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GENERALISED KOSTKA-FOULKES POLYNOMIALS AND COHOMOLOGY OF LINE BUNDLES ON HOMOGENEOUS VECTOR BUNDLES

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CONTENTS

INTRODUCTION

Let G be a semisimple algebraic group with Lie algebra $\mathfrak{g}$. We consider generalisations of Lusztig's q-analogue of weight multiplicity. Fix a maximal torus $T \subset G$. Let m_λ^μ be the multiplicity of weight μ in a simple G-module V_λ with highest weight λ. Lusztig's q-analogues $\mathfrak{m}_\lambda^\mu(q)$ (also known as Kostka-Foulkes polynomials for the root system of G) are certain polynomials in q such that $\mathfrak{m}_\lambda^\mu(1) = m_\lambda^\mu$. A recent survey of their properties, with an eye towards combinatorics, is given in [19]. These polynomials arise in numerous problems of representation theory, geometry, and combinatorics. Work of Lusztig [16] and Kato [12] shows that, for λ and μ dominant, $\mathfrak{m}_\lambda^\mu(q)$ are connected with certain Kazhdan-Lusztig polynomials for the affine Weyl group associated with G. To define $\mathfrak{m}_\lambda^\mu(q)$, one first considers a q-analogue of Kostant's partition function, $\mathcal{P}$. It is conceivable to replace the set of positive roots, Δ^+, occurring in the definition of $\mathcal{P}$ with an arbitrary finite multiset Ψ in the character group $\mathfrak{X}$ of T. If the elements of Ψ belong to an open half-space of $\mathfrak{X} \otimes \mathbb{Q}$ (this is our *first* hypothesis on Ψ), then we still obtain certain polynomials $\mathfrak{m}_{\lambda,\Psi}^\mu(q)$. We always assume that λ is dominant, whereas $\mu \in \mathfrak{X}$ can be arbitrary. In this article, we are interested in the non-negativity problem for the coefficients of $\mathfrak{m}_{\lambda,\Psi}^\mu(q)$. For Lusztig's q-analogues, this problem has been considered by Broer. He proved that $\mathfrak{m}_\lambda^\mu(q)$

has non-negative coefficients for any $\lambda \in \mathfrak{X}_+$ if and only if $(\mu, \alpha^\vee) \geqslant -1$ for all $\alpha \in \Delta^+$ (see [1, Theorem 2.4] and [4, Prop. 2(iii)]).

Our first goal is to provide sufficient conditions for $\mathfrak{m}^{\mu}_{\lambda,\Psi}(q)$ to have non-negative coefficients. Let B be the Borel subgroup of G corresponding to Δ^+ (i.e., the roots of B are positive!) and $\mathfrak{X}_+$ the set of dominant weights. The *second* hypothesis is that Ψ is assumed to be the multiset of weights for a B-submodule N of a G-module V. Then $\mathfrak{m}^{\mu}_{\lambda,\Psi}(q)$ is said to be a *generalised Kostka-Foulkes polynomial*. Let $P \supset B$ be any parabolic subgroup normalising N and $G \times_P N$ the corresponding homogeneous vector bundle on G/P. We obtain a relation between the Euler characteristic of induced line bundles $\mathcal{L}$ on the $G \times_P N$ and generalised Kostka-Foulkes polynomials. Using the collapsing $G \times_P N \to G{\cdot}N \subset V$, we get a vanishing result for $H^i(G \times_P N, \mathcal{L})$, $i \geqslant 1$, and conclude that $\mathfrak{m}^{\mu}_{\lambda,\Psi}(q)$ has non-negative coefficients for all $\lambda \in \mathfrak{X}_+$ if μ is sufficiently large. An explicit lower bound for μ is also given, see Section 3. This approach is based on the Grauert-Riemenschneider vanishing theorem. We also notice that Broer's formula for $\frac{d}{dq}\mathfrak{m}^{\mu}_{\lambda}(q)$ [3] can be generalised to $\mathfrak{m}^{\mu}_{\lambda,\Psi}(q)$. The most natural examples of generalised Kostka-Foulkes polynomials occur if $\Psi \subset \Delta^+$. For instance, one can take N to be a B-stable ideal in $\mathsf{Lie}(B,B) \subset \mathfrak{g}$.

Our second goal is to study in details the special case in which $\Psi = \Delta^+_s$, the set of short positive roots. The required B-submodule, $V^+_{\bar\theta}$, lies in $V_{\bar\theta}$, where $\bar\theta$ is the short dominant root. The polynomials $\overline{\mathfrak{m}}^{\mu}_{\lambda}(q) := \mathfrak{m}^{\mu}_{\lambda,\Delta^+_s}(q)$ are said to be *short q-analogues*. The numbers $\overline{\mathfrak{m}}^{\mu}_{\lambda}(1)$ appeared already in work of Heckman [8], and a geometric interpretation of $\overline{\mathfrak{m}}^{0}_{\lambda}(q)$ given in [24] shows that $\overline{\mathfrak{m}}^{0}_{\lambda}(q)$ have non-negative coefficients. Let Δ^+_l be the set of long positive roots, W_l the (normal) subgroup of W generated by all s_α ($\alpha \in \Delta^+_l$), and ρ_l the half-sum of the long positive roots. Approach of Section 3 enables us to prove that $\overline{\mathfrak{m}}^{\mu}_{\lambda}(q)$ has nonnegative coefficients whenever $\mu + \rho_l \in \mathfrak{X}_+$ (Cor. 4.3). But to obtain exhaustive results, we take another path. We consider the *shifted* (= dot) action of W_l on $\mathfrak{X}$, $(w, \mu) \mapsto w \odot \mu = w(\mu + \rho_l) - \rho_l$, and show that $\overline{\mathfrak{m}}^{w\odot\mu}_{\lambda}(q) = (-1)^{\ell(w)}\overline{\mathfrak{m}}^{\mu}_{\lambda}(q)$. Therefore $\overline{\mathfrak{m}}^{\mu}_{\lambda}(q) \equiv 0$ if μ is not regular relative to the shifted W_l-action, and it suffices to consider $\overline{\mathfrak{m}}^{\mu}_{\lambda}(q)$ only for μ that are dominant with respect to Δ^+_l. For a Δ^+_l-dominant μ, we prove that $\overline{\mathfrak{m}}^{\mu}_{\lambda}(q)$ has non-negative coefficients for all $\lambda \in \mathfrak{X}_+$ if and only if $(\mu, \alpha^\vee) \geqslant -1$ for all $\alpha \in \Delta^+_s$, see Theorem 4.10. This is an extension of Broer's results in [1, Sect. 2]. Again, this stems from a careful study of cohomology of line bundles on $G \times_B V^+_{\bar\theta}$. In these considerations, it is important that W is a semi-direct product $W(\Pi_s) \ltimes W_l$, where the first group is generated by the short simple reflections. Modifying approach of R. Gupta [6], we define analogues of Hall-Littlewood polynomials (Section 5). These polynomials in q, denoted $\overline{P}_\lambda(q)$, are indexed by $\lambda \in \mathfrak{X}_+$ and form a $\mathbb{Z}$-basis for the q-extended character ring $\mathbf{\Lambda}[q]$ of G. Let χ_λ be the character of V_λ and H the connected semisimple subgroup of G whose root system is Δ_l. The polynomials $\overline{P}_\lambda(q)$ interpolate between χ_λ (at $q = 0$) and a certain sum of irreducible characters of H (at $q = 1$). We obtain some orthogonality relations for $\overline{P}_\lambda(q)$ and show that $\chi_\lambda = \sum_{\mu\in\mathfrak{X}_+} \overline{\mathfrak{m}}_\lambda^\mu(q)\overline{P}_\mu(q)$. Moreover, the whole theory developed by R. Gupta in [6, 7] can be extended to this setting. For instance, we prove a version of Kato's identity [12, 1.3] and point out a scalar product in $\mathbf{\Lambda}[q]$ such that $\{\overline{P}_\lambda(q)\}_{\lambda\in\mathfrak{X}_+}$ to be an orthogonal basis. In a sense, the reason for such an extension is that $G{\cdot}V_{\bar\theta} =: \mathfrak{N}(V_{\bar\theta})$ is the null-cone in $V_{\bar\theta}$, and, as well as the nilpotent cone $\mathfrak{N} \subset \mathfrak{g}$, this variety is an irreducible normal complete intersection. On the other hand, Theorem 4.10 yields vanishing of higher cohomology of the structure sheaf $\mathcal{O}_{G\times_B V_{\bar\theta}^+}$, and, together with [15], this implies that $\mathfrak{N}(V_{\bar\theta})$ has only rational singularities.

We conjecture that if μ satisfies vanishing conditions of Theorem 4.10, then $\overline{\mathfrak{m}}_\lambda^\mu(q)$ can be interpreted as the "jump polynomial" associated with a filtration of a subspace of V_λ^μ, see Subsection 6.3. This is inspired by [5].

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1. Notation

Let G be a connected semisimple algebraic group of rank r, with a fixed Borel subgroup B and a maximal torus $T \subset B$. The corresponding triangular decomposition of $\mathfrak{g} = \mathsf{Lie}(G)$ is $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u}$ and $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$. The character group of T is denoted by $\mathfrak{X}$. Let Δ be the root system of (G, T). Then B determines the set of positive roots Δ^+ and the monoid of dominant weights $\mathfrak{X}_+$.

- Π is the set of simple roots in Δ^+;
- $\varphi_1, \dots, \varphi_r$ are the fundamental weights in $\mathfrak{X}_+$.

Write W for the Weyl group and s_α for the reflection corresponding to $\alpha \in \Delta^+$. Set $\mathsf{N}(w) = \{\alpha \in \Delta^+ \mid w\alpha \in -\Delta^+\}$ and $\varepsilon(w) = (-1)^{\ell(w)}$, where $\ell(w) = \#\mathsf{N}(w)$ is the usual length function on W. For $\mu \in \mathfrak{X}$, let μ^+ denote the unique dominant element in $W\mu$. We fix a W-invariant scalar product $(\ ,\ )$ on $\mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{Q}$. As usual, $\alpha^\vee = 2\alpha/(\alpha,\alpha)$ for $\alpha \in \Delta$. For any $\lambda \in \mathfrak{X}_+$, we choose a simple highest weight module V_λ; V_λ^μ is the μ-weight space in V_λ and $m_\lambda^\mu = \dim V_\lambda^\mu$.

We consider two partial orders in $\mathfrak{X}$. For $\mu, \nu \in \mathfrak{X}$,

- the *root order* is defined by letting $\mu \preccurlyeq \nu$ if and only if $\nu - \mu$ lies in the monoid generated by Δ^+; notation $\mu \prec \nu$ means that $\mu \preccurlyeq \nu$ and $\mu \neq \nu$;
- the *dominant order* is defined by letting $\mu \lessdot \nu$ if and only if $\nu - \mu \in \mathfrak{X}_+$.

If Ψ is a finite multiset in $\mathfrak{X}$, then $|\Psi|$ is the sum of all elements of Ψ (with respective multiplicities). Recall that $|\Delta^+|/2 = \varphi_1 + \ldots + \varphi_r$, and this quantitiy is denoted by ρ.

Let P be a parabolic subgroup of G. For a P-module N, let $G \times_P N$ denote the homogeneous G-vector bundle on G/P whose fibre over $\{P\} \in G/P$ is N; we write $\mathcal{L}_{G/P}(V)$ for the locally free $\mathcal{O}_{G/P}$-module of its sections. If N is a submodule of a G-module, then the natural morphism $f : G \times_P N \to G{\cdot}N$ is projective and G-equivariant. It is a *collapsing* in the sense of Kempf [13]. Recall that $G{\cdot}N$ is a closed subvariety of V, since N is P-stable. If $\dim G \times_P N = \dim G{\cdot}N$, then f is said to be *generically finite*. If N' is another P-module, then $G \times_P (N \oplus N')$ is a vector bundle on $G \times_P N$ with sheaf of sections $\mathcal{L}_{G\times_P N}(N')$.

For any graded G-module $\mathcal{C} = \oplus_j \mathcal{C}_j$ with $\dim \mathcal{C}_j < \infty$, its *$G$-Hilbert series* is defined by

$$\mathcal{H}_G(\mathcal{C}; q) = \sum_j \sum_{\lambda\in\mathfrak{X}_+} \dim \mathrm{Hom}_G(V_\lambda, \mathcal{C}_j)e^\lambda q^j \in \mathbb{Z}[\mathfrak{X}][[q]].$$

2. Main definitions and first properties

Let V be a finite-dimensional rational G-module and N a P-stable subspace of V. We assume that the T-weights occurring in N lie in an open half-space of $\mathfrak{X}\otimes_{\mathbb{Z}}\mathbb{Q}$. (This hypothesis implies that all $v \in N$ are unstable vectors in the sense of Geometric Invariant Theory.) Counting each T-weight according to its multiplicity in N, we get a finite multiset Ψ in $\mathfrak{X}$. The *generalised partition function*, $\mathcal{P}_\Psi$, is defined by the series $\dfrac{1}{\prod_{\alpha\in\Psi}(1-e^\alpha)} = \sum_\nu \mathcal{P}_\Psi(\nu)e^\nu$. Accordingly, its q-analogue is defined by

$$\frac{1}{\prod_{\alpha\in\Psi}(1-qe^\alpha)} = \sum_\nu \mathcal{P}_{\Psi,q}(\nu)e^\nu.$$

In view of our assumption on N, the numbers $\mathcal{P}_\Psi(\nu)$ are well-defined, and $\mathcal{P}_{\Psi,q}(\nu)$ is a polynomial in q, with non-negative integer coefficients. Clearly, $\mathcal{P}_{\Psi,q}(\nu)$ counts the "graded occurrences" of ν in the symmetric algebra $\mathcal{S}^\bullet(N)$. That is, $[q^j]\mathcal{P}_{\Psi,q}(\nu) = \dim(S^jN)^\nu$.

For $\lambda \in \mathfrak{X}_+$ and $\mu \in \mathfrak{X}$, define the polynomials $\mathfrak{m}^\mu_{\lambda,\Psi}(q)$ by

$$\mathfrak{m}^\mu_{\lambda,\Psi}(q) = \sum_{w\in W} \varepsilon(w)\mathcal{P}_{\Psi,q}(w(\lambda+\rho)-(\mu+\rho)). \tag{2.1}$$

This definition makes sense for any multiset Ψ. But we require that our Ψ to be always the multiset of weights of a P-submodule of a G-module, since we are going to exploit geometric methods.

For $N = \mathfrak{u} \subset \mathfrak{g}$ and $\Psi = \Delta^+$, one obtains Lusztig's q-analogues of weight multiplicity [16] (= Kostka-Foulkes polynomials for Δ), and $\mathfrak{m}^\mu_{\lambda,\Delta^+}(1) = m^\mu_\lambda$. Therefore, $\mathfrak{m}^\mu_{\lambda,\Psi}(q)$ is said to be a (Ψ, q)*-analogue of weight multiplicity* or *generalised Kostka-Foulkes polynomial*. If $\Psi = \Delta^+$, we will omit the subscript Δ^+ in previous formulae.

As $\mathfrak{m}^\mu_{\lambda,\Psi}(q)$ is a polynomial in q, one might be interested in its derivative. For $\Psi = \Delta^+$, a nice formula for $\frac{d}{dq}\mathfrak{m}^\mu_\lambda(q)$ is found by Broer [3, p. 394]. We notice that his method works in general, and it is more natural to begin with a formula for the derivative of $\mathcal{P}_{\Psi,q}(\nu)$.

Theorem 2.1. $\dfrac{d}{dq}\mathcal{P}_{\Psi,q}(\nu) = \sum_{\gamma\in\Psi}\sum_{n\geqslant 1} q^{n-1}\mathcal{P}_{\Psi,q}(\nu - n\gamma).$

Proof. The derivative $\frac{d}{dq}\mathcal{P}_{\Psi,q}(\nu)$ equals the coefficient of t in the expansion of $\mathcal{P}_{\Psi,q+t}(\nu)$. Let the polynomials $\mathcal{R}_{n,\mu}(q)$ be defined by the generating function

$$\prod_{\alpha\in\Psi}\frac{1-qe^{\alpha}}{1-(q+t)e^{\alpha}} = \frac{\sum_{\nu}\mathcal{P}_{\Psi,q+t}(\nu)e^{\nu}}{\sum_{\nu}\mathcal{P}_{\Psi,q}(\nu)e^{\nu}} =: \sum_{\mu}\sum_{n\geqslant 0}\mathcal{R}_{n,\mu}(q)e^{\mu}t^{n}.$$

It is easy to compute these polynomials for $n = 0, 1$. First, taking $t = 0$, we obtain $\sum_{\mu}\mathcal{R}_{0,\mu}(q)e^{\mu} = 1$. Second, we have

$$\sum_{\mu}\mathcal{R}_{1,\mu}(q)e^{\mu} = \left[\prod_{\alpha\in\Psi}\frac{1-qe^{\alpha}}{1-(q+t)e^{\alpha}}\right]'_{t}|_{t=0} = \sum_{\alpha\in\Psi}\frac{e^{\alpha}}{1-qe^{\alpha}} = \sum_{\alpha\in\Psi}\sum_{n\geqslant 1}q^{n-1}e^{n\alpha}.$$

Hence $\mathcal{R}_{1,\mu}(q) = \begin{cases} q^{n-1} & \text{if } \mu = n\alpha, \alpha\in\Psi \\ 0, & \text{otherwise.}\end{cases}$

Next, $\sum_{\nu}\mathcal{P}_{\Psi,q+t}(\nu)e^{\nu} = \sum_{n,\mu,\gamma}\mathcal{R}_{n,\mu}(q)\mathcal{P}_{\Psi,q}(\gamma)e^{\mu+\gamma}t^{n}$. Hence

$$\mathcal{P}_{\Psi,q+t}(\nu)e^{\nu} = \sum_{n,\mu}\mathcal{R}_{n,\mu}(q)\mathcal{P}_{\Psi,q}(\nu-\mu)t^{n},$$

and extracting the coefficient of t we get the assertion. □

Corollary 2.2. $\dfrac{d}{dq}\mathfrak{m}^{\mu}_{\lambda,\Psi}(q) = \sum_{\gamma\in\Psi}\sum_{n\geqslant 1}q^{n-1}\mathfrak{m}^{\mu+n\gamma}_{\lambda,\Psi}(q).$

It would be nice to have a formula for the degree of these polynomials and necessary conditions for $\mathfrak{m}^{\mu}_{\lambda,\Psi}(q)$ to be nonzero. For Lusztig's q-analogues, it is easily seen that $\mathfrak{m}^{\mu}_{\lambda}(q) \neq 0$ if and only if $\mu \preccurlyeq \lambda$, and $\deg \mathfrak{m}^{\mu}_{\lambda}(q) = \mathsf{ht}(\lambda-\mu)$. However, if Ψ is arbitrary, i.e., there is no relation between Δ^{+} and Ψ, then it is impossible to compare the degrees of different summands in Equation (2.1). The only general assertion we can prove concerns the case in which $\Psi\subset\Delta^{+}$.

Lemma 2.3. *Suppose that $\Psi\subset\Delta^{+}$. Then $\mathfrak{m}^{\lambda}_{\lambda,\Psi}(q) = 1$ and if $\mathfrak{m}^{\mu}_{\lambda,\Psi}(q)\neq 0$, then $\mu\preccurlyeq\lambda$.*

Note that if $\mathfrak{m}^{\mu}_{\lambda,\Psi}(q) \neq 0$, then it is not necessarily true that $\lambda-\mu$ lies in the monoid generated by Ψ.

3. Cohomology of line bundles and generalised Kostka-Foulkes polynomials

3.1. **Statement of main results.** We assume that $P\supset B$ and choose a Levi subgroup $L\subset P$ such that $L\supset T$. Write $\mathfrak{n}$ for the nilpotent radical of $\mathfrak{p} = \mathsf{Lie}(P)$, and $\Delta(\mathfrak{n})$ for the roots of $\mathfrak{n}$; hence $\Delta(\mathfrak{n})\subset\Delta^{+}$. Let $\mathfrak{X}^{P}$ denote the character group of P. Obviously, $\mathfrak{X}^{P}$ is the character group of the central torus in L, and we may identify $\mathfrak{X}^P$ with a subgroup of $\mathfrak{X}$. Then $\mathfrak{X}^P_+ = \mathfrak{X}_+ \cap \mathfrak{X}^P$ is the monoid of P-dominant weights, i.e., the dominant weights λ such that P stabilises a nonzero line in V_λ. Let ρ_P be the sum of those fundamental weights that belong to $\mathfrak{X}^P_+$.

In this section, we prove the following two theorems:

Theorem 3.1. *Set $\boldsymbol{Z} = G \times_P N$. For $\mu \in \mathfrak{X}^P$, let $\mathcal{L}_{\boldsymbol{Z}}(\mu)^\star$ be the dual of the sheaf of sections of the line bundle $G \times_P (N \oplus \mathbb{C}_\mu) \to \boldsymbol{Z}$. Then*

- (i) *$H^i(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star) = 0$ for all $i \geqslant 1$ whenever $\mu \succcurlyeq \rho_P + |\Psi| - |\Delta(\mathfrak{n})|$.*
- (ii) *If the collapsing $\boldsymbol{Z} \to G{\cdot}N$ is generically finite, then $H^i(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star) = 0$ for all $i \geqslant 1$ whenever $\mu \succcurlyeq |\Psi| - |\Delta(\mathfrak{n})|$.*

Theorem 3.2. *Suppose N is P-stable and $\mu \in \mathfrak{X}^P$.*

- (i) *If $\mu \succcurlyeq \rho_P + |\Psi| - |\Delta(\mathfrak{n})|$, then $\mathfrak{m}^\mu_{\lambda,\Psi}(q)$ has non-negative coefficients for any $\lambda \in \mathfrak{X}_+$.*
- (ii) *If the collapsing $G \times_P N \to G{\cdot}N$ is generically finite, then $\mathfrak{m}^\mu_{\lambda,\Psi}(q)$ has non-negative coefficients for any $\lambda \in \mathfrak{X}_+$ whenever $\mu \succcurlyeq |\Psi| - |\Delta(\mathfrak{n})|$.*

(Note that $|\Psi|, |\Delta(\mathfrak{n})| \in \mathfrak{X}^P$. Hence both inequalities concern weights lying in $\mathfrak{X}^P$.)

Actually, Theorem 3.2 follows from Theorem 3.1 and a relation between (Ψ, q)-analogues and cohomology of line bundles, see Theorem 3.9 below. Such an approach to (Ψ, q)-analogues is inspired by work of Broer [1, 2].

3.2. **Algebraic-geometric facts.** For future reference, we recall some standard results in the form that we need below. Let U be the total space of a line bundle on an algebraic variety Z and $\pi : U \to Z$ be the corresponding projection. If $\mathcal{E}$ is a locally free $\mathcal{O}_Z$-module, then $\mathcal{E}^\star$ is its dual.

Lemma 3.3. *Let $\mathcal{F}$ be the sheaf of sections of π.*

- (i) *If $\mathcal{L}$ is a locally free $\mathcal{O}_Z$-module of finite type, then $\pi_*(\pi^*\mathcal{L}) = \bigoplus_{n\geqslant 0}(\mathcal{L} \otimes (\mathcal{F}^{\otimes n})^\star)$.*
- (ii) *If $\mathcal{G}$ is a quasi-coherent sheaf on U, then $H^i(U, \mathcal{G}) = H^i(Z, \pi_*\mathcal{G})$ for all i.*

Proof. (i) Use the "projection formula" and the equality $\pi_*(\mathcal{O}_U) = \bigoplus_{n\geqslant 0}(\mathcal{F}^{\otimes n})^\star$.
(ii) This is true because π is an affine morphism. □

Thus, vanishing of higher cohomology for $\pi^*\mathcal{L}$ will imply that for $\mathcal{L} \otimes (\mathcal{F}^{\otimes n})^\star$ for all $n \geqslant 0$. The following is a special case of the Grauert–Riemenschneider theorem in Kempf's version ([13, Theorem 4]):

Theorem 3.4. *Let ω_U denote the canonical bundle on U. Suppose there is a proper generically finite morphism $U \to X$ onto an affine variety X. Then $H^i(U, \omega_U) = 0$ for all $i \geqslant 1$.*

3.3. **Proof of Theorem 3.1.** Recall that N is a P-submodule of a G-module V, Ψ is the corresponding multiset of weights, and Ψ belongs to an open half-space of $\mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{Q}$. Our goal is to obtain a sufficient condition for vanishing of higher cohomology of line bundles on $\boldsymbol{Z} := G \times_P N$.

For $\mu \in \mathfrak{X}_+^P$, let $\mathbb{C}_\mu$ denote the corresponding one-dimensional P-module. Consider $\boldsymbol{U} = G \times_P (N \oplus \mathbb{C}_\mu)$ with projections $\pi : \boldsymbol{U} \to G \times_P N$ and $\kappa : \boldsymbol{U} \to G/P$. Then π makes $\boldsymbol{U}$ the total space of a line bundle on $\boldsymbol{Z}$. For simplicity, the sheaf of sections of this bundle is often denoted by $\mathcal{L}_{\boldsymbol{Z}}(\mu)$ in place of $\mathcal{L}_{\boldsymbol{Z}}(\mathbb{C}_\mu)$. Note that $\mathcal{L}_{\boldsymbol{Z}}(\mu)^\star = \mathcal{L}_{\boldsymbol{Z}}(-\mu)$. We regard $\mathbb{C}_\mu$ as the highest weight space in the G-module V_μ. Therefore $\boldsymbol{U}$ admits the collapsing into $V \oplus V_\mu$.

Since $\boldsymbol{U}$ is the total space of a G-linearised vector bundle on G/P, the canonical bundle $\omega_{\boldsymbol{U}}$ is a pull-back of a line bundle on G/P. The top exterior power of the cotangent space at $e * \tilde{n} \in \boldsymbol{U}$ ($e \in G$ is the identity and $\tilde{n} \in N \oplus \mathbb{C}_\mu$) is

$$\wedge^{top}(\mathfrak{g}/\mathfrak{p})^* \otimes \wedge^{top} N^* \otimes (\mathbb{C}_\mu)^* = \wedge^{top}\mathfrak{n} \otimes (\wedge^{top} N)^* \otimes (\mathbb{C}_\mu)^*.$$

The corresponding character of P is $\gamma - \mu$, where $\gamma := |\Delta(\mathfrak{n})| - |\Psi|$. Therefore

$$\omega_{\boldsymbol{U}} \simeq \kappa^*\big(\mathcal{L}_{G/P}(\mathbb{C}_{\gamma-\mu})\big) \simeq \pi^*\big(\mathcal{L}_{\boldsymbol{Z}}(\gamma - \mu)\big).$$

By Lemma 3.3, we obtain $\pi_*(\omega_{\boldsymbol{U}}) = \bigoplus_{n\geqslant 0} \mathcal{L}_{\boldsymbol{Z}}(\gamma - \mu) \otimes \mathcal{L}_{\boldsymbol{Z}}(n\mu)^\star$ and hence

$$H^i(\boldsymbol{U}, \omega_{\boldsymbol{U}}) = \bigoplus_{n\geqslant 0} H^i(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}((n{+}1)\mu{-}\gamma)^\star).$$

In order to apply Theorem 3.4, we need sufficient conditions for the collapsing

$$f_\mu : \boldsymbol{U} \to G{\cdot}(N \oplus \mathbb{C}_\mu)$$

to be generically finite. There are two possibilities now.

A) The collapsing $f : \boldsymbol{Z} \to G{\cdot}N$ is generically finite.
It is then easily seen that f_μ is generically finite for any $\mu \in \mathfrak{X}_+$. This yields the following vanishing result:

Proposition 3.5. *If $f_0 : \boldsymbol{Z} \to G{\cdot}N$ is generically finite and $\gamma = |\Delta(\mathfrak{n})| - |\Psi|$, then*

$$H^i(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}((n{+}1)\mu{-}\gamma)^\star) = 0$$

for any $\mu \in \mathfrak{X}_+^P$ and all $n \geqslant 0$, $i \geqslant 1$. In particular, taking $n = 0$ and letting $\nu = \mu - \gamma$, we obtain

$$H^i(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\nu)^\star) = 0 \ \text{ for all } i \geqslant 1$$

if $\nu \in \mathfrak{X}^P$ is such that $\nu \succcurlyeq |\Psi| - |\Delta(\mathfrak{n})|$.

B) The collapsing $f: \boldsymbol{Z} \to G{\cdot}N$ is <u>not</u> generically finite.
Here we have to correct the situation, i.e., choose μ such that f_μ to be generically finite.

Looking at the collapsing $f_\mu : G \times_P (N \oplus \mathbb{C}_\mu) \to G{\cdot}(N \oplus \mathbb{C}_\mu)$ the other way around, we notice that if $\psi_\mu : G \times_P \mathbb{C}_\mu \to G{\cdot}\mathbb{C}_\mu \subset V_\mu$ is generically finite, then so is f_μ. However, ψ_μ is generically finite (in fact, birational) if and only if $\mu \in \mathfrak{X}_+^P$ is a P-regular dominant weight, i.e., $\mu \succcurlyeq \rho_P$. Equivalently, $\mu = \tilde\mu + \rho_P$ for some $\tilde\mu \in \mathfrak{X}_+^P$.

This provides a weaker vanishing result that applies to arbitrary P-submodules.

Proposition 3.6. *Let N be an arbitrary P-submodule. If $\mu \in \mathfrak{X}_+^P$ and $\mu \succcurlyeq \rho_P$, then*

$$H^i(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}((n{+}1)\mu{-}\gamma)^\star) = 0$$

for all $n \geqslant 0$, $i \geqslant 1$. In particular, taking $n = 0$ and letting $\nu = \mu - \gamma$, we obtain

$$H^i(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\nu)^\star) = 0 \textit{ for all } i \geqslant 1$$

whenever $\nu \in \mathfrak{X}^P$ and $\nu \succcurlyeq \rho_P + |\Psi| - |\Delta(\mathfrak{n})|$.

Combining Propositions 3.5 and 3.6, we obtain Theorem 3.1.

Remark 3.7. The estimate in part B) is not optimal, because we do not actually need generic finiteness for ψ_μ. It can happen that both f and ψ_μ are not generically finite, while f_μ is. (See e.g. Theorem 4.2 below.)

3.4. **Proof of Theorem 3.2.** The cohomology groups of $\mathcal{L}_{\boldsymbol{Z}}(\mu) = \mathcal{L}_{G\times_P N}(\mu)$ have a natural structure of a graded G-module by

$$H^i(G \times_P N, \mathcal{L}_{G\times_P N}(\mu)) \simeq \bigoplus_{j=0}^{\infty} H^i(G/P, \mathcal{L}_{G/P}(\mathcal{S}^j N^* \otimes \mathbb{C}_\mu)),$$

where $\mathcal{S}^j N^*$ is the j-th symmetric power of the dual of N. Set $H^i(\mu) := H^i(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star)$. It is a graded G-module with

$$(H^i(\mu))_j = H^i(G/P, \mathcal{L}_{G/P}(\mathcal{S}^j N \otimes \mathbb{C}_\mu)^\star).$$

As $\dim(H^i(\mu))_j < \infty$, the G-Hilbert series of $H^i(\mu)$ is well-defined:

$$\mathcal{H}_G(H^i(\mu); q) = \sum_j \sum_{\lambda\in\mathfrak{X}_+} \dim \operatorname{Hom}_G(V_\lambda, (H^i(\mu))_j) e^\lambda q^j \in \mathbb{Z}[\mathfrak{X}][[q]].$$

We also need the non-graded version of functor $\mathcal{H}_G$. If M is a finite-dimensional G-module, then

$$\mathcal{H}_G(M) = \sum_{\lambda\in\mathfrak{X}_+} \dim \operatorname{Hom}_G(V_\lambda, M) e^\lambda \in \mathbb{Z}[\mathfrak{X}].$$

This extends to virtual G-modules by linearity.

Assume for a while that $P = B$, i.e., $\boldsymbol{Z} = G \times_B N$. By the Borel-Weil-Bott theorem for G/B, we have

$$H^i(G/B, \mathcal{L}_{G/B}(\mu)^\star) = \begin{cases} V_\nu^*, & \text{if } \nu = w(\mu+\rho) - \rho \in \mathfrak{X}_+ \text{ and } \ell(w) = i. \\ 0, & \text{otherwise.} \end{cases}$$

Using the non-graded functor $\mathcal{H}_G$, one can also write

$$\mathcal{H}_G\Big(\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\mu)^\star)\Big) = \begin{cases} \varepsilon(w) e^{\nu^*}, & \text{if } \nu = w(\mu+\rho) - \rho \in \mathfrak{X}_+. \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

The following result is well known in case of Lusztig's q-analogues, see e.g. [5, Lemma 6.1]. For convenience of the reader, we provide a proof of the general statement.

Theorem 3.8. *For any $\mu \in \mathfrak{X}$, we have*

$$\sum_i (-1)^i \mathcal{H}_G\big(H^i(G \times_B N, \mathcal{L}_{G \times_B N}(\mu)^\star); q\big) = \sum_{\lambda \in \mathfrak{X}_+} \mathfrak{m}^\mu_{\lambda,\Psi}(q) e^{\lambda^*}.$$

Proof. Each finite-dimensional B-module M has a B-filtration such that the associated graded B-module, denoted $\widetilde{M}$, is completely reducible. Then

$$\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(M)^\star) = \sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}((\widetilde{M})^\star).$$

We will apply this to the B-modules $\mathcal{S}^j N \otimes \mathbb{C}_\mu$, $j = 0, 1, \dots$.

$$\begin{aligned}
&\sum_i (-1)^i \mathcal{H}_G(H^i(G \times_B N, \mathcal{L}_{G \times_B N}(\mu)^\star; q) \\
&\qquad = \sum_{j=0}^\infty \mathcal{H}_G\big(\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\mathcal{S}^j N \otimes \mathbb{C}_\mu)^\star); q\big) \\
&\qquad = \sum_{j=0}^\infty \mathcal{H}_G\big(\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\widetilde{\mathcal{S}^j N} \otimes \mathbb{C}_\mu)^\star); q\big) \\
&\qquad = \sum_{j=0}^\infty \sum_{\nu \vdash \mathcal{S}^j N} \dim(\mathcal{S}^j N)^\nu q^j \cdot \mathcal{H}_G\big(\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\nu+\mu)^\star\big) \\
&\qquad = \sum_{\nu \vdash \mathcal{S}^\bullet N} \mathcal{P}_{\Psi,q}(\nu) \mathcal{H}_G\big(\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\nu+\mu)^\star\big),
\end{aligned}$$

where notation $\nu \vdash \mathcal{S}^j N$ means that ν is a weight of $\mathcal{S}^j N$. By the BWB-theorem, the weight $\nu + \mu$ contributes to the last sum if and only if $\nu + \mu + \rho$ is regular, i.e., $w(\nu + \mu + \rho) - \rho =$

$\lambda \in \mathfrak{X}_+$ for a unique $w \in W$. Therefore, using Eq. (3.1), we obtain

$$\sum_{\nu} \mathcal{P}_{\Psi,q}(\nu)\mathcal{H}_G\big(\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\nu+\mu)^\star\big) = \\ \sum_{\lambda\in\mathfrak{X}_+}\sum_{w\in W} \varepsilon(w)\mathcal{P}_{\Psi,q}(w^{-1}(\lambda+\rho)-\mu-\rho)e^{\lambda^*} = \sum_{\lambda\in\mathfrak{X}_+} \mathfrak{m}^{\mu}_{\lambda,\Psi}(q)e^{\lambda^*},$$

as required. □

Theorem 3.9. *For any $\mu \in \mathfrak{X}^P$, we have*

$$\sum_i (-1)^i \mathcal{H}_G\big(H^i(G\times_P N, \mathcal{L}_{G\times_P N}(\mu)^\star); q\big) = \sum_{\lambda\in\mathfrak{X}_+} \mathfrak{m}^{\mu}_{\lambda,\Psi}(q)e^{\lambda^*}.$$

Proof. Using the Leray spectral sequence associated to the morphism $G/B \to G/P$, one easily proves that, for any $\mu \in \mathfrak{X}^P$, there is an isomorphism

$$H^i(G/B, \mathcal{L}_{G/B}(\mathcal{S}^j N\otimes \mathbb{C}_\mu)^\star) \simeq H^i(G/P, \mathcal{L}_{G/P}(\mathcal{S}^j N\otimes \mathbb{C}_\mu)^\star).$$

Thus, the assertion reduces to the previous theorem. □

Corollary 3.10. *If $\mu \in \mathfrak{X}^P$ and $H^i(G\times_P N, \mathcal{L}_{G\times_P N}(\mu)^\star) = 0$ for $i \geqslant 1$, then $\mathfrak{m}^{\mu}_{\lambda,\Psi}(q)$ has non-negative coefficients for all $\lambda \in \mathfrak{X}_+$.*

Now, combining this corollary and Propositions 3.5, 3.6, we obtain Theorem 3.2.

Remark 3.11. By Theorem 3.9, if higher cohomology of $\mathcal{L}_{\boldsymbol{Z}}(\mu)^\star$ vanishes, then the polynomial $\mathfrak{m}^{\mu}_{\lambda,\Psi}(q)$ counts occurrences of V^*_λ in the graded G-module $H^0(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star)$. In particular, $\mathfrak{m}^{\mu}_{\lambda,\Psi}(1)$ is the multiplicity of V^*_λ in $H^0(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star)$.

3.5. If we wish to get a generically finite collapsing for a B-stable $N \subset V$, then P must be chosen as large as possible. That is, we have to take $P = \mathsf{Norm}_G(N)$, the normaliser of N in G. However, even this does not guarantee the generic finiteness.

Example 3.12. Let $\mathfrak{c}$ be a B-stable subspace of $\mathfrak{u} \subset \mathfrak{g}$. Actually, $\mathfrak{c}$ is a B-stable ideal of $\mathfrak{u}$. Let $P = \mathsf{Norm}_G(\mathfrak{c})$. The image of the collapsing $G\times_P \mathfrak{c} \to G{\cdot}\mathfrak{c}$ is the closure of a nilpotent orbit. Hence $\dim(G{\cdot}\mathfrak{c})$ is even. However, $\dim(G\times_P \mathfrak{c})$ can be odd. For instance, take $\mathfrak{c} = [\mathfrak{u},\mathfrak{u}]$. If G is simple and $G \ne SL_2$, then $\mathsf{Norm}_G([\mathfrak{u},\mathfrak{u}]) = B$. But $\dim(G\times_B [\mathfrak{u},\mathfrak{u}])$ is even if and only if $\mathsf{rk}(G)$ is. It can be shown that the collapsing $G\times_B [\mathfrak{u},\mathfrak{u}] \to G{\cdot}[\mathfrak{u},\mathfrak{u}]$ is generically finite if and only if $\mathfrak{g} \in \{\mathbf{A}_{2n}, \mathbf{B}_{2n}, \mathbf{C}_{2n}, \mathbf{E}_6, \mathbf{E}_8, \mathbf{F}_4, \mathbf{G}_2\}$.

B-stable (or "ad-nilpotent") ideals of $\mathfrak{u}$ provide the most natural class of examples of generalised Kostka-Foulkes polynomials. There is a rich combinatorial theory of these ideals. In particular, the normalisers of ad-nilpotent ideals has been studied in [21].

Example 3.13. a) For $G = SL_{2n+1}$, consider $\Psi = \{\gamma \in \Delta^+ \mid \mathsf{ht}(\gamma) \geqslant n+1\}$. The corresponding ad-nilpotent ideal is $\mathfrak{u}_n = \underbrace{[\dots[}_{n}\mathfrak{u},\mathfrak{u}],\dots,\mathfrak{u}]$. By direct calculations, $|\Psi| = \rho$. Therefore the normaliser of $\mathfrak{u}_n$ equals B [21, Theorem 2.4(ii)]. Next, $\dim(G \times_B \mathfrak{u}_n) = 2n^2 + 2n + \binom{n}{2}$ and the dense orbit in $G{\cdot}\mathfrak{u}_n$ corresponds to the partition $(2,\dots,2,1)$. Therefore $\dim G{\cdot}\mathfrak{u}_n = 2n^2 + 2n$, and the collapsing is not generically finite unless $n = 1$. By Theorems 3.1(i) and 3.2(i) with $P = B$, we obtain

- $H^i(G \times_B \mathfrak{u}_n, \mathcal{L}_{G\times_B \mathfrak{u}_n}(\mu)^\star) = 0$ for any $\mu \in \mathfrak{X}_+$ and $i \geqslant 1$;
- $\mathfrak{m}^\mu_{\lambda,\Psi}(q)$ has non-negative coefficients for all $\lambda, \mu \in \mathfrak{X}_+$.

b) For $G = SL_{2n}$, consider $\Psi = \{\gamma \in \Delta^+ \mid \mathsf{ht}(\gamma) \geqslant n\}$. The corresponding ad-nilpotent ideal is $\mathfrak{u}_{n-1}$. Since $|\Psi| = \rho + \varphi_n$, the normaliser of $\mathfrak{u}_{n-1}$ equals B. Again, direct calculations show that $\dim(G \times_B \mathfrak{u}_{n-1}) - \dim G{\cdot}\mathfrak{u}_{n-1} = \binom{n}{2}$. Here we have

- $H^i(G \times_B \mathfrak{u}_n, \mathcal{L}_{G\times_B \mathfrak{u}_{n-1}}(\mu)^\star) = 0$ for any $\mu \succcurlyeq \varphi_n$ and $i \geqslant 1$;
- $\mathfrak{m}^\mu_{\lambda,\Psi}(q)$ has non-negative coefficients for all $\lambda \in \mathfrak{X}_+$ and $\mu \succcurlyeq \varphi_n$.

Remark 3.14. For an arbitrary B-stable subspace $N \subset V$, the normaliser of N is fully determined by $|\Psi|$. The proof of [21, Theorem 2.4(i),(ii)] goes thorough verbatim, and it shows that $|\Psi|$ is dominant and

$$\left\{\begin{matrix}\text{the root subspace } \mathfrak{g}_{-\alpha}\ (\alpha \in \Pi) \\ \text{belong to } \mathsf{Lie}(\mathsf{Norm}_G(N))\end{matrix}\right\} \Leftrightarrow \{(\alpha, |\Psi|) = 0\}\,.$$

Equivalently, one can say that $\mathsf{Norm}_G(N) = \mathsf{Norm}_G(\wedge^{\dim N} N)$, where $\wedge^{\dim N} N \subset \wedge^{\dim N} V$.

4. The little adjoint module and short q-analogues

Let G be a simple algebraic group such that Δ has two root lengths. There is a special interesting case in which $\Psi = \Delta^+_s$ is the set of short positive roots. The subscripts 's' and 'l' will be used to mark objects related to short and long roots, respectively. For instance, Δ_l is the set of all long roots, $\Delta^+ = \Delta^+_s \sqcup \Delta^+_l$, and $\Pi_s = \Pi \cap \Delta_s$. Let $\bar\theta$ be the short dominant root. The G-module $V_{\bar\theta}$ is said to be *little adjoint*.

Lemma 4.1. *The set of nonzero weights of $V_{\bar\theta}$ is Δ_s; $m^\nu_{\bar\theta} = 1$ for $\nu \in \Delta_s$ and $m^0_{\bar\theta} = \#\Pi_s$.*

The last equality is proved in [20, Prop. 2.8]; the rest is obvious. It follows that there is a unique B-stable subspace of $V_{\bar\theta}$ whose set of weights is Δ^+_s. Write $V^+_{\bar\theta}$ for this subspace. In the rest of the article, we work with $\Psi = \Delta^+_s$ and the B-stable subspace $N = V^+_{\bar\theta}$. In place of $\mathcal{P}_{\Delta^+_s,q}(\nu)$ and $\mathfrak{m}^\mu_{\lambda,\Delta^+_s}(q)$, we write $\overline{\mathcal{P}}_q(\nu)$ and $\overline{\mathfrak{m}}^\mu_\lambda(q)$, respectively. The polynomials $\overline{\mathfrak{m}}^\mu_\lambda(q)$ are said to be *short q-analogues* (of weight multiplicities).

We have $\mathfrak{X}_+ \cap \Delta^+_s = \{\bar\theta\}$. Set $\rho_s = \frac{1}{2}|\Delta^+_s|$ and $\rho_l = \frac{1}{2}|\Delta^+_l|$. It is easily seen that ρ_s (resp. ρ_l) is the sum of fundamental weights corresponding to Π_s (resp. Π_l). Let H be the connected semisimple subgroup of G that contains T and whose root system is Δ_l. The Weyl group of H is the normal subgroup of W generated by all "long" reflections. It is denoted by W_l. Let $G(\Pi_s)$ (resp. $\mathfrak{g}(\Pi_s)$) denote the simple subgroup of G (subalgebra of $\mathfrak{g}$) whose set of simple roots is Π_s. Then $\mathsf{rk}\,\mathfrak{g}(\Pi_s) = \#\Pi_s$ and $B \cap G(\Pi_s) =: B(\Pi_s)$ is a Borel subgroup of $G(\Pi_s)$. Clearly, $G(\Pi_s){\cdot}T =: L$ is a standard Levi subgroup of G and $G(\Pi_s) = (L, L)$.

The collapsing $G \times_B V_{\bar\theta}^+ \to G{\cdot}V_{\bar\theta}^+$ is not generically finite, and Theorem 3.2(i) (with $\rho_P = \rho$, $\Delta(\mathfrak{n}) = \Delta^+$, and $|\Delta_s^+| = 2\rho_s$) yields the bound $\mu \succcurlyeq 2\rho_s - \rho = \rho_s - \rho_l$ for $\overline{\mathfrak{m}}_\lambda^\mu(q)$. However, in this case there is a better bound, and our first goal is to obtain it. To this end, we need some further properties of little adjoint modules.

The weight structure of $V_{\bar\theta}$ shows that $V_{\bar\theta}|_{G(\Pi_s)}$ contains the adjoint representation of $G(\Pi_s)$. To distinguish the Lie algebra $\mathfrak{g}(\Pi_s)$ sitting in $\mathfrak{g}$ and the adjoint representation of $G(\Pi_s)$ sitting in $V_{\bar\theta}$, the latter will be denoted by $\widehat{\mathfrak{g}}(\Pi_s)$. That is,

$$V_{\bar\theta}|_{G(\Pi_s)} = \widehat{\mathfrak{g}}(\Pi_s) \oplus R,$$

where R is *the* complementary $G(\Pi_s)$-submodule. The above decomposition is L-stable and hence T-stable. We have $R^T = 0$ and the weights of R are those short roots that are not $\mathbb{Z}$-linear combinations of short simple roots. Furthermore, $V_{\bar\theta}^+ = \widehat{\mathfrak{g}}(\Pi_s)^+ \oplus R^+$, where $R^+ \subset R$ and $\mathfrak{g}(\Pi_s)^+ = \mathfrak{g}(\Pi_s) \cap \mathfrak{u}$ is a maximal nilpotent subalgebra of $\mathfrak{g}(\Pi_s)$.

Theorem 4.2. *If $\mu \succcurlyeq \rho_l$, then the collapsing $f_\mu^{(s)} : G \times_B (V_{\bar\theta}^+ \oplus \mathbb{C}_\mu) \to G{\cdot}(V_{\bar\theta}^+ \oplus \mathbb{C}_\mu)$ is birational.*

Proof. Recall that $\mathbb{C}_\mu$ is the line of B-highest weight vectors in V_μ. Obviously, $f_\mu^{(s)}$ is birational if and only if the following property holds: for a generic point $(v, v_\mu) \in V_{\bar\theta}^+ \oplus \mathbb{C}_\mu$, if $g{\cdot}(v, v_\mu) \in V_{\bar\theta}^+ \oplus \mathbb{C}_\mu$ ($g \in G$), then $g \in B$. Let $\tilde P$ denote the standard parabolic subgroup of G whose Levi subgroup is L. If $\mu \succcurlyeq \rho_l$, then the normaliser in G of the line $\langle v_\mu\rangle$ is contained in $\tilde P$. Consequently, if $g{\cdot}(v, v_\mu) \in V_{\bar\theta}^+ \oplus \mathbb{C}_\mu$, then $g \in \tilde P$.

Take $v = v' + r \in V_{\bar\theta}^+$ ($r \in R$) such that v' is a regular nilpotent element of $\widehat{\mathfrak{g}}(\Pi_s)^+$. Write $g = g_1 g_2 \in \tilde P$, where $g_1 \in G(\Pi_s)$ and g_2 lies in the radical of $\tilde P$, $\mathsf{rad}(\tilde P)$. It is easily seen that $\mathsf{rad}(\tilde P)$ preserves R^+ and acts trivially in $V_{\bar\theta}^+/R^+$. Therefore g_2 does not change the $\widehat{\mathfrak{g}}(\Pi_s)$-component of v, i.e., $g_2{\cdot}v = v' + r'$ ($r' \in R^+$). Hence $g{\cdot}v = g_1{\cdot}v' + g_1{\cdot}r'$, and $g_1{\cdot}v' \in \widehat{\mathfrak{g}}(\Pi_s)^+$ is still a regular nilpotent element of $\widehat{\mathfrak{g}}(\Pi_s)$. But the latter is only possible if $g_1 \in B(\Pi_s)$ and hence $g \in B$. □

Corollary 4.3. *If $\nu + \rho_l \in \mathfrak{X}_+$, then*

(i) *$H^i(G \times_B V_{\bar\theta}^+, \mathcal{L}_{G\times_B V_{\bar\theta}^+}(\nu)^\star) = 0$ for $i \geqslant 1$;*
(ii) *$\overline{\mathfrak{m}}_\lambda^\nu(q)$ has non-negative coefficients for all $\lambda \in \mathfrak{X}_+$.*

Proof. (i) Set $\boldsymbol{U} = G \times_B (V_{\bar\theta}^+ \oplus \mathbb{C}_\mu)$ and $\boldsymbol{Z} = G \times_B V_{\bar\theta}^+$. Then $\omega_{\boldsymbol{U}} = \mathcal{L}_{\boldsymbol{U}}(\gamma - \mu)$, where $\gamma = |\Delta^+| - |\Delta_s^+| = 2\rho_l$. By Theorems 3.4 and 4.2, $H^i(\boldsymbol{U}, \omega_{\boldsymbol{U}}) = 0$ for $i \geqslant 1$ whenever $\mu \succcurlyeq \rho_l$.

Hence $H^i(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}((n+1)\mu-\gamma)^\star) = 0$, see Section 3. In particular, $H^i(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\nu)^\star) = 0$, where $\nu = \mu - \gamma$. It remains to observe that $\nu \succcurlyeq -\rho_l$.

(ii) This follows from (i) and Theorem 3.8. □

Remark 4.4. The proof of Corollary 4.3(i) uses (a version of) the Grauert–Riemenschneider theorem. However, for $\nu = 0$ (at least) one can adapt Hesselink's proof of [9, Theorem B], which does not refer to Grauert–Riemenschneider and goes through for any algebraically closed field $\Bbbk$ of characteristic zero. Using this, one can prove the following: Let $\tilde{N}$ be any B-stable subspace of $V_{\bar\theta}$ such that $\tilde{N} \supset V_{\bar\theta}^+$. Then $H^i(G\times_B \tilde{N}, \mathcal{O}_{G\times_B\tilde{N}}) = 0$ for $i \geqslant 1$.

Let us describe a semi-direct product structure of W, which plays an important role below. Consider two subgroups of W:

- W_l is generated by <u>all</u> "long" reflections in W. It is a normal subgroup of W.
- $W(\Pi_s)$ is generated by all <u>simple</u> "short" reflections, i.e., by s_α with $\alpha \in \Pi_s$.

Lemma 4.5. (i) *W is a semi-direct product of W_l and $W(\Pi_s)$: $W \simeq W(\Pi_s) \ltimes W_l$.*
(ii) *$W(\Pi_s) = \{w \in W \mid w(\Delta_l^+) \subset \Delta_l^+\}$.*

Proof. (i) Since W_l is a normal subgroup of W and $W_l \cap W(\Pi_s) = \{1\}$, it suffices to prove that the natural mapping $W(\Pi_s) \times W_l \to W$ is onto. We argue by induction on the length of $w \in W$. Suppose $w \notin W(\Pi_s)$ and $w = w_1 s_\beta w_2 \in W$, $\beta \in \Pi_l$, is a reduced decomposition. Then $w = w_1w_2s_{\beta'}$, where $\beta' = w_2(\beta) \in \Delta_l$, and $\ell(w_1w_2) < \ell(w)$. Thus, all long simple reflections occurring in an expression for w can eventually be moved up to the right.

(ii) Since $s_\alpha(\Delta_l^+) \subset \Delta_l^+$ for $\alpha \in \Pi_s$, $W(\Pi_s) \subset \{w \in W \mid w(\Delta_l^+) \subset \Delta_l^+\}$. On the other hand, if $w(\Delta_l^+) \subset \Delta_l^+$ and $w = w's_\alpha$ is a reduced decomposition, then the equality $\mathsf{N}(w) = s_\alpha(\mathsf{N}(w')) \cup \{\alpha\}$ shows that α is necessarily short, so that we can argue by induction on $\ell(w)$. □

Recall that the *null-cone* of a G-module V, $\mathfrak{N}(V)$, is the zero set of all homogeneous G-invariant polynomials of positive degree. Next proposition summarises invariant-theoretic properties of $V_{\bar\theta}$ and $\mathfrak{N}(V_{\bar\theta})$ required below, which are of independent interest. All the assertions can easily be verified using the classification, but our intention is to present a conceptual proof.

Proposition 4.6. *a)* $\mathfrak{N}(V_{\bar\theta}) = G{\cdot}V_{\bar\theta}^+$. *Hence it is irreducible;*

b) The restriction homomorphisms $\mathbb{C}[V_{\bar\theta}] \to \mathbb{C}[\widehat{\mathfrak{g}}(\Pi_s)] \to \mathbb{C}[V_{\bar\theta}^0]$ induce the isomorphisms $\mathbb{C}[V_{\bar\theta}]^G \xrightarrow{\sim} \mathbb{C}[\widehat{\mathfrak{g}}(\Pi_s)]^{G(\Pi_s)} \xrightarrow{\sim} \mathbb{C}[V_{\bar\theta}^0]^{W(\Pi_s)}$, and $\mathbb{C}[V_{\bar\theta}]^G$ is a polynomial algebra.

c) $\mathfrak{N}(V_{\bar\theta})$ is a reduced normal complete intersection of codimension $\#(\Pi_s)$.

Outline of the proof. We refer to [22] for invariant-theoretic results mentioned below.

a) This follows from the Hilbert-Mumford criterion and the fact any maximal subset of weights of $V_{\bar\theta}$, lying in an open half-space, is W-conjugate to Δ_s^+.

b) The weight structure of $V_{\bar\theta}$ shows that $V_{\bar\theta}^0 = V_{\bar\theta}^H$. If $v \in V_{\bar\theta}^0$ is generic, then $\mathfrak g{\cdot}v + V_{\bar\theta}^0 = V_{\bar\theta}$. Therefore $G{\cdot}V_{\bar\theta}^0$ is dense in $V_{\bar\theta}$ and a generic stabiliser (= *stabiliser in general position*) for $G{:}V_{\bar\theta}$ contains H. Actually, it is not hard to prove that H is a generic stabiliser for $G{:}V_{\bar\theta}$. By the Luna-Richardson theorem, we then have $\mathbb C[V_{\bar\theta}]^G \xrightarrow{\sim} \mathbb C[V_{\bar\theta}^H]^{N_G(H)/H}$, and it is easily seen that $N_G(H)/H \simeq W/W_l \simeq W(\Pi_s)$. Furthermore, the $W(\Pi_s)$-action on $V_{\bar\theta}^0$ is nothing but the standard reflection representation on the Cartan subalgebra of $\mathfrak g(\Pi_s)$.

c) Let $f_1,\dots,f_m$ be basic invariants in $\mathbb C[V_{\bar\theta}]^G \simeq \mathbb C[\widehat{\mathfrak g}(\Pi_s)]^{G(\Pi_s)}$, $m = \#(\Pi_s)$. Let $e \in \widehat{\mathfrak g}(\Pi_s) \subset V_{\bar\theta}$ be regular nilpotent. Then the differentials of the f_i's are linearly independent at $e \in \mathfrak N(V_{\bar\theta})$ [14]. Hence the ideal of $\mathfrak N(V_{\bar\theta})$ is $(f_1,\dots,f_m)$ and $\mathfrak N(V_{\bar\theta})$ is a reduced complete intersection (cf. [14, Lemma 4]). Finally, $\mathfrak N(V_{\bar\theta})$ contains a dense G-orbit whose complement is of codimension $\geqslant 2$. This yields the normality. □

Our ultimate goal is to get a complete characterisation of weights $\mu \in \mathfrak X$ such that $\overline{\mathfrak m}_\lambda^\mu(q)$ has nonnegative coefficients for any $\lambda \in \mathfrak X_+$. To this end, we exploit a different approach that does not use vanishing theorems of Section 3.

A key observation is that short q-analogues obey certain symmetries with respect to the simple reflections $s_\alpha \in W$, $\alpha \in \Pi_l$. Clearly, $s_\alpha(\Delta_s^+) = \Delta_s^+$. Therefore $\overline{\mathcal P}_q(\nu) = \overline{\mathcal P}_q(s_\alpha\nu)$. Using this, we compute

$$\begin{aligned}
(4.1)\quad \overline{\mathfrak m}_\lambda^\mu(q) &= \sum_{w\in W} \varepsilon(w)\overline{\mathcal P}_q(w(\lambda+\rho)-(\mu+\rho)) \\
&= \sum_{w\in W} \varepsilon(w)\overline{\mathcal P}_q(s_\alpha w(\lambda+\rho)-s_\alpha(\mu+\rho)) = -\sum_{w\in W} \varepsilon(w)\overline{\mathcal P}_q(w(\lambda+\rho)-s_\alpha\mu-s_\alpha\rho) \\
&= -\sum_{w\in W} \varepsilon(w)\overline{\mathcal P}_q(w(\lambda+\rho)-(s_\alpha\mu-\alpha+\rho)) = -\overline{\mathfrak m}_\lambda^{s_\alpha(\mu+\alpha)}(q).
\end{aligned}$$

The *shifted action* of W_l on $\mathfrak X$ is defined by

$$w\odot\gamma = w(\gamma+\rho_l)-\rho_l.$$

For $\alpha \in \Pi_l$, one easily recognise $s_\alpha(\mu+\alpha)$ as $s_\alpha\odot\mu$ and hence Eq. (4.1) can be written as $\overline{\mathfrak m}_\lambda^{s_\alpha\odot\mu}(q) = -\overline{\mathfrak m}_\lambda^\mu(q)$. This readily implies the equality

$$\overline{\mathfrak m}_\lambda^{w\odot\mu}(q) = \varepsilon(w)\,\overline{\mathfrak m}_\lambda^\mu(q) \tag{4.2}$$

for any $w \in W_l$. Note that for $w \in W_l$, the length $\ell(w)$ depends on the choice of ambient group, W or W_l, but the parity $\varepsilon(w)$ does not! (This is because $\varepsilon(w) = \det(w)$ for the reflection representation of W in $\mathfrak X\otimes_{\mathbb Z}\mathbb Q$.)

Let $\mathfrak{X}_{+,H}$ denote the monoid of H-dominant weights with respect to Δ_l^+. From (4.2), we immediately deduce that

- it suffices to know $\overline{\mathfrak{m}}_\lambda^\mu(q)$ for $\mu \in \mathfrak{X}_{+,H} - \rho_l$.
- if such a μ is not H-dominant, then it lies on a wall of the shifted dominant Weyl chamber for H, and hence $\overline{\mathfrak{m}}_\lambda^\mu(q) \equiv 0$.
- Thus, the problem is reduced to studying polynomials $\overline{\mathfrak{m}}_\lambda^\mu(q)$ for $\mu \in \mathfrak{X}_{+,H}$.

Short q-analogues enjoy several good interpretations at $q = 1$. Write $\overline{m}_\lambda^\nu$ in place of $\overline{\mathfrak{m}}_\lambda^\nu(1)$.

(1) As already observed in Remark 3.11, if higher cohomology of $\mathcal{L}_{\boldsymbol{Z}}(\nu)^\star$ vanish, then $\overline{m}_\lambda^\nu$ is the multiplicity of V_λ^* in $H^0(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\nu)^\star)$.
(2) If $\nu \in \mathfrak{X}_{+,H}$ and $V_\nu^{(H)}$ is a simple H-module with highest weight ν, then $\overline{m}_\lambda^\nu$ is the multiplicity of $V_\nu^{(H)}$ in $V_\lambda|_H$, denoted $\mathsf{mult}(V_\nu^{(H)}, V_\lambda|_H)$, see [8, Lemma 3.1].

(Our $\overline{m}_\lambda^\nu$ is $m_\lambda^{G,H}(\nu)$ in the notation of [8]. In fact, Heckman works in a general situation, where $H \subset G$ is an arbitrary connected reductive group.) Furthermore, the numbers $\overline{m}_\lambda^\nu$ are naturally defined for all $\lambda, \nu \in \mathfrak{X}$ and they satisfy the relation

$$\overline{m}_{w(\lambda+\rho)-\rho}^{\bar{w}(\nu+\rho_l)-\rho_l} = \varepsilon(w)\varepsilon(\bar{w})\overline{m}_\lambda^\nu, \quad w \in W,\ \bar{w} \in W_l. \tag{4.3}$$

(See Equation (3.7) in [8].) The semi-direct product structure of W provides an extra symmetry to this picture that is absent in the general setting of [8]. Namely, if ν is H-dominant, then so is $w\nu$ for any $w \in W(\Pi_s)$. Using this one easily proves that $\overline{m}_\lambda^\nu = \overline{m}_\lambda^{w\nu}$ for all $\lambda \in \mathfrak{X}_+$ and $w \in W(\Pi_s)$.

Recall that $\{\mu^+\} = W\mu \cap \mathfrak{X}_+$. Let w_μ denote the unique element of minimal length such that $w_\mu(\mu) = \mu^+$.

Lemma 4.7. *If $\mu \in \mathfrak{X}_{+,H}$, then $w_\mu \in W(\Pi_s)$ and hence $\mu^+ - \mu$ is a nonnegative $\mathbb{Z}$-linear combination of short simple roots.*

Proof. It is known that $\mathsf{N}(w_\mu) = \{\gamma \in \Delta^+ \mid (\gamma, \mu) < 0\}$, see [4, Prop. 2(i)]. Since μ is H-dominant, $\mathsf{N}(w_\mu) \subset \Delta_s^+$, and we conclude by Lemma 4.5(ii). □

Proposition 4.8. *Let $\mu \in \mathfrak{X}_{+,H}$.*

1) Suppose that there is $\nu \in \mathfrak{X}_+$ such that $\mu \preccurlyeq \nu \prec \mu^+$. Then $\overline{\mathfrak{m}}_\nu^\mu(q) \neq 0$ and $\overline{m}_\nu^\mu = 0$. In particular, $\overline{\mathfrak{m}}_\nu^\mu(q)$ has both positive and negative coefficients.

2) If $V_{\mu^+}^*$ occurs in $H^0(G/B, \mathcal{L}_{G/B}(\widetilde{\mathcal{S}^j(V_{\bar{\theta}}^+)} \otimes \mathbb{C}_\mu)^\star)$, then $j \geqslant \mathsf{ht}(\mu^+ - \mu)$. Furthermore, for $j = \mathsf{ht}(\mu^+ - \mu)$, $H^0(\dots)$ contains a unique copy of $V_{\mu^+}^*$.

Proof. 1) Since $w_\mu \in W(\Pi_s)$, we have $\overline{m}_\nu^\mu = \overline{m}_\nu^{\mu^+}$, and the latter equals zero, because $\nu \prec \mu^+$. (Obviously, the H-module with highest weight μ^+ cannot occur in $V_\nu|_H$.)

Since $\mu \preccurlyeq \nu \prec \mu^+$ and $\mu^+ - \mu$ is a nonnegative $\mathbb{Z}$-linear combination of short simple roots, the latter holds for $\nu - \mu$ as well. Set $a = \mathsf{ht}(\nu - \mu)$. By definition,

$$\overline{\mathfrak{m}}^{\mu}_{\nu}(q) = \sum_{w \in W} \varepsilon(w) \overline{\mathcal{P}}_q(w(\nu + \rho) - (\mu + \rho)).$$

As $\nu - \mu \in \mathsf{Span}(\Pi_s)$, the summand $\overline{\mathcal{P}}_q(w(\nu+\rho)-(\mu+\rho))$ can be nonzero only if $w \in W(\Pi_s)$. For $w = 1$, we have $\overline{\mathcal{P}}_q(\nu-\mu) = q^a +$ (lower terms). If $w \neq 1$, then $\deg \overline{\mathcal{P}}_q(w(\nu+\rho)-(\mu+\rho)) < a$. Hence the highest term of $\overline{\mathfrak{m}}^{\mu}_{\nu}(q)$ is q^a, and we are done.

2) This readily follows from the BWB-theorem and Lemma 4.7. $\square$

Our main result on non-negativity for short q-analogues is a converse to the first claim of the previous proposition. For the proof of the main theorem, we need a technical lemma.

Lemma 4.9. 1) *Suppose that V^*_{ν} occurs in $H^i(G/B, \mathcal{L}_{G/B}(\wedge^j(V_{\bar\theta}/V^+_{\bar\theta}) \otimes \mathbb{C}_{\mu})^\star)$. Then $\nu \preccurlyeq \mu^+$.*
2) *(For $\nu = \mu^+$.) If $V^*_{\mu^+}$ occurs in $H^i(G/B, \mathcal{L}_{G/B}(\wedge^j(\widetilde{V_{\bar\theta}/V^+_{\bar\theta}}) \otimes \mathbb{C}_{\mu})^\star)$, then $j \geqslant i \geqslant \ell(w_{\mu})$.*

Proof. Set $M_j = \wedge^j(V_{\bar\theta}/V^+_{\bar\theta}) \otimes \mathbb{C}_{\mu}$.

1) If V^*_{ν} occurs in $H^i(G/B, \mathcal{L}_{G/B}(M_j)^\star)$, then it also occurs in $H^i(G/B, \mathcal{L}_{G/B}(\tilde{M}_j)^\star)$. By the BWB-theorem, there is then a weight γ of M_j and $w \in W$ such that $\ell(w) = i$ and $w(\gamma + \rho) - \rho = \nu$. All weights of M_j are of the form $\mu - |A|$ for some $A \subset \Delta^+_s$, where $\#(A) \leqslant j$. Hence $w(\mu + \rho - |A|) = \rho + \nu$. Clearly, $w(\rho - |A|) = \rho - |C|$ for some $C \subset \Delta^+_s$ depending on w and A. Thus, $w(\mu + \rho - |A|) \preccurlyeq w(\mu) + \rho$ and $\nu \preccurlyeq w(\mu) \preccurlyeq \mu^+$.

2) If $V^*_{\mu^+}$ occurs in $H^i(G/B, \mathcal{L}_{G/B}(\tilde{M}_j)^\star)$, then, by the first part of the proof, we must have $w(\mu + \rho - |A|) = \rho + \mu^+$, where $A \subset \Delta^+_s$ and $\ell(w) = i$. Hence $w(\mu) = \mu^+$ and $w(\rho - |A|) = \rho$. Therefore $A = \mathsf{N}(w)$ and $i = \ell(w) = \#(A) \geqslant \ell(w_{\mu})$. Since $\#(A) \leqslant j$ as well, we are done. $\square$

The following is the main result of this section.

Theorem 4.10. *For $\mu \in \mathfrak{X}_{+,H}$, the following conditions are equivalent:*

- (i) $H^i(G \times_B V^+_{\bar\theta}, \mathcal{L}_{G\times_B V^+_{\bar\theta}}(\mu)^\star) = 0$ *for all* $i \geqslant 1$;
- (ii) $\overline{\mathfrak{m}}^{\mu}_{\lambda}(q)$ *has nonnegative coefficients for any* $\lambda \in \mathfrak{X}_+$;
- (iii) *If* $\mu \preccurlyeq \nu \preccurlyeq \mu^+$ *for* $\nu \in \mathfrak{X}_+$, *then* $\nu = \mu^+$;
- (iv) $(\mu, \alpha^\vee) \geqslant -1$ *for all* $\alpha \in \Delta^+_s$.

Proof. By Corollary 3.10, (i) implies (ii); and Proposition 4.8 shows that (ii) implies (iii). Since ν is already assumed to be H-dominant, (iii) and (iv) are equivalent in view of [4, Prop. 2(iii)].

It remains to prove the implication (iii) $\Rightarrow$ (i). Our argument is an adaptation of Broer's proof of [1, Theorem 2.4]. We construct a similar Koszul complex and consider its spectral sequence of hypercohomology.

The pull-back vector bundle $G \times_B (V_{\bar\theta} \oplus (V_{\bar\theta}/V_{\bar\theta}^+))$ on $\boldsymbol{X} := G \times_B V_{\bar\theta}$ has the global G-equivariant section $g * v \mapsto g * (v, \bar v)$ whose scheme of zeros is exactly $\boldsymbol{Z} = G \times_B V_{\bar\theta}^+$. Here $\bar v$ is the image of $v \in V_{\bar\theta}$ in $V_{\bar\theta}/V_{\bar\theta}^+$. Let $\iota : \boldsymbol{Z} \to \boldsymbol{X}$ denote the inclusion. The dual of this section gives rise to a locally free Koszul resolution of $\mathcal{O}_{\boldsymbol{Z}}$ regarded as $\mathcal{O}_{\boldsymbol{X}}$-module:

$$\cdots \to \mathcal{F}^{-1} \to \mathcal{F}^0 \to \iota_* \mathcal{O}_{\boldsymbol{Z}} \to 0$$

with $\mathcal{F}^{-j} = \mathcal{L}_{\boldsymbol{X}}(\wedge^j (V_{\bar\theta}/V_{\bar\theta}^+)^\star[-j]$. Here the brackets '$[-j]$' denote the degree shift of a graded module. (That is, if $\mathcal{M} = \oplus \mathcal{M}_i$, then $\mathcal{M}[r]_i = \mathcal{M}_{r+i}$.) Therefore the generators of the locally free $\mathcal{O}_{\boldsymbol{X}}$-module $\mathcal{F}^{-j}$ have degree j. Tensoring this complex with the invertible sheaf $\mathcal{L}_{\boldsymbol{X}}(\mathbb{C}_\mu)^\star = \mathcal{L}_{\boldsymbol{X}}(\mu)^\star$, we get a locally free resolution of graded $\mathcal{O}_{\boldsymbol{X}}$-modules

$$\mathcal{F}(\mu)^\bullet \to \iota_* \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star \to 0, \tag{4.4}$$

where $\mathcal{F}(\mu)^{-j} = \mathcal{L}_{\boldsymbol{X}}(\wedge^j (V_{\bar\theta}/V_{\bar\theta}^+) \otimes \mathbb{C}_\mu)^\star[-j]$. Since $\boldsymbol{X} \simeq G/B \times V_{\bar\theta}$, we have the isomorphism

$$H^i(\boldsymbol{X}, \mathcal{L}_{\boldsymbol{X}}(\wedge^j (V_{\bar\theta}/V_{\bar\theta}^+) \otimes \mathbb{C}_\mu)^\star) \simeq \mathbb{C}[V_{\bar\theta}] \otimes H^i(G/B, \mathcal{L}_{G/B}(\wedge^j (V_{\bar\theta}/V_{\bar\theta}^+) \otimes \mathbb{C}_\mu)^\star)$$

of graded $\mathbb{C}[V_{\bar\theta}]$-modules. For the spectral sequence of hypercohomology associated to the Koszul complex (4.4), we have

$$''E_2^{kl} = H^k(\boldsymbol{X}, \mathcal{H}^l(\mathcal{F}(\mu)^\bullet)) = \begin{cases} H^k(\boldsymbol{X}, \iota_* \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star) = H^k(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star), & \text{if } l = 0; \\ 0, & \text{if } l \neq 0. \end{cases}$$

and

$$'E_1^{kl} = H^l(\boldsymbol{X}, \mathcal{F}(\mu)^k) = \mathbb{C}[V_{\bar\theta}][k] \otimes H^l(G/B, \mathcal{L}_{G/B}(\wedge^{-k} (V_{\bar\theta}/V_{\bar\theta}^+) \otimes \mathbb{C}_\mu)^\star).$$

(See [25, 5.7] for basic facts on hypercohomology.) It follows that there is a spectral sequence of graded $\mathbb{C}[V_{\bar\theta}]$-modules

$$'E_1^{-j,i} = \mathbb{C}[V_{\bar\theta}][-j] \otimes H^i(G/B, \mathcal{L}_{G/B}(\wedge^j (V_{\bar\theta}/V_{\bar\theta}^+) \otimes \mathbb{C}_\mu)^\star) \Rightarrow H^{i-j}(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star). \tag{4.5}$$

Let $i - j$ be maximal with $H^i(G/B, \mathcal{L}_{G/B}(\wedge^j (V_{\bar\theta}/V_{\bar\theta}^+) \otimes \mathbb{C}_\mu)^\star) \neq 0$. If V_ν^* occurs in this cohomology group, then $\nu \preccurlyeq \mu^+$, by Lemma 4.9(1). A basis for V_ν^* corresponds to some free generators of $\mathbb{C}[V_{\bar\theta}]$-module $'E_1^{-j,i}$ of degree j. Since $i - j$ is maximal, these generators are in the kernel of $d_1^{-j,i}$. But they are not in the image of $d_1^{-j-1,i}$, as all elements of $'E_1^{-j-1,i}$ are of degree $> j$. Hence these generators correspond to nonzero generators of $'E_2^{-j,i}$. Likewise, their images in $'E_k^{-j,i}$ do not vanish. In view of convergence of the above spectral sequence, this implies that the multiplicity of V_ν^* in $H^{i-j}(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star)$ is at least one. It follows that, for some $m \in \mathbb{N}$, the multiplicity of V_ν^* in $H^{i-j}(G/B, \mathcal{L}_{G/B}(\widetilde{\mathcal{S}^m(V_{\bar\theta}^+)} \otimes \mathbb{C}_\mu)^\star)$ is also at least one. Any weight of $\mathcal{S}^m(V_{\bar\theta}^+) \otimes \mathbb{C}_\mu$ is of the form $\mu + \gamma$ with $\gamma \succcurlyeq 0$. Hence $\nu + \rho = w(\mu + \gamma + \rho)$ for some $w \in W$ with $\ell(w) = i - j$. Consequently, $\nu \succcurlyeq \mu$ and altogether $\mu \preccurlyeq \nu \preccurlyeq \mu^+$. Hence $\nu = \mu^+$. Now, Lemma 4.9(2) yields $i = j \geqslant \ell(w_\mu)$. In particular, condition (i) holds. $\square$

The following is an analogue of [1, Prop. 2.6].

Proposition 4.11. *Suppose that $\mu \in \mathfrak{X}_{+,H}$ satisfies vanishing conditions of Theorem 4.10. Then the graded $\mathbb{C}[V_{\bar\theta}]$-module $H^0(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star)$ is generated by the unique copy of $V^*_{\mu^+}$ sitting in degree $\mathsf{ht}(\mu^+ - \mu)$.*

Proof. Eq. (4.5) an the last part of the proof of Theorem 4.10 shows that

- The generators of the $\mathbb{C}[V_{\bar\theta}]$-module $H^0(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star)$ arise from G-modules sitting in $H^i(G/B, \mathcal{L}_{G/B}(\wedge^i(V_{\bar\theta}/V^+_{\bar\theta}) \otimes \mathbb{C}_\mu)^\star)$, with $i \geqslant \ell(w_\mu)$;
- $H^i(G/B, \mathcal{L}_{G/B}(\wedge^i(V_{\bar\theta}/V^+_{\bar\theta}) \otimes \mathbb{C}_\mu)^\star)$ only contains G-modules of type $V^*_{\mu^+}$.

It follows that the degree of generators of $H^0(\boldsymbol{Z}, \mathcal{L}_{\boldsymbol{Z}}(\mu)^\star)$ is at least $\ell(w_\mu)$. On the other hand, if $H^0(G/B, \mathcal{L}_{G/B}(\widetilde{\mathcal{S}^j(V^+_{\bar\theta})} \otimes \mathbb{C}_\mu)^\star)$ contains a G-submodule of type $V^*_{\mu^+}$, then $j \leqslant \mathsf{ht}(\mu^+ - \mu)$ by Proposition 4.8(2). Therefore, there cannot be generators of degree larger than $\mathsf{ht}(\mu^+ - \mu)$. It only remains to prove that if $\mu \in \mathfrak{X}_{+,H}$ satisfies the vanishing condition, then $\ell(w_\mu) = \mathsf{ht}(\mu^+ - \mu)$. Clearly, $\ell(w_\mu) \leqslant \mathsf{ht}(\mu^+ - \mu)$. Assume the inequality is strict. Then there is a $w \in W$ and a simple reflection s_i such that $\mu \preccurlyeq w(\mu) \prec s_i w(\mu) \preccurlyeq \mu^+$ and $s_i w(\mu) = w(\mu) + k\alpha_i$ with $k \geqslant 2$. Then $\nu := w(\mu) + \alpha_i$ belongs to the convex hull of $w(\mu)$ and $s_i w(\mu)$; hence $\mu \prec \nu^+ \prec \mu^+$, which contradicts the vanishing condition. □

Finally, we mention that above two interpretations of numbers $\overline{m}^\nu_\lambda$ and Theorem 4.10 lead to an interesting equality.

Proposition 4.12. *If $\nu \in \mathfrak{X}_{+,H}$ and $(\nu, \alpha^\vee) \geqslant -1$ for all $\alpha \in \Delta^+_s$, then $H^0(G/H, \mathcal{L}_{G/H}(V^{(H)*}_\nu))$ and $H^0(G \times_B V^+_{\bar\theta}, \mathcal{L}_{G\times_B V^+_{\bar\theta}}(\nu)^\star)$ are isomorphic G-modules. In particular, for $\nu = 0$, we obtain $\mathbb{C}[G/H] \simeq \mathbb{C}[G \times_B V^+_{\bar\theta}]$ as G-modules.*

Proof. By Frobenius reciprocity,

$$\mathsf{mult}(V^*_\lambda, H^0(G/H, \mathcal{L}_{G/H}((V^{(H)*}_\nu)) = \mathsf{mult}(V^{(H)}_\nu, V_\lambda|_H).$$

Hence the multiplicity of V^*_λ in both spaces $H^0(..)$ under consideration is equal to $\overline{m}^\nu_\lambda$. □

5. Short Hall-Littlewood polynomials

In this section, we define "short" analogues of Hall-Littlewood polynomials and establish their basic properties. Recall that Δ is a reduced irreducible root system, and $\Delta^+ = \Delta^+_s \sqcup \Delta^+_l$, $\Pi = \Pi_s \sqcup \Pi_l$, etc. It is convenient to assume that in the simply-laced case all roots are short and $\Pi_l = \varnothing$. Then the following can be regarded as a generalisation of Gupta's theory [6, 7].

The character ring $\boldsymbol{\Lambda}$ of finite-dimensional representations of G is identified with $\mathbb{Z}[\mathfrak{X}]^W$. For $\lambda \in \mathfrak{X}_+$, let χ_λ denote the character of V_λ, i.e., $\chi_\lambda = \mathsf{ch}(V_\lambda) = \sum_\mu m^\mu_\lambda e^\mu$. By Weyl's character formula, $\chi_\lambda = J(e^{\lambda+\rho})/J(e^\rho)$, where $J = \sum_{w\in W} \varepsilon(w)w$ is the skew-symmetrisation operator. Weyl's denominator formula says that $J(e^\rho) = e^\rho \prod_{\alpha>0}(1-e^{-\alpha})$. The usual scalar product $\langle\ ,\ \rangle$ on $\mathbf{\Lambda} = \mathbb{Z}[\mathfrak{X}]^W$ is given by $\langle \chi_\lambda, \chi_\nu\rangle = \delta_{\lambda,\nu}$.

The projection $j : \mathbb{Z}[\mathfrak{X}] \to \mathbb{Z}[\mathfrak{X}]^W$ is given by $j(f) := J(f)/J(e^\rho)$.

Set $t_\lambda^{(\Pi_s)}(q) = \sum q^{\ell(w)}$, where the summation is over $w \in W(\Pi_s)_\lambda$, the stabiliser of λ in $W(\Pi_s)$.

We will work in the q-extended character ring $\mathbf{\Lambda}[[q]]$ or its subring $\mathbf{\Lambda}[q]$ and agree to extend our operators and form q-linearly. We first put

$$\tilde{\mathbf{\Delta}}_q^{(s)} = \frac{e^\rho}{\prod_{\alpha\in\Delta_s^+}(1-qe^{\alpha})}, \quad \mathbf{\Delta}_q^{(s)} = e^\rho \prod_{\alpha\in\Delta_s^+}(1-qe^{-\alpha}).$$

For $\lambda, \mu \in \mathfrak{X}_+$, define :

$$\overline{E}_\mu(q) = j(e^\mu{\cdot}\tilde{\mathbf{\Delta}}_q^{(s)}), \quad \overline{P}_\lambda(q) = \frac{1}{t_\lambda^{(\Pi_s)}(q)} j(e^\lambda{\cdot}\mathbf{\Delta}_q^{(s)}).$$

Clearly, $\overline{E}_\mu(q) \in \mathbf{\Lambda}[[q]]$ and $t_\lambda^{(\Pi_s)}(q){\cdot}\overline{P}_\lambda(q) \in \mathbf{\Lambda}[q]$. It will immediately be shown that $\overline{P}_\lambda(q)$ is a well-defined element of $\mathbf{\Lambda}[q]$, i.e., $t_\lambda^{(\Pi_s)}(q)$ divides $j(e^\lambda{\cdot}\mathbf{\Delta}_q^{(s)})$ in $\mathbf{\Lambda}[q]$. We say that $\overline{P}_\lambda(q)$ is a *short Hall-Littlewood polynomial*. (For, if $\Delta_s^+ = \Delta^+$ or if Δ_s^+ and Π_s are replaced with Δ^+ and Π in the above definition, then one obtains the usual Hall-Littlewood polynomials $P_\lambda(q)$ for Δ.)

Proposition 5.1.

$$\overline{P}_\lambda(q) = J\big(e^{\lambda+\rho} \prod_{\alpha\in\Delta_s^+,\,(\alpha,\lambda)>0} (1-qe^{\alpha})\big) J(\rho)^{-1}.$$

Proof. 1) First consider the case in which $\lambda = 0$. Here

$$J(e^\rho){\cdot}j(e^0{\cdot}\mathbf{\Delta}_q^{(s)}) = J\big(\sum_{A\subset\Delta_s^+} (-q)^{\#A} e^{\rho-|A|}\big).$$

It is known that $\rho - |A|$ is regular if and only if $A = \mathsf{N}(w)$ for some $w \in W$ [17]. Since $A \subset \Delta_s^+$, Lemma 4.5(ii) shows that actually $w \in W(\Pi_s)$. Hence

$$J\big(\sum_{A\subset\Delta_s^+} (-q)^{\#A} e^{\rho-|A|}\big) = \sum_{w\in W(\Pi_s)} (-q)^{\ell(w)} J(e^{w^{-1}\rho}) = \sum_{w\in W(\Pi_s)} q^{\ell(w)}{\cdot}J(e^\rho) = t_0^{(\Pi_s)}(q)J(e^\rho).$$

This proves that $\overline{P}_0(q) = 1$.

2) For an arbitrary $\lambda \in \mathfrak{X}_+$, we notice that $\sum_{w\in W_\lambda} \varepsilon(w) w(e^{\lambda+\rho}\prod_{\alpha\in\Delta_s^+}(1-qe^{-\alpha}))$ is divisible by $t_\lambda^{(\Pi_s)}(q)$, by the first part of proof.
(One has to consider the splitting $\prod_{\alpha\in\Delta_s^+}(1-qe^{-\alpha}) = \prod_{\alpha:\,(\alpha,\lambda)=0}(\dots)\prod_{\alpha:\,(\alpha,\lambda)>0}(\dots)$, and use the fact that $w(\prod_{\alpha:\,(\alpha,\lambda)>0}(1-qe^{-\alpha})) = \prod_{\alpha:\,(\alpha,\lambda)>0}(1-qe^{-\alpha})$ for any $w \in W_\lambda$.)

This is already sufficient to conclude that $\overline{P}_\lambda(q)$ belongs to $\mathbf{\Lambda}[q]$. Further easy calculations that require a splitting $W \simeq W^\lambda \times W_\lambda$ are left to the reader. □

Remark 5.2. Our proof is inspired by the remark in [6, p. 70, last paragraph], where R. Gupta refers to Macdonald's argument for the Hall-Littlewood symmetric functions.

Remark 5.3. The Hall-Littlewood polynomials $P_\lambda(q)$ interpolate between the irreducible characters χ_λ (if $q = 0$) and orbital sums $\frac{1}{\#(W_\lambda)}\sum_{w\in W} e^{w\lambda}$ (if $q = 1$). For the short Hall-Littlewood polynomials $\overline{P}_\lambda(q)$, we still have $\overline{P}_\lambda(0) = \chi_\lambda$. At $q = 1$, we obtain a linear combination of irreducible characters for H. Namely, if $\chi_\mu^{(H)}$ denote the character of $V_\mu^{(H)}$, $\mu \in \mathfrak{X}_{+,H}$, then

$$\overline{P}_\lambda(1) = \frac{1}{\#(W(\Pi_s)_\lambda)} \sum_{w\in W(\Pi_s)} \chi_{w\lambda}^{(H)}.$$

An easy proof uses the semi-direct product structure of W (Lemma 4.5) and Weyl's character formula for H. (Note that if $\lambda \in \mathfrak{X}_+$, then $w\lambda \in \mathfrak{X}_{+,H}$ for any $w \in W(\Pi_s)$.)

Theorem 5.4. *In $\mathbf{\Lambda}[[q]]$, the following relations hold:*

(1) $\langle \overline{E}_\mu(q), \overline{P}_\lambda(q)\rangle = \delta_{\lambda,\mu}$;

(2) $\overline{E}_\mu(q) = \dfrac{t_\mu^{(\Pi_s)}(q)}{\prod_{\alpha\in\Delta_s}(1-qe^\alpha)}\overline{P}_\mu(q)$ *and* $\overline{E}_0(q) = \dfrac{t_0^{(\Pi_s)}(q)}{\prod_{\alpha\in\Delta_s}(1-qe^\alpha)}$.

Proof. (1) We mimic Gupta's proof of [6, Theorem 2.5]. The plan is as follows:

(i) If χ_π occurs in $\overline{E}_\mu(q) = j(e^\mu{\cdot}\tilde{\mathbf{\Delta}}_q^{(s)})$, then $\pi \succcurlyeq \mu$; and the coefficient of χ_μ equals 1;

(ii) If χ_π occurs in $j(e^\lambda{\cdot}\mathbf{\Delta}_q^{(s)})$, then $\pi \preccurlyeq \lambda$; and the coefficient of χ_λ equals $t_\lambda^{(\Pi_s)}(q)$;

(iii) Put $c_{\lambda,\mu} = \langle j(e^\lambda{\cdot}\mathbf{\Delta}_q^{(s)}), j(e^\mu{\cdot}\tilde{\mathbf{\Delta}}_q^{(s)})\rangle$. Then $c_{\lambda,\mu} = c_{\mu,\lambda}$ and hence $t_\mu^{(\Pi_s)}(q){\cdot}\langle \overline{E}_\lambda(q), \overline{P}_\mu(q)\rangle = t_\lambda^{(\Pi_s)}(q){\cdot}\langle \overline{E}_\mu(q), \overline{P}_\lambda(q)\rangle$.

It will then follow that $c_{\lambda,\mu} = \delta_{\lambda,\mu}{\cdot}t_\lambda^{(\Pi_s)}(q)$ proving the assertion.

For (i): By Weyl's character formula, the coefficient of χ_π in $j(e^\mu{\cdot}\tilde{\mathbf{\Delta}}_q^{(s)})$ equals the coefficient of $e^{\pi+\rho}$ in (the expansion of)

$$J(e^\rho)\overline{E}_\mu(q) = \sum_{w\in W} \varepsilon(w)w\left(\frac{e^{\mu+\rho}}{\prod_{\alpha\in\Delta_s^+}(1-qe^\alpha)}\right).$$

This coefficient equals $\sum_{w,B}\varepsilon(w)q^{\#B}$, where the summation is over $w \in W$ and multi-sets B of Δ_s^+ such that $\pi+\rho = w(\mu+\rho+|B|)$. Then $\pi+\rho \succcurlyeq w^{-1}(\pi+\rho) = \mu+\rho+|B| \succcurlyeq \mu+\rho$. Hence $\pi \succcurlyeq \mu$. If $\pi = \mu$, then the only possibility is $w = 1$ and $B = \varnothing$.

For (ii): Now, we are interested in the coefficient of $e^{\pi+\rho}$ in

$$\sum_{w\in W}\varepsilon(w)w\big(e^{\lambda+\rho}\prod_{\alpha\in\Delta_s^+}(1-qe^{-\alpha})\big)$$

It is equal to $\sum_{w,A}\varepsilon(w)(-q)^{\#A}$, where the summation is over $w \in W$ and subsets $A \subset \Delta_s^+$ such that $\pi+\rho = w(\lambda+\rho-|A|)$. Since $w\lambda \preccurlyeq \lambda$ and $w(\rho-|A|) \preccurlyeq \rho$, we obtain $\pi+\rho \preccurlyeq \lambda+\rho$. Moreover, in case of equality we have $w\lambda = \lambda$ and $\rho - w^{-1}\rho = |A|$. This means that $w \in W_\lambda$

and $\mathsf{N}(w) = A \subset \Delta_s^+$. By Lemma 4.5(ii), we conclude that $w \in W(\Pi_s)$. Thus, $\#A = \ell(w)$ and the coefficient of $e^{\lambda+\rho}$ equals $\sum_{w\in W(\Pi_s)_\lambda} q^{\ell(w)} = t_\lambda^{(\Pi_s)}(q)$.

For (iii): Set $\overline{\xi} = \dfrac{1}{\prod_{\alpha\in\Delta_s}(1-qe^\alpha)}$. It is a W-invariant element of $\mathbf{\Lambda}[[q]]$ and $\mathbf{\Delta}_q^{(s)}\overline{\xi} = \tilde{\mathbf{\Delta}}_q^{(s)}$. Hence $j(e^\mu{\cdot}\mathbf{\Delta}_q^{(s)})\overline{\xi} = j(e^\mu{\cdot}\tilde{\mathbf{\Delta}}_q^{(s)})$. But $\overline{\xi}$ is also a self-dual character. Thus, we have

$$c_{\lambda,\mu} = \langle j(e^\lambda{\cdot}\mathbf{\Delta}_q^{(s)}), j(e^\mu{\cdot}\mathbf{\Delta}_q^{(s)})\overline{\xi}\rangle = \langle j(e^\lambda{\cdot}\mathbf{\Delta}_q^{(s)})\overline{\xi}, j(e^\mu{\cdot}\mathbf{\Delta}_q^{(s)})\rangle = c_{\mu,\lambda}.$$

(2) The equality $\overline{E}_\mu(q) = t_\mu^{(\Pi_s)}(q)\overline{\xi}{\cdot}\overline{P}_\mu(q)$ is essentially proved in (iii). Taking $\mu = 0$ yields the rest. □

Proposition 5.5. $\overline{E}_\mu(q) = \sum_{\lambda\in\mathfrak{X}_+} \overline{\mathfrak{m}}_\lambda^\mu(q)\chi_\lambda$.

Proof. By definition, $J(e^\rho)\overline{E}_\mu(q) = J\left(\dfrac{e^{\mu+\rho}}{\prod_{\alpha\in\Delta_s^+}(1-qe^\alpha)}\right) = \sum_\nu \overline{\mathcal{P}}_q(\nu)J(e^{\mu+\nu+\rho})$.

The weight $\mu+\nu+\rho$ contributes to the last sum if and only if $\mu+\nu+\rho = w(\lambda+\rho)$ for some $\lambda\in\mathfrak{X}_+$ and $w\in W$. Hence

$$\begin{aligned}\sum_\nu \overline{\mathcal{P}}_q(\nu)J(e^{\mu+\nu+\rho}) &= \sum_{\lambda\in\mathfrak{X}_+}\sum_{w\in W}\overline{\mathcal{P}}_q(w(\lambda+\rho)-(\mu+\rho))J(e^{w(\lambda+\rho)}) \\ &= \sum_{\lambda\in\mathfrak{X}_+}\sum_{w\in W}\varepsilon(w)\overline{\mathcal{P}}_q(w(\lambda+\rho)-(\mu+\rho))J(e^{\lambda+\rho}) = \sum_{\lambda\in\mathfrak{X}_+}\overline{\mathfrak{m}}_\lambda^\mu(q)J(e^{\lambda+\rho}).\end{aligned}$$

□

Part 1(ii) in the proof of Theorem 5.4 shows that $\{\overline{P}_\lambda(q)\}_{\lambda\in\mathfrak{X}_+}$ is a $\mathbb{Z}$-basis in $\mathbf{\Lambda}[q]$. Furthermore, Theorem 5.4(1) and Proposition 5.5 readily imply that

$$\chi_\pi = \sum_{\lambda\in\mathfrak{X}_+}\overline{\mathfrak{m}}_\pi^\lambda(q)\overline{P}_\lambda(q). \tag{5.1}$$

Note that this sum is finite, since $\overline{\mathfrak{m}}_\pi^\lambda(q) = 0$ unless $\lambda \preccurlyeq \pi$. Let us transform the expression for $\overline{P}_\lambda(q)$ given by definition:

$$\begin{aligned}J(e^\rho){\cdot}t_\lambda^{(\Pi_s)}(q){\cdot}\overline{P}_\lambda(q) &= J\big(e^{\lambda+\rho}\prod_{\alpha\in\Delta_s^+}(1-qe^{-\alpha})\big) \\ &= J\left(e^\lambda\frac{\prod_{\alpha\in\Delta_s^+}(1-qe^{-\alpha})}{\prod_{\alpha>0}(1-e^{-\alpha})}\cdot e^\rho\prod_{\alpha>0}(1-e^{-\alpha})\right) = \sum_{w\in W} w\left(e^\lambda\frac{\prod_{\alpha\in\Delta_s^+}(1-qe^{-\alpha})}{\prod_{\alpha>0}(1-e^{-\alpha})}\right)\cdot J(e^\rho).\end{aligned}$$

Hence $\overline{P}_\lambda(q) = \dfrac{1}{t_\lambda^{(\Pi_s)}(q)}\sum_{w\in W} w\left(e^\lambda\dfrac{\prod_{\alpha\in\Delta_s^+}(1-qe^{-\alpha})}{\prod_{\alpha>0}(1-e^{-\alpha})}\right)$, and substituting this in Equation (5.1) we obtain a generalisation of an identity of Kato (cf. [6, Theorem 3.9]):

$$\chi_\pi = \sum_{\lambda\in\mathfrak{X}_+}\overline{\mathfrak{m}}_\pi^\lambda(q)\frac{1}{t_\lambda^{(\Pi_s)}(q)}\sum_{w\in W} w\left(e^\lambda\frac{\prod_{\alpha\in\Delta_s^+}(1-qe^{-\alpha})}{\prod_{\alpha>0}(1-e^{-\alpha})}\right). \tag{5.2}$$

Taking $q=1$, we obtain

$$\chi_\pi=\sum_{\lambda\in\mathfrak X_+}\overline{m}_\pi^\lambda\cdot\frac{1}{\#W(\Pi_s)_\lambda}\cdot\sum_{w\in W}w\left(\frac{e^\lambda}{\prod_{\alpha\in\Delta_l^+}(1-e^{-\alpha})}\right).$$

Taking into account that $W=W(\Pi_s)\rtimes W_l$ and $\overline{m}_\pi^\lambda=\overline{m}_\pi^{w\lambda}$ for any $w\in W(\Pi_s)$, this specialisation is equivalent to the formula $\chi_\pi=\sum_{\lambda\in\mathfrak X_{+,H}}\overline{m}_\pi^\lambda\chi_\lambda^{(H)}$.

We introduce another bilinear form in $\mathbf{\Lambda}[q]$ such that $\{\overline{P}_\lambda(q)\}$ to be an orthogonal basis. To this end, the null-cone in $V_{\bar\theta}$ plays the same role as the nilpotent cone $\mathfrak N\subset\mathfrak g$ for the Hall-Littlewood polynomials $P_\lambda(q)$, cf. [7, § 2].

For a graded G-module $\mathcal M=\oplus_i\mathcal M_i$ with $\dim\mathcal M_i<\infty$, the graded character of $\mathcal M$, $\mathsf{ch}_q(\mathcal M)$, is the formal sum $\sum_i\mathsf{ch}(\mathcal M_i)q^i\in\mathbf{\Lambda}[[q]]$.

Proposition 5.6. *The graded character of the graded G-algebra $\mathbb C[\mathfrak N(V_{\bar\theta})]$ equals*

$$\mathsf{ch}_q(\mathbb C[\mathfrak N(V_{\bar\theta})])=\frac{t_0^{(\Pi_s)}(q)}{\prod_{\alpha\in\Delta_s}(1-qe^\alpha)}=t_0^{(\Pi_s)}(q)\cdot\bar\xi=\overline{E}_0(q).$$

Proof. The weight structure of $V_{\bar\theta}$ (Lemma 4.1) shows that the graded character of $\mathbb C[V_{\bar\theta}]$ equals $\mathsf{ch}_q(\mathbb C[V_{\bar\theta}])=\dfrac{1}{(1-q)^{\#\Pi_s}\prod_{\alpha\in\Delta_s}(1-qe^\alpha)}$. We know that $\mathfrak N(V_{\bar\theta})$ is a complete intersection of codimension $m:=\#\Pi_s$ and the ideal of $\mathfrak N(V_{\bar\theta})$ is generated by algebraically independent generators of $\mathbb C[V_{\bar\theta}]^G$. Furthermore, if $d_1,\dots,d_m$ are the degrees of these generators, then $d_1-1,\dots,d_m-1$ are the exponents of $W(\Pi_s)$ (Prop. 4.6). Thus,

$$\mathsf{ch}_q(\mathbb C[\mathfrak N(V_{\bar\theta})])=\frac{\prod_{i=1}^m(1-q^{d_i})}{(1-q)^m\prod_{\alpha\in\Delta_s}(1-qe^\alpha)}=\frac{\prod_{i=1}^m(1+q+\dots+q^{d_i-1})}{\prod_{\alpha\in\Delta_s}(1-qe^\alpha)},$$

and it is well known that $t_0^{(\Pi_s)}(q)=\prod_{i=1}^m(1+q+\dots+q^{d_i-1})$. □

Combining Propositions 5.5 and 5.6 yields

$$\mathsf{ch}_q(\mathbb C[\mathfrak N(V_{\bar\theta})])=\sum_{\lambda\in\mathfrak X_+}\overline{\mathfrak m}_\lambda^0(q)\chi_\lambda,$$

which is [24, Theorem 4]. In other words, $\sum_{i\geqslant0}\dim\bigl(\mathrm{Hom}_G(V_\lambda,\mathbb C[\mathfrak N(V_{\bar\theta})]_i)\bigr)q^i=\overline{\mathfrak m}_\lambda^0(q)$ for every $\lambda\in\mathfrak X_+$.

Define a new bilinear form in $\mathbf{\Lambda}[q]$ by letting

$$\langle\!\langle\chi_\lambda,\chi_\mu\rangle\!\rangle=\langle\chi_\lambda\chi_\mu^*,\ t_0^{(\Pi_s)}(q)\cdot\bar\xi\rangle=\langle\chi_\lambda,\ t_0^{(\Pi_s)}(q)\cdot\bar\xi\chi_\mu\rangle.$$

In view of Proposition 5.6, $\langle\!\langle\chi_\lambda,\chi_\mu\rangle\!\rangle$ is a polynomial in q that counts graded occurrences of the G-module $V_\lambda\otimes V_\mu^*$ in $\mathbb C[\mathfrak N(V_{\bar\theta})]$.

Theorem 5.7. $\langle\!\langle\overline{P}_\lambda(q),\overline{P}_\mu(q)\rangle\!\rangle=\dfrac{t_0^{(\Pi_s)}(q)}{t_\mu^{(\Pi_s)}(q)}\delta_{\lambda,\mu}$.

Proof. By definition and Theorem 5.4, we have

$$\langle\!\langle \overline{P}_\lambda(q), \overline{P}_\mu(q)\rangle\!\rangle = \langle \overline{P}_\lambda(q), t_0^{(\Pi_s)}(q)\cdot\overline{\xi}\cdot\overline{P}_\mu(q)\rangle = \langle \overline{P}_\lambda(q), \frac{t_0^{(\Pi_s)}(q)}{t_\mu^{(\Pi_s)}(q)}\overline{E}_\mu(q)\rangle = \frac{t_0^{(\Pi_s)}(q)}{t_\mu^{(\Pi_s)}(q)}\delta_{\lambda,\mu}.$$

Here we also use the fact that $\mathbf{\Delta}_q^{(s)}\overline{\xi} = \tilde{\mathbf{\Delta}}_q^{(s)}$ and hence $t_\mu^{(\Pi_s)}(q)\cdot\overline{\xi}\cdot\overline{P}_\mu(q) = \overline{E}_\mu(q)$. □

Finally, using Eq. (5.1), we obtain

$$\langle\!\langle \chi_\lambda, \chi_\mu\rangle\!\rangle = \sum_{\pi\in\mathfrak{X}_+} \overline{\mathfrak{m}}_\lambda^\pi(q)\,\overline{\mathfrak{m}}_\mu^\pi(q)\,\frac{t_0^{(\Pi_s)}(q)}{t_\pi^{(\Pi_s)}(q)}.$$

6. Miscellaneous remarks

6.1. It is noticed in [6, 5.1] that Lusztig's q-analogues $\mathfrak{m}_\lambda^\mu(q)$ satisfy the identity

$$\sum_{\mu\in\mathfrak{X}}\mathfrak{m}_\lambda^\mu(q)e^\mu = \frac{J(e^{\lambda+\rho})}{e^\rho\prod_{\alpha>0}(1-qe^{-\alpha})} = \chi_\lambda\cdot\prod_{\alpha>0}\frac{(1-e^{-\alpha})}{(1-qe^{-\alpha})}. \tag{6.1}$$

This can be regarded as quantisation of the equality $\chi_\lambda = \sum_\mu m_\lambda^\mu e^\mu$, which describes V_λ as T-module. In the context of short q-analogues, we wish to have a quantisation of the equality $\chi_\lambda = \sum_{\mu\in\mathfrak{X}_{+,H}}\overline{m}_\lambda^\mu\chi_\mu^{(H)}$, which describes V_λ as H-module [8, § 3]. The desired quantisation is

Proposition 6.1. $\displaystyle\sum_{\mu\in\mathfrak{X}_{+,H}}\overline{\mathfrak{m}}_\lambda^\mu(q)\chi_\mu^{(H)} = \chi_\lambda\cdot\frac{1}{\#W_l}\sum_{w\in W_l} w\left(\prod_{\alpha\in\Delta_s^+}\frac{1-e^{-\alpha}}{1-qe^{-\alpha}}\right).$

Proof. Using Weyl's formula, the function $(\mu\in\mathfrak{X}_{+,H})\mapsto\chi_\mu^{(H)}$ can be extended to the whole of $\mathfrak{X}$ such that it will satisfy the identity $\chi_{w\odot\mu}^{(H)} = \varepsilon(w)\chi_\mu^{(H)}$, $w\in W_l$. Recall that '$\odot$' stands for the shifted action of W_l. Since the same identity holds for $\overline{\mathfrak{m}}_\lambda^\mu(q)$, see Eq. (4.2), the left hand side can be replaced with $\displaystyle\frac{1}{\#W_l}\sum_{\mu\in\mathfrak{X}}\overline{\mathfrak{m}}_\lambda^\mu(q)\chi_\mu^{(H)}$. The rest can by achieved via routine transformations of this sum, using the definition of $\overline{\mathfrak{m}}_\lambda^\mu(q)$ and Weyl's character formulae for H and G. □

Yet another quantisation, which is easier to prove, is

$$\sum_{\mu\in\mathfrak{X}}\overline{\mathfrak{m}}_\lambda^\mu(q)e^\mu = \frac{J(e^{\lambda+\rho})}{e^\rho\prod_{\alpha\in\Delta_s^+}(1-qe^{-\alpha})} = \chi_\lambda\cdot\frac{\prod_{\alpha>0}(1-e^{-\alpha})}{\prod_{\alpha\in\Delta_s^+}(1-qe^{-\alpha})}. \tag{6.2}$$

Comparing Equations (6.1) and (6.2), we obtain a relation between Lusztig's and short q-analogues:

$$\prod_{\alpha\in\Delta_l^+}(1-qe^{-\alpha})\sum_{\mu\in\mathfrak{X}}\mathfrak{m}_\lambda^\mu(q)e^\mu = \sum_{\nu\in\mathfrak{X}}\overline{\mathfrak{m}}_\lambda^\nu(q)e^\nu.$$

Whence $\overline{\mathfrak{m}}_\lambda^\mu(q) = \sum_{A\subset\Delta_l^+} (-q)^{\#A}\mathfrak{m}_\lambda^{\mu+|A|}(q)$. Or, conversely, $\mathfrak{m}_\lambda^\mu(q) = \sum_B q^{\#B}\overline{\mathfrak{m}}_\lambda^{\mu+|B|}(q)$, where B ranges over the finite multisets in Δ_l^+. In particular, taking $q=1$ and $\mu=0$, we obtain

$$\dim V_\lambda^H = \overline{m}_\lambda^0 = \sum_{A\subset\Delta_l^+} (-1)^{\#A} m_\lambda^{|A|}.$$

Example. If $G = Sp_{2n}$, then $H=(SL_2)^n$ and $\Delta_l^+ = \{2\varepsilon_1,\dots,2\varepsilon_n\}$. Here $\varepsilon_{i_1}+\ldots+\varepsilon_{i_k}$ is W-conjugate to $\varphi_k = \varepsilon_1+\ldots+\varepsilon_k$ and the previous relation becomes

$$\dim V_\lambda^H = \sum_{k=0}^n (-1)^k \binom{n}{k} m_\lambda^{2\varphi_k}.$$

6.2. It is well known that, for λ strictly dominant, the Hall-Littlewood polynomials $P_\lambda(q)$ have a nice specialisation at $q=-1$: If $\lambda \gtrdot \rho$, then $P_\lambda(-1) = \chi_{\lambda-\rho}\chi_\rho$. (See [23, 7.4] for a generalisation to symmetrisable Kac-Moody algebras.) For Δ of type $\mathbf{A}_n$, $P_\lambda(-1)$ is a classical Schur's Q-function [18, III.8]. A similar phenomenon occurs for short Hall-Littlewood polynomials.

Proposition 6.2. *Suppose $\lambda \gtrdot \rho_s$ and G is of type $\mathbf{B}_n, \mathbf{C}_n$, or $\mathbf{F}_4$. Then $\overline{P}_\lambda(-1) = \chi_{\lambda-\rho_s}\chi_{\rho_s}$.*

Proof. If $\lambda \gtrdot \rho_s$, then $t_\lambda^{(\Pi_s)}(q) = 1$ and

$$\overline{P}_\lambda(-1) = J\big(e^{\lambda+\rho}\prod_{\lambda\in\Delta_s^+}(1+e^{-\alpha})\big)J(e^\rho)^{-1} =$$
$$\sum_{w\in W}\varepsilon(w)w(e^{\lambda-\rho_s+\rho})(e^{\rho_s}\prod_{\lambda\in\Delta_s^+}(1+e^{-\alpha}))\cdot J(e^\rho)^{-1} = \chi_{\lambda-\rho_s}\cdot\prod_{\lambda\in\Delta_s^+}(e^{\alpha/2}+e^{-\alpha/2}).$$

For G is of type $\mathbf{B}_n, \mathbf{C}_n$, or $\mathbf{F}_4$, it is known that $\chi_{\rho_s} = \prod_{\lambda\in\Delta_s^+}(e^{\alpha/2}+e^{-\alpha/2})$ [20, Theorem 2.9]. □

Remark 6.3. The proof of equality $\chi_{\rho_s} = \prod_{\lambda\in\Delta_s^+}(e^{\alpha/2}+e^{-\alpha/2})$ in [20] is only based on the assumption that $\|\text{long}\|^2/\|\text{short}\|^2 = 2$, i.e., it does not refer to classification. For $\mathbf{G}_2$, the true equality is $\prod_{\lambda\in\Delta_s^+}(e^{\alpha/2}+e^{-\alpha/2}) = \chi_{\rho_s}+1$.

6.3. Ranee Brylinski proved that Lusztig's q-analogues $m_\lambda^\mu(q)$ can be computed via a principal filtration on V_λ^μ whenever $H^i(G\times_B\mathfrak{u}, \mathcal{L}_{G\times_B\mathfrak{u}}(\mathbb{C}_\mu)^\star) = 0$ for all $i \geqslant 1$. Namely, $\mathfrak{m}_\lambda^\mu(q)$ coincides with the "jump polynomial" of the principal filtration, see [5] for details. Another approach to her results can be found in [11].

I hope that a similar description exists for short q-analogues. First, we need a subspace of V_λ whose dimension equals $\overline{m}_\lambda^\mu = \mathsf{mult}(V_\mu^{(H)}, V_\lambda)$. Let $V_\lambda^{U(H)}$ be the subspace of H-highest vectors in V_λ with respect to Δ_l^+. Then $V_\lambda^{U(H),\mu} = V_\lambda^{U(H)} \cap V_\lambda^\mu$ has the required dimension. For $\alpha\in\Delta^+$, let e_α be a nonzero root vector of $\mathfrak{g}$. Brylinski's principal filtration is determined by the principal nilpotent element $e = \sum_{\alpha\in\Pi} e_\alpha$. In the context of short q-analogues, we consider $e_s = \sum_{\alpha\in\Pi_s} e_\alpha$ and the corresponding filtration of $V_\lambda^{U(H),\mu}$. That is, we set

$$J^p_{e_s}(V_\lambda^{U(H),\mu}) = \{v \in V_\lambda^{U(H),\mu} \mid e_s^{p+1}{\cdot}v = 0\}.$$

The jump polynomial is defined to be

$$\bar{r}^\mu_\lambda(q) = \sum_{p\geqslant 0} \dim\bigl(J^p_{e_s}(V_\lambda^{U(H),\mu})/J^{p-1}_{e_s}(V_\lambda^{U(H),\mu})\bigr)q^p.$$

Conjecture 6.4. *If $\mu \in \mathfrak{X}_{+,H}$ satisfies vanishing conditions of Theorem 4.10, then $\bar{r}^\mu_\lambda(q) = \overline{\mathfrak{m}}^\mu_\lambda(q)$.*

6.4. Although the collapsing $f : \boldsymbol{Z} = G \times_B V^+_{\bar\theta} \to \mathfrak{N}(V_{\bar\theta})$ is not generically finite, it can be used for deriving useful properties of the null-cone. Let $\varrho : \mathcal{O}_{\mathfrak{N}(V_{\bar\theta})} \to Rf_*\mathcal{O}_{\boldsymbol{Z}}$ be the corresponding natural morphism. Since f is projective, $H^0(\boldsymbol{Z}, \mathcal{O}_{\boldsymbol{Z}})$ is a finite $\mathbb{C}[\mathfrak{N}(V_{\bar\theta})]$-module; and there is the trace map $H^0(\boldsymbol{Z}, \mathcal{O}_{\boldsymbol{Z}}) \to \mathbb{C}[\mathfrak{N}(V_{\bar\theta})]$ because $\mathfrak{N}(V_{\bar\theta})$ is normal. The trace map determines a morphism (in the derived category of $\mathcal{O}_{\mathfrak{N}(V_{\bar\theta})}$-modules) $\varrho' : Rf_*\mathcal{O}_{\boldsymbol{Z}} \to \mathcal{O}_{\mathfrak{N}(V_{\bar\theta})}$. By Theorem 4.10, $H^i(\boldsymbol{Z}, \mathcal{O}_{\boldsymbol{Z}}) = 0$ for $i \geqslant 1$, i.e., $R^if_*\mathcal{O}_{\boldsymbol{Z}} = 0$ for $i \geqslant 1$. Hence $\varrho' \circ \varrho$ is a quasi-isomorphism of $\mathcal{O}_{\mathfrak{N}(V_{\bar\theta})}$ with itself. Therefore, by [15, Theorem 1], $\mathfrak{N}(V_{\bar\theta})$ has only rational singularities.

Clearly, this argument works in a more general context and yields the following:

Proposition 6.5. *Let N be a P-stable subspace in a G-module V. If $H^i(G \times_P N, \mathcal{O}_{G\times_P N}) = 0$ for all $i \geqslant 1$, then the normalisation of $G{\cdot}N$ has only rational singularities.*

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