SUBGROUP SEPARABILITY IN RESIDUALLY FREE GROUPS

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Abstract. We prove that the finitely presentable subgroups of residually free groups are separable and that the subgroups of type FP_∞ are virtual retracts. We describe a uniform solution to the membership problem for finitely presentable subgroups of residually free groups.

1. Introduction

The importance of residual finiteness and subgroup separability have long been recognized, both in group theory and topology. The importance of residual freeness, on the other hand, has only come into focus recently despite being studied for several decades, cf. [1]. This increased awareness of residually free groups is due to the central role that they play in the resolution of Tarski's problems on the logic of free groups (see [9], [10] *et seq.* and [6], [7] *et seq.*). In that body of work, the central objects of study are the finitely generated *fully residually free* groups, also known as *limit groups*.

Fix a non-abelian free group $\mathbb{F}$. A group G is *residually free* if, given an element $g \in G \smallsetminus 1$, there exists a homomorphism $f : G \to \mathbb{F}$ with $f(g) \neq 1$. And G is *fully residually free* if, for every finite subset $X \subseteq G \smallsetminus 1$, there exists a homomorphism $f : G \to \mathbb{F}$ with $1 \notin f(X)$. Sela [9] and Baumslag–Myasnikov–Remeslennikov [2] show that every finitely generated residually free group is a subgroup of a direct product of finitely many fully residually free groups.

Free groups and free abelian groups are fully residually free. So too are the fundamental groups of compact surfaces of Euler characteristic at most -2. A direct product of residually free groups is residually free but the direct product of a non-abelian group and a non-trivial group is never fully residually free. The reader may wish to keep direct products of free and surface groups in mind as examples of residually free groups.

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The second author [13] established strict constraints on the manner in which finitely generated subgroups can sit inside limit groups (see Theorem 4), while the first author, in joint work with Howie, Miller and Short [4], proved that the finiteness properties of residually free groups govern the manner in which they can sit inside direct products of limit groups (see Theorems 6 and 8). Following some reductions, the results of [4] allow one to treat an arbitrary finitely presented residually free group as if it were a subdirect product of limit groups that contains a term of the lower central series. Thus, provided that one has lemmas to cover the required reductions, one can prove interesting facts about residually free groups by pulling back results about nilpotent groups. In the context that interests us here—the results of [13] concerning subgroup separability and virtual retractions—we shall see that the required reductions pose little problem.

Theorem A. *If G is a finitely generated residually free group and $H \subset G$ is a subgroup of type FP_∞ over $\mathbb{Q}$ then H is a virtual retract of G.*

Theorem B. *If G is a finitely generated residually free group and H is a finitely presentable subgroup of G then H is separable in G.*

Theorem B is new even in the case where G is a direct product of free groups or surface groups.

We remind the reader of the meaning and origin of the terms appearing in these theorems. A subgroup $H \subset G$ is a *virtual retract* if there exists a finite-index subgroup $V \subset G$, containing H, and a homomorphism $\rho : V \to H$ that restricts to the identity on H. Following Long and Reid [8], if every finitely generated subgroup of G is a virtual retract then we say that G has *local retractions* or *property LR*.

A finitely generated group G is *of type* FP_n *over* $\mathbb{Q}$ if $\mathbb{Q}$, viewed as a trivial $\mathbb{Q}G$-module, has a projective resolution in which the first $(n+1)$ resolving modules are finitely generated. The group G is *of type* FP_∞ *over* $\mathbb{Q}$ if it is FP_n over $\mathbb{Q}$ for all n. If G has a compact $K(G,1)$ then it is of type FP_∞, and the converse holds for residually free groups [4].

A subgroup $H \subset G$ is termed *separable* if, for every $g \in G \smallsetminus H$, there exists a finite-index subgroup $K \subset G$ containing H but not g.

Remark 1. *If $f : G' \to G$ is an epimorphism of groups then $H \subset G$ is separable in G if and only if $f^{-1}(H)$ is separable in G'.*

If every finitely generated subgroup of G is separable then G is called *locally extended residually finite*, or *LERF*. An old theorem of Mal'cev shows that finitely generated virtually nilpotent groups are LERF.

Already in the direct product of three free groups one finds finitely subgroups that are not of type FP_∞; the first examples are due to Stallings [12]. Thus Theorem A would not remain valid if we assumed only that H was finitely presented. Likewise, the hypothesis of Theorem B cannot be weakened to allow for finitely generated subgroups, because such subgroups are not separable in general, even in the direct product of two free groups, as the following example shows.

Example 2. *Let $Q = \langle a_1, \dots, a_m | r_1, \dots, r_n \rangle$ be a finite presentation for a non-residually finite group, for example*

$$BS(2,3) = \langle a, b |\ a^{-1}b^2a = b^3 \rangle,$$

and let $q : F \to Q$ be the corresponding surjection from a free group $F = \langle a_1, \dots, a_n \rangle$. The diagonal subgroup Δ is not separable in $Q \times Q$. Thus $H = (q \times q)^{-1}(\Delta)$ is not separable in $F \times F$, by Remark 1. But H is generated by the finite set of elements $\{(a_i, a_i) | 1 \le i \le m\} \cup \{(r_j, 1) | 1 \le j \le n\} \subset F \times F$.

Finally, we would like to point out that although it is far from view in the present article, the two trains of thought that we harness to prove our results ([13] and [4]) both draw crucially on ideas that originate in Stallings' proof [11] of Marshall Hall's Theorem, i.e. Theorem A for free groups.

2. Reduction to subdirect products

The purpose of this section is to reduce Theorems A and B to the special case where G is a direct product of limit groups and $H \subset G$ is a subdirect product. (We remind the reader that a subgroup of a direct product $S \subset G_1 \times \cdots \times G_n$ is termed a *subdirect product* if its projection to each G_i is onto.)

Our results rely crucially on the following result of Sela [9, Claim 7.5] and Baumslag–Myasnikov–Remeslennikov [2, Corollary 19].

Theorem 3. *A finitely generated group G is residually free if and only if it is a subdirect product of finitely generated, fully residually free groups.*

If $H \subset G$ is separable in (respectively, is a virtual retract of) G and $K \subset G$ is any subgroup then $H \cap K$ is separable in (respectively, is a virtual retract of) K. It follows that we need only consider the case when

$$G = \Lambda_1 \times \cdots \times \Lambda_n$$

where each Λ_i is a limit group. Write $\pi_i : G \to \Lambda_i$ for projection onto the ith factor.

In order to apply the results of [4] we must also arrange that the subgroups $H \subset G$ that we consider are subdirect products. This is achieved by using the following lemma to replace G by $\hat{H} := \pi_1(H) \times \cdots \times \pi_n(H)$. The proof of this lemma applies to subdirect products of arbitrary groups with property LR. Its applicability in our situation rests on work of the first author.

Theorem 4 (Theorems A & B of [13])**.** *If G is a finitely generated, fully residually free group and H is a finitely generated subgroup of G then H is separable in G and a virtual retract of G.*

Lemma 5. *A subgroup H is separable in (respectively, is a virtual retract of) G if and only if it is separable in (respectively, is a virtual retract of) $\hat{H}$.*

Proof. It is easy to see that, if H is separable in (respectively, is a virtual retract of) G then H is separable in (respectively, is a virtual retract of) $\hat{H}$. So we will concentrate on the converse assertions.

For each i, $\pi_i(H)$ is a virtual retract of Λ_i, so there exists a finite-index subgroup $V_i \subset \Lambda_i$ and a retraction $\rho_i : V_i \to \pi_i(H)$. The product retraction

$$\rho = \rho_1 \times \cdots \times \rho_n : V_1 \times \cdots \times V_n \to \hat{H}.$$

exhibits $\hat{H}$ as a virtual retract. It follows immediately that if H is a virtual retract of $\hat{H}$ then H is a virtual retract of G. Suppose H is separable in $\hat{H}$ and $g \notin H$. There are two cases to consider. If $g \notin \hat{H}$ then, for some i, $\pi_i(g) \notin \pi_i(H)$. As $\pi_i(H)$ is separable in Λ_i it follows that g can be separated from H. If $g \in \hat{H}$ then, by the hypothesis that H is separable in $\hat{H}$, there exists a finite-index subgroup $\hat{V} \subset \hat{H}$ containing H but not g. Now $\rho^{-1}(\hat{V})$ is a finite-index subgroup of G containing H but not g. □

With this lemma in hand, we shall henceforth assume that $G = \Lambda_1 \times \cdots \times \Lambda_n = \hat{H}$.

The following notation will be useful in what follows. Let $A_i = \Lambda_1 \times \cdots \times \Lambda_i$, let $B_i = \Lambda_{i+1} \times \cdots \times \Lambda_n$, and write

$$p_i : G = A_i \times B_i \to A_i$$

for the projection that kills B_i and

$$q_i : G = A_i \times B_i \to B_i$$

for the projection that kills A_i.

3. Virtual retractions

The reductions of the previous section allow us to appeal to the following theorem of Bridson, Howie, Miller and Short [4].

Theorem 6. *Let $D = \Lambda_1 \times \cdots \times \Lambda_n$ be a direct product of non-abelian limit groups. If a subdirect product $S \subset D$ is of type FP_n over $\mathbb{Q}$ and $\Lambda_i \cap S \neq 1$ for $i = 1, \ldots, n$, then S has finite index in D.*

The following lemma completes the proof of Theorem A.

Lemma 7. *Let $G = \Lambda_1 \times \cdots \times \Lambda_n$ be a product of limit groups and let $H \subset G$ be a subgroup that is of type FP_n over $\mathbb{Q}$. Then H is a virtual retract of G.*

Proof. According to Lemma 5, we may assume that H is a subdirect product. We reorder the factors so that, for some i, $A_{i-1} \cap H$ is trivial and, for all $j \geq i$, $H \cap \Lambda_j \neq 1$. There is no loss of generality in assuming that for at most one $j \geq i$, say $j = i$, Λ_j is abelian. Decompose Λ_j as $P \times Q$ where $H \cap P = 1$ and $H \cap Q$ is of finite index in Q. We relabel the factors again so that P is subsumed into A_{i-1} and Q becomes Λ_i.

We have now arranged that $G = \Lambda_1 \times \cdots \times \Lambda_n$ where, for some i, $A_{i-1} \cap H = 1$, Λ_i is abelian (possibly trivial) with $H \cap \Lambda_i$ of finite index in Λ_i, and B_i is a direct product of non-abelian limit groups that intersect H non-trivially. The projection q_{i-1} is injective on H; let $\iota : q_{i-1}(H) \to G$ be the inverse of $q_{i-1}|_H$.

We claim that the intersection $S := q_{i-1}(H) \cap B_i$ is finitely generated. Indeed $I := q_{i-1}(H) \subset \Lambda_i \times B_i$ is finitely generated and intersects Λ_i in a subgroup of finite index, so $I_0 := (I \cap \Lambda_i) \times S$, being of finite index in I, is also finitely generated, and S is a quotient of I_0.

Now S satisfies the hypotheses of Theorem 6 in the direct product $B_i = \Lambda_{i+1} \times \cdots \times \Lambda_n$, and hence is of finite-index. Therefore $q_{i-1}(H)$ has finite index in $\Lambda_i \times B_i = B_{i-1}$ and the restriction of $\iota \circ q_{i-1}$ defines a retraction $A_{i-1} \times q_{i-1}(H) \to H$. □

4. Separability

The proof that finitely presented subgroups are separable requires a further result from [4].

Theorem 8 (cf. Theorem 4.2 of [4])**.** *Let $D = \Lambda_1 \times \cdots \times \Lambda_n$ be a direct product of limit groups and suppose that $S \subset D$ is a finitely presented subdirect product with $S \cap \Lambda_i \neq 1$ for each i. Then $S \cap \Lambda_i \lhd \Lambda_i$ and $\Lambda_i/(S \cap \Lambda_i)$ is virtually nilpotent for each i.*

That $S \cap \Lambda_i$ is normal in Λ_i follows directly from the fact that S is a subdirect product. In the argument of [4], Theorem 8 appears under some additional hypotheses: that $n \geq 2$; that each Λ_i is non-abelian; and that each Λ_i splits as an HNN-extension over a cyclic subgroup, with stable letter $t_i \in S \cap \Lambda_i$. But in the way we have stated the theorem, each of these hypotheses can be removed. If $n = 1$ then, because S is a subdirect product, $S = \Lambda_i$ so the quotient is trivial. If Λ_i is abelian then $\Lambda_i/(S \cap \Lambda_i)$ is abelian and therefore nilpotent. As observed in the proof of part 5 of Proposition 3.1 of [4], each Λ_i contains a finite-index subgroup Δ_i that decomposes as an HNN-extension of the required form. As $\Delta_i/(S \cap \Delta_i)$ is virtually nilpotent, $\Lambda_i/(S \cap \Lambda_i)$ is also virtually nilpotent.

Corollary 9. *If D and S satisfy the hypotheses of Theorem 8 then S is separable in D.*

Proof. Let $N = (S \cap \Lambda_1) \times \cdots \times (S \cap \Lambda_n)$, a normal subgroup of D. By Theorem 8, D/N is virtually nilpotent. Let $q : D \to D/N$ be the quotient map. As virtually nilpotent groups are subgroup separable, $q(S)$ is separable in D/N. Since $N \subset S$ we have $S = q^{-1}(q(S))$, so the result follows by Remark 1. □

In the light of Lemma 5, the following lemma completes the proof of Theorem B.

Lemma 10. *Let $G = \Lambda_1 \times \cdots \times \Lambda_n$ be a direct product of limit groups and let $H \subset G$ be a finitely presentable subdirect product. Then H is separable in G.*

Proof. Renumbering the factors, as before, we may assume there exists an integer i so that $q_i|_H$ is a monomorphism and $q_i(H) \cap \Lambda_j \neq 1$ for each $j > i$. Let $\iota : q_i(H) \to H$ be the inverse of $q_i|_H$. Let $g \in G \smallsetminus H$. We aim to find a finite-index subgroup of G containing H but not g.

Suppose that $g \notin A_i \times q_i(H)$. As $q_i(H) \subset B_i$ satisfies the hypotheses of Corollary 9, there exists a finite-index subgroup $K \subset B_i$ containing $q_i(H)$ but not $q_i(g)$, so $q_i^{-1}(K)$ separates g from H in G.

Suppose now that $g \in A_i \times q_i(H)$. We first claim that there exists a finite-index subgroup V of $A_i \times q_i(H)$ containing H but not g. To see this, we write H as the graph of ι:

$$H = \{(a, b) \in A_i \times q_i(H) |\ a = \iota(b)\}.$$

Consider the map $f : A_i \times q_i(H) \to A_i \times A_i$ defined by $(a, b) \mapsto (a, \iota(b))$. The pre-image of the diagonal subgroup of $A_i \times A_i$ under f is precisely

H—in particular, $f(g) \notin f(H)$. The claim now follows because diagonal subgroups of direct products of residually finite groups are always separable. For if $f(g) = (\alpha_1, \alpha_2)$ with $\alpha_1 \neq \alpha_2$, then since A_i is residually finite there exists a map ϕ from A_i to a finite group Q such that $\phi(\alpha_1) \neq \phi(\alpha_2)$, so

$$\phi \times \phi : A_i \times A_i \to Q \times Q$$

is a map to a finite group and $(\phi \times \phi) \circ f(g) \notin (\phi \times \phi) \circ f(H)$. If V is the pre-image of the diagonal subgroup of $Q \times Q$ under the map $(\phi \times \phi) \circ f$, then V is a finite-index subgroup of $A_i \times q_i(H)$ containing H but not g.

It now suffices to separate V from g in G. Notice that V is a finitely presented subgroup of G and, for each i, $V \cap \Lambda_i \neq 1$. Furthermore, since H is a subdirect product of G and $H \subset V$, we have that V is also a subdirect product. Therefore V satisfies the hypotheses of Theorem 8, so is separable. □

5. The membership problem

A solution to the membership problem for an arbitrary finitely presentable subgroup of a product of free and surface groups is given in section 5.2 of [5]. The results of [4] allow one to extend this to finitely presented subgroups of residually free groups [3]. However, that solution is not uniform—the algorithm depends on additional information about the subgroup. Here we provide a uniform solution.

Corollary 11. *There is an algorithm that, given a finite presentation for a residually free group $G \cong \langle A \mid R \rangle$, a finite set of words S in the alphabet $A \cup A^{-1}$ and an additional word g, will determine whether or not g lies in the subgroup of G generated by S provided that S is finitely presentable. (If S is not finitely presentable, the algorithm will terminate with the correct conclusion if $g \in \langle S \rangle$ but may fail to terminate if $g \notin \langle S \rangle$.)*

The proof is a standard argument about separable subgroups, which we outline here. The algorithm consists of two processes run in parallel, one of which will reach a conclusion: on the one hand, one enumerates the finite quotients of G and checks to see if the image of g lies in the image of $\langle S \rangle$, stopping if it does not and declaring that $g \notin \langle S \rangle$; on the other hand, working in the free group $F(A)$, one proceeds along an enumeration u_n of the words in the free monoid on $S \cup S^{-1}$ and a naïve enumeration of the products p_m in $F(A)$ of conjugates of relators $r \in R$, checking to see if each $g^{-1}u_n$ is freely equal to p_m. (One proceeds diagonally through the enumerations u_n and p_m.)

A limitation of the utility of this algorithm comes from the observation that in general there is no way of knowing if $\langle S \rangle$ is finitely presentable. This is the case when G is the direct product of two free groups, for example.

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