_3 \otimes \dots \otimes u_{n+1}) + \dots + (-1)^{n+2} f(u_1 \otimes \dots \otimes u_n u_{n+1}),$$

where $f \in \text{Hom}_{A\text{-mod-}A}(\mathcal{B}, A)$.

When we pass to $\text{Hom}_{\mathbb{K}}(\mathcal{B}, A)$ an element $h \in \text{Hom}_{\mathbb{K}}(\mathcal{B}, A)$ is defined by $h(v_1 \otimes \dots \otimes v_n) = f(1 \otimes v_1 \otimes \dots \otimes v_n \otimes 1)$, since f is an A -bimodule morphism. Thus

$$(D_{\mathbb{K}}^* h)(v_1 \otimes \dots \otimes v_{n-1}) = v_1 h(v_2 \otimes \dots \otimes v_{n-1}) - h(v_1 v_2 \otimes v_2 \dots \otimes v_{n-1}) + \dots$$

$$(-1)^{n-2} h(v_1 \otimes \dots \otimes v_{n-2} v_{n-1}) + (-1)^{n-1} h(v_1 \otimes \dots \otimes v_{n-2}) v_{n-1},$$

where $h \in \text{Hom}_{\mathbb{K}}(\mathcal{B}, A)$.

Doing the same for the tensor product of N copies of the bar complex $\mathcal{B}$ and dualising it by $\text{Hom}_{A^{\otimes N}\text{-mod-}A^{\otimes N}}(-, A_{\text{cycl}}^{\otimes N})$, we obtain the expression for the differential in the higher cyclic Hochschild complex.

The differential obtained from the differential of the tensor product of bar complexes after dualising by $\text{Hom}_{A^{\otimes N}\text{-mod-}A^{\otimes N}}(-, A_{\text{cycl}}^{\otimes N})$ is the following:

$$D^* h(v_1, \dots, v_N) = \sum_{\alpha=1}^n (-1)^{s_1 + \dots + s_{\alpha-1}} D_{\alpha}^* h(v_1, \dots, v_N),$$

where $v_\alpha = x_1^\alpha \dots x_{s_\alpha}^\alpha \in A^{\otimes s_\alpha} \subset \mathcal{B}$, and

$$\begin{aligned} D_\alpha^* h(v_1, \dots, v_N) & \sum_{j=1}^{s_\alpha-1} (-1)^j h(v_1 \otimes \dots \otimes v_{\alpha-1} \otimes x_1^\alpha \otimes \dots \otimes x_j^\alpha x_{j+1}^\alpha \otimes \dots \otimes x_{s_\alpha}^\alpha \otimes v_{\alpha+1} \otimes \dots \otimes v_N) + \\ & (1 \otimes \dots \otimes 1 \otimes x_1^\alpha \otimes \dots \otimes 1) \bullet h(v_1 \otimes \dots \otimes v_{\alpha-1} \otimes x_2^\alpha \otimes \dots \otimes x_{s_\alpha}^\alpha \otimes v_{\alpha+1} \otimes \dots \otimes v_N) + \\ & (-1)^{s_\alpha} h(v_1 \otimes \dots \otimes v_{\alpha-1} \otimes x_1^\alpha \otimes \dots \otimes x_{s_\alpha-1}^\alpha \otimes v_{\alpha+1} \otimes \dots \otimes v_N) \bullet (1 \otimes \dots \otimes 1 \otimes x_{s_\alpha}^\alpha \otimes \dots \otimes 1). \end{aligned}$$

Here the element $(1 \otimes \dots \otimes 1 \otimes x_1^\alpha \otimes \dots \otimes 1)$ has x_1^α in the place α , the element $1 \otimes \dots \otimes 1 \otimes x_{s_\alpha}^\alpha \otimes \dots \otimes 1$ has $x_{s_\alpha}^\alpha$ in place $\alpha + 1 \pmod{N}$.

Note that we need the small complex $\zeta^{(n)}$ to calculate cohomologies of the higher Hochschild complex, so we want to ensure they do coincide. It is indeed the case for good enough algebras, this is why we choose to work with formally smooth algebras at this stage.

When we substitute the small complex $R_{\min}$ with its n -th tensor power and apply the functor $\text{Hom}_{(A^e)}^{\otimes n}$, we get subcomplex $\zeta^{(n)}$ of the higher Hochschild complex, which has the same cohomologies.

Proposition 5.1. *Let A be a formally smooth algebra (in particular, free associative algebra). Then $HC^{(n)}(A[1], A[1]) = H\zeta^{(n)}$.*

Proof. Since both complexes are projective resolutions of $A^{\otimes n}$, and A is smooth, we have that the statement is true. $\square$

The goal for the next section is to describe how the Lie bracket defined on the higher Hochschild complex $\mathbf{g} = C^\bullet(A[1], A[1])$ acts on elements of the basis consisting of $\xi\partial$ -monomials. In previous section we seen how $\xi\partial$ -monomials interpret as elements of $\mathbf{g}$. That is, which element of $\text{Hom}_{\mathbb{K}}(A^{\otimes n}, A^{\otimes k})$ corresponds to a given $\xi\partial$ -monomial. By specifying this correspondence we gave the concrete embedding of $\zeta^{(n)}$ into the higher cyclic Hochschild complex $C^{(\bullet)}(A)$. Now we will use this correspondence to show that the necklace bracket of two $\xi\partial$ -monomials again sitting in the linear span of $\xi\partial$ -monomials, and how this new $\xi\partial$ -monomials are constructed, which forms a foundation of $\xi\partial$ -calculus.

6 Lie bracket on $\zeta^{(\bullet)}$

We give here a constructive description of the bracket in the small subcomplex $\zeta_{\mathbb{K}}^{(N)}$ of the Hochschild complex in terms of $\xi\partial$ calculus which makes it into a Lie subalgebra of $C_{\mathbb{K}}^{(N)}$.

Theorem 6.1. *I. The above described embedding $\zeta_{\mathbb{K}} \rightarrow C^\bullet(A[1], A[1])$ is an embedding of complexes, whose image is a Lie subalgebra of $\mathbf{g} = C^\bullet(A[1], A[1])$ equipped with the generalised necklace bracket.*

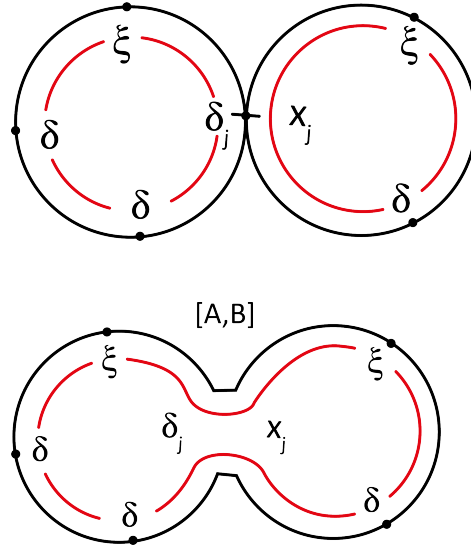
II. Precise combinatorial description of this bracket is given by () (Pic. 3).*

Proof. To prove this we need to show that the bracket of the Lie algebra $\mathbf{g} = C^\bullet(A[1], A[1])$ applied to $\xi\partial$ -monomials yields a member of ζ , that is, a linear combination of $\xi\partial$ -monomials again.

Let A and B be two $\xi\partial$ -monomials. We perform composition of corresponding operations $U \circ W$ from the Hochschild complex according to the necklace bracket rule. We will see that we can not

express the resulting operation $U \circ W$ via $\xi\partial$ monomials, but we can do it for the operation $[U, V] = U \circ W - W \circ U$. Perform first $U \circ W$. This composition of operations from $C^\bullet(A)$ is realised as application of $\xi\partial$ -monomial A to the input (according to the procedure described by pic.2), and then application of $\xi\partial$ -monomial B to the output of the first operation. As an output of this composition we will get linear combination of monomials from $A \otimes t_i$. We call such a monomial *non-essential* if it is obtained as a result of gluing some letter x_i from the input of operation A to some ∂_i . Gluing letters x_j from the inside of operation A (red arcs in pic.2) to the ∂_j will result in obtaining *essential* monomials in the output of composition $U \circ W$. Note that the copy of the same monomial present in composition $U \circ W$ can be essential or not, depending on how it is obtained. Thus to be essential is not a property of the monomial, but it just characterises the way it got into the output of the composition of these operations.

We claim that non-essential output monomials for the operations $U \circ W$ and $W \circ U$ will be the same (with the same coefficients) and therefore they will cancel out in the bracket $[U, W]$ formed only by the essential outputs meaning exactly that it is described by the operation following operation (*) on $\xi\partial$ words. That is, letters x_i from the monomial A (red in Pic. 3) are getting inserted in ∂_i of the second $\xi\partial$ word B (and other way around for $W \circ U$). Insertions of this kind alone define $[A, B]$.



Pic.3

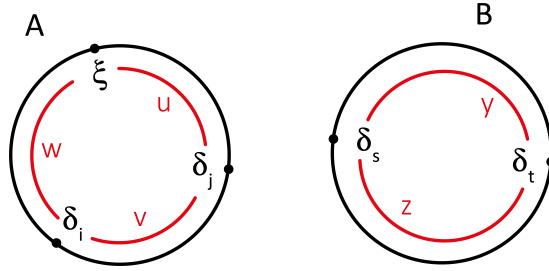
Thus the operation $(*)$ on ∂ -monomials is described as follows.

The complete proof of this claim consists of consideration of 8 cases depending on which combinations of ξ and ∂_j surrounds the place of insertion as well as whether the two letters of the input involved in the two compositions come from the same input or from different ones. $\square$

We illustrate the proof by the following example.

Example

Consider the following two operations from Pic. 4.

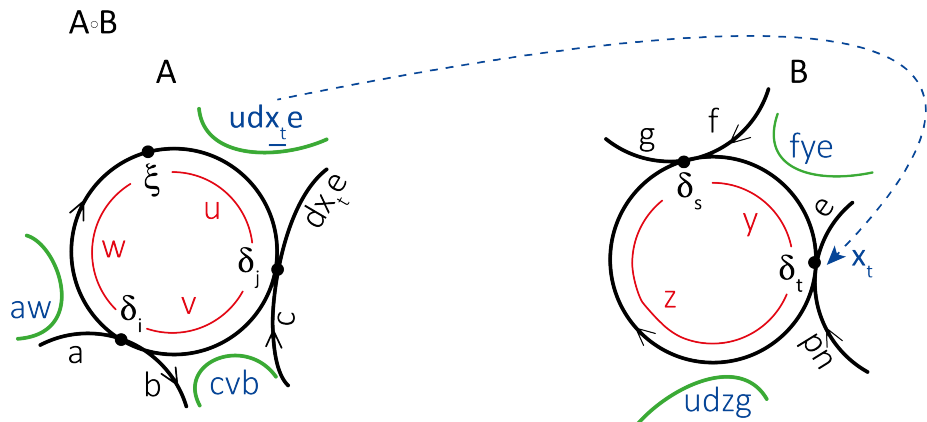


Pic.4

Namely, let $A = \xi u \partial_i v \partial_j w \xi$ and $B = \partial_3 y \partial_4 z$. As an input consider three monomials w_1 , w_2 and w_3 of the form

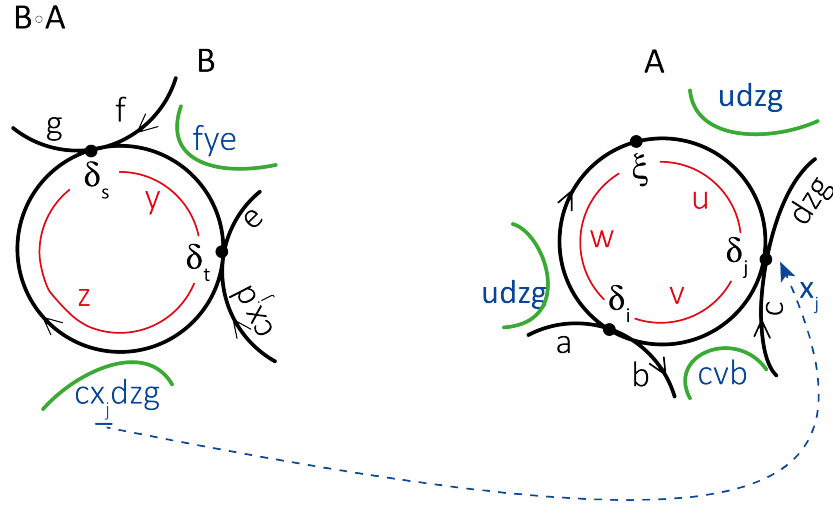
$$w_1 = ax_i b, \quad w_2 = cx_j dx_t e, \quad w_3 = fx_s g.$$

We fixed certain points in them, to construct coupling monomials which will cancel in $[U, W]$. Consider first operation corresponding to a $\xi \partial$ monomial A . After inserting x_i of w_1 and x_j of w_2 into ∂_i and ∂_j of A , we get three outputs aw , $udx_t e$ and cvb . After inserting x_t from one of those outputs $udx_t e$, and x_s of w_3 into the operation corresponding to B , we get four outputs for the composition aw , cvb , $udzg$ and fye . These are non-essential monomials since we started from letter x_i in the input w_1 of operation U , not from the internal letter in the $\xi \partial$ -presentation of operation U .



Pic.5

Next, consider composition of operations $W \circ U$. If we insert x_s of w_3 and x_t of w_2 into the operation corresponding to B , we get two outputs fye and $ucx_j dzg$. Now inserting x_i of w_1 , and x_j of the second output into the operation defined by A , we get the same four outputs for the composition: aw , cvb , $udzg$ and fye .



Pic.6

This example demonstrate that compositions in different order exhibit the same non-essential output monomials. As a matter of fact, the same argument works for every non-essential output monomial, so we can see that all non-essential outputs cancel in $[U, W] = U \circ W - W \circ U$.

7 Homological purity and formality

7.1 Homological purity of the higher Hochschild complex

The goal of this section will be to prove homological purity of the small complex $\zeta = \zeta_{\mathbb{K}}^{(N)} = \text{Hom}_{\mathbb{K}}(R_{min}^{\otimes N}, A_{cycl}^{\otimes N})^{\mathbb{Z}_N}$ considered in previous sections. It is a subcomplex of a complex $\bar{\zeta} = \text{Hom}_{\mathbb{K}}(R_{min}^{\otimes N}, A_{cycl}^{\otimes N})$, which is the version of the higher Hochschild complex, where we do not take invariants under $\mathbb{Z}_N$ -action, or in other words operations do have a fixed point. This complex can be described as $\bar{\zeta} = \{\text{monomials } u \in \mathbb{K}\langle \xi, x_i, \partial_i \rangle, i = 1, \dots, r, \text{ starting from } \xi \text{ or } \partial_i\}_{\mathbb{K}}$. It will be instrumental in the proof of purity for ζ . Our proof, using Gröbner bases techniques in the ideals of free algebras, for

example those generated by one element associated to a differential, can be considered in a way as a construction of corresponding homotopy map.

We naturally have a bigrading on $\zeta = \oplus \zeta_m^k$: the grading by ∂ -degree, and by degree with respect to ξ and ∂_i th, $i = 1, \dots, r$, which we call *weight*. That is, $u \in \zeta_m^k$, if $\deg_{\xi, \partial} u =: w(u) = m$, and $\deg_{\partial} u =: g(u) = k$. Essential for our considerations will be the *cohomological grading* by ξ -degree: $\zeta = \oplus \zeta(l)$, where $\zeta(l) = \bigoplus_{m-k=l} \zeta_m^k$.

$$\begin{array}{ccccccc}
 & & \partial & & \partial\partial & & \partial\partial\partial & & \partial\partial\partial\partial \\
 & \nearrow & | & \nearrow & | & \nearrow & & \nearrow & \\
 \xi & & \xi\partial & & \xi\partial\partial & & \xi\partial\partial\partial & & \\
 & \nearrow & | & \nearrow & | & \nearrow & & & \\
 \xi\xi & & \xi\xi\partial & & \xi\xi\partial\partial & & & & \\
 & \nearrow & | & \nearrow & | & & & & \\
 \xi\xi\xi & & \xi\xi\xi\partial & & & & & & \\
 & \nearrow & | & & & & & & \\
 \xi\xi\xi\xi & & & & & & & &
 \end{array}$$

If ζ_m is a subcomplex of ζ , namely a slice consisting of elements of weight m , then we will use also splitting $\zeta_m = \oplus \zeta_m^{(s)}$, where s is an x -degree: $u \in \zeta_m^{(s)}$, if $w(u) = m$, and $\deg_{x_1, \dots, x_r}(u) = s$.

The differential in bimodule Hom explained in section 5, in terms of $\xi\partial$ -monomials boils down to the following differential on ζ :

$$d(u_1 \xi u_2 \xi \dots u_n) = \sum (-1)^{g(u_1 \xi u_2 \xi \dots u_i)} u_1 \xi u_2 \xi \dots u_i \Delta u_{i+1} \dots u_n$$

if $u_1 \neq \emptyset$ (u_1 starting with ∂_i), here $\Delta = \sum_{i=1}^r \partial_i x_i - x_i \partial_i$, and

$$d(\xi u_1 \xi u_2 \xi \dots u_n) = \xi d(u_1 \xi u_2 \xi \dots u_n) + \sum_{i=1}^r [\partial_i x_i u_1 \xi u_2 \xi \dots u_n - \partial_i u_1 \xi u_2 \xi \dots u_n x_i]$$

if the monomial starts with ξ . This is a differential in the tensor product of bar complexes dualised by $Hom(-, A_{cycl}^{\otimes N})$.

Theorem 7.1. *The homology of the complex $\zeta = (\zeta, d)$, $\zeta = \oplus \zeta_m^k$ is sitting in the diagonal $k = m$, consequently, the complex $\zeta = \oplus \zeta(l)$, $\zeta(l) = \bigoplus_{m-k=l} \zeta_m^k$ is pure, that is the homology is sitting only in the last place of the complex with respect to cohomological grading by ξ -degree.*

Proof. To prove this, we consider related complex $\widehat{\zeta}$ with the following differential:

$$d_{\widehat{\zeta}}(u_1 \xi u_2 \xi \dots u_n) = \sum (-1)^{g(u_1 \xi u_2 \xi \dots u_i)} u_1 \xi u_2 \xi \dots u_i \Delta u_{i+1} \dots u_n,$$

where $\Delta = \sum_{i=1}^r \partial_i x_i - x_i \partial_i$.

We first prove that homologies are sitting in one place in the complex $(\widehat{\zeta}, d_{\widehat{\zeta}})$ and then reduce the situation for (ζ, d) to this. After that simple argument of Lemma 7.12 shows that for the subcomplex $(\zeta, d) \subset (\widehat{\zeta}, d)$ the homology is also sitting in one place, if it is the case for $(\widehat{\zeta}, d)$.

Theorem 7.2. *The m -th slice of the complex $(\widehat{\zeta}, d_{\widehat{\zeta}})$,*

$$\widehat{\zeta}_m = \{u \in \mathbb{K}\langle \xi, x_i, \partial_i \rangle : w(u) = \deg_{\xi, \partial} u = m\}$$

for each $m \geq 2$ has non-trivial homology only in the last place with respect to cohomological grading by ξ -degree.

Proof. Induction by m . We will first need to consider the following particular case: the case of one ξ .

Lemma 7.3. *Consider the place in $(\widehat{\zeta}_m, d_{\widehat{\zeta}})$ for any $m \geq 2$, where $\deg_{\xi} u = 1$, $u \in \widehat{\zeta}_m$ (one but last place in the complexes $\widehat{\zeta}_m$). Then the homology in this place is trivial.*

Proof. Let $d_{\widehat{\zeta}}(u) = 0$ for $u \in \widehat{\zeta}$ with $\deg_{\xi} u = 1$. We show that $u \in \text{Im } d_{\widehat{\zeta}}$. Since $\deg_{\xi} u = 1$, u has the shape

$$u = \sum a_i \xi b_i, \quad a_i, b_i \in \mathbb{K}\langle x_1 \dots, x_r, \partial_1 \dots \partial_r \rangle.$$

Then

$$d_{\widehat{\zeta}} u = \sum (-1)^{g(a_j)} a_j \Delta b_j = 0.$$

Consider the ideal I in $\mathbb{K}\langle x_1 \dots, x_r, \partial_1 \dots \partial_r \rangle$ generated by Δ : $I = \text{Id}(\Delta)$.

We will use the following lemma and notions of Gröbner bases theory to describe when this above equality might happen.

Definition 7.4. Monomials $u, v \in \mathbb{K}\langle Y \rangle$ form an ambiguity (u, v) , if for some $w \in \mathbb{K}\langle Y \rangle$, $uw = vw$.

Suppose in $\mathbb{K}\langle Y \rangle$ we have fixed some well-ordering compatible with multiplication, for example, (left-to-right) degree-lexicographical ordering: we fix an order on variables, say $y_1 < \dots < y_n$, and compare monomials on Y lexicographically (from left to right). Polynomials are compared by their highest terms.

Definition 7.5. Let u, v be two monomials u, v , which are highest terms of the elements U, V from the ideal $I \in \mathbb{K}\langle Y \rangle$: $U = u + \tilde{u}, V = v + \tilde{v}$, where $\tilde{u}, \tilde{v} \in \mathbb{K}\langle Y \rangle$, smaller then $u, v \in \langle Y \rangle$ respectively: $\tilde{u} < u, \tilde{v} < v$. Then the resolution of the ambiguity (u, v) formed by monomials u, v is a polynomial $Uw - wV = \tilde{u}w - w\tilde{v}$, which is reducible to zero modulo generators of an ideal.

Definition 7.6. A reduction on $\mathbb{K}\langle Y \rangle$ modulo generators of an ideal $f_i = \bar{f}_i + \tilde{f}_i$, where $\bar{f}_i$ is a highest term of f_i , is a collection of linear maps defined on monomials as follows: $r_{u\bar{f}_i v}(w) = u\tilde{f}_i v$, if $w = u\bar{f}_i v$, and w otherwise.

The polynomial is called reducible to zero if there exists a sequence of reductions modulo generators of an ideal, which results in zero.

Lemma 7.7. (Version of Diamond Lemma [18]) *Let $A = \mathbb{K}\langle y_1 \dots, y_n \rangle / \text{Id}(r_1, \dots, r_m)$. Let M be the syzygy module of the relations $r_1, \dots, r_m$, that is M is the submodule of the free $\mathbb{K}\langle y_1 \dots, y_n \rangle$ -bimodule generated by the symbols $\widehat{r}_1, \dots, \widehat{r}_m$ consisting of $\sum f_i \widehat{r}_{s_i} g_i$ such that $\sum f_i r_{s_i} g_i = 0$.*

Then M is generated by trivial syzygies $\widehat{r}_i u r_j - r_i u \widehat{r}_j$ and the syzygies obtained by resolutions of ambiguities between highest terms of relations (with respect to some ordering).

Let us fix the ordering $\partial_1 > \partial_2 > \dots > x_1 > x_2 > \dots$. Then the leading term of the polynomial Δ is $\partial_1 x_1$. It does not produce any ambiguities. Hence by Lemma 7.7 (version of Diamond Lemma), the corresponding syzygy module M is generated by trivial syzygies, and therefore

$$(*) \quad \sum a_j \widehat{\Delta} b_j = \sum u_k (\widehat{\Delta} v_k \Delta - \Delta v_k \widehat{\Delta}) w_k$$

After we know this we can construct an element

$$g = \sum \gamma_k u_k \xi v_k \xi w_k$$

where $\gamma_k \in \mathbb{C}$ are chosen in such a way that in the following sum all summands have positive signs

$$d_{\widehat{\zeta}}(g) = \sum (u_k \xi v_k \Delta w_k - u_k \Delta v_k \xi w_k)$$

We can see then that the latter expression is just the same as the above formula (*) with $\widehat{\Delta}$ substituted by ξ , hence

$$\sum (u_k \xi v_k \Delta w_k - u_k \Delta v_k \xi w_k) = \sum a_j \xi b_j = u.$$

And we finally have

$$d_{\widehat{\zeta}}(g) = \sum a_j \xi b_j = u.$$

□

To continue the proof of Theorem 7.2, we need a basis of induction. So we prove that in the complex $(\widehat{\zeta}_2, d_{\widehat{\zeta}})$

$$0 \rightarrow \dots \xi \dots \xi \dots \rightarrow \dots \xi \dots \partial \dots \rightarrow \dots \partial \dots \partial \dots \rightarrow 0$$

the homology is sitting in the last place.

Since we already have Lemma 7.3, which deals with the case of one ξ it is equivalent to proving exactness only in one term, where ξ -degree is equal to two. That is, we need to show that if $\deg_{\xi} u = 2$, $\deg_{\partial_i} u = 0$, and $d_{\widehat{\zeta}}(u) = 0$, then $u = 0$. Write $u = \xi u_0 + v$, where v does not have ξ on the first position. Then we have

$$0 = d_{\widehat{\zeta}}(u) = \xi d_{\widehat{\zeta}}(u_0) + \Delta u_0 + d_{\widehat{\zeta}}(v).$$

The only term starting with ξ is $\xi d_{\widehat{\zeta}}(u_0)$, so $d_{\widehat{\zeta}}(u_0) = 0$. Since $\deg_{\xi} u_0 = 1$, we are in situation of Lemma 7.3 and $u_0 = d_{\widehat{\zeta}}(v_0)$. Since u_0 is free from ∂_i , we have $u_0 = 0$. Thus $u = v = \sum x_i u_i$ and $0 = d_{\widehat{\zeta}}(u) = \sum x_i d_{\widehat{\zeta}}(u_i)$ implies $d_{\widehat{\zeta}}(u_i) = 0$. Applying the same argument to u_i repeatedly, we arrive at $u = 0$, as required.

Step of induction in the proof of Theorem 7.2. Let $\deg_{\xi, \partial_i} u = m$ and u is homogeneous with respect to ξ as well as with respect to x_i, ξ, ∂_i and u is not in the last term of the complex: $\deg_{\xi} u \geq 1$. We need to show that $u \in \text{Im}(d_{\widehat{\zeta}})$ provided $d_{\widehat{\zeta}}(u) = 0$. We present $u = \xi u_0 + v$, where v is not starting from ξ . Then

$$d_{\widehat{\zeta}}(u) = \Delta u_0 + \xi d_{\widehat{\zeta}}(u_0) + d_{\widehat{\zeta}}(v) = 0.$$

The only term starting with ξ can not cancel with anything, so $d_{\widehat{\zeta}}(u_0) = 0$. Now $\deg_{\xi, \partial_i}(u_0) = m-1$. By induction hypothesis and Lemma 7.3 (if $m = 2$), $u_0 = d_{\widehat{\zeta}}(w)$. Consider $d_{\widehat{\zeta}}(\xi w) = \Delta w + \xi d_{\widehat{\zeta}}(w)$. Then

$$u' = u - d_{\widehat{\zeta}}(\xi w) = \xi u_0 + v - \Delta w - \xi d_{\widehat{\zeta}}(w) = v - \Delta w.$$

Thus u' equals u modulo $\text{Im } d_{\widehat{\zeta}}$ and does not have ξ in the first position:

$$u' = \sum x_j \xi u_j + \sum \partial_j \xi v_j + v,$$

where ξ is absent from v in the first two positions. Then

$$0 = d_{\bar{\zeta}}(u') = \sum x_j \Delta u_j - \sum \partial_j \Delta v_j + \sum x_j \xi d_{\bar{\zeta}}(u_j) + \sum \partial_j \xi d_{\bar{\zeta}}(v_j) + d_{\bar{\zeta}}(v).$$

Considering terms with ξ in the second position, we deduce $d_{\bar{\zeta}}(u_j) = d_{\bar{\zeta}}(v_j) = 0$ for all j . By the induction hypothesis $u_j = d_{\bar{\zeta}}(w_j)$ and $v_j = d_{\bar{\zeta}}(s_j)$. Now u'' equals u' modulo $\text{Im} d_{\bar{\zeta}}$ and u'' has no ξ in the first two positions, where

$$u'' = u - d_{\bar{\zeta}}(\sum x_j \xi w_j + \sum \partial_j \xi s_j).$$

After repeating this procedure, at the end we get $u = t\xi^m$ modulo $\text{Im} d_{\bar{\zeta}}$, where $\deg_{\xi} t = 0$. Now $0 = d_{\bar{\zeta}}(t\xi^m) = td_{\bar{\zeta}}(\xi^m)$ and therefore $t = 0$ since $d_{\bar{\zeta}}(\xi^m) = \Delta \xi^{m-1} + \dots + \xi^{m-1} \Delta \neq 0$. Hence $u \in \text{Im} d_{\bar{\zeta}}$. $\square$

Now we prove the theorem for $(\bar{\zeta}, d)$.

Theorem 7.8. *The m -th slice of the complex $(\bar{\zeta}, d)$,*

$$\bar{\zeta}_m = \{u \in \mathbb{K}\langle \xi, x_i, \partial_i \rangle : w(u) = \deg_{\xi, \partial} u = m\}$$

for each $m \geq 2$ has non-trivial homology only in the last place with respect to cohomological grading by ξ -degree.

Proof. First we need preliminary exactness result for the case of one ξ .

Lemma 7.9. *Consider the place in $(\bar{\zeta}_m, d)$ for any $m \geq 2$, where $\deg_{\xi} u = 1$, $u \in \bar{\zeta}_m$ (one but last place in the complex). Then the homology in this place is trivial.*

Proof. Let $u \in \bar{\zeta}$ be such that $\deg_{\xi} u = 1$, $\deg_{\xi, \partial_i} u \geq 2$ and $d_{\bar{\zeta}}(u) = 0$. We have to show that $u \in \text{Im}(d)$. Write

$$u = \xi u_0 + \sum a_i \xi b_i, \quad a_i \neq \text{const.}$$

Then

$$0 = d(u) = \sum \partial_i [x_i, u_0] + \sum (-1)^{\sigma} a_i \Delta b_i.$$

Thus the following equality holds in $A = \mathbb{K}\langle x_1 \dots, x_r, \partial_1 \dots \partial_r \rangle / \text{Id}(\Delta)$:

$$0 = d(u) = \sum \partial_i [x_i, u_0]. \tag{7.1}$$

Lemma 7.10. *The equality $\sum \partial_i [x_i, u] = 0$ in A implies $[x_i, u] = 0$ in A for any i .*

Proof. Let us consider ordering $x_1 > x_2 > \dots > \partial_1, \partial_2 > \dots$, then Δ forms a Gröbner basis. Take a normal form $N([x_i, u]) \in \mathbb{K}\langle x_1 \dots, x_r, \partial_1 \dots \partial_r \rangle / \mathbb{K}\langle XD \rangle$ with respect to the Gröbner basis of the ideal $\text{Id}(\Delta)$. In other words, we present the element $[x_i, u] \in \mathbb{K}\langle x_1 \dots, x_r, \partial_1 \dots \partial_r \rangle /$ as a sum of monomials which does not contain $x_1 \partial_1$ as a submonomial. Then element $N(\sum \partial_i [x_i, u]) = \sum \partial_i N([x_i, u]) = 0$ in $\mathbb{K}\langle XD \rangle$, hence $N[x_i, u] = 0$ in $\mathbb{K}\langle XD \rangle$, which means $[x_i, u] = 0$ in A . $\square$

Lemma 7.11. *(Centralizer) If in $A = \mathbb{K}\langle x_1 \dots, x_r, \partial_1 \dots \partial_r \rangle / \text{Id}(\Delta)$, $[u, x_i] = 0$ for all i , then $u \in \mathbb{K}$.*

Proof. Fix the ordering $\partial_1 > \partial_2 > \dots > x_1 > x_2 > \dots$. The highest term of Δ is $\partial_1 x_1$. Then the set N of corresponding Normal words (those which do not contain $\partial_1 x_1$) is closed under multiplication by x_2 on either side: $x_2 N \subset N$ and $N x_2 \subset N$.

Let $u \in A$ and $[u, x_i] = 0$ for all i . As every element of A , u can be written as a linear combination of normal words: $u = \sum c_j w_j$, where $w_j \in N$ are pairwise distinct and c_j are non-zero constants. Then $0 = [u, x_2] = \sum c_j (w_j x_2 - x_2 w_j)$. Since $w_j x_2, x_2 w_j \in N$, the last equality holds if and only if it holds in the free algebra. Hence $\sum c_j w_j$ commutes with x_2 in the free algebra and therefore $u \in \mathbb{K}[x_2]$. The same holds for any other x_j , $j \neq 1$ (they enter the game symmetrically) and therefore u is in the intersection of $\mathbb{K}[x_j]$ as subalgebras of A . Since this intersection is $\mathbb{K}$, $u \in \mathbb{K}$. $\square$

From (7.1), $[x_j, u_0] = 0$ in A for all i , according to Lemma 7.10. By the centralizer lemma u_0 is a constant in A . Since $m \geq 2$, $u_0 = 0$ in A . Hence

$$u_0 = \sum s_i \Delta t_i$$

in the free algebra. Thus

$$u = \sum \xi s_i \Delta t_i + \sum a_i \xi b_i.$$

Now we substitute u with $u' = u \pmod{\text{Im}(d)}$, where

$$u' = u - d\left(\sum (-1)^{\deg_{\partial} s_i} \xi s_i \xi t_i\right).$$

After cancelations, we get

$$u' = \sum a_i \xi b_i - \sum_{i,j} (-1)^{\sigma} \partial_j [x_j, s_i \xi t_i]$$

and therefore u' has no terms starting with ξ . Thus

$$u' = \sum \partial_i u_i$$

and we fall into the situation of the differential $d_{\hat{\zeta}}$ on the complex $\hat{\zeta}$:

$$d(u') = \sum \partial_i d_{\hat{\zeta}}(u_i) \iff d_{\hat{\zeta}}(u_i) = 0.$$

By Theorem 7.2, $u_i = d_{\hat{\zeta}}(w_i)$ and

$$u' = \sum \partial_i d_{\hat{\zeta}}(w_i) = d(-\sum \partial_i w_i),$$

which yields that u' and therefore u belongs to $\text{Im } d$. $\square$

Now let $\deg_{\xi} u \geq 2$. Suppose $du = 0$. We will show that $u \in \text{Im } d$. As before present it as $u = \xi u_0 + v$, where v does not start with ξ . Then $0 = du = \xi d_{\hat{\zeta}} u_0 + v'$, where v' does not start with ξ , hence $d_{\hat{\zeta}} u_0 = 0$. By Theorem 7.2 $u_0 = d_{\hat{\zeta}} s$ for some s . Thus take $u' = u - d(\xi s) = u - \xi d_{\hat{\zeta}} s - \dots = \xi d_{\hat{\zeta}} s + v - \xi d_{\hat{\zeta}} s - \dots$, and we have a presentation of u modulo the ideal $\text{Im } d$ as an element with no ξ at the first position: $u = \sum \partial_i u_i$. Thus, $du = d_{\hat{\zeta}} u$ and we can use Theorem 7.2 to ensure that $u \in \text{Im } d$. Indeed, since $0 = du = d_{\hat{\zeta}} u$ and $d_{\hat{\zeta}} u = -\sum \partial_i d_{\hat{\zeta}} u_i$, $d_{\hat{\zeta}} u_i = 0$ for all i . Since $\deg_{\xi} u_i \geq 1$, by Theorem 7.2 we have $u_i = d_{\hat{\zeta}}(w_i)$ for some w_i . Thus $u = \sum \partial_i u_i = -\sum \partial_i d_{\hat{\zeta}} w_i = d(\sum \partial_i w_i)$. The latter equality $d(\sum \partial_i w_i) = -\sum \partial_i d_{\hat{\zeta}} w_i$ holds because $\partial_i w_i$ not starting with ξ . So $u \in \text{Im } d$, and this completes the proof of Theorem 7.8. $\square$

Lemma 7.12. *If the complex $(\bar{\zeta}, d)$ is homologically pure, and homology is sitting in the last place w.r.t cohomological grading by ξ -degree, the same is true for the complex (ζ, d) .*

Proof. The statement about ζ follows from the fact the cyclization of the complex commutes with the differential in our case. This in turn deduced from the fact that the differential, given precisely at the beginning of this section obviously commute with $\mathbb{Z}_N$ action. $\square$

This lemma together with Theorem 7.8 completes the proof of Theorem 7.1. $\square$

7.2 Formality

The important consequence of the result on the homological purity of the higher cyclic Hochschild complex is that we derive formality for these complexes in L_∞ sense [12]. Various aspects of formality have been studied extensively (for example [5, 1, 2, 23, 20]), some of them are famously difficult.

Definition 7.13. The complex (C, d) is formal if it is quasi-isomorphic to its cohomologies $(H^\bullet C, 0)$, considered with zero differential, as L_∞ -algebra.

Theorem 7.14. *The higher cyclic Hochschild complex $C = \prod_N C_{cycl}^{(N)}(A)$ is formal.*

Proof. Remind that in the higher cyclic Hochschild complex $C = \prod_N C_{cycl}^{(N)}(A)$ we have the following grading and the subcomplex ζ quasi-isomorphic to this complex is situated with respect to this grading in the following way: $C = \oplus_{i \in \mathbb{Z}} C_i$, where i is a number of inputs minus number of outputs of corresponding operation. $\zeta \subseteq C$ in such a way that $\zeta_0 \subseteq C_0$ and ζ is $0 \rightarrow \dots \rightarrow \zeta_0 \rightarrow 0$. Hence our main Theorem 7.1 ensures that the homology of C is sitting in the zero place of the grading. Let us consider the group action on C induced by scaling, namely, $\mathbb{C}^*$ acts by $\lambda(u) = \lambda^m u$ for $u \in C_m$. This means that the action uniquely defines the grading.

Now consider $(H^\bullet C, \infty)$, the L_∞ -structure on the homologies obtained by the homotopy transfer of Kadeishvili [9], constructive description of which is given in [11], one can find explanations also in [24]. Since we had a reductive group action on $(C, 0)$ this action can be pulled through to $(H^\bullet C, \infty)$ and so will be compatible with the new L_∞ -structure on $H^\bullet C$ again. Thus the grading on $(H^\bullet C, \infty)$, being defined by this action, is also natural, i.e. only zero component of it will be nontrivial.

Obviously, if there is only one component in the grading of L_∞ -algebra, only one multiplication from L_∞ -structure can be non-zero. Since we shown that homology $(H^\bullet C, \infty)$ is sitting in zero component only, and we are using convention where binary multiplication in infinity structure has degree zero, only multiplication m_2 can be present. Thus in the L_∞ -structure of $(H^\bullet C, \infty)$ $m_n = 0$ for $n \geq 3$, and this implies formality. Indeed, for formality we need to show that (C, d) is quasi-isomorphic to $(H^\bullet C, 0)$. Since for the L_∞ -structure it is always true that (C, d) is qiso to $(H^\bullet C, \infty)$, it is enough to show that $(H^\bullet C, 0)$ is qiso to $(H^\bullet C, \infty)$, and this is the case when $m_n = 0, n \geq 3$. $\square$

References

- [1] M.Bordemann, O.Elchinger, S.Gutt, A.Makhlouf, *L_∞ -Formality check for the Hochschild complex of certain universal enveloping algebras*, arxiv1807.0308
- [2] M.Bordemann, A.Makhlouf, *Formality and deformations of universal enveloping algebras*, Int.J.Theor.Phys. **47** (2008), 311-332.

- [3] C. Brav, T. Dyckerhoff, *Relative Calabi-Yau structures*, arXiv:1606.00619
- [4] J. Cuntz, D. Quillen, *Algebra extensions and nonsingularity*, Journal of AMS, **8** (1995), N2, 251–289
- [5] Deligne P., Griffiths P., Morgan J., Sullivan D., *The real homotopy theory of Kaehler manifolds*, Invent. Math., **29** (1975), p. 245–274.
- [6] M.Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math. **78** (1964), 59–103.
- [7] N.Iyudu, *Examples of Pre-Calabi-Yau algebras, associated operads, and homology*, preprint IHES M/17/02 (2017).
- [8] N.Iyudu, M.Kontsevich *Pre-Calabi-Yau algebras and double Poisson structures*, preprint IHES M/18/04 (2018).
- [9] T. V. Kadeishvili, The algebraic structure in the homology of an A_∞ -algebra, Soobshch. Akad. Nauk Gruzin. SSR 108(1982), no. 2, 249–252 (1983).
- [10] M.Kontsevich, Y.Soibelman, *Notes on A_∞ -algebras, A_∞ -categories and non-commutative geometry*, ArXiv MathRA/0606241 (2006)
- [11] M.Kontsevich, Y.Soibelman, *Homological mirror symmetry and torus fibrations*, arxiv:math/0011041 (2001)
- [12] M.Kontsevich, *Deformation quantization of Poisson manifolds, I*, Letters Math.Physics **66**(2003), no 3, 157–216.
- [13] M.Kontsevich, *Formal non-commutative symplectic geometry*, in 'The Gelfand mathematical Seminars 1990–1992', Birkhäuser, (1993), 173–187.
- [14] M.Kontsevich, *Weak Calabi-Yau algebras*, Conference on Homological Mirror Symmetry, Miami, 2013
- [15] M.Kontsevich, Y.Vlassopoulos, *Pre-Calabi-Yau algebras and topological quantum field theories*, preprint.
- [16] M.Kontsevich, A.Rosenberg, *Noncommutative smooth spaces*, in 'The Gelfand mathematical Seminars 1996–1999', Birkhäuser, (1993), 85–108.
- [17] J-L.Loday, B.Valette, *Algebraic operads*, Springer 2013
- [18] T.Mora, *Seven variations on standard bases*, Proceedings (1988)
- [19] P.Seidel, *Fukaya A_∞ structures associated to Lefschetz fibrations I*, ArXiv MathSG/0912.3932v2
- [20] B.Shoikhet, *Tsygan formality and Duflo formula*, Math.Res.Lett, **10** (2003), 763–775.
- [21] J.Stasheff, *J. Homotopy associativity of H-spaces I, II*, Trans. Amer. Math. Soc. **108**(1963), 275–312.
- [22] T.Tradler and M.Zeinalian, *Algebraic string operations*, K-Theory **38** (2007), 59–82
- [23] B.Tsygan, *Formality conjecture for chains*, AMS Trans. ser. **194** (1998), 261–274
- [24] B.Valette, *Algebra-homotopy-operad*, arxiv:1202.3245
- [25] M.Van den Bergh, *Double Poisson algebras*, Trans. Amer. Math. Soc. **360** (2008), 5711–5769.

NATALIA IYUDU, SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, JCMB, THE KING'S BUILDINGS, EDINBURGH, SCOTLAND EH9 3FD

E-MAIL ADDRESS: N.IYUDU@ED.AC.UK, IYUDU@IHES.FR

MAXIM KONTSEVICH, INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, 35 ROUTE DE CHARTRES, F - 91440 BURES-SUR-YVETTE

E-MAIL ADDRESS: MAXIM@IHES.FR